Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 3, No. 1, pp. 193 − 212, January 2007

RADIANT AND STAR-SHAPED FUNCTIONS

A.M. Rubinov and A.P. Shveidel

Abstract: We study some properties of functions with star-shaped (and, in particular, radiant) and closed epigraph. We show that these functions can be characterized as abstract convex with respect to a certain class of min-type functions and give conditions that guarantee the non-emptiness of the subdifferential with respect to this class.

Key words: *radiant set, star-shaped set, min-type function, abstract convex function, subdifferential, lower Dini derivative*

Mathematics Subject Classification: *26B25, 52A01, 52A30*

1 Introduction

In this paper we study functions whose epigraph is a star-shaped set.

Recall that a set $U \subset \mathbb{R}^n$ is called star-shaped if the set

$$\operatorname{kern} U = \{\bar{u} : u + \alpha(\bar{u} - u) \in U \quad \text{for all} \quad u \in U, \alpha \in [0,1]\} \tag{1.1}$$

is not empty. A star-shaped set is a natural generalization of a convex set (a set U is convex if and only if $U = \operatorname{kern} U$), so a function with a star-shaped epigraph can be considered as a natural generalization of a function with a convex epigraph, that is, a convex function.

A star-shaped set U is called radiant if $0 \in \operatorname{kern} U$. Clearly each star-shaped set is a shift of a radiant set. It follows from this that the examination of star-shaped functions can be reduced to the examination of functions whose epigraph is radiant. Such functions are called radiant. Analytically, a function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is radiant if $f(\lambda x) \le \lambda f(x)$ for all $x \in \mathbb{R}^n$ and $\lambda \in [0,1]$.

The first part of the paper contains a description of radiant functions in terms of their lower Dini derivatives $f_D^{\downarrow}(x,x)$ at a point x in the direction x: in particular, given a finite function f whose restrictions to each ray are continuous, then f is radiant if and only if $f_D^{\downarrow}(x,x) \ge f(x)$ for all x. A radiant function f is called strictly radiant at a point x_0 if $\mu' x_0 \in \operatorname{dom} f$ for some $\mu' > 1$ and $f_D^{\downarrow}(x_0,x_0) > f(x_0)$. We show that a locally Lipschitz radiant function is strictly radiant at a point x_0 if and only if

$$(x_0, f(x_0)) \notin \Gamma((x_0, f(x_0)), \operatorname{epi} f), \tag{1.2}$$

where $\Gamma((x_0, f(x_0)), \operatorname{epi} f)$ is the Bouligand tangent cone to the set $\operatorname{epi} f$ at the point $(x_0, f(x_0))$. Using differential properties of radiant and star-shaped functions we show that the class of star-shaped functions is very broad: for example, a Lipschitz function defined on a star-shaped compact set U is the restriction to U of a star-shaped function $f : \mathbb{R}^n \to \mathbb{R}$.

The second part of the paper contains the description of radiant functions in terms of abstract convexity.

One of the main result of convex analysis states that a lower semicontinuous function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is convex if and only if this function can be represented as the upper envelope of a set of affine functions. The presentation of functions as the upper envelope of a set of not necessarily affine functions is examined in abstract convex analysis (see [3, 4, 6] for details). Many functions $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ can be represented as the upper envelope

$$f(x) = \sup_{h \in U} h(x) \tag{1.3}$$

of a set U of the so-called min-type functions that is functions h of the form

$$h(x) = \min_{i=1}^{p} \{\langle a_i, x\rangle - c_i\}, \qquad a_i \in \mathbb{R}^n, \ \ c_i \in \mathbb{R}. \tag{1.4}$$

The choice of a set $\mathcal{H}$ of min-type functions (1.4) in (1.3) (mainly, the choice of constants c_i in (1.4)) depends on the properties of f. We describe functions (1.4) that can be used for the supremal representation (1.3) of radiant and star-shaped functions.

Another question from abstract convexity that we study in this paper is a description of conditions that guarantee the non-emptiness of the $\mathcal{L}$-subdifferential $\partial_{\mathcal{L}} f(x_0)$ of a radiant function f at a point x_0. Here $\mathcal{L}$ is a set of positively homogeneous min-type functions (i.e. functions (1.4) with $c_i = 0$) corresponding to the set $\mathcal{H}$ which supremally generates radiant functions. In other words we are interested in cases where there exists a positively homogeneous min-type function l such that $f(x) \geq l(x) - l(x_0) + f(x_0)$ for all $x \in \mathbb{R}^n$. It turns out that the required conditions can be expressed in terms of the Bouligand cone to the epigraph $\operatorname{epi} f$ at $(x_0, f(x_0))$. In particular for locally Lipschitz functions these conditions follows from (1.2), so for strictly radiant functions the $\mathcal{L}$-subdifferential is not empty.

We use the following notation:

$\mathbb{R}$ is the real line, $\quad \mathbb{R}_{+\infty} = \mathbb{R} \cup \{+\infty\}$;

$\mathbb{R}^n$ is the n-dimensional Euclidean space with the inner product $\langle \cdot, \cdot \rangle$.

$\|x\| = \sqrt{\langle x, x\rangle}$ is the norm of an element x.

For $x \in \mathbb{R}^n$, $x \neq 0$ we will use the notation:

$$[0, x] = \{\nu x : 0 \leq \nu \leq 1\}, \quad (0, x] = \{\nu x : 0 < \nu \leq 1\}, \quad (0, x) = \{\nu x : 0 < \nu < 1\};$$

$\operatorname{dom} f = \{x : |f(x)| < +\infty\}$ is the domain of a function $f : X \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$;

$\operatorname{epi} f = \{(x, \mu) \in \mathbb{R}^n \times \mathbb{R} : \mu \geq f(x)\}$ is the epigraph of a function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$.

2 Radiant Functions

Let X be a vector space and $f : X \to \mathbb{R}_{+\infty}$ be a function whose epigraph is star-shaped. Let $(x, \gamma) \in \operatorname{kern} \operatorname{epi} f$. Then $\lambda(y, f(y)) + (1 - \lambda)(x, \gamma) \in \operatorname{epi} f$ for all $y \in \operatorname{dom} f$ and $\lambda \in [0, 1]$. This means that

$$\lambda f(y) + (1 - \lambda)\gamma \geq f(\lambda y + (1 - \lambda)x), \qquad y \in \operatorname{dom} f, \ \lambda \in [0, 1]. \tag{2.1}$$

It easy to check that (2.1) implies $(x, \gamma) \in \operatorname{kern} \operatorname{epi} f$. Hence the function f has a star-shaped epigraph if and only if there exists a point $x \in \operatorname{dom} f$ and a number $\gamma \geq f(x)$ such that (2.1) holds. Making, if it is necessary, the change of variables $y \mapsto y - x$ where x is the point from (2.1), we will consider functions f such that $(0, \gamma) \in \operatorname{kern} f$. Then (2.1) has the form

$$\lambda f(y) + (1 - \lambda)\gamma \geq f(\lambda y), \qquad y \in \operatorname{dom} f,\ \lambda \in [0, 1]. \tag{2.2}$$

Assume for the sake of simplicity that γ is also equal to zero. Then $0 \in \operatorname{dom} f$ and (2.2) can be rewritten as

$$f(\lambda y) \leq \lambda f(y), \qquad y \in \operatorname{dom} f, \quad \lambda \in [0, 1]. \tag{2.3}$$

Definition 2.1 *Let X be a vector space. A function $f : X \to \mathbb{R}_{+\infty}$ is called radiant if $0 \in \operatorname{dom} f$ and (2.3) holds.*

Remark 2.1 A function $f : X \to \mathbb{R} \cup \{-\infty\}$ is called co-radiant if $0 \in \operatorname{dom} f$ and $f(\lambda x) \geq \lambda f(x)$ for all $x \in \operatorname{dom} f$ and $\lambda \in [0, 1]$. Clearly f is co-radiant if and only if $-f$ is radiant. So all results that are valid for radiant functions can be naturally reformulated for co-radiant functions.

It follows from the above that f is radiant if and only if the epigraph $\operatorname{epi} f$ is a radiant set. It is easy to check that $f : X \to \mathbb{R}_{+\infty}$ with $0 \in \operatorname{dom} f$ is radiant if and only if $f(\nu x) \geq \nu f(x)$ for all $x \in \mathbb{R}^n$ and $\nu \geq 1$. Indeed, let $\nu \geq 1$, $\lambda = 1/\nu$ and $y = \nu x$. Then $f(x) \leq \lambda f(y)$ which is equivalent to $f(\nu x) \geq \nu f(x)$.

If f is radiant then

1) the set $\operatorname{dom} f$ is radiant;

2) $0 \in \operatorname{dom} f$ and $f(0) \leq 0$. Indeed, the latter follows from $|f(0)| < +\infty$ and $f(0) = f\left(\frac{1}{2} \cdot 0\right) \leq \frac{1}{2} f(0)$.

We need the following definition. If $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ and $U \subset \operatorname{dom} f$ then the function $f + \delta_U$ is called the restriction of f to U. Here

$$\delta_U(x) = \begin{cases} 0 & x \in U \\ +\infty & x \notin U \end{cases}$$

is the indicator function of the set U.

It is easy to check that the restriction of a radiant function f to a radiant subset of $\operatorname{dom} f$ is a radiant function.

Denote by R the class of radiant functions defined on $\mathbb{R}^n$ and mapping into $\mathbb{R}_{+\infty}$. This class is very broad. We indicate some properties of R.

1) If T is an arbitrary family of indices and $f_t \in R$ for all $t \in T$ then the function $t \mapsto \sup_{t \in T} f_t(x)$ belongs to R; if $\inf_{t \in T} f_t(x) > -\infty$ for all $x \in \mathbb{R}^n$ then the function $t \mapsto \inf_{t \in T} f_t(x)$ also belongs to R;

2) if $f_1, f_2 \in R$ and $\lambda_1, \lambda_2 > 0$ then $\lambda_1 f_1 + \lambda_2 f_2 \in R$.

3) R is closed under pointwise convergence.

4) Let $f \in R$ and $g(x) = f(x) - \gamma$ $(x \in \mathbb{R}^n)$, where $\gamma > 0$. Then $g \in R$. Indeed, $g(\lambda x) = f(\lambda x) - \gamma \leq \lambda f(x) - \gamma \leq \lambda(f(x) - \gamma) = \lambda g(x)$ for all $x \in \operatorname{dom} f$ and $\lambda \in (0, 1)$.

A function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is called convex-along-rays if the function of one variable $t \mapsto f(tx)$, $t \in [0, +\infty)$ is convex for each x. The class R contains all convex-along-rays functions $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ such that $f(0) \le 0$. Indeed, if $x \in X, \lambda \in (0,1)$ then

$$f(\lambda x) = f(\lambda x + (1-\lambda)0) \le \lambda f(x) + (1-\lambda) f(0) \le \lambda f(x).$$

In particular each convex function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ with $f(0) \le 0$ is radiant. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a positively homogeneous function of degree $2k$, $k = 1, \dots$ with $0 \in \operatorname{dom} f$. Then $f(tx) = t^{2k} f(x)$ and the function $t \mapsto f(tx)\,, t \in [0, +\infty)$ is convex if and only if $f(x) \ge 0$. Thus if f is nonnegative then f is convex-along-rays; since also $f(0) = 0$, it follows that f is a radiant function. Let $k_1, \dots, k_n$ be nonnegative integers. Consider the monomial

$$f_{k_1,\dots,k_n}(x) = x_1^{k_1} \cdots x_n^{k_n} \qquad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

We have $f_{k_1,\dots,k_n}(tx) = t^{k_1+\dots+k_n} f_{k_1,\dots,k_n}(x)$. If each k_i is an even number then $f_{k_1,\dots,k_n}$ is a convex-along-rays, hence radiant, function. It follows from this that a series

$$f(x) = a_0 + \sum_{i=1}^{n} a_i x_i + \sum_{k_1,\dots,k_n} a_{k_1,\dots,k_n} f_{k_1,\dots,k_n}(x)$$

with even k_i, $a_0 \le 0$ and $a_{k_1,\dots,k_n} \ge 0$ is a radiant function.

We now give three more examples of radiant functions.

Example 2.1 Let $X = \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ be an upper semicontinuous function which possesses the following property: there exists a quadratic function q such that $f(x) \le q(x)$. Then (see [4]) there exists a family $(f_t)_{t \in T}$ of convex functions such that $f(x) = \inf_t f_t(x)$. If $f_t(0) \le 0$ for each function f_t from this family then f is radiant.

Example 2.2 Let $A \subset \mathbb{R}^n$ be a radiant set. Then the distance function

$$d_A(x) = \inf\{\|x - y\| : y \in A\}, \qquad (x \in \mathbb{R}^n)$$

to the set A is radiant. Indeed, let $\nu \ge 1$. Then

$$\begin{aligned} d_A(\nu x) &= \inf\{\|\nu x - y\| : y \in A\} \\ &= \nu \inf\{\|x - \frac{y}{\nu}\| : y \in A\} \\ &\ge \nu \inf\{\|x - y\| : y \in A\} = \nu d_A(x). \end{aligned}$$

If d_A is a radiant function and A is closed then A is a radiant set, since $A = \{x : d_A(x) = 0\}$.

Example 2.3 A set $\Omega \subset \mathbb{R}^n$ is radiant if and only if δ_Ω is a radiant function.

3 Differential Properties of Radiant Functions

In this section we describe differential properties of radiant functions. Let $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ with $0 \in \operatorname{dom} f$ and let $x \in \operatorname{dom} f$. Let $u \in \mathbb{R}^n$. We will use the lower Dini derivative at x in the direction $u \in \mathbb{R}^n$:

$$f_D^{\downarrow}(x,u) = \liminf_{\alpha \to +0} \frac{f(x + \alpha u) - f(x)}{\alpha}$$

and the lower Hadamard derivative at x in the direction u:

$$f_H^{\downarrow}(x,u) = \liminf_{\alpha \to +0, u' \to u} \frac{f(x + \alpha u') - f(x)}{\alpha}.$$

In particular, $f_D^{\downarrow}(x,0) = f_H^{\downarrow}(x,0) = 0$.

Definition 3.1 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$, $x \in \mathbb{R}^n$, $x \neq 0$. We say that f is radiant on the segment $[0, x]$ if there exists $\mu' > 1$ such that*

$$[0, \mu' x) \subset \operatorname{dom} f,\ f(\lambda y) \leq \lambda f(y) \text{ for all } \lambda \in [0,1] \text{ and } y = \mu x \text{ with } \mu \in [0, \mu').$$

It is obvious that if there exists a segment on which a function f is radiant, than $f(0) \leq 0$. It is also clear that if f is a radiant function, $x \neq 0$, $\mu' > 1$, $\mu' x \in \operatorname{dom} f$, then f is radiant on the segment $[0, x]$.

Definition 3.2 *We say that $U \subset \mathbb{R}^n$ is open-along-rays if the intersection $U \cap \mathbb{R}_x$ is open in $\mathbb{R}_x$ for each $x \in \mathbb{R}^n$, $x \neq 0$.*

Proposition 3.1 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$, U be an open-along-rays radiant subset of* $\operatorname{dom} f$. *Then $f + \delta_U$ is radiant if and only if f is radiant on $[0, x]$ for all $x \in U$, $x \neq 0$.*

Proof. Necessity. Let $f + \delta_U$ be radiant, $x \in U$, $x \neq 0$. Since U is open-along-rays, there exists $\mu' > 1$ such that $\mu' x \in U$. Since U is radiant, we have $[0, \mu' x) \subset U \subset \operatorname{dom} f$. Let $\mu \in [0, \mu')$, $y = \mu x$, $\lambda \in [0, 1]$. Then y, $\lambda y \in U = \operatorname{dom}(f + \delta_U)$ and we have

$$f(\lambda y) = (f + \delta_U)(\lambda y) \leq \lambda (f + \delta_U)(y) = \lambda f(y).$$

Hence, according to Definition 3.1, f is radiant on $[0, x]$.
Sufficiency. Let f be radiant on $[0, x]$ for all $x \in U$, $x \neq 0$. Let $x \in \operatorname{dom}(f + \delta_U)$, $\lambda \in [0, 1]$. Since $\operatorname{dom}(f + \delta_U) = U$, there exists $\mu' > 1$ such that $[0, \mu' x) \subset \operatorname{dom} f$ and $f(\lambda y) \leq \lambda f(y)$ for $\lambda \in [0, 1]$ and $y = \mu x$ with $\mu \in [0, \mu')$. Then for $\mu = 1$ we have $f(\lambda x) \leq \lambda f(x)$. Since U is radiant, 0 and λx belong to U. Therefore $(f + \delta_U)(0) = f(0) \leq 0 < +\infty$,

$$(f + \delta_U)(\lambda x) = f(\lambda x) \leq \lambda f(x) = \lambda (f + \delta_U)(x).$$

Hence $f + \delta_U$ is a radiant function. □

Proposition 3.2 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$.*

1) *If $x \in \mathbb{R}^n$, $x \neq 0$ and f is radiant on $[0, x]$, then there exists $\mu' > 1$ such that $f_D^{\downarrow}(y, y) \geq f(y)$ for each $y \in [0, \mu' x)$;*
2) *Let $x \neq 0$, $\mu' > 1$, $[0, \mu' x) \subset \operatorname{dom} f$, $f_x(\lambda) = f(\lambda x)$ be continuous on $(0, \mu')$ and $f_D^{\downarrow}(y, y) \geq f(y)$ for each $y \in [0, \mu' x)$. Then f is radiant on $[0, x]$.*

Proof. 1) Let f be radiant on $[0, x]$. Then there exists $\mu' > 1$ such that $[0, \mu' x) \subset \operatorname{dom} f$ and $f(\lambda y) \leq \lambda f(y)$ for for all $y \in [0, \mu' x)$ and for all $\lambda \in [0, 1]$. Since $0 \in \operatorname{dom} f$, we have $f_D^{\downarrow}(0, 0) = 0$. Since $f(0) \leq 0$, we have $f_D^{\downarrow}(y, y) \geq f(y)$ for $y = 0$. Let now $y = \mu x$ with $\mu \in (0, \mu')$. For ε small enough we have $y + \alpha y \in \operatorname{dom} f$ if $\alpha \in (0, \varepsilon)$. Then

$$\frac{f(y + \alpha y) - f(y)}{\alpha} \geq \frac{(1 + \alpha) f(y) - f(y)}{\alpha} = f(y),$$

whence it follows

$$\inf_{0 < \alpha < \varepsilon} \frac{f(y + \alpha y) - f(y)}{\alpha} \geq f(y),$$

and thereby

$$f_D^{\downarrow}(y, y) = \lim_{\varepsilon \downarrow 0} \inf_{0 < \alpha < \varepsilon} \frac{f(y + \alpha y) - f(y)}{\alpha} \geq f(y).$$

2) Let $x \neq 0$, $\mu' > 1$, $[0, \mu' x) \subset \operatorname{dom} f$, $f_x(\lambda) = f(\lambda x)$ be continuous on $(0, \mu')$ and $f^{\downarrow}_D(y, y) \geq f(y)$ for each $y \in [0, \mu' x)$. Since $0 \in \operatorname{dom} f$, we have $f^{\downarrow}_D(0, 0) = 0$. Since $f^{\downarrow}_D(0, 0) \geq f(0)$, we have $f(0) \leq 0$. Hence $f(\lambda \cdot 0) \leq \lambda f(0)$ for all $\lambda \in [0, 1]$. Let $y = \mu x$ with $\mu \in (0, \mu')$. We put

$$\phi_y(\lambda) = \frac{f(\lambda y)}{\lambda} \qquad \left(\lambda \in \left(0, \frac{\mu'}{\mu}\right)\right).$$

Since

$$\begin{aligned}(\phi_y)^{\downarrow}_D(\lambda, 1) &= \lim_{\varepsilon \downarrow 0} \inf_{0<\alpha<\varepsilon} \left(\frac{f(\lambda y + \alpha y) - f(\lambda y)}{(\lambda + \alpha)\alpha} - \frac{f(\lambda y)}{(\lambda + \alpha)\lambda} \right) \\ &= \lim_{\varepsilon \downarrow 0} \inf_{0<\alpha<\varepsilon} \left(\frac{f(\lambda y + \alpha y) - f(\lambda y)}{(\lambda + \alpha)\alpha} \right) - \frac{f(\lambda y)}{\lambda^2} \\ &= \frac{1}{\lambda} \lim_{\varepsilon \downarrow 0} \inf_{0<\alpha<\varepsilon} \left(\frac{f(\lambda y + \alpha y) - f(\lambda y)}{\alpha} \right) - \frac{f(\lambda y)}{\lambda^2} \\ &= \frac{f^{\downarrow}_D(\lambda y, y)\lambda - f(\lambda y)}{\lambda^2} = \frac{f^{\downarrow}_D(\lambda y, \lambda y) - f(\lambda y)}{\lambda^2},\end{aligned}$$

we have $(\phi_y)^{\downarrow}_D(\lambda, 1) \geq 0$ for $\forall \lambda \in (0, \frac{\mu'}{\mu})$. Let $0 < \lambda_0 < 1$. The function $\phi_y(\lambda)$ is continuous on $[\lambda_0, 1]$ and $(\phi_y)^{\downarrow}_D(\lambda, 1) \geq 0$ on $(\lambda_0, 1)$. Hence by Lemma 3.1 from [1] $\phi_y(\lambda_0) \leq \phi_y(1)$, whence it follows that $f(\lambda_0 y) \leq \lambda_0 f(y)$. For $\lambda_0 = 0$ we have $f(0 \cdot y) = f(0) \leq 0 = 0 \cdot f(y)$. Thus for $y = \mu x$ with $\mu \in (0, \mu')$ and $\lambda \in [0, 1]$ we have $f(\lambda y) \leq \lambda f(y)$. Hence f is radiant on $[0, x]$. □

Corollary 3.1 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$, $U \subset \operatorname{dom} f$ be an open-along-rays set.*

1) *If $f + \delta_U$ is a radiant function, then $f^{\downarrow}_D(x, x) \geq f(x)$ for $\forall x \in U$;*

2) *If U is a radiant set, $f^{\downarrow}_D(x, x) \geq f(x)$ and $f_x(\lambda) = f(\lambda x)$ is continuous on $\operatorname{dom} f_x \setminus \{0\}$ for $\forall x \in U$, $x \neq 0$, then $f + \delta_U$ is a radiant function.*

Remark 3.1 Proposition 3.2 can be considered as a generalization of the following result that has been proved in [2]: a function $u : \mathbb{R}^n_+ \to \mathbb{R}_{+\infty}$ is co-radiant if and only if

$$u(x) \geq u^{\downarrow}_D(x, x) \qquad \forall x \in \mathbb{R}^n_+.$$

We use the scheme of the proof from [2].

Let $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be a differentiable function and let $G \subset \operatorname{dom} f$ be a radiant open set. It follows from Proposition 3.2 that the restriction f to G is a radiant function if and only

$$\langle \nabla f(x), x \rangle \geq f(x), \qquad x \in G. \tag{3.1}$$

In view of Euler theorem, the equality

$$\langle \nabla f(x), x \rangle = f(x), \qquad x \in \mathbb{R}^n$$

holds if and only if f is a positively homogeneous of degree one function defined on $\mathbb{R}^n$.

In the sequel we need the notion of the Bouligand cone. Let $U \subset \mathbb{R}^n$ be a closed set and $x \in U$. The Bouligand cone $\Gamma(x, U)$ to U at x consists of all vectors u such that there exist sequences $\alpha_k \to 0+$ and $u_k \to u$ such that $x + \alpha_k u_k \in U$. If U is a cone and $x \in U, x \neq 0$ then $x \in \Gamma(x, U)$. The following statement can be found in [1] (see Proposition 3.1):

Proposition 3.3 *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$, $x \in \operatorname{dom} f$ *and* $y = (x, f(x))$. *Then*

1) $((x,\lambda) \in \Gamma(y, \operatorname{epi} f), \lambda' \geq \lambda) \implies (x, \lambda') \in \Gamma(y, \operatorname{epi} f)$;

2) $f_H^{\downarrow}(x,u) = \inf\{\mu : (x,\mu) \in \Gamma(y, \operatorname{epi} f\}$.

It is interesting to compare Proposition 3.2 with the following statement.

Theorem 3.1 *Let* $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ *and* $x \in \operatorname{dom} f$. *Then*

$$f_H^{\downarrow}(x,x) > f(x) \iff (x, f(x)) \notin \Gamma((x, f(x)), \operatorname{epi} f) \tag{3.2}$$

Proof. The cone $\Gamma((x, f(x)), \operatorname{epi} f)$ is closed. Applying Proposition 3.3 we can easily check that $\Gamma(x, f(x)) = \operatorname{epi}(f_H^{\downarrow}(x, \cdot)$. This implies that

$$(x, f(x)) \notin \Gamma(((x, f(x)), \operatorname{epi} f)) \iff f(x) < f_H^{\downarrow}(x,x).$$

□

Let $x \in \mathbb{R}^n$, $x \neq 0$. If a function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is radiant on $[0, x]$, then there exists $\mu' > 1$ such that $[0, \mu' x) \subset \operatorname{dom} f$ and $f_D^{\downarrow}(y,y) \geq f(y)$ for all $y \in [0, \mu' x)$. In particular $f_D^{\downarrow}(x,x) \geq f(x)$.

Definition 3.3 *Let* $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$, $x \in \mathbb{R}$, $x \neq 0$. *We say that* f *is strictly radiant at the point* x *if there exists* $\mu' > 1$ *such that* $[0, \mu' x) \subset \operatorname{dom} f$ *and* $f_D^{\downarrow}(x,x) > f(x)$.

If $\operatorname{dom} f$ is radiant (it is so, for example, for a radiant function) and $\mu' x \in \operatorname{dom} f$, then $[0, \mu' x) \subset \operatorname{dom} f$.

Proposition 3.4 *Let* $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ *be locally Lipschitz at* $x \in \operatorname{dom} f$, $x \neq 0$. *If* $\operatorname{dom} f$ *is a radiant set, then*

$$(x, f(x)) \notin \Gamma((x, f(x)), \operatorname{epi} f) \iff f \quad \textit{is strictly radiant at the point} \quad x.$$

Proof. Since f is locally Lipschitz at $x \in \operatorname{dom} f$, it is finite on some ball $B(x, r)$ and $f_H^{\downarrow}(x,x) = f_D^{\downarrow}(x,x)$. Since $\operatorname{dom} f$ is radiant we have that $[0, (1 + r/2\|x\|)x) \subset \mathit{dom} f$. Applying Theorem 3.1 we have

$$(x, f(x)) \notin \Gamma((x, f(x)), \operatorname{epi} f) \iff f_D^{\downarrow}(x,x) = f_H^{\downarrow}(x,x) > f(x).$$

□

4 Abstract Convexity of Radiant Functions

Let X be a set and H be a set of functions $h : X \to \mathbb{R}$ defined on X. A function $f : X \to \mathbb{R}_{+\infty}$ is called abstract convex with respect to H if there is a set $U \subset H$ such that $f(x) = \sup\{h(x) : h \in U\}$.

In this section we describe properties of radiant functions in terms of abstract convexity. We show that the set $\mathcal{R}$ of all lower semicontinuous radiant functions can be described as the set of all abstract convex functions with respect to the set $\mathcal{H}_{n+1}$ that consists of min-type functions h of the form $h(x) = \min\limits_{i=1}^{p}\{\langle a_i, x\rangle - c_i\}$ where $p \leq n+1$, $a_i \in \mathbb{R}^n$ and $c_i \geq 0$, $i \in 1 : p$. We start with the examination of some properties of min-type functions.

Lemma 4.1 *Let h and g be functions defined on $\mathbb{R}^n$ by $h(x) = \min_{i=1}^{p}\{\langle a_i, x\rangle - c_i\}$ and $g(x) = \min_{i=1}^{q}\{\langle b_i, x\rangle - d_i\}$, respectively. Then $h = g$ if and only if*

$$\mathrm{co}\{(a_i, -c_i)_{i=1}^{p}\} + \{(0, \mu) : \mu \geq 0\} = \mathrm{co}\{(b_i, -d_i)_{i=1}^{q}\} + \{(0, \mu) : \mu \geq 0\} \qquad (4.1)$$

Proof. Consider the superlinear functions

$$\bar{h}(x, \lambda) = \min_{i=1}^{p}\{\langle a_i, x\rangle - c_i\lambda\}, \qquad \bar{g}(x, \lambda) = \min_{i=1}^{q}\{\langle b_i, x\rangle - d_i\lambda\}, \qquad (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}_+.$$

Then $h(x) = \bar{h}(x, 1)$, $g(x) = \bar{g}(x, 1)$, so

$$h(x) = g(x) \iff \bar{h}(x, \lambda) = \bar{g}(x, \lambda) \quad \text{for all} \quad \lambda > 0 \iff \bar{h}(x, \lambda) = \bar{g}(x, \lambda) \quad \text{for all} \quad \lambda \geq 0.$$

Superdifferentials at zero of these functions, $\bar{\partial}\bar{h}(0)$ and $\bar{\partial}\bar{g}(0)$, respectively, have the form:

$$\bar{\partial}\bar{h}(0) = \mathrm{co}\{(a_i, -c_i)_{i=1}^{p}\}, \qquad \bar{\partial}\bar{g}(0) = \mathrm{co}\{(b_i, -d_i)_{i=1}^{q}\}.$$

Let $K = \{(x, \lambda) : x \in \mathbb{R}^n, \lambda \geq 0\}$ and let δ_K be the indicator function of K. Then the cone K^*, conjugate to K , has the form $K^* = \{(0, \mu) : \mu \geq 0\}$ and the superdifferential $\bar{\partial}(-\delta_K)(0)$ of the superlinear function $-\delta_K$ coincides with K^*. Let $\bar{h}_* = \bar{h} - \delta_K$, $\bar{g}_* = \bar{g} - \delta_K$. Since $\bar{h}_*$ and $\bar{g}_*$ are upper semicontinuous superlinear functions, it follows that $\bar{h}_* = \bar{g}_*$ if and only if $\bar{\partial}\bar{h}_*(0) = \bar{\partial}\bar{g}_*(0)$. We have

$$\bar{\partial}\bar{h}_*(0) = \bar{\partial}\bar{h}(0) + \bar{\partial}(-\delta_K) = \mathrm{co}\{(a_i, -c_i)_{i=1}^{p}\} + \{(0, \mu) : \mu \geq 0\},$$

$$\bar{\partial}\bar{g}_*(0) = \bar{\partial}\bar{g}(0) + \bar{\partial}(-\delta_K) = \mathrm{co}\{(b_i, -d_i)_{i=1}^{q}\} + \{(0, \mu) : \mu \geq 0\}.$$

Thus $h = g$ if and only if (4.1) holds. □

Corollary 4.1 *Consider vectors $(a_i, -c_i) \in \mathbb{R}^n \times \mathbb{R}$ $(i = 1, \dots, m)$. Then*

$$\min_{i=1}^{p}\{\langle a_i, x\rangle - c_i\} = \min_{i=1, i\neq k}^{p}\{\langle a_i, x\rangle - c_i\} \quad \textit{for all} \quad x \in \mathbb{R}^n \qquad (4.2)$$

if and only if

$$(a_k, -c_k) \in \mathrm{co}\{(a_i, -c_i)_{i=1, i\neq k}^{p}\} + \{(0, \mu) : \mu \geq 0\}. \qquad (4.3)$$

Proof. Indeed, the inclusion

$$\mathrm{co}\{(a_i, -c_i)_{i=1}^{p}\} + \{(0, \mu) : \mu \geq 0\} \supset \mathrm{co}\{(a_i, -c_i)_{i=1}^{p}, i \neq k\} + \{(0, \mu) : \mu \geq 0\} \qquad (4.4)$$

is always valid. The equality in (4.4) implies that (4.3) holds. On the other hand it follows from (4.3) that (4.1) is true. In view of Lemma 4.1 we get (4.2). □

Lemma 4.2 *Let $h(x) = \min_{i=1}^{p}\{\langle a_i, x\rangle - c_i\}$, $x \in \mathbb{R}^n$. Let $\tilde{a}_i = (a_i, -1)$, $i \in 1 : p$. Then*

$$\mathrm{epi}\, h = \bigcup_{i=1}^{p}\{z \in \mathbb{R}^{n+1} : \langle \tilde{a}_i, z\rangle \leq c_i\}.$$

Proof. A vector $z = (z_1, \dots, z_n, z_{n+1}) \in \mathbb{R}^{n+1}$ can be presented in the form $z = (z_*, z')$ where $z_* = (z_1, \dots, z_n) \in \mathbb{R}^n$, $z' = z_{n+1} \in \mathbb{R}$. We have

$$\begin{aligned} \operatorname{epi} h &= \{z = (z_*, z') \in \mathbb{R}^{n+1} : z' \geq h(z_*)\} \\ &= \bigcup_{i=1}^{p} \{(z_*, z') \in \mathbb{R}^{n+1} \times \mathbb{R} : z' \geq \langle a_i, z_* \rangle - c_i\} = \bigcup_{i=1}^{p} \{z : \langle \tilde{a}_i, z \rangle \leq c_i\}. \end{aligned}$$

□

Corollary 4.2 *Consider vectors* $(a_i, -c_i) \in \mathbb{R}^n \times \mathbb{R}$ $(i = 1, \dots, p)$. *Then*

$$\min_{i=1}^{p} \{\langle a_i, x \rangle - c_i\} = \min_{i=1, i \neq k}^{p} \{\langle a_i, x \rangle - c_i\} \quad \text{for all} \quad x \in \mathbb{R}^n \tag{4.5}$$

if and only if

$$\bigcup_{i=1}^{p} \{z \in \mathbb{R}^{n+1} : \langle \tilde{a}_i, z \rangle \leq c_i\} = \bigcup_{i=1, i \neq k}^{p} \{z \in \mathbb{R}^{n+1} : \langle \tilde{a}_i, z \rangle \leq c_i\},$$

where $\tilde{a}_i = (a_i, -1)$.

We need the following assertions:

Proposition 4.1 *([5]) Let* $a_i \in \mathbb{R}^n$, $\mu_i \in \mathbb{R}$, $i \in 1 : p$ *and let*

$$A := \bigcup_{i=1}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle \leq \mu_i\} \neq \mathbb{R}^n. \tag{4.6}$$

Assume that $A \neq \bigcup_{i=1, i \neq k}^{p} \{x \in \mathbb{R}^n : \langle a_i, x \rangle \leq \mu_i\}$ *for all* $k \in 1 : p$. *Then* A *is star-shaped if and only if*

$$\tilde{A} = \bigcap_{i=1}^{p} \{x :\in \mathbb{R}^n : \langle a_i, x \rangle > 0\} \neq \emptyset. \tag{4.7}$$

If the set $\tilde{A}$ *is not empty then*

$$\operatorname{kern} A = \bigcap_{i=1}^{p} \{z \in \mathbb{R}^n : \langle a_i, z \rangle \leq \mu_i\}.$$

Proposition 4.2 *([4], Proposition 5.32) Let* $Q \subset \mathbb{R}^n$ *be a solid cone and* $x \in \operatorname{int} Q$. *Then there exist* n *linearly independent vectors* $a_1, \dots, a_n$ *such that the cone* $T = \bigcap_{i=1}^{n} \{x \in \mathbb{R}^n : \langle a_i, x \rangle < 0\}$ *is located into* $\operatorname{int} Q$ *and* $\langle a_i, x \rangle = -1$ *for all* $i = 1, \dots, n$.

Proposition 4.3 *Let* $a_1, \cdots, a_p \in \mathbb{R}^n$, $c_1, \cdots, c_p \in \mathbb{R}$, *and let*

$$h(x) = \min_{i=1}^{p} \{\langle a_i, x \rangle - c_i\} \quad (x \in \mathbb{R}^n). \tag{4.8}$$

If $c_i \geq 0$ *for all* $i \in 1 : p$, *then epi* h *is a radiant set. If*

$$(a_k, -c_k) \notin \operatorname{co}\{(a_i, -c_i)_{i=1, i \neq k}^{p}\} + \{(0, \mu) : \mu \geq 0\} \tag{4.9}$$

for each $k \in 1 : p$ *then epi* h *is radiant if and only if* $c_i \geq 0$ *for all* $i \in 1 : p$.

Proof. Let $c_i \geq 0,\ \lambda \in [0,1]$ and $(x,\mu) \in \operatorname{epi} h$. Then

$$\lambda\mu \geq \lambda \min_{i=1}^{p}\{\langle a_i, x\rangle - c_i\} \geq \min_{i=1}^{p}\{\langle a_i, \lambda x\rangle - c_i\} = h(\lambda x).$$

This means that $(\lambda x, \lambda\mu) \in \operatorname{epi} h$ and therefore $\operatorname{epi} h$ is a radiant set. Assume now that for every $k \in 1:p$ (4.9) holds. Let $\tilde{a}_i = (a_i, -1),\ i \in 1:p$. In view of Corollary 4.2 it holds:

$$\bigcup_{i=1}^{p}\{z \in \mathbb{R}^{n+1} : \langle \tilde{a}_i, z\rangle \leq c_i\} \neq \bigcup_{\substack{i=1 \\ i\neq k}}^{p}\{z \in \mathbb{R}^{n+1} : \langle \tilde{a}_i, x\rangle \leq c_i\}$$

for every $k \in 1:p$. According to Proposition 4.1, the set $\operatorname{epi} h$ is radiant if and only if

$$\bigcap_{i=1}^{p}\{z \in \mathbb{R}^{n+1} : \langle \tilde{a}_i, z\rangle < 0\} \neq \emptyset, \tag{4.10}$$

and

$$(0,0) \in \bigcap_{i=1}^{p}\{z \in \mathbb{R}^{n+1} : \langle \tilde{a}_i, z\rangle \leq c_i\}. \tag{4.11}$$

Since $\langle \tilde{a}_i, (x,\mu)\rangle = \langle a_i, x\rangle - \mu$, it is clear that (4.10) is always valid. At the same time (4.11) holds if and only if $c_i \geq 0$ for all $i \in 1:p$. □

Consider the subset $\mathcal{H}$ of the set $\mathcal{R}$ of all lower semicontinuous radiant functions $f : \mathbb{R}^n \to (-\infty, +\infty]$ that consists of all functions h of the form (4.8) with $c_i \geq 0$. In other words

$$\mathcal{H} = \{h : h(x) = \min_{i=1}^{p}\{\langle a_i, x\rangle - c_i\},\ a_i \in \mathbb{R}^n,\ c_i \geq 0,\ \ i \in 1:p\} \tag{4.12}$$

In view of Proposition 4.3 each function $h \in \mathcal{H}$ is radiant. In order to give an explicit description of $\mathcal{H}$-convex functions we need the following assertion.

Lemma 4.3 *Let $K \subset \mathbb{R}^{n+1}$ be a solid cone such that $(0,-1) \in \operatorname{int} K$. Let $x_0 \in \mathbb{R}^n$, $x_0 \neq 0$ be a point such that $(x_0,\mu) \in \operatorname{int} K$ for some $\mu \in \mathbb{R}$. Then there exist $n+1$ linearly independent vectors $a_1, \cdots, a_{n+1} \in \mathbb{R}^{n+1}$ such that*

$$(0,-1), (x_0,\mu) \in \operatorname{int} T, \quad T \setminus \{(0,0)\} \subset \operatorname{int} K,$$

where

$$T = \bigcap_{i=1}^{n+1}\{z \in \mathbb{R}^{n+1} : \langle a_i, z\rangle \leq 0\}.$$

Proof. Let $P = \{\omega_t : \omega_t = t(0,-1) + (1-t)(x_0,\mu),\ (t \in \mathbb{R})\}$ be the straight line passing through the points $(0,-1)$ and (x_0,μ). Since $(0,0) \notin P$, there exists a hyperplane H_1 such that $P \subset H_1,\ (0,0) \notin H_1$. Since $\omega_0 = (x_0,\mu) \in \operatorname{int} K$ and $\omega_1 = (0,-1) \in \operatorname{int} K$, there exists $t_0 > 0$ such that $\omega_{1+t_0}, \omega_{-t_0} \in \operatorname{int} K$. Consider the hyperplane

$$H_2 = \{z \in \mathbb{R}^{n+1} : \langle \omega_0 - \omega_1, z\rangle = \langle \omega_0 - \omega_1, \omega_{1+t_0}\rangle\}.$$

Let $M = H_1 \cap H_2$. Since $H_1 + H_2 = \mathbb{R}^{n+1}$ and $H_1 \cap H_2 \neq \emptyset$, we have

$$\dim M = \dim H_1 + \dim H_2 - \dim(H_1 + H_2) = n + n - (n+1) = n - 1.$$

So there exist n affinely independent vectors $y_1, \cdots, y_n \in M \cap (\operatorname{int} K)$ such that

$$\omega_{1+t_0} \in \operatorname{ri}(\operatorname{co}\{y_1, \cdots, y_n\}) \tag{4.13}$$

Let $S = \operatorname{co}\{\omega_{-t_0}, y_1, \cdots, y_n\}$. Since $\omega_{-t_0} \notin H_2$, the vectors $\omega_{-t_0}, y_1, \cdots, y_n$ are affinely independent. So S is a simplex. Since $\omega_{-t_0} \in P \subset H_1$ and $y_1, \ldots, y_n \in M \subset H_1$, we conclude that the affine hull $\operatorname{aff} S$ of S coincides with H_1. Therefore $(0,0) \notin \operatorname{aff} S = H_1$. It follows from (4.13) and the definition of S that (x_0, μ), $(0,-1) \in \operatorname{ri} S$. It also follows from the definition of S that $S \subset \operatorname{int} K$. There exist $n+1$ linear independent vectors $a_i \in \mathbb{R}^{n+1}$ such that $S = \{x \in H_1 : \langle a_i, x\rangle \leq 0\}$. Let $T = \operatorname{cone} S$ be the cone hull of S. Then $T = \{x \in \mathbb{R}^{n+1} : \langle a_i, x\rangle \leq 0,\ i = 1, \ldots, n+1\}$. Since $S \subset \operatorname{int} K$, it follows that $T \setminus \{(0,0)\} \subset \operatorname{int} K$. Since $(0,-1), (x_0, \mu) \in \operatorname{ri} S$, we obtain that (x_0, μ), $(0,-1) \in \operatorname{int} T$. □

It is obvious that a $\mathcal{H}$-convex function belongs to $\mathcal{R}$. We denote by $\mathcal{H}_{n+1}$ the set of all functions $h \in \mathcal{H}$ with $p \leq n+1$.

Theorem 4.1 *If $f \in \mathcal{R}$, then f is $\mathcal{H}_{n+1}$-convex.*

Proof. We need to show that for each point $x_0 \in \mathbb{R}^n$ and each $\mu < f(x_0)$ there exists a function $h \in \mathcal{H}_{n+1}$ such that $l(x) \leq f(x)$ for all x and $h(x_0) \geq \mu$. First we consider $x_0 = 0$ and then $x_0 \neq 0$.

1) $x_0 = 0$. For each positive integer m consider the set $C_m = (0,-1) + \dfrac{1}{m} B$, where B is the unit ball of $\mathbb{R}^{n+1}$. Let

$$K_m = \bigcup_{\alpha \geq 0} \alpha C_m. \tag{4.14}$$

Since $f \in \mathcal{R}$, it follows that $f(0) > -\infty$. Let $-\infty < \mu < f(0)$. Suppose that

$$[(0,\mu) + K_m] \cap \operatorname{epi} f \neq \emptyset, \qquad m = 1, \ldots\ . \tag{4.15}$$

Let $\omega_m = (x_m, \mu_m) \in [(0,\mu) + K_m] \cap \operatorname{epi} f$. Then there exist $\alpha_m > 0$ and $(\tilde{x}_m, \tilde{\mu}_m) \in B$ such that

$$\omega_m = (x_m, \mu_m) = (0, \mu) + \alpha_m \left((0,-1) + \frac{1}{m}(\tilde{x}_m, \tilde{\mu}_m) \right). \tag{4.16}$$

We have $x_m = \frac{\alpha_m}{m}\tilde{x}_m$, $\mu_m = \mu - \alpha_m + \frac{\alpha_m}{m}\tilde{\mu}_m$ and therefore $\mu - \alpha_m + \dfrac{\alpha_m}{m}\tilde{\mu}_m \geq f\left(\dfrac{\alpha_m}{m}\tilde{x}_m\right)$. Let $y_m = \alpha_m \left((0,-1) + \dfrac{1}{m}(\tilde{x}_m, \tilde{\mu}_m) \right)$. Consider two possible cases.

(i) The sequence $\{y_m\}_{m=1}^{\infty}$ is unbounded. Then we can assume without loss of generality that $\|y_m\| \underset{m\to\infty}{\to} \infty$. Let $\lambda > 0$. Since f is a radiant function, it follows that $\operatorname{epi} f$ is a radiant set. Hence $\frac{\lambda}{\|y_m\|}\omega_m \in \operatorname{epi} f$ for sufficiently large m. It follows from (4.16) that

$$\frac{\lambda}{\|y_m\|}\omega_m = \frac{\lambda}{\|y_m\|}(0,\mu) + \frac{\lambda}{\|y_m\|} y_m \underset{m\to\infty}{\to} (0, -\lambda),$$

so $(0,-\lambda) \in \operatorname{epi} f$ for all $\lambda > 0$. This is impossible, so the sequence y_m can not be unbounded.

(ii) The sequence $\{y_m\}_{m=1}^{\infty}$ is bounded. Assume without loss of generality that $y_m \underset{m\to\infty}{\to} y$. Since

$$\alpha_m = \frac{\|y_m\|}{\|(0,-1) + \frac{1}{m}(\tilde{x}_m, \tilde{\mu}_m)\|} \underset{m\to\infty}{\to} \|y\|,$$

we have $(0,\mu) + \|y\|(0,-1) = (0,\mu - \|y\|) \in \operatorname{epi} f$. On the other hand μ was chosen in such a way that $\mu < f(0)$. We got a contradiction, so the sequence $\{y_m\}_{m=1}^{\infty}$ cannot be bounded. It follows from (i) and (ii) that (4.15) does not hold, so there exists m_0 such that

$$[(0,\mu) + K_{m_0}] \cap \operatorname{epi} f = \emptyset$$

Applying Proposition 4.2 to the cone K_{m_0} and the vector $(0,-1) \in \operatorname{int} K_{m_0}$ we conclude that there exist $n+1$ affinely independent vectors $a_1, \dots, a_{n+1}$ such that

$$\bigcap_{i=1}^{n+1} \{(x,\nu) \in \mathbb{R}^{n+1} : \langle a_i, x\rangle + \nu < 0\} \subset \operatorname{int} K_{m_0}.$$

We have

$$\left[(0,\mu) + \bigcap_{i=1}^{n+1} \{(x,\nu) \in \mathbb{R}^{n+1} : \langle a_i, x\rangle + \nu < 0\}\right] \cap \operatorname{epi} f = \emptyset,$$

so

$$\operatorname{epi} f \subset \bigcup_{i=1}^{n+1} \{(x,\nu) \in \mathbb{R}^{n+1} : \nu \geq \langle -a_i, x\rangle - |\mu|\}.$$

(Since f is radiant, it follows that $\mu < 0$.) Set

$$h_i(x) = \langle -a_i, x\rangle - |\mu| \quad (x \in \mathbb{R}^n,\ i \in 1:n+1),$$

$$h(x) = \min_{i=1}^{n+1} h_i(x) \quad (x \in \mathbb{R}^n).$$

Then $h \in \mathcal{H}_{n+1},\ h(x) \leq f(x)$ for all $x \in \mathbb{R}^n$, and $h(0) = \mu$. Since μ is an arbitrary number less than $f(0)$, it follows that $f(0) = \sup\{h(0) : h \in \mathcal{H}_{n+1}, h \leq f\}$.

2) Let now $x_0 \in \mathbb{R}^n,\ x_0 \neq 0,\ \mu \in \mathbb{R},\ (x_0,\mu) \notin \operatorname{epi} f$. Consider the set $(x_0,\mu) + \frac{1}{m}B$ and let P_m be the cone hull of this set:

$$P_m = \bigcup_{\beta \geq 0} \beta\left((x_0,\mu) + \frac{1}{m}B\right)$$

Assume that

$$[(x_0,\mu) + K_m + P_m] \cap \operatorname{epi} f \neq \emptyset \tag{4.17}$$

for all natural m. Then for every m there exist $\alpha_m, \beta_m \geq 0$ and $(u_m,\mu_m), (v_m,\nu_m) \in B$ such that

$$\omega_m = (x_0,\mu) + \alpha_m\left((0,-1) + \frac{1}{m}(u_m,\mu_m)\right) + \beta_m\left((x_0,\mu) + \frac{1}{m}(v_m,\nu_m)\right) \in \operatorname{epi} f. \tag{4.18}$$

Since $\mu < f(x_0)$, it follows that $\alpha_m + \beta_m > 0$ for every m. Let

$$\tilde{y}_m = \alpha_m\left((0,-1) + \frac{1}{m}(u_m,\mu_m)\right) + \beta_m\left((x_0,\mu) + \frac{1}{m}(v_m,\nu_m)\right),$$

$$z_m = \gamma_m\left((0,-1) + \frac{1}{m}(u_m,\mu_m)\right) + (1-\gamma_m)\left((x_0,\mu) + \frac{1}{m}(v_m,\nu_m)\right),$$

where

$$\gamma_m = \frac{\alpha_m}{\alpha_m + \beta_m}.$$

Then $\omega_m - (x_0, \mu) = \tilde{y}_m = (\alpha_m + \beta_m) z_m$. Without loss of generality we assume that $\gamma_m \underset{m\to\infty}{\to} t \in [0,1]$. Then $z_m \underset{m\to\infty}{\to} t(0,-1) + (1-t)(x_0,\mu) = z$. Since $x_0 \neq 0$, we have $z \neq 0$. We again consider two possible cases:

(i) The sequence $\{\tilde{y}_m\}_{m=1}^{\infty}$ is unbounded. Then we can assume without loss of generality that $\alpha_m + \beta_m \underset{m\to\infty}{\to} \infty$. Let $\lambda > 0$. Since f is a radiant function, it follows that $\operatorname{epi} f$ is a radiant set. Hence $\frac{\lambda}{\alpha_m+\beta_m}\omega_m \in \operatorname{epi} f$ for sufficiently large m. It follows from (4.18) that

$$\frac{\lambda}{\alpha_m + \beta_m}\omega_m = \frac{\lambda}{\alpha_m + \beta_m}(x_0, \mu) + \lambda z_m \underset{m\to\infty}{\longrightarrow} \lambda z.$$

Since f is a l.s.c. function, $\lambda z = \lambda((1-t)x_0, (1-t)\mu - t) \in \operatorname{epi} f$ for all $\lambda > 0$. Since $f(0) \in \mathbb{R}$, it follows from this that $t \neq 1$. Putting $\lambda = \frac{1}{1-t}$ we obtain that $(x_0, \mu - \frac{t}{1-t}) \in \operatorname{epi} f$. This is impossible, so the sequence $\{\bar{y}_m\}_{m=1}^{\infty}$ can not be unbounded.

(ii) The sequence $\{\tilde{y}_m\}_{m=1}^{\infty}$ is bounded. Assume without loss of generality that $\tilde{y}_m \underset{m\to\infty}{\to} y$. Let $q = \frac{\|y\|}{\|z\|}$. Then $q \geq 0$, $\alpha_m + \beta_m = \frac{\|y_m\|}{\|z_m\|} \underset{m\to\infty}{\to} q$, and therefore

$$(x_0, \mu) + qt(0, -1) + q(1-t)(x_0, \mu) = (1 + q(1-t))(x_0, \mu) + (0, -qt) \in \operatorname{epi} f.$$

This implies that $(1+q(1-t))(x_0, \mu) \in \operatorname{epi} f$. Since the set $\operatorname{epi} f$ is radiant and $1+q(1-t) \geq 1$ we obtain $(x_0, \mu) \in \operatorname{epi} f$, which is impossible. Since both (i) and (ii) are wrong, we conclude that (4.17) is not valid and there exists m_0 such that

$$[(x_0, \mu) + K_{m_0} + P_{m_0}] \cap \operatorname{epi} f = \emptyset \tag{4.19}$$

for some m_0. Set $K = K_{m_0} + P_{m_0}$. Then $(0, -1) \in \operatorname{int} K$ and $(x_0, \mu) \in \operatorname{int} K$. According to Lemma 4.3 there exist linear independent vectors $(a_1, c_1), \cdots, (a_{n+1}, c_{n+1}) \in \mathbb{R}^{n+1}$ such that

$$(0, -1) \in \bigcap_{i=1}^{n+1} \{(x, \nu) \in \mathbb{R}^{n+1} : \langle a_i, x\rangle + c_i\nu < 0\}, \tag{4.20}$$

$$(x_0, \mu) \in \bigcap_{i=1}^{n+1} \{(x, \nu) \in \mathbb{R}^{n+1} : \langle a_i, x\rangle + c_i\nu < 0\}, \tag{4.21}$$

and

$$\left(\bigcap_{i=1}^{n+1} \{(x, \nu) \in \mathbb{R}^{n+1} : \langle a_i, x\rangle + c_i\nu \leq 0\}\right) \setminus \{(0, 0)\} \subset \operatorname{int} K. \tag{4.22}$$

It follows from (4.20) and (4.21), respectively, that $c_i > 0$ and $\langle a_i, x_0\rangle + c_i\mu < 0$ for all $i \in 1 : n+1$. In view of definition of m_0 and (4.22) we obtain

$$\left((x_0, \mu) + \bigcap_{i=1}^{n+1} \{(x, \nu) \in \mathbb{R}^{n+1} : \langle a_i, x\rangle + c_i\nu < 0\}\right) \cap \operatorname{epi} f = \emptyset.$$

Hence

$$\operatorname{epi} f \subset \bigcup_{i=1}^{n+1} \left\{(x, \nu) \in \mathbb{R}^{n+1} : \nu \geq \left\langle -\frac{a_i}{c_i}, x\right\rangle - \frac{|\langle a_i, x_0\rangle + c_i\mu|}{c_i}\right\}.$$

Let

$$h_i(x) = \left\langle -\frac{a_i}{c_i}, x\right\rangle - \frac{|\langle a_i, x_0\rangle + c_i\mu|}{c_i} \quad (x \in \mathbb{R}^n,\ i \in 1 : n+1),$$

and

$$h(x) = \min_{i=1}^{n+1} h_i(x) \quad (x \in \mathbb{R}^n)$$

Then

$$h \in \mathcal{H}_{n+1}, \quad h(x) \leq f(x) \;\forall\; x \in \mathbb{R}^n, \quad h(x_0) = \mu. \tag{4.23}$$

In view of (4.23), the set supp $(f, \mathcal{H}_{n+1}) = \{h \in \mathcal{H}_{n+1} : h \leq f\}$ is nonempty. Since μ is an arbitrary number with the property $\mu < f(x_0)$ we obtain from (4.23) that $f(x_0) = \{\sup h(x_0) : l \in \text{supp } (f, \mathcal{H}_{n+1})\}$. □

Let

$$\mathcal{L}_{n+1} = \{l \in \mathcal{H}_{n+1} : l(x) = \min_{i=1}^{p}\{\langle a_i, x\rangle - c_i\},\; a_i \in \mathbb{R}^n,\; c_i = 0 \;\forall\; i \in 1:p,\; p \leq n+1\} \tag{4.24}$$

Corollary 4.3 *A function $f : \mathbb{R}^n \to (-\infty, +\infty]$ is $\mathcal{L}_{n+1}$-convex if and only if f is lower semicontinuous and positively homogeneous.*

Another proof of this result can be found in [4].

5 $\mathcal{L}$-subdifferentiability of Radiant Functions

In this section we examine conditions for the non-emptiness of the $\mathcal{L}_{n+1}$-subdifferential, where $\mathcal{L}_{n+1}$ is the set defined by (4.24). Recall that the $\mathcal{L}_{n+1}$- subdifferential of a function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ at a point $x_0 \in \text{dom}\, f$ is the set of $l \in \mathcal{L}_{n+1}$ such that

$$f(x) \geq f(x_0) + l(x) - l(x_0), \qquad x \in \mathbb{R}^n.$$

The proofs of the following theorems are slight modifications of the proof of Theorem 4.1.

Theorem 5.1 *Let $f \in \mathcal{R}$, $x_0 \in \mathbb{R}^n$, $x_0 \neq 0$. If $(0, -1) \notin \Gamma((x_0, f(x_0)), epi\, f)$ and $(x_0, f(x_0)) \notin \Gamma((x_0, f(x_0)), epi\, f)$, then $\partial_{\mathcal{L}_{n+1}} f(x_0) \neq \emptyset$.*

Proof. Let

$$K_m = \bigcup_{\alpha>0} \alpha\left((0,-1) + \frac{1}{m}B\right), \quad P_m = \bigcup_{\beta>0} \beta\left((x_0, f(x_0)) + \frac{1}{m}B\right), \quad m = 1, \ldots$$

Suppose that

$$[(x_0, f(x_0)) + K_m + P_m] \cap \text{epi}\, f \neq \emptyset \tag{5.1}$$

for all m. Then for for each m there exist $\alpha_m, \beta_m > 0$ and $(u_m, \mu_m), (v_m, \nu_m) \in B$ such that

$$\omega_m = (x_0, f(x_0)) + \alpha_m\left((0,-1) + \frac{1}{m}(u_m, \mu_m)\right) + \beta_m\left((x_0, f(x_0)) + \frac{1}{m}(v_m, \nu_m)\right) \in \text{epi}\, f.$$

Set as in the proof of Theorem 4.1

$$\tilde{y}_m = \alpha_m\left((0,-1) + \frac{1}{m}(u_m, \mu_m)\right) + \beta_m\left((x_0, \mu) + \frac{1}{m}(v_m, \nu_m)\right),$$

$$z_m = \frac{\alpha_m}{\alpha_m + \beta_m}\left((0,-1) + \frac{1}{m}(u_m, \mu_m)\right) + \frac{\beta_m}{\alpha_m + \beta_m}\left((x_0, \mu) + \frac{1}{m}(v_m, \nu_m)\right).$$

Then $\tilde{y}_m = (\alpha_m + \beta_m)z_m$ for all m and we can assume that $\frac{\alpha_m}{\alpha_m+\beta_m} \underset{m\to\infty}{\to} t \in [0,1]$. Then $z_m \underset{m\to\infty}{\to} t(0,-1) + (1-t)(x_0, f(x_0)) = z \neq 0$. First assume that the sequence $\{\tilde{y}_m\}_{m=1}^{\infty}$ is unbounded. Arguing as in the proof of Theorem 4.1 we obtain that $\lambda z \in \operatorname{epi} f$ for all $\lambda > 0$. If $t = 1$ then we get that $(0, -\lambda) \in \operatorname{epi} f$ for all $\lambda > 0$ which is impossible. If $t = 0$ then $\lambda(x_0, f(x_0)) \in \operatorname{epi} f$ for all $\lambda > 0$. This implies that $(x_0, f(x_0)) \in \Gamma((x_0, f(x_0)), \operatorname{epi} f)$ which is also impossible, so $0 < t < 1$. Putting $\lambda = \frac{1}{1-t}$ we obtain that $\left(x_0, f(x_0) - \frac{t}{1-t}\right) \in \operatorname{epi} f$, which contradicts the definition of the epigraph. Therefore the assumption that the sequence $\{\tilde{y}_m\}_{m=1}^{\infty}$ is unbounded does not hold. Assume now that this sequence is bounded. Then without loss of generality we can consider that $\tilde{y}_m \underset{m\to\infty}{\to} y$. If $y = 0$, we would have $\alpha_m + \beta_m \underset{m\to\infty}{\to} 0$. Since $(x_0, f(x_0)) + (\alpha_m + \beta_m)z_m = \omega_m \in \operatorname{epi} f$, $\alpha_m + \beta_m \to 0$, and $z_m \to z$, it follows that $z \in \Gamma((x_0, f(x_0)), \operatorname{epi} f)$. Since $(0,-1) \notin \Gamma((x_0, f(x_0)), \operatorname{epi} f)$, then $t \neq 1$. Note that

$$z = t(0,-1) + (1-t)(x_0, f(x_0) = (1-t)\left(x_0, f(x_0) - \frac{t}{1-t}\right).$$

Since $\Gamma((x_0, f(x_0)), \operatorname{epi} f)$ is a cone, it follows that

$$\frac{z}{1-t} = \left(x_0, f(x_0) - \frac{t}{1-t}\right) \in \Gamma((x_0, f(x_0)), \operatorname{epi} f) \tag{5.2}$$

The set $\operatorname{epi} f$ possesses the following property: $((x,\lambda) \in \operatorname{epi} f, \lambda' \geq \lambda) \implies ((x,\lambda') \in \operatorname{epi} f)$. Then the cone $\Gamma((x_0, f(x_0)), \operatorname{epi} f)$ enjoys the same property. It follows from (5.2) that $(x_0, f(x_0)) \in \Gamma((x_0, f(x_0)), \operatorname{epi} f)$ which is impossible. Thus $y \neq 0$. Set

$$q = \frac{\|y\|}{\|z\|} = \lim_m (\alpha_m + \beta_m).$$

Then $q > 0$. We have

$$\omega_m = (x_0, f(x_0)) + \tilde{y}_m = (x_0, f(x_0)) + (\alpha_m + \beta_m)z_m \in \operatorname{epi} f.$$

Hence $(x_0, f(x_0)) + qz = (x_0, f(x_0)) + tq(0,-1) + q(1-t)(x_0, f(x_0)) \in \operatorname{epi} f$. If $t = 1$, we would have that $(x_0, f(x_0) - q) \in \operatorname{epi} f$, which is impossible. Therefore $t < 1$. Then $(x_0, f(x_0)) + q(1-t)(x_0, f(x_0)) \in \operatorname{epi} f$. Let $\alpha_k \to 0+$. Since $\operatorname{epi} f$ is radiant, it follows that

$$(x_0, f(x_0)) + \alpha_k q(1-t)(x_0, f(x_0)) \in \operatorname{epi} f,$$

so $(x_0, f(x_0)) \in \Gamma((x_0, f(x_0)), \operatorname{epi} f)$. We arrive at a contradiction, which shows that (5.1) does not hold. Hence there exists m_0 such that

$$[(x_0, f(x_0)) + K_{m_0} + P_{m_0}] \cap \operatorname{epi} f = \emptyset$$

and we obtain (4.17) with $\mu = f(x_0)$. Hence we can claim that there exists $l \in \mathfrak{L}_{n+1}$ such that $l(x) \leq f(x)$ for all $x \in \mathbb{R}^n$, and $l(x_0) = f(x_0)$. This means that $l \in \partial_{\mathfrak{L}_{n+1}} f(x_0)$. □

We now show that the condition $(0,-1) \notin \Gamma((x_0, f(x_0)), \operatorname{epi} f)$ holds under very mild assumptions.

Proposition 5.1 *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $x_0 \in \operatorname{dom} f$. Assume that $f_H^{\downarrow}(x_0, u) > -\infty$ for all $u \neq 0$. Then $(0,-1) \notin \Gamma((x_0, f(x_0)), \operatorname{epi} f)$.*

Proof. Assume that $(0,-1) \in \Gamma((x_0, f(x_0)), \operatorname{epi} f)$. Then there exist sequences $\mu_k \to 0+$ and $(x_k, \lambda_k) \to (0,-1)$ such that $(x_0, f(x_0)) + \mu_k(x_k, \lambda_k) \in \operatorname{epi} f$. The latter means that

$$f(x_0) + \mu_k\lambda_k \geq f(x_0 + \mu_k x_k), \quad k = 1, \dots ,$$

i.e.

$$\lambda_k \geq \frac{f(x_0 + \mu_k x_k) - f(x_0)}{\mu_k} = \frac{f\left(x_0 + \alpha_k \frac{x_k}{\|x_k\|}\right) - f(x_0)}{\alpha_k}\|x_k\|,$$

where $\alpha_k = \mu_k\|x_k\|$. Without loss of generality assume that $u_k = \frac{x_k}{\|x_k\|} \longrightarrow u \neq 0$ and $\nu_k = \frac{f(x_0+\alpha_k u_k)-f(x_0)}{\alpha_k} \longrightarrow \nu$. Then $\nu_k < 0$ for all k large enough and $\nu \geq f_H^{\downarrow}(x_0, u) > -\infty$. Hence the sequence $\{\nu_k\}_{k=1}^{\infty}$ is bounded. Therefore $-1 = \lim \lambda_k \geq \lim_k \nu_k\|x_k\| = 0$, which is impossible. □

Corollary 5.1 *Let $f \in \mathcal{R}$ be locally Lipschitz and strictly radiant at x_0. Then $\partial_{\mathcal{L}_{n+1}} f(x_0) \neq \emptyset$.*

Indeed it follows from Proposition 3.4, Theorem 5.1 and Proposition 5.1

Remark 5.1 Theorem 5.1 cannot be applied for positively homogeneous functions since $(x_0, p(x_0)) \in \Gamma((x_0, p(x_0)), \operatorname{epi} p)$ for each positively homogeneous function p and $x_0 \in \operatorname{dom} p$, $x_0 \neq 0$. The non-emptiness of the $\mathcal{L}_{n+1}$-subdifferential for positively homogeneous function p has been examined in [4].

We now turn to the non-emptiness of the subdifferential at the point zero. We need the following auxiliary result.

Lemma 5.1 *Let $f \in \mathcal{R}$ and $x_0 \in dom f$ be a point such that $\partial_{\mathcal{L}_{n+1}} f(x_0)$ is not empty. Then*

$$(0,-1) \notin \Gamma((x_0, f(x_0)), epi\, f).$$

Proof. Let $l \in \partial_{\mathcal{L}_{n+1}} f(x_0)$. Consider the function $\tilde{l}$, where

$$\tilde{l}(x) = l(x) - (l(x_0) - f(x_0)) \quad (x \in \mathbb{R}^n).$$

Then

$$\tilde{l}(x) = \min_{i=1}^{p}\{\langle a_i, x\rangle - c_i\} \quad (x \in \mathbb{R}^n)$$

for some $a_1, \cdots, a_p \in \mathbb{R}^n$, $c_1, \cdots, c_p \in \mathbb{R}$. We also have: $\tilde{l}(x_0) = f(x_0)$, $\tilde{l}(x) \leq f(x)$ for all $x \in \mathbb{R}^n$. Assume that $(0,-1) \in \Gamma((x_0, f(x_0)), \operatorname{epi} f)$. Then there exist sequences $(x_k, \mu_k) \underset{k\to\infty}{\to} (0,-1)$ and $\lambda_k \downarrow 0$ such that $(x_0, f(x_0)) + \lambda_k(x_k, \mu_k) \in \operatorname{epi} f$ for all k. This means that

$$f(x_0) + \lambda_k\mu_k \geq f(x_0 + \lambda_k x_k) \geq \tilde{l}(x_0 + \lambda_k x_k) \quad \text{for all} \quad k.$$

Choosing if necessary a subsequence, we can find an index $i \in 1:p$ such that $\tilde{l}(x_0 + \lambda_k x_k) = \langle a_i, x_0 + \lambda_k x_k\rangle - c_i$ for all k. We have

$$f(x_0) + \lambda_k\mu_k \geq \langle a_i, x_0\rangle + \lambda_k\langle a_i, x_k\rangle - c_i \geq \tilde{l}(x_0) + \lambda_k\langle a_i, x_k\rangle,$$

which implies that $\mu_k \geq \langle a_i, x_k\rangle$ for all k. Letting $k \to \infty$ we obtain $-1 \geq 0$. Hence our assumption is wrong and $(0,-1)) \notin \Gamma((x_0, f(x_0)), \operatorname{epi} f)$. □

Theorem 5.2 *Let $f \in \mathcal{R}$. Then*

$$\partial_{\mathcal{L}_{n+1}} f(0) \neq \emptyset \iff (0,-1) \notin \Gamma((0, f(0)), epi\, f).$$

Proof. Necessity follows from Lemma 5.1. So it remains to prove that

$$((0,-1) \notin \Gamma((0, f(0)), \operatorname{epi} f)) \implies (\partial_{\mathcal{L}_{n+1}} f(0) \neq \emptyset).$$

For this purpose we put

$$K_m = \bigcup_{\alpha>0} \alpha \left((0,-1) + \frac{1}{m} B \right).$$

for every natural m and suppose that

$$[(0, f(0)) + K_m] \cap \operatorname{epi} f \neq \emptyset$$

for all m. Let $\alpha_m > 0$ and $(\tilde{x}_m, \tilde{\mu}_m) \in B$ be such that

$$\omega_m = (x_m, \mu_m) = (0, f(0)) + \alpha_m \left((0,-1) + \frac{1}{m} (\tilde{x}_m, \tilde{\mu}_m) \right) \in \operatorname{epi} f,$$

and let $y_m = \alpha_m \left((0,-1) + \frac{1}{m} (\tilde{x}_m, \tilde{\mu}_m) \right)$. The same argument as in the proof of Theorem 4.1 shows that the sequence $\{y_m\}_{m=1}^{\infty}$ is bounded. We can assume that $y_m \underset{m\to\infty}{\to} y$. Again arguing as in the proof of Theorem 4.1 we obtain that $\alpha_m \underset{m\to\infty}{\to} \|y\|$, and consequently $(0, f(0) - \|y\|) \in \operatorname{epi} f$. Hence $y = 0$ and since $(0,-1) + \frac{1}{m}(\tilde{x}_m, \tilde{\mu}_m) \underset{m\to\infty}{\to} (0,-1)$, we obtain that $(0,-1) \in \Gamma((0, f(0)), \operatorname{epi} f)$. Thus our assumption is wrong and

$$[(0, f(0)) + K_{m_0}] \cap \operatorname{epi} f = \emptyset$$

for some $m_0 > 1$. Applying Proposition 4.2 to the cone K_{m_0} with

$$\operatorname{int} K_{m_0} = \bigcup_{\alpha>0} \alpha \left((0,-1) + \frac{1}{m_0} \operatorname{int} B \right)$$

and the vector $(0,-1) \in \operatorname{int} K_{m_0}$ and arguing as at the end of the proof of Theorem 4.1 we conclude that there exists $l \in \mathcal{L}_{n+1}$ such that $l(x) \leq f(x)$ for all $x \in \mathbb{R}^n$, and $l(0) = f(0)$. But l with this properties belongs to $\partial_{\mathcal{L}_{n+1}} f(0)$. □

6 Star-shaped Functions

Recall that a function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is called star-shaped if the epigraph $\operatorname{epi} f$ is a star-shaped set. This is equivalent to the following: there exists $(x_0, \gamma) \in \operatorname{epi} f$ such that

$$\lambda f(x) + (1-\lambda)\gamma \geq f(\lambda x + (1-\lambda) x_0), \qquad x \in \operatorname{dom} f,\ \lambda \in [0,1].$$

A function f is star-shaped if and only if there exists $(x_0, \gamma) \in \mathbb{R}^{n+1}$ such that $\operatorname{epi} f - (x_0, \gamma)$ is a radiant set. Let

$$\bar{f}(x) = f(x + x_0) - \gamma \qquad (x \in \mathbb{R}^n). \tag{6.1}$$

Then

$$\begin{aligned} \operatorname{epi} \bar{f} &= \{(y,\nu) : \nu \geq \bar{f}(y)\} = \{(x-x_0, \lambda-\gamma) : \lambda-\gamma \geq \bar{f}(x-x_0)\} \\ &= \{(x-x_0, \lambda-\gamma) : \lambda \geq f(x)\} \\ &= \{(x,\lambda) : \lambda \geq f(x)\} - (x_0,\gamma) = \operatorname{epi} f - (x_0,\gamma). \end{aligned}$$

Thus f is star-shaped with $(x_0,\gamma) \in \operatorname{kern} f$ if and only if $\bar{f}$ is radiant. Using (6.1) we can describe properties of star-shaped functions.

We need the following notation: if $V \subset \mathbb{R}^n \times \mathbb{R}$ then

$$\pi(V) = \{x : \exists \lambda \text{ such that } (x,\lambda) \in V\}.$$

If $x, x_0 \in \mathbb{R}^n$ and $\lambda > 0$ then $f_{x,x_0}(\lambda) = f(x_0 + \lambda x)$.

Let U be a star-shaped set and $x_0 \in \operatorname{kern} U$. We say that U is open-along-rays at x_0 if the radiant set $U - x_0$ is open-along-rays.

Theorem 6.1 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$, $U \subset \operatorname{dom} f$ be open-along-rays at a point $x_0 \in U$.*

1) *If $f + \delta_U$ is a star-shaped function with $x_0 \in \pi(\operatorname{kern}\operatorname{epi}(f+\delta_U))$, then there exists $\gamma \in \mathbb{R}$ such that*
$$f_D^{\downarrow}(x, x-x_0) \geq f(x) - \gamma \quad \textit{for all } x \in U.$$

2) *If U is a star-shaped set with $x_0 \in \operatorname{kern} U$, there exists $\gamma \in \mathbb{R}$ such that*
$$f_D^{\downarrow}(x, x-x_0) \geq f(x) - \gamma \quad \textit{for all } x \in U$$
and $f_{x,x_0}(\lambda)$ is continuous on $\operatorname{dom} f_{x,x_0} \setminus \{0\}$, then $f + \delta_U$ is a star-shaped function with $(x_0,\gamma) \in \operatorname{kern}\operatorname{epi}(f+\delta_U)$.

Proof. 1) Let $(x_0,\gamma) \in \operatorname{kern}\operatorname{epi}(f+\delta_U)$. For the function

$$\overline{f+\delta_U}(x) = (f+\delta_U)(x+x_0) - \gamma \quad (x \in \mathbb{R}^n)$$

we have

$$\operatorname{epi}(\overline{f+\delta_U}) = \operatorname{epi}(f+\delta_U) - (x_0,\gamma), \qquad \operatorname{dom}(\overline{f+\delta_U}) = -x_0 + U. \tag{6.2}$$

Hence $\overline{f+\delta_U}$ is a radiant function and its domain is an open-along-rays radiant set. Then (see Corollary 3.1)

$$(\overline{f+\delta_U})_D^{\downarrow}(x,x) \geq \overline{f+\delta_U}(x) \text{ for all } x \in -x_0 + U.$$

We have

$$\overline{f+\delta_U}(x) = (\bar{f} + \delta_{(-x_0+U)})(x) \quad (x \in \mathbb{R}^n). \tag{6.3}$$

Indeed,

$$\begin{aligned} \overline{f+\delta_U}(x) &= (f+\delta_U)(x+x_0) - \gamma \\ &= \begin{cases} f(x+x_0) - \gamma, & \text{if } x + x_0 \in U, \\ +\infty, & \text{otherwise} \end{cases} = \begin{cases} \bar{f}(x), & \text{if } x \in -x_0 + U, \\ +\infty, & \text{otherwise} \end{cases} \\ &= (\bar{f} + \delta_{(-x_0+U)}))(x) \quad (x \in \mathbb{R}^n). \end{aligned}$$

Let $x \in U$. Then we have

$$\overline{f+\delta_U}(x-x_0) = (\bar f + \delta_{-x_0+U}(x-x_0) = \bar f(x-x_0) = f(x)-\gamma,$$

$$\begin{aligned}
&(\overline{f+\delta_U})^{\downarrow}_D(x-x_0, x-x_0) = (\bar f + \delta_{(-x_0+U)})^{\downarrow}_D(x-x_0, x-x_0)\\
&= \lim_{\varepsilon\to 0}\ \inf_{0<\alpha<\varepsilon} \frac{(\bar f+\delta_{(-x_0+U)})((x-x_0)+\alpha(x-x_0)) - (\bar f+\delta_{(-x_0+U)})(x-x_0)}{\alpha}\\
&= \lim_{\varepsilon\to 0}\ \inf_{0<\alpha<\varepsilon} \frac{\bar f((x-x_0)+\alpha(x-x_0)) - \bar f(x-x_0)}{\alpha}\\
&= (\bar f)^{\downarrow}_D(x-x_0, x-x_0) = \bar f^{\downarrow}_D(x, x-x_0) = f^{\downarrow}_D(x, x-x_0).
\end{aligned}$$

Therefore $f^{\downarrow}_D(x, x-x_0) \geq f(x)-\gamma$.
2) Set as above

$$\bar f(y) = f(y+x_0)-\gamma \quad (y \in \mathbb{R}^n)$$

Then $\operatorname{dom} \bar f = -x_0 + \operatorname{dom} f \supset -x_0+U$ and the set $-x_0+U$ is open-along-rays and radiant. Let $x \in U,\ x \neq 0$. The function $\bar f_x(\lambda) = \bar f(\lambda x) = f(x_0+\lambda x)-\gamma$ is continuous on $\operatorname{dom} \bar f_x \setminus \{0\}$ and

$$\bar f^{\downarrow}_D(x-x_0, x-x_0) = f^{\downarrow}_D(x, x-x_0) \geq f(x)-\gamma = \bar f(x-x_0).$$

Hence (see Corollary 3.1) $\bar f + \delta_{(-x_0+U)}$ is a radiant function. According to (6.3) $\overline{f+\delta_U}$ is a radiant function. According to (6.2) $f+\delta_U$ is a star-shaped function with $(x_0,\gamma) \in \ker \operatorname{epi}(f+\delta_U)$. □

Corollary 6.1 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{+\infty}$ and let $U \subset \operatorname{dom} f$ be an open-along-rays star-shaped bounded set with $x_0 \in \ker U$. Suppose that f is a Lipschitz function on U. Then $f+\delta_U$ is a star-shaped function.*

Proof. Let $C > 0$ be such that

$$|f(x_1) - f(x_2)| \leq C\|x_1 - x_2\| \text{ for all } x_1, x_2 \in U.$$

Then $|f(x)| \leq |f(x_0)| + C\|x-x_0\|$ for all $x \in U$. Since U is bounded, there exists $\bar C$ such that $|f(x)| \leq \bar C$ for all $x \in U$. If $x \in U$ and α is small enough we have

$$f(x+\alpha(x-x_0)) - f(x) \geq -|\alpha| C\|x-x_0\|,$$

so $f^{\downarrow}_D(x, x-x_0) \geq -C\|x-x_0\|$ for all $x \in U$. The set U is bounded, therefore there exists $K \in \mathbb{R}$ such that $f^{\downarrow}_D(x, x-x_0) \geq K$ for all $x \in U$. Set $\gamma = \bar C - K$. Then

$$f^{\downarrow}_D(x, x-x_0) \geq K = \bar C - \gamma \geq f(x)-\gamma \text{ for all } x \in U.$$

By theorem 6.1 f is a star-shaped function. □

Theorem 4.1 allows us to describe the set of min-type functions that generate star-shaped functions. Let $\mathcal{H}^{\gamma}_{n+1}(x_0)$ be the set of all min-type functions h of the form

$$h(x) = \min_{i=1}^{p}\{\langle a_i, x-x_0\rangle - c_i\}, \qquad a_i \in \mathbb{R}^n,\ c_i \geq \gamma,\ p \leq n+1 \tag{6.4}$$

Theorem 6.2 *Let $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be a lower semicontinuous star-shaped function and $x_0 \in \pi(\operatorname{kern} \operatorname{dom} f)$. Then there exists $\gamma \in \mathbb{R}$ such that f is $\mathcal{H}^{\gamma}_{n+1}(x_0)$-convex.*

In order to get this result we should apply Theorem 4.1 to the function $\bar f$ defined by (6.1).

Theorem 5.1 and Theorem 5.2 can be easily reformulated for the case of star-shaped functions too.

Acknoledgement

We are thankful to the anonymous referee whose valuable comments helped us to improve an earlier version of the paper.

References

[1] V.F Demyanov and A.M. Rubinov, *Constructive Nonsmooth Analysis*, Peter Lang, Frankfurt am Main, 1995.

[2] J E. Martinez Legaz, A.M. Rubinov and S. Shaible, Increasing quasiconcave co-radiant functions with applications in mathematical economics, *Mathematical Methods in Operation Research* 61 (2005) 261–280.

[3] D. Pallaschke and S. Rolewicz, *Foundations of Mathematical Optimization*, Kluwer Academic Publishers, Dordrechet, 1997.

[4] A.M. Rubinov, *Abstract convexity and global optimization*, Kluwer Academic Publishers, Boston, 2000.

[5] A.P Shveidel, Separability of star-shaped sets and its application to an optimization problem, *Optimization* 40 (1997) 207–227.

[6] I. Singer, *Abstract Convex Analysis*, John Wiley & Sons, New York, 1997.

Manuscript received 9 December 2005
revised 29 August 2006
accepted for publication 31 August 2006

A.M. Rubinov

A.P. Shveidel
Department of Mathematics, Karaganda State University, Karaganda 100026, Kazakstan
E-mail address: shveidel@mail.ru