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MODELS OF BRIDGE LANE DIRECTION SPECIFICATION

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Abstract: An important matter in the management of some freeways that include a city bridge is specifying the direction of the bridge lanes in each time period of a given, congested planning horizon. We propose and analyse some dynamic optimisation models and solution techniques for this problem.

Key words: *bridge lanes, dynamic optimization, models, positive linear systems*

Mathematics Subject Classification: *90, 90C10, 90C90, 93, 93B05*

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1 Introduction

There has been much research into highway management, including that by: Bunker [1], Daganzo [2, 3], Hall and Lam [9], Helbing [10], Helbing et al. [11], Papageoriou et al. [14], Ran et al. [15] and Schach [18]. However, as far as the authors are aware, apart from some early articles (Foulds [4, 5, 6, 7, 8]) and one report on lane sign control (Schaefer et al. [19]), there has been little of significance reported in the open literature concerning the important carriageway management issue of lane direction specification. We attempt to fill this gap, at least for congested freeways that include bridges, by proposing dynamic models of the problem of lane direction specification with the objective of maximising the throughput time of all vehicles that travel over the bridge during a specified (congested) period of time. The models are motivated by three issues: (i) the need for policies, (ii) the contribution of deductive analysis, and (iii) the need to take temporal considerations into account. The models are developed for a given section of a freeway (F) that includes a multi-laned bridge (B), under the following assumptions:

- It is possible to specify the direction of each lane of B in each time period of a given planning horizon by relocating a moveable barrier that divides the two directions of travel on B.
- At least one lane must be left open in each direction in each period.
- The estimated number of vehicles that enter F, in each direction in each time period, is dependent upon the number of lanes specified in each direction. (This is based on electronically posted estimates of the time to traverse F.)
- The objective is to specify the direction of each lane of B in each period so as to maximise the throughput time of all vehicles on F that travelled over B for a given planning horizon.

We now develop models of the scenario just described.

2 The Models

We introduce first some notation.

α		denotes a direction of travel on F, with $\alpha = 1$, (2) denoting one direction, say north to south (south to north).
m	$=$	the total number of lanes of B.
T	$=$	$0, 1, 2, \ldots, T$; the time periods of the planning horizon.
x_t^α	$=$	the number of all vehicles that have crossed B in direction α since the beginning of the planning horizon until the beginning of time period t (the total throughput in direction α over t periods of time).
u_t^α	$=$	the number of lanes (the *decision variables*) specified as direction α in period t, that is, the number of lanes in which drivers can travel in direction α, $\alpha = 1, 2$ and $t = 0, 1, 2, \ldots, T-1$.
b_t^α	$=$	the net increase (number of motor vehicles) in traffic per lane travelling in direction α during period t, taking into account all vehicles arriving at and departing from B. (Note that $b_t^\alpha \geq 0; \alpha = 1, 2; t = 0, 1, 2, \ldots, T-1$, and it is assumed that T covers a busy and congested time span when traffic volumes in both directions are queuing up on F to cross B.)
β_t^α	$=$	increase in traffic (number of motor vehicles) travelling in direction α during period t. (Clearly, $\beta_t^\alpha \geq 0; \alpha = 1, 2; t = 0, 1, 2, \ldots, T-1$).
x_T^α	$=$	the total throughput of all vehicles that have travelled through B in direction α over the entire planning horizon.
J	$=$	the total throughput of all vehicles using B over the planning horizon. (Clearly, $J = x_T^1 + x_T^2$.)

Two models are developed and analysed in the paper. The first model assumes that the total traffic volume in each direction is linearly dependent upon the number of lanes specified in that direction. The increase in traffic (number of motor vehicles) β_t^α travelling in direction α during period t in the second model is a non-linear function of the number of lanes.

2.1 Linear Model

A simple model of congested bridge lane direction specification (Model 1) is described below.

Model 1

$$Maximize \quad J = x_T^1 + x_T^2 \tag{1}$$

Subject to:
Traffic flow balance equations (linear dynamics):

$$x_{t+1}^\alpha = x_t^\alpha + b_t^\alpha u_t^\alpha \qquad \alpha = 1, 2;\ t = 0, 1, 2, \ldots, T-1 \tag{2}$$

Given initial conditions on the initial flow in each direction:

$$x_0^1 \qquad \text{and} \qquad x_0^2 \tag{3}$$

Restrictions on the number of lanes:

$$u_t^1 + u_t^2 = m \qquad t = 0, 1, 2, \ldots, T-1 \tag{4}$$

The fact that least one lane must remain open in each direction:

$$u_t^\alpha \geq 1 \qquad \alpha = 1, 2;\ t = 0, 1, 2, \ldots, T-1. \tag{5}$$

2.2 A Model that Allows for Arbitrary Traffic Volume Relationships

In Model 1, it is assumed that the total traffic volumes in each direction are linearly dependent (although this dependence might vary in time) upon the number of lanes specified in that direction. We now make the assumption of a non-linear relationship between traffic volume and lane specification that also depends upon the direction α, and the time period t, namely the increase in traffic (number of motor vehicles) travelling in direction α during period t is a non-linear function β_t^α, of the number of lanes u_t^α, for $\alpha = 1, 2$, and $t = 0, 1, 2, \ldots, T-1$. Following the same reasoning as in the formulation of Model 1:

Model 2

$$Maximize \qquad J = x_T^1 + x_T^2 \tag{6}$$

Subject to:
Non-linear traffic flow balance equations :

$$x_{t+1}^\alpha = x_t^\alpha + \beta_t^\alpha(u_t^\alpha) \qquad \alpha = 1, 2;\ t = 0, 1, 2, \ldots, T-1 \tag{7}$$

$$x_0^1 \qquad \text{and} \qquad x_0^2 \tag{8}$$

$$u_t^1 + u_t^2 = m \qquad t = 0, 1, 2, \ldots, T-1 \tag{9}$$

$$u_t^\alpha \geq 1 \qquad \alpha = 1, 2;\ t = 0, 1, 2, \ldots, T-1. \tag{10}$$

It is often assumed in traffic engineering (see for example, Foulds [7]) that β_t^α is a quadratic function of u_t^α, of the form:

$$\beta_t^\alpha = b_t^\alpha [u_t^\alpha]^2 \tag{11}$$

For studies that involve more general functions of β_t^α, the reader is referred to Daganzo [2, 3].

2.3 Discussion

Models 1 and 2 described above, exhibit positive systems behaviour. The common property of positive systems is that their trajectories lie entirely in the non-negative orthant whenever the initial state is non-negative, (see for example, Kaczorek [12] or Luenberger [13]). Control theory of positive systems is a rapidly developing area of research and many properties of positive systems have recently been revealed. Controllability (Rumchev and James [16]) is a property of the system that shows its ability to move in space and has direct implications in many optimization problems. The two-point boundary-value optimal control problem, for example, might not have a solution if the system is not controllable. Model 1 above represents a time-variant discrete-time positive linear system with a free terminal state. Reachability and controllability properties of such systems have been studied by Rumchev and Adeanu [17], who have developed reachability, null-controllability, and controllability criteria to test these properties. Criteria for testing reachability and controllability properties of discrete-time positive non-linear systems (like Model 2, for example) have not been established yet. The main focus in the next section is on the solution of the optimal control problems formulated above.

3 The Optimal Control Problems

As mentioned in Section 2, Equations (2) with the given initial condition (3), subject to the restrictions (4) and (5), represent a time-varying discrete-time positive system. Using the standard concepts and notation of control theory, Model 1 can be rewritten as:

Optimal Control Problem 1 (OC1)

$$\max_{u} J(\mathbf{x_o}, \mathbf{u}) = \mathbf{1}.\mathbf{x_T} \tag{12}$$

Subject to:

$$\mathbf{x}_{t+1} = \mathbf{x}_t + \mathbf{B}\mathbf{u}_t \qquad t = 0, 1, 2, \ldots, T-1, \tag{13}$$

$$\text{with} \qquad \mathbf{x}(0) = \mathbf{x}_o \tag{14}$$

$$\mathbf{1}.\mathbf{u_t} = m, \qquad \mathbf{1} = (1, 1) \tag{15}$$

and

$$\mathbf{u}_t \geq \mathbf{1}', \qquad \text{integer vectors.} \tag{16}$$

where $\mathbf{x}_t = (x_t^1, x_t^2)'$ is the state of the system, $\mathbf{u}_t = (u_t^1, u_t^2)' \geq 0$ is the decision (control) vector, $\mathbf{u} = (u_0^1, u_0^2, u_1^1, u_1^2 \ldots, u_{T-1}^1, u_{T-1}^2)'$ is the expanded decision vector, $\mathbf{B} = diag\{b_t^1, b_t^2\}$ is the control matrix, and the symbol " $'$ " denotes the transpose.

Problem (12) – (16) is an optimal control problem of a time-variant discrete-time linear system with integer decision variables. It can be approached by general schemes based on Pontryagin's discrete maximum principle (with relaxation on the integral constraints) or on the principle of optimality of dynamic programming (Sethi and Thompson [20]). Because of the specific structure of the constraints (13) – (16), OC1 has an integer solution, provided b_t^α are integers. Moreover, it is possible to decompose this optimal control problem into T small-scale linear programming problems with simple solutions, as shown below.

A substitution of (13) – (16) into the objective functional $J(\mathbf{x}_o, \mathbf{u})$ reduces OC1 to the following linear program:

$$\max_{u} \{J(\mathbf{x_o}, \mathbf{u}) = x_0^1 + x_0^2 + b_0^1 u_0^1 + b_0^2 u_0^2 + b_1^1 u_1^1 + b_1^2 u_1^2 + \ldots \\ + b_{T-2}^1 u_{T-2}^1 + b_{T-2}^2 u_{T-2}^2 + b_{T-1}^1 u_{T-1}^1 + b_{T-1}^2 u_{T-1}^2 | \text{ (15) and (16)}\} \tag{17}$$

Consider, now, the problem represented by (17) above. Constraints (15) – (16) represent identical line segments $[(m-1, 1), (1, m-1)]$ in the non-negative quadrants, on the (u_t^1, u_t^2) – planes. Thus, all admissible u_t^1 and u_t^2 are positive. Also, the initial conditions x_0^1 and x_0^2 are non-negative. The objective function in (17), to be maximized, is a linear function in u_t^1 and u_t^2 with positive coefficients. Therefore, problem (17) is equivalent to the following T problems in two variables.

$$\max_{u_t^1, u_t^2} J_t(u_t^1, u_t^2) = b_t^1 u_t^1 + b_t^2 u_t^2$$

Subject to:

$$u_t^1 + u_t^2 = m \qquad u_t^1, u_t^2 \geq 1 \qquad t = 0, 1, 2, \ldots, T-1, \tag{18}$$

with integer decision variables u_t^1 and u_t^2.

It is straightforward to show that the following solutions are optimal:

$$\begin{aligned} u_t^1 &= 1 \ \text{ and } \ u_t^2 = m-1, \ \text{ if } \ b_t^1 < b_t^2, \\ u_t^1 &= m-1 \ \text{ and } \ u_t^2 = 1, \ \text{ if } \ b_t^1 > b_t^2, \ \text{and} \\ u_t^1 &= \lfloor m/2 \rfloor \ \text{ and } \ u_t^2 = \lceil m/2 \rceil, \ \text{ if } \ b_t^1 = b_t^2. \end{aligned} \tag{19}$$

We conclude this subsection with the observation that the discrete-time maximum principle (with relaxation on the integer variables) and the dynamic programming approach lead to the same decomposition of the original optimal control problem into a number of small-scale, static LP problems.

Model 2 is now rewritten in a similar way. It is an optimal control problem of time-varying discrete-time positive non-linear system with a free terminal state.

Optimal Control Problem 2 (**OC2**)

$$\max_{u_t^1, u_t^2} J(\mathbf{x_o}, \mathbf{u}) = \mathbf{1}.\mathbf{x_T} \tag{20}$$

Subject to:

$$\mathbf{x}_{t+1} = \mathbf{x}_t + \mathbf{B}(\mathbf{u_t}) \qquad t = 0, 1, 2, \ldots, T-1, \tag{21}$$

$$\text{with} \qquad \mathbf{x}(0) = \mathbf{x}_o \tag{22}$$

$$\mathbf{1}.\mathbf{u_t} = m, \qquad \mathbf{1} = (1,1) \tag{23}$$

and

$$\mathbf{u}_t \geq \mathbf{1}', \qquad \text{integer vectors.} \tag{24}$$

where $\mathbf{x}_t = (x_t^1, x_t^2)'$ is the state of the system, $\mathbf{u}_t = (u_t^1, u_t^2)' \geq 0$ is the decision (control) vector, $\mathbf{u} = (u_0^1, u_0^2, u_1^1, u_1^2 \ldots, u_{T-1}^1, u_{T-1}^2)'$ is the expanded decision vector, $\mathbf{B}(\mathbf{u_t}) = diag\{b_t^1(u_t^1), b_t^2(u_t^2)\}$ is the control matrix, and the symbol " $'$ " denotes the transpose.

The substitution of the dynamic equations (21) with the initial condition (22) into the objective function (20) reduces OC2 to the following non-linear optimization problem.

$$\begin{aligned} \max_{u_t^1, u_t^2} \{J(\mathbf{x_0^1}, \mathbf{x_0^2}, \mathbf{u}) &= x_0^1 + x_0^2 + b_0^1(u_0^1) + b_0^2(u_0^2) + b_1^1(u_1^1) + b_1^2(u_1^2) + \ldots \\ &+ b_{T-2}^1(u_{T-2}^1) + b_{T-2}^2(u_{T-2}^2) + b_{T-1}^1(u_{T-1}^1) + b_{T-1}^2(u_{T-1}^2) \end{aligned} \tag{25}$$

Subject to (23) and (24)$\}$

Following a similar reasoning to that used earlier, it is straightforward to establish that the large-scale non-linear optimization problem (25) is equivalent to T smaller, non-linear optimization problems of the following type:

$$\max_{u_t^1, u_t^2} \{J_t(u_t^1, u_t^2) = b_t^1(u_t^1) + b_t^2(u_t^2) | u_t^1 + u_t^2 = m; \ u_t^1, u_t^2 \geq 1, \ t = 0, 1, 2, \ldots, T-1\} \tag{26}$$

The constraints on the controls in (26) are identical line segments $[(m-1, 1), (1, m-1)]$ in the non-negative quadrants on the (u_t^1, u_t^2) – planes. Thus, (26) can be reduced to:

$$\max_{u_t^1} \ \{J_t(u_t^1) = b_t^1(u_t^1) + b_t^2(m - u_t^1) | u_t^1 \in [1, m], \ t = 0, 1, 2, \ldots, T-1\}. \tag{27}$$

Assuming that the functions $b_t^\alpha(u_t^\alpha)$ are differentiable for $\alpha = 1, 2$ and $t = 0, 1, 2, \ldots, T-1$, all critical points in the interior, that is $u_t^1 \in (1, m)$, are the solutions $\{\overline{u_t^1}\}$ of the equation:

$$\frac{d}{du_t^1}(b_t^1(u_t^1)) + \frac{d}{du_t^1} b_t^2(m - u_t^1) = 0 \qquad \text{on the interval } (1, m). \tag{28}$$

Since the intervals in (27) are closed and bounded, the maximal value of the objective function $J_t(u_t^1)$ on $[1, m]$ can be found by comparing the objective function values at the critical points $\overline{u_t^1}$ in the interior and at the end-points of the interval, i.e. at $u_t^1 = 1$ and at $u_t^1 = m$ (a finite number of points). If $b_t^\alpha(u_t^\alpha)$ are quadratic functions, the equation (28) is clearly linear, so it is straightforward to obtain its solution. Otherwise, (28) is a non-linear equation and, in general, numerical methods are usually used to find the critical points. The maximal value of $J_t(u_t^1)$ on $[1, m]$ can also be determined by using any of a number of well-known one-dimensional search methods, such as bisection search or Fibonacci search. These methods often work well even when the functions $b_t^\alpha(u_t^\alpha)$ are not differentiable on $[(m-1, 1), (1, m-1)]$. Further, adding the integral constraint (24) to (27) simplifies the finding of the minimal value of $J_t(u_t^1)$.

Problem OC2 is a relaxed optimal control problem; that is, it is equivalent to Model 2 without the integral constraints (7). Also, the application of the discrete maximum principle or principle of optimality of dynamic programming to the same optimal control problem leads to the same decomposition of the original problem into T static problems of the form of (27). However, the scheme based on the discrete maximum principle cannot incorporate the integral constraint, as is done above.

4 Numerical Illustrations

A 5-laned bridge, which we denote as usual by 'B', connecting the north part (N) of a city to the south part (S) of the city experiences different levels of traffic in different time periods. Table 4 below gives detailed information about the traffic present in the weekday morning periods.

Table 1: Detailed information about the numerical illustrations

Time Period (t=0,1,2,3,4,5)	Traffic level towards N ($\alpha = 1$)	Traffic level towards S ($\alpha = 2$)	Net increase in traffic towards N(b_t^1)	Net increase in traffic towards S(b_t^2)
6.30am – 7am (t=0)	Low	High	700	2900
7am – 7.30am (t=1)	High	Low	2500	1800
7.30am – 8am (t=2)	Very High	Low	4000	1500
8am – 8.30am (t=3)	High	Low	2500	1400
8.30am – 9am (t=4)	High	Low	2000	1200
9am – 9.30am (t=5)	Low	Low	1500	1000

Table 2: The optimal solution to the first numerical illustration

Time period	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
	1	4	4	4	4	4
	4	1	1	1	1	1
	500	1200	11200	27200	37200	45200
	1500	13100	14900	16400	17800	19000

Note that the traffic during a given time interval is categorised as low traffic if the net increase of the number of vehicles per lane is up to 1999; as high traffic if the number of vehicles per lane is between 2000 and 2999 and very high when this number is over 3000 vehicles per lane. At the beginning of zero period ($t = 0$) 500 and 1500 vehicles were observed on F at the entrance of B, travelling towards N and S respectively, so that $x_0^1 = 500$ and $x_0^2 = 1500$. The net increase (see Table 4) in traffic on B towards N and S was 700 and 2900 respectively, that is $b_0^1 = 700$ and $b_0^2 = 2900$.

The time-variant discrete-time optimal control problem OC1 for the set of data in Table 4 was solved first as a linear program (17). The solution, that is the numbers of lanes in each direction and the corresponding throughputs obtained by LINDO, is given in Table 4.

At the beginning of period 6, $x_6^1 = 51200$ and $x_6^2 = 20000$ so that the maximal value of the objective function (the maximal total throughput over the horizon of planning) is $J = 71200$. Then the problem with the same input data was decomposed into six smaller-scale LP problems (18), which were solved again by LINDO. The results were exactly the same as those in Table 4 and the maximal total throughput for the given planning horizon was once again $J = 71200$, as expected. Suppose that, a result of rapidly increased business and industrial activities in the city, the traffic volumes have significantly increased. The new measurements have indicated an increase in the number of motor vehicles travelling in both directions and the relationships between the increases in the traffic in terms of the number of lanes have been identified as quadratic functions of the form of (11):

$$\beta_t^\alpha = b_t^\alpha (u_t^\alpha)^2, \alpha = 1, 2 \text{ and } t = 0, 1, 2, \dots, 5,$$

while the increases in the number of vehicles per lane, that is b_t^α, have remained the same, they are very much determined by the construction of the "freeway-bridge-freeway" system. Now, to specify the directions of lanes in the different time intervals in the new scenario, the operation analysis team has to solve optimal control problem OC 2. The number of vehicles observed at the beginning of period $t = 0$ is the same as before, that is, $x_0^1 = 500$ travelling towards N and $x_0^2 = 1500$ travelling towards S. Since the increases in the number of vehicles per lane have remained the same, Table 4 represents the data needed to solve problem OC 2.

The time-variant discrete-time optimal control problem OC 2 for the set of data in Table 4 was solved first as a non-linear program (3.12–3.14). The optimal numbers of lanes in each direction and the corresponding throughputs obtained by AMPL are given in Table 4.

At the beginning of period 6, $x_6^1 = 201200$ and $x_6^2 = 54800$. Thus, the maximal total throughput over the planning horizon is $J = 256000$. The problem with the same input data was decomposed into six smaller non-linear problems (26), which were solved again by AMPL. The results are the same as those in Table 4 and the maximal total throughput for the given planning horizon is again $J = 256000$.

The above computational experience shows that numerical instances of practical size

Table 3: The optimal solution to the second numerical illustration

Time period	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
	1	4	4	4	4	4
	4	1	1	1	1	1
	500	1200	41200	105200	145200	177200
	1500	47900	49700	51200	52600	53800

are amenable to solution by available commercial software. This software makes sensitivity analysis possible. The decomposition procedure enables the solution of large-scale problems, which is particularly important in the case of Model 2.

5 Conclusions

We have developed dynamic models of bridge lane direction specification. The models are motivated, not only by the need for policies, but also by the need to incorporate temporal considerations and to open the way for deductive analysis. Some interesting characterizations of lane direction management that are important in traffic planning and control appear in the models. These characteristics can be quite helpful in the decision-making process.

Model 1 belongs to the class of time-variant discrete-time positive linear systems. The theory for such systems is still under development. Model 1 gives rise to a number of interesting theoretical questions such as: (i) How do the integral values of controls affect the reachability and controllability criteria developed for systems with continuous controls?, (ii) How do additional linear constraints on controls affect the reachability and controllability properties?, and (iii) What is a reachable set of positive systems with integer controls? A solution technique for solving the optimal control problem of maximizing the total throughput over a given planning horizon (OC 1) is developed in the present article. Due to the structure of the relaxed problem the solution is integer. Moreover, it is shown that the optimal control problem can be decomposed into a number of small static integer linear programming problems. The decomposition approach may be of importance in large-scale problems which could arise when the number of periods in the planning horizon is very large and also in the case of real-time management.

Model 2 represents a discrete-time positive non-linear system. A solution technique for solving the optimal control problem of maximizing the total throughput over a given planning horizon (OC 2) is proposed. It is shown that the optimal control problem can be decomposed into a number of problems of maximizing a non-linear objective function of a single variable over closed and bounded sets (line segments). To study the reachability and controllability properties of Model 2, which are important in certain optimal control problems, the properties of the functions representing the net increase in traffic per lane travelling in each direction in each time period must be specified.

It has been demonstrated that it is possible to solve large problems and perform sensitivity analysis by means of commercial software that is readily available. For very long bridges, such as the Ponte de Rio-Niteroi in Rio de Janeiro, Brasil, which is over 15km in length, the traversal time becomes significant and must be taken into account. Incorporating the long traversal time in the dynamic model of bridge lane specification leads to a positive non-linear system with time-delays. We plan to publish the solution techniques that we have developed for these problems elsewhere.

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