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IMPOSSIBILITY AND POSSIBILITY THEOREMS FOR SOCIAL CHOICE FUNCTIONS ON INCOMPLETE PREFERENCE PROFILES

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Dedicated to Professor Masakazu Kojima on the occasion of his 60th birthday.

Abstract: Given a set of individuals and a set of alternatives, an ordinary social choice function is a rule to choose one alternative aggregating preference orders on the alternatives reported by the individuals. We generalize this notion of social choice function where some individual's preference order may be incomplete: each individual reports his preference on his alternative set that is a subset of the whole set of alternatives. Axioms like strategy-proofness and dictatorship are suitably redefined and we show that these two notions are incompatible. In addition, we weaken the definition of strategy-proofness as well as dictatorship, and show the existence of a social choice function with desirable properties.

Key words: *social choice function, impossibility theorem, mutual evaluation, strong positive association, strategy-proofness, dictatorship, independence of irrelevant alternatives, Pareto principle*

Mathematics Subject Classification: *91B14, 91B08, 91B12*

1 Introduction

Given a finite set N of individuals and a finite set X of alternatives, a social choice function is a rule to choose one alternative aggregating preference orders on X reported by the individuals $i \in N$, where we assume that a preference order is a complete and transitive binary relation. Such an aggregation rule is required to meet some reasonable criteria. A social choice function is strategy-proof if no individual can manipulate the social choice to obtain a better outcome by misreporting his preference over the alternatives. Also, a social choice function satisfies non-dictatorship if there is no individual such that the social choice is always among the best alternatives with respect to the preference order he reports. Gibbard [2] and Satterthwaite [4] showed an impossibility theorem that there exists no social choice function satisfying both non-dictatorship and strategy-proofness.

In some practical contexts, it is perhaps unrealistic to expect each individual in the society to report a preference order on the whole set of alternatives: some individual may not know about or may not be interested in some alternatives. Also, in some context it may be the case that some individual is allowed to express his preference order only on a proper subset of the alternatives. In this paper we study existence of a social choice function with desirable properties when some individual's preference order may be incomplete: each individual i reports his preference on his alternative set X_i that is a subset of the whole set of alternatives.

We propose two notions of strategy-proofness according to the context. If we suppose each individual is interested only in his alternative set and so he reports his preference order only on that set, then we have a weak notion of strategy-proofness: if the current social choice is in X_i for some individual i, then individual i cannot manipulate the social choice to obtain a better outcome by misreporting his preference over X_i. On the other hand, if we suppose each individual is interested in all the alternatives but is allowed to report his preference order only on X_i, then we are led to a strong notion of strategy-proofness, where we assume, in addition to the condition for the strategy-proofness in the weak sense, that if the current social choice is not in X_i, then individual i cannot *change* the social choice and if the current social choice is in X_i, then the social choice remains in X_i whatever preference order he may report.

We first show that a social choice function satisfying strategy-proofness (in the strong sense) is dictatorial. It is also shown that generalized strong positive association and strategy-proofness are equivalent. We then consider the weaker notion of strategy-proofness, which we call weak strategy-proofness, and show the existence of a social choice function which is weakly strategy-proof and non-dictatorial. We finally consider a property called independence of irrelevant alternatives and show that a weakly strategy-proof social choice function that has this property is necessarily dictatorial.

The rest of this paper is organized as follows. In Section 2, we review the Gibbard-Satterthwaite theorem and introduce the framework and notations. In Section 3, we give an impossibility theorem when some individual's preference order is incomplete. In Section 4, we focus on generalized strong positive association and show its equivalence to strategy-proofness. In Section 5, we introduce the notion of weak strategy-proofness as well as weak dictatorship and present a possibility theorem. In Section 6, we show that any weakly strategy-proof social choice function that satisfies independence of irrelevant alternatives satisfies generalized strong positive association, and hence, is dictatorial. Section 7 summarizes the results.

2 The Framework and Notations

Let us consider a society consisting of a finite number, say n, of individuals. Each individual in the society has his own preference on the set X of a finite number of alternatives. The problem faced by the society is to aggregate the individual preferences into a society's choice. A rule of choice is called a *social choice function.* In the framework of Gibbard [2] and Satterthwaite [4], each individual in the society is interested in the whole set X of alternatives, and his preference is defined as a preference order on X. A *preference order*, denoted by $\succsim$, is a binary relation on X satisfying

(i) completeness: $x \succsim y$, $y \succsim x$ or both hold for any pair of alternatives $x, y \in X$, and

(ii) transitivity: if $x \succsim y$ and $y \succsim z$, then $x \succsim z$ holds for any alternatives $x, y, z \in X$.

We say that x is *weakly preferred* to y when $x \succsim y$. We write $x \sim y$ when both $x \succsim y$ and $y \succsim x$ hold and say that x is *indifferent* to y. When $x \succsim y$ and $y \not\succsim x$ we write $x \succ y$, reading that x is *strictly preferred* to y. For a subset $Y \subseteq X$, we denote by $\succsim|Y$ the restriction of binary relation $\succsim$ to Y, i.e., $\succsim|Y$ is defined on $Y \times Y$ and $x \succsim|Y y$ if and only if $x \succsim y$ and $x, y \in Y$.

Let $\mathcal{W}$ denote the set of all preference orders on X and let $\mathcal{W}^n$ denote its n-ary Cartesian product. We call an element, denoted by $\succsim^p$ or simply by p, of $\mathcal{W}^n$ a *profile*, which is a combination of preference orders $\succsim_i^p$ of individuals $i \in N$. Then, a social choice function is a function that assigns an alternative of X to each profile. Throughout this paper we denote the set of individuals by $N = \{1, 2, \ldots, n\}$ and assume that $n \geq 2$ except in the context of mutual evaluation, where $n \geq 3$ is assumed.

Gibbard and Satterthwaite require strategy-proofness as one of the properties that a social choice function should have. Strategy-proofness and dictatorship are defined by Gibbard and Satterthwaite as follows.

Definition 2.1 (Strategy-proofness in Gibbard-Satterthwaite's Framework). We say that a social choice function $C\colon \mathcal{W}^n \to X$ is *strategically manipulable* by individual $i \in N$ at profile $p \in \mathcal{W}^n$ if there is a preference order $\succsim \in \mathcal{W}$ such that

$$C(p/_{-i} \succsim) \succ_i^p C(p) \tag{2.1}$$

holds, where

$$p/_{-i} \succsim = (\succsim_1^p, \ldots, \succsim_{i-1}^p, \succsim, \succsim_{i+1}^p, \ldots, \succsim_n^p).$$

When C is not strategically manipulable by any individual at any profile, it is said to be *strategy-proof.*

A social choice function is required to avoid being strategically manipulable because otherwise some individual can profit from a misrepresentation of his preference order.

For a social choice function $C\colon \mathcal{W}^n \to X$ let $R(C)$ denote the *range* of C defined by

$$R(C) = \{\, x \in X \mid x = C(p) \text{ for some } p \in \mathcal{W}^n \,\}.$$

Definition 2.2 (Dictatorship). For a social choice function $C\colon \mathcal{W}^n \to X$ an individual $k \in N$ is said to be a *dictator* for C if

$$C(p) \in \{\, x \in R(C) \mid x \succsim_k^p y \text{ for all alternatives } y \in R(C) \,\}$$

holds for each profile $p \in \mathcal{W}^n$. In other words, the society always chooses an alternative out of those that the dictator prefers best among those that are chosen at some profile. A social choice function that admits a dictator is said to be *dictatorial.*

Gibbard [2] and Satterthwaite [4] independently proved that strategy-proofness and non-dictatorship are incompatible.

Theorem 2.3 (Gibbard-Satterthwaite's Theorem). *If a social choice function $C\colon \mathcal{W}^n \to X$ satisfies strategy-proofness in Definition 2.1 and $|R(C)| \geq 3$, then it is dictatorial in the sense of Definition 2.2.*

In some practical contexts, it is perhaps unrealistic to expect each individual in the society to report his preference order on the whole set of alternatives: some individual may not know about or may not be interested in some alternatives. Also, in some context it may be the case that some individual is allowed to express his preference order only on a proper subset of the alternatives. To formulate such a situation we introduce the alternative set on which individual i expresses his preference. For $i \in N$ let X_i be a subset of X, which we call *individual i's alternative set.* We assume that $|X_i| \geq 2$ for all $i \in N$ and each alternative $x \in X$ belongs to some X_i, i.e., $X = \bigcup_{i \in N} X_i$. For each $i \in N$ let $\mathcal{W}_i$ denote the set of all preference orders on X_i. We write $\mathcal{P} = \mathcal{W}_1 \times \mathcal{W}_2 \times \cdots \times \mathcal{W}_n$. Then, a *social choice function* for this situation is a function $D\colon \mathcal{P} \to X$, i.e., D assigns an alternative in X to each profile $p \in \mathcal{P}$.

Now we introduce several definitions for such social choice functions D.

Definition 2.4 (Strategy-proofness (SP)). We say that a social choice function $D\colon \mathcal{P} \to X$ is *strategically manipulable* by individual $i \in N$ at profile $p \in \mathcal{P}$ if there is a preference order $\succsim \in \mathcal{W}_i$ such that

$$\begin{aligned}&D(p/_{-i} \succsim) \neq D(p)\\ &\text{and}\\ &D(p/_{-i} \succsim) \succ_i^p D(p) \text{ when } \{D(p/_{-i} \succsim), D(p)\} \subseteq X_i\end{aligned}$$

holds, where

$$p/_{-i} \succsim = (\succsim_1^p, ..., \succsim_{i-1}^p, \succsim, \succsim_{i+1}^p, ..., \succsim_n^p).$$

When D is not strategically manipulable by any individual at any profile, it is said to be *strategy-proof*.

We assume that each individual i is interested in the whole set of alternatives but is allowed to express his preference only on the alternatives in X_i. When $D(p)$ or $D(p/_{-i} \succsim)$ is outside X_i, there is a possibility that he might have preference $D(p/_{-i} \succsim) \succ_i^p D(p)$ at profile p, which he is not allowed to express. If this is the case, then individual i can manipulate D by misreporting his preference order $\succsim$. We adopt the above definition of strategy-proofness to exclude such a manipulation.

Definition 2.5 (Dictatorship). For a social choice function $D\colon \mathcal{P} \to X$ an individual $k \in N$ is called a *dictator* for D if

$$D(p) \in \{\, x \in R(D) \mid x \succsim_k^p y \text{ for all alternatives } y \in R(D) \,\}$$

holds for each profile $p \in \mathcal{P}$, where

$$R(D) = \{\, x \in X \mid x = D(p) \text{ for some } p \in \mathcal{P} \,\}.$$

A social choice function that admits a dictator is said to be *dictatorial*.

This definition is indeed the same as Definition 2.2, but it requires the dictator's alternative set be large enough to contain the range of the social choice function.

Definition 2.6. For a subset Y of X, $N(Y)$ denotes the set of individuals whose alternative set contains Y, i.e.,

$$N(Y) = \{\, i \in N \mid Y \subseteq X_i \,\}.$$

Definition 2.7 (Generalized Strong Positive Association (GSPA)). A social choice function $D\colon \mathcal{P} \to X$ is said to satisfy *generalized strong positive association* if it satisfies the following condition for any pair of distinct profiles p and $q \in \mathcal{P}$.

> If there exists a (possibly empty) subset of individuals $M \subseteq N(\{D(p)\})$ such that
> for all $i \in M, D(p) \succsim_i^p x$ implies $D(p) \succ_i^q x$ for all $x \in X_i \setminus \{D(p)\}$ and
> for all $j \in N(\{D(p)\}) \setminus M$, $\succsim_j^p = \succsim_j^q$,
> then $D(q) = D(p)$.

This means that if an alternative chosen by the society at profile p receives no worse evaluation from all individuals at profile q, then it is chosen also at profile q.

$\boxed{3}$ Impossibility Theorem

The main theorem of this section below states that strategy-proofness is incompatible with non-dictatorship on the domain of $\mathcal{P}$.

Theorem 3.1. *Suppose that a social choice function $D\colon \mathcal{P} \to X$ satisfies strategy-proofness (SP) in Definition 2.4 and $|R(D)| \geq 3$. Then, it is dictatorial.*

The key tools to prove the theorem are the restriction $r(p)$ of profile $p = (\succsim_1^p, \succsim_2^p, \ldots, \succsim_n^p) \in \mathcal{W}^n$ which is defined as

$$r(p) = (\succsim_1^p|X_1,\ \succsim_2^p|X_2,\ \ldots,\ \succsim_n^p|X_n), \tag{3.1}$$

and function $C\colon \mathcal{W}^n \to X$ derived from social choice function $D\colon \mathcal{P} \to X$ as

$$C(p) = D\left(r(p)\right) \text{ for each } p \in \mathcal{W}^n. \tag{3.2}$$

Lemma 3.2. *Suppose $D\colon \mathcal{P} \to X$ satisfies strategy-proofness (SP) in Definition 2.4. Then, the function $C\colon \mathcal{W}^n \to X$ defined by* (3.2) *satisfies strategy-proofness in Gibbard-Satterthwaite's sense, Definition 2.1.*

Proof. We start the proof by assuming that C does not satisfy strategy-proofness in Gibbard-Satterthwaite's sense, i.e., there exist individual $i \in N$, profile $p \in \mathcal{W}^n$ and preference order $\succsim \in \mathcal{W}$ such that

$$C(p/_{-i} \succsim) \succ_i^p C(p).$$

We will show that this assumption leads to a contradiction.

First note that

$$r(p/_{-i} \succsim) = r(p)/_{-i}(\succsim|X_i) \tag{3.3}$$

by the definitions of profile $p/_{-i} \succsim$ and restriction r. When $\{C(p), C(p/_{-i} \succsim)\} \subseteq X_i$, we have $D(r(p/_{-i} \succsim)) \succ_i^{r(p)} D(r(p))$. Therefore, by (3.3) we see $D(r(p)/_{-i}(\succsim|X_i)) \succ_i^{r(p)} D(r(p))$, which contradicts strategy-proofness (SP) of D in Definition 2.4.

When $\{C(p), C(p/_{-i} \succsim)\} \not\subseteq X_i$, we have $\{D(r(p)), D(r(p/_{-i} \succsim))\} \not\subseteq X_i$. We see $D(r(p)) \neq D(r(p/_{-i} \succsim))$ since $C(p) \neq C(p/_{-i} \succsim)$. By (3.3), these facts again contradict (SP) of D in Definition 2.4. $\square$

Proof of Theorem 3.1.

Since

$$\{\, r(p) \mid p \in \mathcal{W}^n \,\} = \mathcal{P}, \tag{3.4}$$

we see $R(C) = R(D)$ and $|R(C)| = |R(D)| \geq 3$. This and Lemma 3.2 show that C defined by (3.2) is dictatorial by Gibbard-Satterthwaite Theorem, Theorem 2.3. Namely, there is a dictator $k \in N$ for C, which satisfies

$$C(p) \in \{\, x \in R(C) \mid x \succsim_k^p y \text{ for all } y \in R(C) \,\}$$

for any profile $p \in \mathcal{W}^n$. We will show that this individual k belongs to $N(R(D))$ and is a dictator for D.

Assume that $R(C) \not\subseteq X_k$, choose arbitrarily an alternative $x \in R(C) \setminus X_k$, and consider a profile $p_1 \in \mathcal{W}^n$ such that

$$x \succ_k^{p_1} y \text{ for any } y \in R(C) \setminus \{x\}.$$

Since individual k is a dictator for C, we see that $C(p_1) = x$. Consider another profile $p_2 \in \mathcal{W}^n$ such that

$$
\begin{aligned}
&y \succ_k^{p_2} x \text{ for some } y \in R(C) \setminus \{x\},\\
&\succsim_k^{p_2}|(X \setminus \{x\}) = \succsim_k^{p_1}|(X \setminus \{x\}) \text{ and}\\
&\succsim_j^{p_2} = \succsim_j^{p_1} \text{ for all } j \in N \setminus \{k\}.
\end{aligned}
$$

Again by the dictatorship of individual k, we see $C(p_2) \neq x$. Then, we have $D(r(p_1)) \neq D(r(p_2))$.

On the other hand, we see

$$\succsim_j^{r(p_1)} = \succsim_j^{r(p_2)} \text{ for all } j \in N \setminus \{k\}$$

since $\succsim_j^{p_1} = \succsim_j^{p_2}$ for all $j \in N \setminus \{k\}$, and also

$$\succsim_k^{r(p_1)} = \succsim_k^{r(p_2)}$$

by the construction of profiles p_1 and p_2 and the fact that $x \notin X_k$. Therefore, $r(p_1) = r(p_2)$, implying $D(r(p_1)) = D(r(p_2))$. This is a contradiction, and hence, we conclude that $R(C) \subseteq X_k$, i.e., $k \in N(R(D))$.

Therefore, we see that

$$D(r(p)) \in \{\, x \in R(D) \mid x \succsim_k^{r(p)} y \text{ for all } y \in R(D) \,\}$$

for each profile $p \in \mathcal{W}^n$. By (3.4), we conclude that for each profile $q \in \mathcal{P}$

$$D(q) \in \{\, x \in R(D) \mid x \succsim_k^{q} y \text{ for all } y \in R(D) \,\},$$

i.e., individual k is a dictator for D. □

$\boxed{4}$ Equivalence of Generalized Strong Positive Association and Strategy-proofness

We will prove in this section the equivalence of generalized strong positive association (GSPA) and strategy-proofness (SP).

Theorem 4.1. *A social choice function $D \colon \mathcal{P} \to X$ satisfies strategy-proofness (SP) in Definition 2.4 if and only if it satisfies generalized strong positive association (GSPA) in Definition 2.7.*

Proof of the Sufficiency. Suppose that there exists a social choice function D that satisfies (GSPA) but not (SP). Then, there exist individual $i \in N$, profile $p \in \mathcal{P}$ and preference $\succsim \in \mathcal{W}_i$ such that

$$
\begin{aligned}
&D(p/_{-i} \succsim) \neq D(p)\\
&\text{and}\\
&D(p/_{-i} \succsim) \succ_i^p D(p) \text{ when } \{D(p/_{-i} \succsim), D(p)\} \subseteq X_i.
\end{aligned}
$$

Let us denote the profile $p/_{-i} \succsim$ by q for the sake of simplicity.

First we consider the case where $\{D(q), D(p)\} \subseteq X_i$ and $D(q) \succ_i^p D(p)$. Partition X_i into two subsets

$$U = \{\, x \in X_i \mid x \succ_i^p D(p) \,\} \quad \text{and} \quad V = \{\, x \in X_i \mid D(p) \succsim_i^p x \,\}.$$

Note that $D(p) \in V$ and $D(q) \in U$. Then, choose a preference $\succsim' \in \mathcal{W}_i$ such that

$$D(q) \succ' x \text{ for all } x \in X_i \setminus \{D(q)\} \quad \text{and} \quad D(p) \succ' x \text{ for all } x \in V \setminus \{D(p)\}.$$

Let us denote $p/_{-i} \succsim'$ by q', and consider social choice $D(q')$ at profile q'. Since $D(p) \succsim_i^p x$ implies $D(p) \succ' x$ for all $x \in X_i \setminus \{D(p)\}$ and $\succsim_j^p = \succsim_j^{q'}$ for all $j \in N(\{D(p)\}) \setminus \{i\}$, we see

$$D(q') = D(p) \tag{4.1}$$

by (GSPA) in Definition 2.7. Also, observe that $D(q) \succsim_i^q x$ implies $D(q) \succ' x$ for all $x \in X_i \setminus \{D(q)\}$ and $\succsim_j^q = \succsim_j^{q'}$ for all $j \in N(\{D(q)\}) \setminus \{i\}$. These imply by (GSPA) in Definition 2.7 that

$$D(q') = D(q),$$

which by (4.1) yields $D(p) = D(q)$, a contradiction.

Next, we consider the case where $\{D(p), D(q)\} \not\subseteq X_i$. When $D(p) \notin X_i$, we have $i \notin N(\{D(p)\})$. Note that

$$\succsim_j^q = \succsim_j^p \text{ for all } j \in N(\{D(p)\})$$

since $\succsim_j^p = \succsim_j^q$ for all $j \in N \setminus \{i\}$. Applying (GSPA) in Definition 2.7, we see that $D(q) = D(p)$, which is a contradiction. When $D(q) \notin X_i$, we have $i \notin N(\{D(q)\})$, implying that

$$\succsim_j^p = \succsim_j^q \text{ for all } j \in N(\{D(q)\}).$$

Therefore, we have $D(p) = D(q)$ by (GSPA) in Definition 2.7, which again contradicts $D(p) \neq D(q)$. □

Proof of the Necessity. Suppose that there exists a social choice function D that satisfies (SP) but not (GSPA). Then, there are two distinct profiles $p, q \in \mathcal{P}$ and a subset $M \subseteq N(\{D(p)\})$ of individuals such that

$$\begin{aligned} &D(p) \succsim_i^p x \text{ implies } D(p) \succ_i^q x \text{ for all } x \in X_i \setminus \{D(p)\} \text{ and for all } i \in M, \\ &\succsim_j^p = \succsim_j^q \text{ for all } j \in N(\{D(p)\}) \setminus M, \text{ and} \\ &D(q) \neq D(p). \end{aligned}$$

Let $M = \{1, \ldots, m\}$ and $N \setminus N(\{D(p)\}) = \{m+1, \ldots, l\}$ $(n \geq l \geq m \geq 0)$ by renumbering if necessary. For $j = 0, 1, \ldots, l$ define $r_j \in \mathcal{P}$ by

$$r_j = (\succsim_1^q, \ldots, \succsim_{j-1}^q, \succsim_j^q, \succsim_{j+1}^p, \ldots, \succsim_{l-1}^p, \succsim_l^p, \ldots, \succsim_n^p) \tag{4.2}$$

and consider the sequence $(r_0, r_1, \ldots, r_l)$ of $l+1$ profiles. Since $\succsim_i^p = \succsim_i^q$ for all $i \in N(\{D(p)\}) \setminus M = \{l+1, \ldots, n\}$, we have $r_l = q$, and hence, $D(r_l) = D(q)$. Then, there exists an individual, say $k \in M \cup (N \setminus N(\{D(p)\})) = \{1, \ldots, l\}$, such that $D(r_{k-1}) = D(p)$ and $D(r_k) \neq D(p)$. Let $D(r_k) = w$ and note that w might be equal to $D(q)$. Concerning k, the following two Cases A and B are possible.

Case A: $k \in M$.

Three possibilities should be considered for the preference of individual k at profile r_{k-1} between $D(p)$ and w: $D(p) \succsim_k^{r_{k-1}} w$, $w \succ_k^{r_{k-1}} D(p)$ and $\{D(p), w\} \not\subseteq X_k$. We will show that each of the three possibilities leads to a contradiction.

1. If $D(p) \succsim_k^{r_{k-1}} w$, then we see $D(p) \succ_k^{r_k} w$ since $D(p) \succsim_k^p x$ implies $D(p) \succ_k^q x$ for all $x \in X_k \setminus \{D(p)\}$. This means that $D(r_k/_{-k} \succsim_k^p) = D(r_{k-1}) \succ_k^{r_k} D(r_k)$, which is contrary to (SP), Definition 2.4.
2. If $w \succ_k^{r_{k-1}} D(p)$, then we have that $D(r_{k-1}/_{-k} \succsim_k^q) = D(r_k) \succ_k^{r_{k-1}} D(r_{k-1})$, which is also contrary to (SP), Definition 2.4.
3. If $\{D(p), w\} \not\subseteq X_k$, then this together with $D(r_{k-1}) = D(p) \neq w = D(r_k)$ contradicts (SP), Definition 2.4 since $\succsim_j^{r_{k-1}} = \succsim_j^{r_k}$ for all $j \in N \setminus \{k\}$.

Case B: $k \in N \setminus N(\{D(p)\})$.

Since $D(r_{k-1}) = D(p)$, we see that $D(r_{k-1}) \notin X_k$, meaning that $\{D(r_{k-1}), D(r_k)\} \not\subseteq X_k$. This together with $D(r_{k-1}) \neq D(r_k)$ contradicts (SP), Definition 2.4 since $\succsim_j^{r_{k-1}} = \succsim_j^{r_k}$ for all $j \in N \setminus \{k\}$. □

By the equivalence of the two properties we have the following corollary from Theorem 3.1.

Corollary 4.2. *Suppose that a social choice function $D\colon \mathcal{P} \to X$ satisfies generalized strong positive association (GSPA) in Definition 2.7 and $|R(D)| \geq 3$. Then, it is dictatorial.*

We close this section by noting that a surjective social choice function satisfying generalized strong positive association has a property called the weak Pareto principle.

Definition 4.3 (Weak Pareto Principle (WPP)). A social choice function $D\colon \mathcal{P} \to X$ is said to satisfy the *weak Pareto principle* if it holds that

$$x \succ_k^p y \text{ for all } k \in N(\{x, y\}) \text{ implies } D(p) \neq y$$

for any pair of distinct alternatives $x, y \in N$ and for any profile $p \in \mathcal{P}$.

Lemma 4.4. *If a social choice function $D\colon \mathcal{P} \to X$ satisfies (GSPA) in Definition 2.7 and $R(D) = X$, then it satisfies (WPP) in Definition 4.3.*

Proof. We suppose that there exists a social choice function D that satisfies (GSPA) and $R(D) = X$ but not (WPP). Then, there exist a profile $p \in \mathcal{P}$ and distinct alternatives $x, y \in X$ such that

$$x \succ_k^p y \text{ for all } k \in N(\{x, y\}) \text{ and } D(p) = y.$$

Consider a profile $q \in \mathcal{P}$ such that

$$\begin{array}{ll} x \succ_k^q z & \text{for all } k \in N(\{x\}) \setminus N(\{x, y\}) \text{ and for all } z \in X_k \setminus \{x\}, \\ y \succ_k^q z & \text{for all } k \in N(\{y\}) \setminus N(\{x, y\}) \text{ and for all } z \in X_k \setminus \{y\}, \text{ and} \\ x \succ_k^q y \succ_k^q z & \text{for all } k \in N(\{x, y\}) \text{ and for all } z \in X_k \setminus \{x, y\}. \end{array}$$

Observe that $y \succsim_k^p z$ implies $y \succ_k^q z$ for all $k \in N(\{y\})$ and $z \in X_k \setminus \{y\}$. Hence, by applying (GSPA) to profiles p and q we obtain

$$D(q) = D(p). \tag{4.3}$$

Since $x \in X = R(D)$, there exists a profile $s \in \mathcal{P}$ such that $D(s) = x$. Between s and q we observe that $x \succsim_k^s z$ implies $x \succ_k^q z$ for all $k \in N(\{x\})$ and $z \in X_k \setminus \{x\}$. Therefore, by applying (GSPA) to profiles s and q we have

$$D(q) = D(s). \tag{4.4}$$

From (4.3) and (4.4) we have $x = y$, which is a contradiction. □

$\boxed{5}$ Weak Strategy-Proofness

We have assumed thus far that each individual i is interested in all the alternatives but is allowed to express his preference order only on his set X_i of alternatives. In some context, however, it is natural to assume that each individual is interested only in his own set X_i of alternatives. For such a situation we redefine strategy-proofness (SP) in Definition 2.4 as well as dictatorship in Definition 2.5.

Definition 5.1 (Weak Strategy-proofness (WSP)). We say that a social choice function $D\colon \mathcal{P} \to X$ is *strongly strategically manipulable* by individual $i \in N$ at profile $p \in \mathcal{P}$ if there is a preference order $\succsim \in \mathcal{W}_i$ such that

$$D(p/_{-i} \succsim) \succ_i^p D(p)$$

holds, where

$$p/_{-i} \succsim = (\succsim_1^p, ..., \succsim_{i-1}^p, \succsim, \succsim_{i+1}^p, ..., \succsim_n^p).$$

When D is not strongly strategically manipulable by any individual at any profile, it is said to be *weakly strategy-proof.*

Note that D is not strongly strategically manipulable by individual i at profile p when $D(p)$ or $D(p/_{-i} \succsim)$ is outside of X_i.

Definition 5.2 (Weak Dictator). For a social choice function $D\colon \mathcal{P} \to X$ an individual $k \in N$ is called a *weak dictator for D* if

$$D(p) \in \{\, x \in R(D) \mid \text{there does not exist } y \in R(D) \text{ such that } y \succ_k^p x \,\}$$

holds for each profile $p \in \mathcal{P}$.

This definition, which does not require the dictator's alternative set contains the range of the social choice function, forms a contrast to Definition 2.5.

Since (SP) in Definition 2.4 implies (WSP) in Definition 5.1, the set of social choice functions satisfying (WSP) contains the set of those satisfying (SP). In the meanwhile, the set of social choice functions that do not admit a weak dictator in Definition 5.2 is a subset of those that are not dictatorial in the sense of Definition 2.5. We have seen in Theorem 3.1 that the dotted rectangle and the dotted ellipsoid do not intersect in Figure 1. This remains true when we confine the situation to the following "mutual evaluation."

Consider a signed ballot voting procedure for the chairperson of a committee where each member is not allowed to vote for himself. The set of alternatives is identical to the set of members, i.e., $X{=}N$, and each member reports a preference order on all the members but himself, that is,

$$X_i = N \setminus \{i\}.$$

We call such a situation *mutual evaluation.* To avoid confusion we denote a social choice function for mutual evaluation by D_m.

In contrast, the following theorem claims that there is a social choice function that satisfies (WSP) and does not admit a weak dictator.

Theorem 5.3. *Suppose that $X = N$ and $X_i = N \setminus \{i\}$ for each $i \in N$. Then there exists a weakly strategy-proof social choice function $D\colon \mathcal{P} \to X$ with $R(D) = X$ which does not admit a weak dictator.*

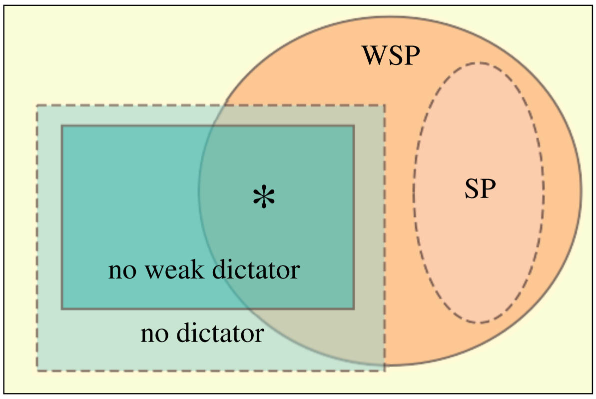

Figure 1: Set of social choice functions

Proof. We will prove this theorem by inductively constructing a social choice function having the claimed properties for each $n \geq 3$. For $h \geq 3$, let $N^h = \{1, \ldots, h\}$, $\mathcal{W}_i^h$ be the set of all preference orders defined on $N^h \setminus \{i\}$ for $i \in N^h$, and $\mathcal{P}^h = \mathcal{W}_1^h \times \cdots \times \mathcal{W}_h^h$. We will denote the social choice function $D\colon \mathcal{P}^h \to N^h$ by D_m^h to clarify the number of individuals h.

Table 1 in the Appendix demonstrates an example* of a social choice function D_m^3 that satisfies (WSP) and $R(D_m^3) = N^3$ and does not admit a weak dictator. That is, the theorem has been proved when $n = 3$.

Assuming the existence of a social choice function $D_m^h\colon \mathcal{P}^h \to N^h$ which satisfies (WSP) and $R(D_m^h) = N^h$ and does not admit a weak dictator, we will show that there exists a social choice function $D_m^{h+1}\colon \mathcal{P}^{h+1} \to N^{h+1}$ with the desired properties.

For each profile $p \in \mathcal{P}^{h+1}$, let

$$r(p) = (\succsim_1^p|(N^h \setminus \{1\}), \succsim_2^p|(N^h \setminus \{2\}), \ldots, \succsim_h^p|(N^h \setminus \{h\})),$$

i.e., $r(p)$ is the restriction of profile $p \in \mathcal{P}^{h+1}$ to $\mathcal{P}^h$. We define $D_m^{h+1} : \mathcal{P}^{h+1} \to N^{h+1}$ as

$$D_m^{h+1}(p) = \begin{cases} D_m^h(r(p)) & \text{when } \succsim_{h+1}^p = (1 \succ 2 \succ \cdots \succ h), \\ h+1 & \text{otherwise} \end{cases} \tag{5.1}$$

for each profile $p \in \mathcal{P}^{h+1}$.

First, to show that D_m^{h+1} defined by (5.1) satisfies (WSP) we suppose the contrary, i.e., there exist an individual $i \in N^{h+1}$, profile $p \in \mathcal{P}^{h+1}$ and preference $\succsim \in \mathcal{W}_i^{h+1}$ such that

$$D_m^{h+1}(p/_{-i} \succsim) \succ_i^p D_m^{h+1}(p). \tag{5.2}$$

Letting $q = p/_{-i} \succsim$ for the sake of notational simplicity, we consider the following two possible cases.

Case A: $i \in N^h$.

If $\succsim_{h+1}^p \neq (1 \succ 2 \succ \cdots \succ h)$, then $D_m^{h+1}(p) = D_m^{h+1}(p/_{-i} \succsim) = h+1$ by the definition (5.1) of D_m^{h+1}. Since this fact contradicts (5.2), we have $\succsim_{h+1}^p = (1 \succ 2 \succ \cdots \succ h)$, implying that $D_m^{h+1}(p) = D_m^h(r(p))$ and $D_m^{h+1}(q) = D_m^h(r(q))$. Then, we have

$$D_m^h(r(q)) \succ_i^{r(p)} D_m^h(r(p)).$$

*We carried out exhaustive enumeration for the case of $N = \{1, 2, 3\}$ and found more than 1.6 billion different social choice functions having the properties of Theorem 5.3 .

However, since $\succsim_j^p = \succsim_j^q$ for all $j \in N^{h+1} \setminus \{i\}$,

$$\succsim_j^{r(p)} = \succsim_j^{r(q)} \text{ for all } j \in N^h \setminus \{i\}.$$

These contradict (WSP) of D_m^h.

Case B: $i = h + 1$.

Since $\succsim_{h+1}^p$ is a preference order on $N^{h+1} \setminus \{h+1\} = N^h$, we see that both $D_m^{h+1}(p)$ and $D_m^{h+1}(q)$ are elements of N^h from (5.2). Therefore, we have by the construction (5.1) of D_m^{h+1} that $\succsim_{h+1}^p = \succsim = (1 \succ 2 \succ \cdots \succ h)$. This implies that $q = p/_{-i} \succsim = p$, which contradicts (5.2).

Next, we will show $R(D_m^{h+1}) = N^{h+1}$. We know from (5.1) that there exists a profile $p \in \mathcal{P}^{h+1}$ such that $D_m^{h+1}(p) = h + 1$. Since $\{\, r(p) \mid p \in \mathcal{P}^{h+1}$ such that $\succsim_{h+1}^p = (1 \succ 2 \succ \cdots \succ h) \,\} = \mathcal{P}^h$ and $R(D_m^h) = N^h$ from the induction assumption, we have $N^h \subseteq R(D_m^{h+1})$. Therefore, we see that $R(D_m^{h+1}) = N^{h+1}$.

Finally, we will show that D_m^{h+1} does not admit a weak dictator. Let j be an arbitrary individual in N^h. Since D_m^h does not admit a weak dictator, we see that there exist a profile $p_1 \in \mathcal{P}^h$ and $k, l \in N^h \setminus \{j\}$ such that

$$k \succ_j^{p_1} l \text{ and } D_m^h(p_1) = l.$$

Take a profile $q_1 \in \mathcal{P}^{h+1}$ such that

$$r(q_1) = p_1 \text{ and } \succsim_{h+1}^{q_1} = (1 \succ 2 \succ \cdots \succ h).$$

Then, we have $D_m^{h+1}(q_1) = D_m^h(r(q_1)) = D_m^h(p_1) = l$ and $k \succ_j^{q_1} l$ by the construction of profile q_1, implying that individual $j \in N^h$ is not a weak dictator for D_m^{h+1}.

Let p_2 be a profile of $\mathcal{P}^h$ such that $D_m^h(p_2) = 2$.[†] Consider a profile $q_2 \in \mathcal{P}^{h+1}$ such that

$$r(q_2) = p_2 \text{ and } \succsim_{h+1}^{q_2} = (1 \succ 2 \succ \cdots \succ h).$$

We see that $D_m^{h+1}(q_2) = D_m^h(r(q_2)) = D_m^h(p_2) = 2$ and $1 \succ_{h+1}^{q_2} 2$. These facts imply that individual $h + 1$ is not a weak dictator for D_m^{h+1}, either. Therefore, we conclude that D_m^{h+1} does not admit a weak dictator. □

We have shown the existence of a weakly strategy-proof social choice function that does not admit a weak dictator in the context of mutual evaluation. When we consider strategy-proofness in the strong sense, Definition 2.4, instead of weak strategy-proofness, we will see in the following theorem that strategy-proofness is so a strong requirement that a strategy-proof social choice function for mutual evaluation does not exist.

Theorem 5.4. *There does not exist a social choice function $D_m \colon \mathcal{P} \to N$ for mutual evaluation that satisfies both (SP) in Definition 2.4 and $R(D_m) = N$.*

Proof. Suppose that $D_m \colon \mathcal{P} \to X$ satisfies both (SP) in Definition 2.4 and $R(D_m) = N$. Then, by Theorem 3.1 there exists a dictator in $N(X)$ for D_m, which is impossible since $N(X) = \emptyset$. □

[†] $D_m^h(p_2)$ can be any natural number between 2 and h.

Remark 5.5. We have an alternative proof of Theorem 5.4. Let $D_m \colon \mathcal{P} \to N$ be a social choice function satisfying both (SP) in Definition 2.4 and $R(D_m) = N$. Let us take the "cyclic" profile c defined by

$$\begin{aligned}
&2 \succ_1^c 3 \succ_1^c \cdots \succ_1^c n,\\
&i+1 \succ_i^c i+2 \succ_i^c \cdots \succ_i^c n-1 \succ_i^c n \succ_i^c 1 \succ_i^c \cdots \succ_i^c i-1 \quad \text{for } i = 2, \ldots, n-1,\\
&1 \succ_n^c 2 \succ_n^c \cdots \succ_n^c n-1.
\end{aligned} \tag{5.3}$$

and consider what alternative D_m should choose at this profile. Take an individual i out of $N \setminus \{n\}$, then $i \succ_j^c i+1$ for all $j \in N \setminus \{i, i+1\}$, which means

$$D_m(c) \neq i+1 \text{ for } i = 1, 2, \ldots, n-1$$

by Lemma 4.4. Concerning individual 1 we see that $n \succ_j^c 1$ for all $j \in N \setminus \{1, n\}$, meaning

$$D_m(c) \neq 1.$$

Therefore $D_m(c)$ cannot be any alternative, and we conclude that such a social choice function D_m satisfying both (SP) in Definition 2.4 and $R(D_m) = N$ does not exist.

It is pointed out in [1] that a social *welfare* function for mutual evaluation degenerates or does not exist due to the presence of the cyclic profile in $\mathcal{P}$. To exclude the cyclic profile Ohbo et al. [3] introduced individuals who are entitled to evaluate all individuals in the society and proved the existence of a dictator among the introduced individuals. Concerning the social choice function, we readily see, by applying Theorem 3.1, that every social choice function satisfying (SP) in Definition 2.4 is dictatorial and one of the introduced individuals is a dictator.

$\boxed{6}$ Independence of Irrelevant Alternatives

In this section we introduce independence of irrelevant alternatives, and discuss a relation between weak strategy-proofness and generalized strong positive association.

Definition 6.1 (Independence of Irrelevant Alternatives (IIA)). A social choice function $D \colon \mathcal{P} \to X$ is said to satisfy *independence of irrelevant alternatives* if for any profiles $p, q \in P$

$$\text{for all } j \in N(\{D(p)\}) \text{ and for all } x \in X_j \cap R(D),\ \succsim_j^p |\{D(p), x\} = \succsim_j^q |\{D(p), x\}$$

implies $D(q) = D(p)$.

(IIA) requires that if we have $D(p) \neq D(q)$ for some profiles $p, q \in P$, then there must exist an individual $j \in N(\{D(p)\})$ who changed his preference order between $D(p)$ and some $x \in R(D)$ and an individual $j' \in N(\{D(q)\})$ who changed his preference order between $D(q)$ and some $x' \in R(D)$. Note that the social choice function D_m given in Table 1 does not satisfy (IIA) as we can see the condition in Definition 6.1 is violated for $p = p_8$ and $q = p_5$.

Lemma 6.2. *If a social choice function $D \colon \mathcal{P} \to X$ satisfies (WSP) in Definition 5.1 and (IIA) in Definition 6.1, then it satisfies (GSPA) in Definition 2.7.*

Proof. On the contrary, suppose that $D\colon \mathcal{P} \to X$ does not satisfy (GSPA). Then, there are two distinct profiles $p, q \in \mathcal{P}$ and a subset $M \subseteq N(\{D(p)\})$ of individuals such that

$$D(p) \succsim_i^p x \text{ implies } D(p) \succ_i^q x \text{ for all } x \in X_i \setminus \{D(p)\} \text{ and for all } i \in M,$$
$$\succsim_j^p = \succsim_j^q \text{ for all } j \in N(\{D(p)\}) \setminus M, \text{ and}$$
$$D(q) \neq D(p).$$

As we did in the proof of the necessity of Theorem 4.1, let $M = \{1, \ldots, m\}$ and $N \setminus N(\{D(p)\}) = \{m+1, \ldots, l\}$ $(n \geq l \geq m \geq 0)$ by renumbering if necessary, and define $r_j \in \mathcal{P}$ by (4.2) for $j = 0, 1, \ldots, l$. Consider the sequence $(r_0, r_1, \ldots, r_l)$ of $l+1$ profiles. Since $\succsim_i^p = \succsim_i^q$ for all $i \in N(\{D(p)\}) \setminus M = \{l+1, \ldots, n\}$, we have $r_l = q$, and hence, $D(r_l) = D(q)$. Then, there exists an individual $k \in M \cup (N \setminus N(\{D(p)\})) = \{1, \ldots, l\}$ such that $D(r_{k-1}) = D(p)$ and $D(r_k) \neq D(p)$. We have the same four cases: A(1), A(2), A(3) and B as in the proof of the necessity of Theorem 4.1 and Cases A(1) and A(2) are treated similarly except that we use (WSP) instead of (SP).

Let us consider remaining Cases A(3): $k \in M$, $\{D(r_{k-1}), D(r_k)\} \not\subseteq X_k$ and B: $k \in N \setminus N(\{D(p)\})$.

In Case A(3), we have $D(p) = D(r_{k-1}) \in X_k$ and $D(r_k) \notin X_k$. Consider Definition 6.1, where we put $p = r_k$ and $q = r_{k-1}$. Then, since $k \notin N(\{D(r_k)\})$ and $\succsim_j^{r_{k-1}} = \succsim_j^{r_k}$ for $j \in N \setminus \{k\}$, we have for all $j \in N(\{D(r_k)\})$ and for all $x \in R(D) \cap X_j$

$$\succsim_j^{r_k} |\{D(r_k), x\} = \succsim_j^{r_{k-1}} |\{D(r_k), x\}.$$

Therefore, we must have $D(r_{k-1}) = D(r_k)$, which is a contradiction.

In Case B, we have $D(p) = D(r_{k-1}) \notin X_k$. Consider Definition 6.1, where we put $p = r_{k-1}$ and $q = r_k$ this time. Then, since $k \notin N(\{D(r_{k-1})\})$, we have a contradiction by a similar argument to Case A(3). □

Theorem 6.3. *If a social choice function $D\colon \mathcal{P} \to X$ satisfies (WSP), (IIA) and $|R(D)| \geq 3$, then it is dictatorial.*

Proof. Lemma 6.2 shows that D satisfies (GSPA), and hence it is dictatorial by Corollary 4.2. □

Concerning the mutual evaluation we have the following theorem from Theorem 4.1, Theorem 5.4 and Lemma 6.2.

Theorem 6.4. *There exists no social choice function $D_m\colon \mathcal{P} \to N$ for mutual evaluation satisfying (WSP), (IIA) and $R(D_m) = N$.*

7 Summary

We considered social choice functions on incomplete preference profiles, where each individual's alternative set is a subset of the whole set of alternatives. We have shown an impossibility theorem: a social choice function satisfying strategy-proofness is dictatorial. We also have shown the equivalence of generalized strong positive association and strategy-proofness. Then, we weakened the condition of strategy-proofness as well as dictatorship, and showed the existence of a weakly strategy-proof social choice function that does not admit a weak dictator in a constructive manner. We finally considered independence of irrelevant alternatives and showed that it is not compatible with weak strategy-proofness in mutual evaluation context.

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Appendix

individual	preference
1	$2 \succ_1^{p_1} 3$
2	$1 \succ_2^{p_1} 3$
3	$1 \succ_3^{p_1} 2$

$D_m(p_1) = 1$

individual	preference
1	$2 \succ_1^{p_2} 3$
2	$1 \succ_2^{p_2} 3$
3	$2 \succ_3^{p_2} 1$

$D_m(p_2) = 1$

individual	preference
1	$2 \succ_1^{p_3} 3$
2	$1 \succ_2^{p_3} 3$
3	$1 \sim_3^{p_3} 2$

$D_m(p_3) = 1$

individual	preference
1	$2 \succ_1^{p_4} 3$
2	$3 \succ_2^{p_4} 1$
3	$1 \succ_3^{p_4} 2$

$D_m(p_4) = 1$

individual	preference
1	$2 \succ_1^{p_5} 3$
2	$3 \succ_2^{p_5} 1$
3	$2 \succ_3^{p_5} 1$

$D_m(p_5) = 1$

individual	preference
1	$2 \succ_1^{p_6} 3$
2	$3 \succ_2^{p_6} 1$
3	$1 \sim_3^{p_6} 2$

$D_m(p_6) = 1$

individual	preference
1	$2 \succ_1^{p_7} 3$
2	$1 \sim_2^{p_7} 3$
3	$1 \succ_3^{p_7} 2$

$D_m(p_7) = 1$

individual	preference
1	$2 \succ_1^{p_8} 3$
2	$1 \sim_2^{p_8} 3$
3	$2 \succ_3^{p_8} 1$

$D_m(p_8) = 2$

individual	preference
1	$2 \succ_1^{p_9} 3$
2	$1 \sim_2^{p_9} 3$
3	$1 \sim_3^{p_9} 2$

$D_m(p_9) = 2$

individual	preference
1	$3 \succ_1^{p_{10}} 2$
2	$1 \succ_2^{p_{10}} 3$
3	$1 \succ_3^{p_{10}} 2$

$D_m(p_{10}) = 1$

individual	preference
1	$3 \succ_1^{p_{11}} 2$
2	$1 \succ_2^{p_{11}} 3$
3	$2 \succ_3^{p_{11}} 1$

$D_m(p_{11}) = 1$

individual	preference
1	$3 \succ_1^{p_{12}} 2$
2	$1 \succ_2^{p_{12}} 3$
3	$1 \sim_3^{p_{12}} 2$

$D_m(p_{12}) = 1$

individual	preference
1	$3 \succ_1^{p_{13}} 2$
2	$3 \succ_2^{p_{13}} 1$
3	$1 \succ_3^{p_{13}} 2$

$D_m(p_{13}) = 2$

individual	preference
1	$3 \succ_1^{p_{14}} 2$
2	$3 \succ_2^{p_{14}} 1$
3	$2 \succ_3^{p_{14}} 1$

$D_m(p_{14}) = 3$

individual	preference
1	$3 \succ_1^{p_{15}} 2$
2	$3 \succ_2^{p_{15}} 1$
3	$1 \sim_3^{p_{15}} 2$

$D_m(p_{15}) = 3$

individual	preference
1	$3 \succ_1^{p_{16}} 2$
2	$1 \sim_2^{p_{16}} 3$
3	$1 \succ_3^{p_{16}} 2$

$D_m(p_{16}) = 2$

individual	preference
1	$3 \succ_1^{p_{17}} 2$
2	$1 \sim_2^{p_{17}} 3$
3	$2 \succ_3^{p_{17}} 1$

$D_m(p_{17}) = 2$

individual	preference
1	$3 \succ_1^{p_{18}} 2$
2	$1 \sim_2^{p_{18}} 3$
3	$1 \sim_3^{p_{18}} 2$

$D_m(p_{18}) = 2$

individual	preference
1	$2 \sim_1^{p_{19}} 3$
2	$1 \succ_2^{p_{19}} 3$
3	$1 \succ_3^{p_{19}} 2$

$D_m(p_{19}) = 1$

individual	preference
1	$2 \sim_1^{p_{20}} 3$
2	$1 \succ_2^{p_{20}} 3$
3	$2 \succ_3^{p_{20}} 1$

$D_m(p_{20}) = 3$

individual	preference
1	$2 \sim_1^{p_{21}} 3$
2	$1 \succ_2^{p_{21}} 3$
3	$1 \sim_3^{p_{21}} 2$

$D_m(p_{21}) = 1$

individual	preference
1	$2 \sim_1^{p_{22}} 3$
2	$3 \succ_2^{p_{22}} 1$
3	$1 \succ_3^{p_{22}} 2$

$D_m(p_{22}) = 1$

individual	preference
1	$2 \sim_1^{p_{23}} 3$
2	$3 \succ_2^{p_{23}} 1$
3	$2 \succ_3^{p_{23}} 1$

$D_m(p_{23}) = 3$

individual	preference
1	$2 \sim_1^{p_{24}} 3$
2	$3 \succ_2^{p_{24}} 1$
3	$1 \sim_3^{p_{24}} 2$

$D_m(p_{24}) = 2$

individual	preference
1	$2 \sim_1^{p_{25}} 3$
2	$1 \sim_2^{p_{25}} 3$
3	$1 \succ_3^{p_{25}} 2$

$D_m(p_{25}) = 2$

individual	preference
1	$2 \sim_1^{p_{26}} 3$
2	$1 \sim_2^{p_{26}} 3$
3	$2 \succ_3^{p_{26}} 1$

$D_m(p_{26}) = 2$

individual	preference
1	$2 \sim_1^{p_{27}} 3$
2	$1 \sim_2^{p_{27}} 3$
3	$1 \sim_3^{p_{27}} 2$

$D_m(p_{27}) = 2$

Table 1: A weakly strategy-proof social choice function that does not admit a weak dictator for $N = \{1, 2, 3\}$