Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 2, No. 3, pp. 627 – 655, September, 2006

SUBSTITUTES AND COMPLEMENTS FOR PRODUCTION PLANNING IN ASSEMBLY SYSTEMS

Margarida P. Mello*

Abstract: Optimal multiperiod production planning in assembly systems is studied in which the variables are the production and inventory levels in each period at each facility. A parameter is associated with each variable and the cost is a sum of functions, each being convex in one variable, subadditive in the corresponding variable-parameter pair and independent of the other pairs. The coefficient matrix is known to be Leontief. A new combinatorial characterization is given of the associated elementary vectors, i.e., elements of the null space of the coefficient matrix having minimal support. An optimal value of a variable is increasing (resp., decreasing) in a second variable's parameter if the two variables are complements (resp., substitutes), i.e., the product of the two variables is nonnegative (resp., nonpositive) in every elementary vector. Apart from first- or last-period variables, only the following distinct pairs are always complements: inventory at a facility in a period and either production there in the period or at its immediate successor in the following period; inventories in a period at distinct facilities with common immediate successor; inventories at the assembly facility in different periods. Apart from first- or last-period variables, only the following distinct pairs are always substitutes: production in a period at a facility and production or inventory there in the preceding period; inventory at a facility in a period and production or inventory then at its immediate successor.

Key words: *assembly systems, substitutes and complements, lattice programming, production planning, inventory control*

Mathematics Subject Classification: *06B35, 90B10, 90B30, 90B50, 90C99*

---

1 Introduction

Consider an assembly system that consists of a group of facilities organized into a production hierarchy divided in levels, with each facility on one level producing a single product and consuming outputs from facilities on the next higher level. Classify facilities as *parts*, *subassembly* or *assembly* facilities according to whether they consume no other products, or consume other products and their product is used in the production of others, or consume other products but their product is not consumed in the production of others. Furthermore, the facility tree representing the flow of material between facilities is a rooted tree with all arcs directed towards the root. Each facility is a node; the parts facilities are leaf nodes; the (unique) assembly facility is the root and the subassembly facilities are the remaining nodes. A facility that (directly) consumes the product of (resp., furnishes input for) another one is the *successor* (resp., *predecessor*) thereof. The assembly facility is at level 0, its predecessors

*This work was supported by CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior) under Grant 3991/82-4. The author also acknowledges partial support by the National Science Foundation. The article is a revision of Chapter 3 of [9].

are at level 1, in general the predecessors of a facility at level m are at level $m+1$. The assumption that each facility has at most one successor is somewhat restrictive since real-life assembly systems often involve products that are used as inputs in the production of more than one other product. Nevertheless, if the production and holding costs of such products and their predecessors are linear, each facility producing such a product and its predecessors may be split into as many copies as the number of its successors, yielding an equivalent assembly system that satisfies the assumption. Figure 1 below depicts the facility tree of an assembly system. The arcs linking the nodes represent the flow of material between the associated facilities. The flows corresponding to the external demand for product at each facility are not explicitly represented. Thus for each product there are two kinds of demands, the internal demand generated by the production at its successors and the external demand originated in the market. Without loss of generality we choose the units of each product so that the production of any given product consumes one unit each of the inputs thereof. Assembly systems with only two levels are called *star assembly systems*.

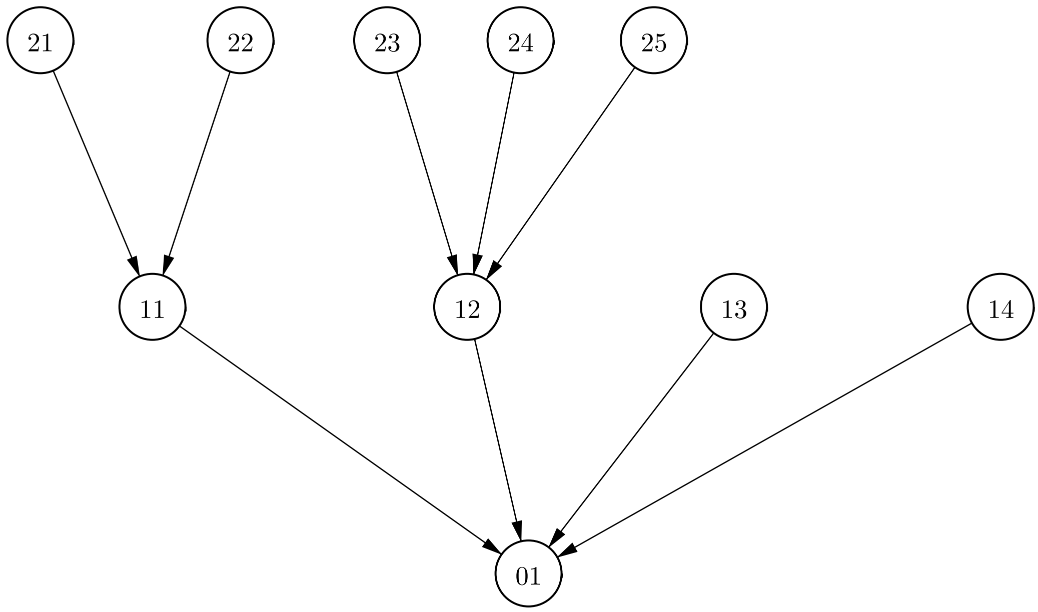


Figure 1: Facility tree for assembly system with three levels.

Consider a multi-period problem whose variables are the levels of production in each period and inventory at the end of each period for each product. There are two types of constraints, nonnegativity and stock-conservation constraints for each product in each period. Initial and final inventories are given and assumed to be zero without loss of generality. The matrix of coefficients of such a system is Leontief and the system described by the constraint set would be a Leontief substitution system were the demands nonnegative [15]. However the demands are allowed to be unrestricted in sign and thus the system is referred to as a *generalized Leontief substitution system*. Alternately, the constraint set can be viewed as describing an *assembly network-flow problem* in which each node of the corresponding network is associated with a subset of the flow-conservation equations instead of a single such equation as is the case in ordinary network-flow problems. This results from the

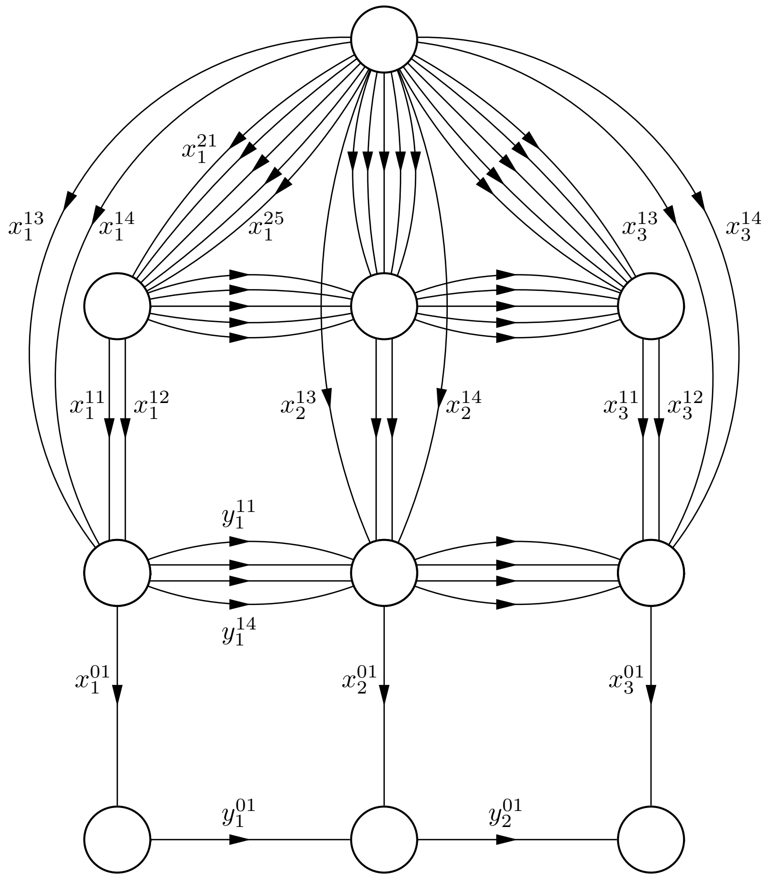


Figure 2: Graph $\mathcal{G}$ of three-period problem associated with facility tree in Figure 1.

fact that in the production process the inputs are combined in fixed proportions, rather than added. The underlying (dynamic) graph of the assembly network-flow problem associated with the assembly system problem is illustrated in Figure 2 for the facility tree in Figure 1 with three periods. The x-labeled (resp., y-labeled) arcs represent production levels (resp., end-of-period inventory levels). The superscripts identify the facility and the subscripts the period. Consider for instance the set of equations associated with the leftmost second-from-bottom node. The outgoing arcs are the production level at the assembly facility in period one and the inventory at the end of period one at each of its four predecessors. The incoming arcs are the production levels in period one at each of the assembly facility's four predecessors. Associated with the node there are four equations assuring the satisfaction of the first period demand (both internal and external) for the product of each of the assembly facility's predecessors.

There are production and storage costs. The problem is to plan production of each product in each period in order to satisfy demand over n periods at minimum cost. The objective of the present work is to establish monotonicity of the optimal solution to the above problem with respect to the parameters of the cost function. The results use the ideas of lattice programming and depend heavily on characterizing the pairs of variables that are conformal. The *elementary vectors* of a subspace of $I\!R^n$ are the nonnull vectors in the subspace with minimal support, cf. [11]. Two variables in a system of linear equations are *conformal* (resp., *complements, substitutes*) if their product in each elementary vector of the associated homogeneous system has common sign (resp., is nonnegative, is nonpositive). This is the natural extension to general linear systems of the concept of conformal arcs used in [3], or substitutes and complements used in [8] and in [2]. In particular, in a single-commodity network, two variables are conformal (resp., complements, substitutes) if every cycle (with no repeated nodes) containing them orients the arcs consistently (resp., similarly, oppositely). Special cases where conformality is easily established are the variables corresponding to arcs that are incident to the same node or, if the network is planar, that lie on a common face [8].

Monotonicity results of this type were first presented for ordinary network-flow problems in [14] and published in [8]. These are extended to take into account one additional linear constraint in [5] and to generalized network-flow problems in [6]. In [7] further results are obtained regarding more general problems. Gale and Politof [4] examine a special case of the problem studied in [8], namely maximizing the weighted circulation in a capacitated directed network. They establish subadditivity properties of the optimal value function with respect to certain parameter pairs that are instances of some of those obtained in [8] for the general case, but do not deal with monotonicity issues. Granot and Veinott were aware that many of the concepts, techniques and results of [8] extend in a straightforward way to much more general problems, e.g., the concept of conformal variables. The main difficulty is that one must determine which pairs of variables are conformal for each new class of problems. Provan [10] studies the problem of complements and substitutes in generic linear models but gives a complete characterization only for network-flow problems. This was done in [2] for multicommodity network flows. We carry out a similar investigation for assembly systems. We do this by first developing a combinatorial characterization of the elementary vectors of the homogeneous system associated with the equality constraints of the problem and then applying this result to identify the conformal pairs of variables.

In order to describe and motivate the combinatorial characterization of the elementary vectors of the homogeneous system associated with the equality constraints of the problem, we need to introduce a few definitions. If we eliminate from the graph associated with the problem all arcs except those associated with the facilities that lie on the (unique) path from a leaf of the facility tree to the root and consider the subset of equations involving the corresponding variables, then the *slice subgraph* and subset of equations obtained are those of a *facilities-in-series network-flow problem* [20]. Figure 3 illustrates the slice subgraph $\mathcal{G}^{\downarrow 23}$ of the graph $\mathcal{G}$ in Figure 2 associated with parts facility 23 of Figure 1. A vector in the subspace of solutions to the associated homogeneous system is commonly called a *circulation* in the network. Thus vectors of the homogeneous system associated with the equations of the entire assembly-system problem are called *assembly circulations*. The elementary circulations of a network are well known to be those circulations whose induced subgraph is a cycle. It is easy to see that this must be so since the cycle links the variables in the support of the circulation so that conservation of flow on the nodes of the cycle implies that all flows in the arcs of the cycle must be of the same magnitude. Thus setting the flow in one of the arcs to zero must result in setting the flows in the remaining arcs of the cycle to zero.

The combinatorial characterization of the elementary assembly circulations of the assembly-system problem generalizes this linking notion. In this problem the induced subgraph may contain several cycles, but must also be *sequentially strongly biconnected*, a different kind of linking that is appropriate for this problem. Thus an assembly circulation is elementary if and only if it is a multiple of a $0, \pm 1$ vector whose induced subgraph is sequentially strongly biconnected. From this and [11] it follows that an integer assembly circulation may be decomposed as a sum of conformal elementary integer assembly circulations. For the case of facilities in series, this is an instance of a known result for networks [1] and is used to prove the Ripple Theorem in [9]. It is also possible to show [9] the unimodularity of the constraint matrix of the system and the integrality of the extreme points of the corresponding feasible set. In the star-assembly-system case, the characterization of elementary assembly circulations can be sharpened. This leads to a linear-time algorithm for checking whether or not an assembly circulation is elementary.

Next we characterize the conformal pairs of variables of the problem. These are the pairs of variables that either (i) lie on the boundary of a common face in some slice subgraph, (ii) are inventories associated with the same period of distinct facilities that belong to the same level and have a common successor or (iii) are production levels at distinct facilities other than the assembly facility in the first or last period. Furthermore, if they satisfy (i) and are oriented in the same (resp., opposite) way in the cycle that forms the boundary of the face, then they are complements (resp., substitutes). If they satisfy (ii), they are complements. If they satisfy (iii) and belong to the same period (resp., distinct periods), then they are complements (reps., substitutes). As the name suggests, two variables that are complements should reinforce each other. Thus, if the optimal value of one increases then the optimal value of the other will certainly not decrease. The opposite is true for variables that are substitutes. Of course subadditivity and convexity conditions on the cost functions need to be imposed in order to guarantee this behavior.

Having identified the conformal pairs of the assembly system and established the integral sum decomposition of the integer assembly circulations, it is straightforward [9]to generalize the results of [8], namely the Ripple, Ripple Selection, Monotone Optimal-Flow Selection, Smoothing, Subadditivity of Minimum Cost in Parameters of Substitutes, and Monotone Optimal-Flow Selection with Nonconvex Flow Costs Theorems. Finally some applications are given.

$\boxed{2}$ Characterization of the Elementary Vectors

A facility is at level m if the path in the facility tree from the facility's corresponding node to the root has m arcs. It is convenient to denote the k^{th} facility at level m by the ordered pair (m, k), or briefly mk when no ambiguity results, the number of facilities at level m by f_m and the successor of facility mk by $s(mk)$. In particular the assembly facility is denoted by 01. Eventually we will need to work with the iterated successor function, defined as follows: $s^0(mk) \equiv mk$ and $s^t(mk) \equiv s(s^{t-1}(mk))$, for $t = 1, \ldots, m$, so that $s^i(mk)$ is the i^{th} facility in the path from mk to 01, the assembly facility. For the system shown in Figure 1, $f_1 = 4$ and $s(23) = 12$. Facility mk is a predecessor of facility $s(mk)$. Occasionally ir will be necessary to refer to a *nonimmediate successor* (resp., *nonimmediate predecessor*) of a facility, say mk, which is a facility that indirectly uses the output of (resp., whose output is indirectly used in the production at) mk. The assembly system has $\ell + 1$ levels if there is a facility at level ℓ and the longest path with respect to the number of arcs from any leaf node of the tree to the root has at most ℓ arcs and $\ell + 1$ nodes. Figure 1 depicts a 3-level system. If the assembly system has only two levels, call it a *star assembly system*.

Denote by $x_i^{mk}, y_i^{mk}, d_i^{mk}$ the production, inventory and demand, respectively, at facility mk in period i, and let $x^{mk} \equiv (x_i^{mk})$, $y^{mk} \equiv (y_i^{mk})$, $L = \{1, \ldots, \ell\}$, $F_m = \{1, \ldots, f_m\}$ and $N = \{1, \ldots, n\}$, where n is the number of periods considered and the assembly system has

$\ell + 1$ levels. Then $z \equiv (x^{01}, y^{01}, x^{11}, y^{11}, \ldots, x^{\ell f_\ell}, y^{\ell f_\ell})$ satisfies the constraints

$$x_i^{mk} + y_{i-1}^{mk} - y_i^{mk} - x_i^{s(mk)} = d_i^{mk} \quad m \in L,\ k \in F_m,\ i \in N \tag{1a}$$

$$x_i^{01} + y_{i-1}^{01} - y_i^{01} = d_i^{01} \quad i \in N \tag{1b}$$

$$z \geq 0 \tag{1c}$$

where $y_0^{01} \equiv y_n^{01} \equiv y_0^{mk} \equiv y_n^{mk} \equiv 0$, for $m \in L$ and $k \in F_m$.

Let A be the matrix of coefficients of the constraint set (1a–b). In this section we give a combinatorial characterization of the elementary vectors of $\mathcal{N} = \{z \mid Az = 0\}$.

The directed graph $\mathcal{G}$ associated with an $(\ell + 1)$-level assembly system and n-period planning horizon has $(\ell + 1)n + 1$ nodes. The *level m intermediate node* for period i is associated with the set of equations in (1a) for the given (m, i) pair and all k in F_m. The *bottom* (*level* 0) *node* for period i is associated with the (unique) equation in (1b) for the given i. Each arc in $\mathcal{G}$ corresponds to a variable in z having tail (resp., head) node associated with the set of equations in which the variable has nonpositive (resp., nonnegative) coefficients with at least one being nonzero and the variable represents the flow along the corresponding arc. A *top node* is added to serve as the tail node of all the production variables x_i^{mk} associated with a parts facility mk, for each mk. The equations associated with the top node, namely $-\sum_{i \in N} x_i^{mk} = -\sum_{i \in N} \sum_{t=0}^{m} d_i^{s^t(mk)}$, for all mk corresponding to a leaf node, are redundant and so are not appended to (1a–b). The graph $\mathcal{G}$ corresponding to the assembly structure depicted in Figure 1 for $n = 3$ is illustrated in Figure 2.

Two kinds of subsets of facilities and the variables associated therewith will play a special role in the sequel: the facilities on the path from a parts facility to the assembly facility (i.e., the set of immediate and nonimmediate successors of a parts facility), a *slice subgraph*, and the maximal subtree of the facility tree whose root is a given facility (i.e., the set of immediate and nonimmediate predecessors of a given facility), a *predecessor subtree*. The slice subgraph generated by the parts facility mk is characterized by the vector $S(mk) = (01, \ldots, mk) = (S_0, \ldots, S_m)$, which contains the facilities on the leaf-to-root node path in reverse order. For example, $S(23) = (01, 12, 23)$ in the facility tree of Figure 1. Thus $S_i = s^{(m-i)}(mk)$. Denote by $z^{\downarrow mk} \equiv (x^{S_0}, y^{S_0}, \ldots, x^{S_m}, y^{S_m})$ the *slice vector* of variables associated with the facilities of the slice $S(mk)$. Denote by $\mathcal{G}^{\downarrow mk}$ the subgraph of $\mathcal{G}$ obtained by deleting all arcs except those associated with the variables in the slice vector $z^{\downarrow mk}$. The subgraph $\mathcal{G}^{\downarrow mk}$ is called a *slice subgraph*. The slice subgraph $\mathcal{G}^{\downarrow 23}$ of the graph $\mathcal{G}$ in Figure 2 appears in Figure 3. Throughout this work the planar embedding of any slice subgraph is fixed analogous to that in Figure 3. The top part is a collection of triangles and the bottom part is a grid, with the production (vertical) arcs and the inventory (horizontal) arcs displayed in increasing order with respect to the period number from left to right and with increasing order with respect to level from bottom up. Notice that if the number of periods is at least three and the number of levels is at least two, the subgraph obtained by doing a series reduction on the pairs of arcs x_1^{01}, y_1^{01} and y_{n-1}^{01}, x_n^{01} of any slice subgraph is 3-connected and thus has a unique planar embedding, see [18, 19]. On the other hand, the above two pairs of arcs must always be incident to the same pair of faces since the node to which they are incident has degree two. Since the planar embedding of the resulting 3-connected graph is unique, there really is no loss of generality in fixing the embedding. The subset of the constraints (1a–b) that involve only the variables in $z^{\downarrow mk}$ constitute precisely the constraint set [20] of an $(m + 1)$-*facilities-in-series network-flow problem*, which is henceforth called a *slice network-flow problem*.

Facility ij's *predecessor subtree* is the maximal rooted subtree of the facility tree whose root is facility ij, i.e., whose other nodes are the facilities mk situated at level $m > i$ such that $ij = s^{(m-i)}(mk)$. Figure 5 shows facility 12's predecessor subtree of the facility tree shown in Figure 1. Denote by $z^{\uparrow ij}$ the vector containing the variables associated with the assembly subsystem corresponding to facility ij's predecessor subtree. Notice that the subvectors $(x^{01}, y^{01}), z^{\uparrow 11}, \ldots, z^{\uparrow 1f_1}$ constitute a partition of z. In particular, again for Figure 5, $z^{\uparrow 12} = (x^{12}, y^{12}, x^{23}, y^{23}, x^{24}, y^{24}, x^{25}, y^{25})$. Denote by $\mathcal{G}^{\uparrow ij}$ the subgraph of $\mathcal{G}$ associated with the variables in $z^{\uparrow ij}$. The notation adopted conveys the methods of generating the various subvectors and subgraphs. The slices are obtained starting from a leaf node and going down towards the root, thus the downarrow in the superscript, and the predecessor subtrees are obtained starting from a node and fanning up the tree, motivating the uparrow in the superscript.

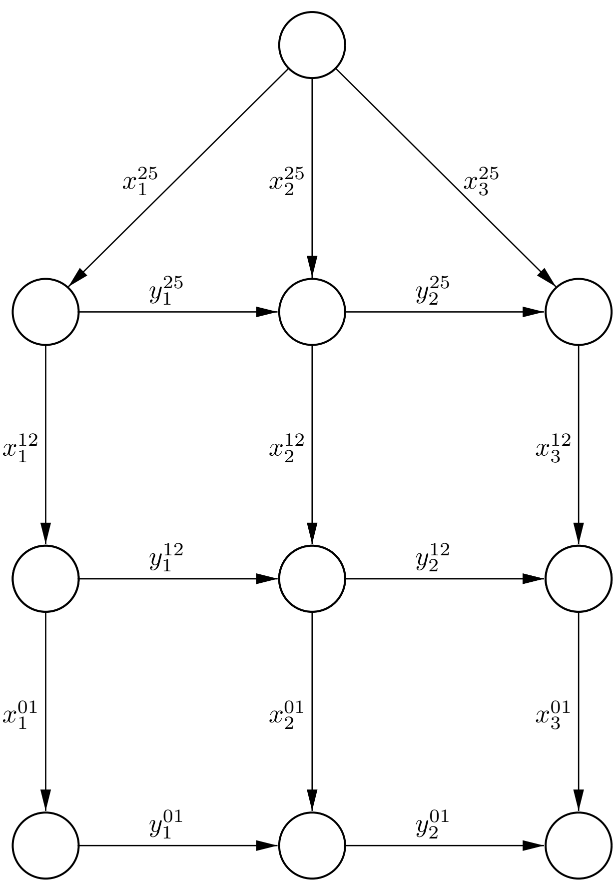
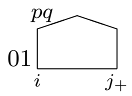
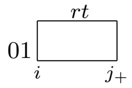


Figure 3: Slice Subgraph $\mathcal{G}^{\downarrow 23}$.

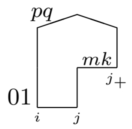
We take advantadge of the fixed embedding of slice subgraphs to introduce notation that simplifies the description of circulations. Notice that the production arcs incident into nodes on a common row belong to the same facility and all production arcs incident into nodes on a common column belong to the same period. Thus if we imagine the nodes as elements of a matrix, we may identify the arcs adjacent to them by row (facility) and column (period) labels. This allows the shortened pictorial representation of cycles in a slice subgraph by diagrams consisting of polygons with labels giving the location of its *direction changing node*, i.e., the nodes at which the cycle changes direction. Examples are given in the following figure. We use notation m_+ for $m+1$ and m_- for $m-1$ in order to simplify the labeling of the polygons.



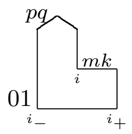
$$\{x_i^{pq}, x_i^{s(pq)}, \ldots, x_i^{01}, y_i^{01}, \ldots, y_j^{01}, x_{j_+}^{01}, \ldots, x_{j_+}^{s(pq)}, x_{j_+}^{pq}\}$$



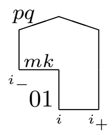
$$\{x_i^{s(rt)}, \ldots, x_i^{01}, y_i^{01}, \ldots, y_j^{01}, x_{j_+}^{01}, \ldots, x_{j_+}^{s(rt)}, y_j^{rt}, \ldots, y_i^{rt}\} \quad (rt = s^{p-r}(pq))$$



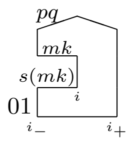
$$\{x_i^{pq}, x_i^{s(pq)}, \ldots, x_i^{01}, y_i^{01}, \ldots, y_{j_-}^{01}, x_j^{01}, \ldots, x_j^{s(mk)}, y_j^{mk}, x_{j_+}^{mk}, \ldots, x_{j_+}^{pq}\}$$



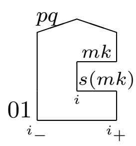
$$\{x_{i_-}^{pq}, \ldots, x_{i_-}^{01}, y_{i_-}^{01}, y_i^{01}, x_{i_+}^{01}, \ldots, x_{i_+}^{s(mk)}, y_i^{mk}, x_i^{mk}, \ldots, x_i^{pq}\}$$



$$\{x_{i_-}^{pq}, \ldots, x_{i_-}^{mk}, y_{i_-}^{mk}, x_i^{s(mk)}, \ldots, x_i^{01}, y_i^{01}, x_{i_+}^{01}, \ldots, x_{i_+}^{pq}\}$$



$$\{x_{i_-}^{pq}, \ldots, x_{i_-}^{mk}, y_{i_-}^{mk}, x_i^{s(mk)}, y_{i_-}^{s(mk)}, x_{i_-}^{s^2(mk)}, \ldots, x_{i_-}^{01}, y_i^{01}, y_{i_+}^{01}, x_{i+2}^{01}, \ldots, x_{i+2}^{pq}\}$$



$$\{x_{i_-}^{pq}, \ldots, x_{i_-}^{01}, y_{i_-}^{01}, y_i^{01}, x_{i_+}^{01}, \ldots, x_{i_+}^{s^2(mk)}, y_i^{s(mk)}, x_i^{s(mk)}, y_i^{mk}, x_{i_+}^{mk}, \ldots, x_{i_+}^{pq}\}$$

Figure 4: Some cycles in slice subgraph $S(pq)$.

For any z, denote by G the subgraph of $\mathcal{G}$ induced by (the nonzero elements of) z and by $G^{\downarrow mk}$ the *induced slice subgraph* of G induced by the slice vector $z^{\downarrow mk}$. Likewise, denote by $G^{\uparrow ij}$ the subgraph of G induced by $z^{\uparrow ij}$. Note that we suppress the dependence of G, $G^{\downarrow \cdot}$ and $G^{\uparrow \cdot}$ on z for simplicity. Call a vector z in $\mathcal{N}$ an *assembly circulation.* Then $z^{\downarrow mk}$ is an ordinary circulation in $G^{\downarrow mk}$. A *z-directed cycle* is a cycle induced by an elementary circulation in $G^{\downarrow mk}$ that is conformal with z for some slice $S(mk)$ and some mk. For illustrative purposes it is useful to consider the graph obtained from G by reversing the orientation of arcs with negative flow. Then a set of arcs and nodes in $G^{\downarrow mk}$ constitutes a z-directed cycle if the corresponding set in the newly oriented graph constitutes a directed cycle. Figure 6 provides a few examples. Figure 6(a) exhibits the facility tree of the assembly system under consideration as well as the line patterns used to draw the arcs associated with

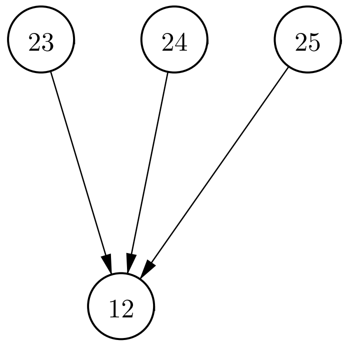
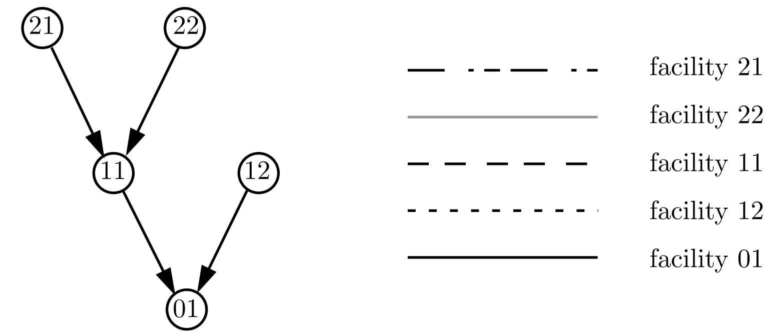


Figure 5: Facility 12's predecessor subtree in facility tree of Figure 1.

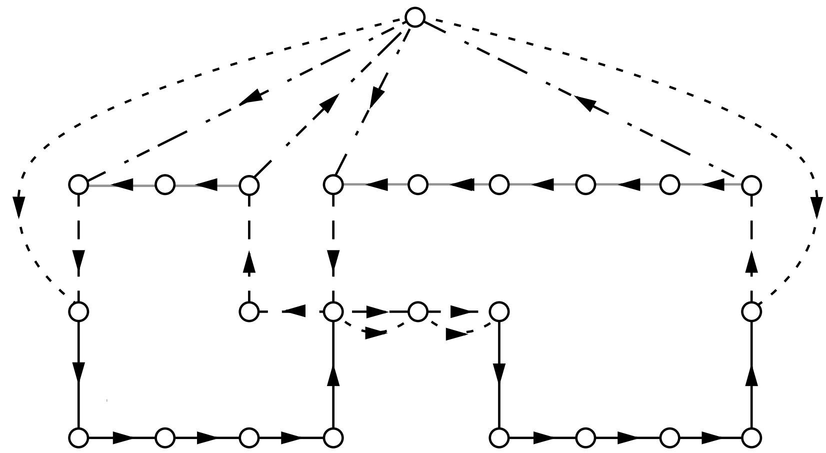
the different facilities. Figure 6(b) shows the subgraph induced by an assembly circulation z, with orientation reversed on arcs with negative flow. The cycle in Figure 6(c) is an example of a z-directed cycle, whereas as the one in Figure 6(d) is not. The reason is that the latter contains arcs from facility 21 and 12 and thus is not a subset of an induced slice subgraph. The induced slice subgraphs of G given in Figure 6(b) are shown in Figure 7.

Consider an assembly circulation z for which it is possible to partition (though not necessarily uniquely) the arcs of each slice subgraph into a set of z-directed cycles. Notice that subgraphs $G^{\downarrow 22}$ and $G^{\downarrow 12}$ in Figure 7 admit only one partition. On the other hand, subgraph $G^{\downarrow 21}$ in the same figure admits two partitions, shown in Figure 8. A *slice-partition* of the subgraph G induced by z is a collection of such partitions, one for each induced slice subgraph. Two arcs in G are *sequentially strongly biconnected with respect to a given slice-partition* if there is a sequence of z-directed cycles in the slice-partition with the first cycle containing one of the arcs, the last cycle containing the other arc, and each successive pair of cycles sharing at least one arc (and so must belong to distinct slices). This relation is symmetric, transitive and reflexive, and thus partitions the arcs of G into equivalence classes. Furthermore, G *is sequentially strongly biconnected with respect to a given slice-partition* if all arcs of G belong to the same equivalence class; and G *is sequentially strongly biconnected* if that is so with respect to every slice-partition. For instance, the production arcs of facility 21 associated with the first and last periods are sequentially strongly biconnected for G in Figure 6(b). In this case the subgraph G admits only two distinct slice-partitions, since only one of the induced slice subgraphs admits two partitions. If the partition in Figure 8(a) is chosen, then the cycle shown on the left of Figure 8(a), the cycle on the bottom of Figure 7 and the cycle shown on the right of Figure 8(a) constitute a legitimate sequence of z-directed cycles linking the given arcs (they share arcs associated with the assembly facility). On the other hand, if the partition in Figure 8(b) is chosen, then the cycle on the right already contains the given arcs, and so they are trivially sequentially strongly biconnected. Thus the production arcs of facility 21 associated with the first and last periods are certainly sequentially strongly biconnected in G of Figure 6(b). It is not difficult to verify that G is in fact sequentially strongly biconnected in this case.

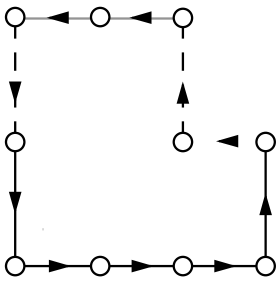
Theorem 1 (Characterization of Elementary Assembly Circulations of Assembly Systems). *Let z be a nonnull assembly circulation and G be the subgraph induced by z. Then z is elementary if and only if the absolute flow in each arc of G is the same and G is sequentially strongly biconnected.*



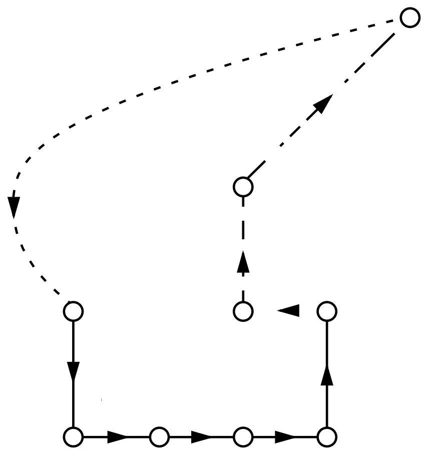
(a) Facility tree representing 3-level assembly system. Line patterns exhibited on the right are used to draw the arcs associated with the respective facilities.



(b) Subgraph induced by assembly circulation z where orientation of arcs with negative flow has been reversed.

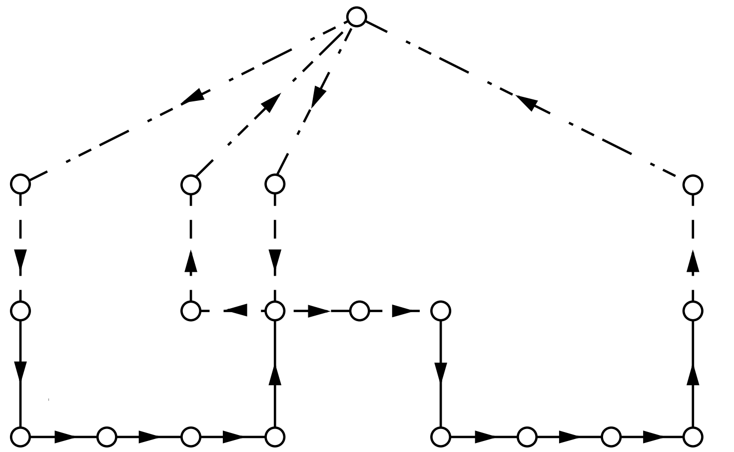


(c) z-directed cycle of subgraph in b) contained in induced slice subgraph $G^{\downarrow 22}$.

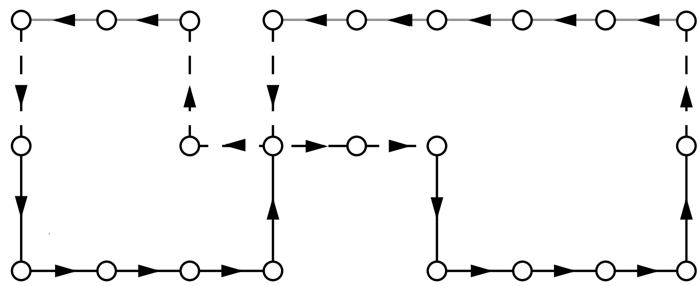


(d) Not a z-directed cycle since arcs do not belong to a common induced slice subgraph.

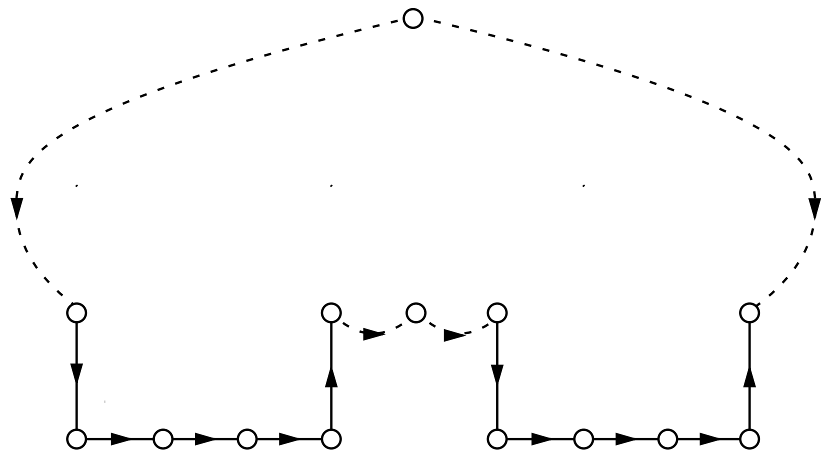
Figure 6: Example illustrating z-directed cycles, instance with 9 periods.



induced slice subgraph $G^{\downarrow 21}$

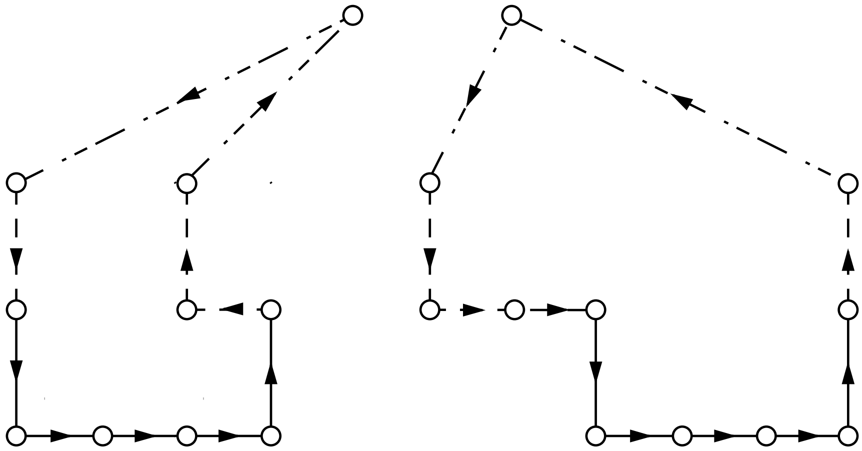


induced slice subgraph $G^{\downarrow 22}$

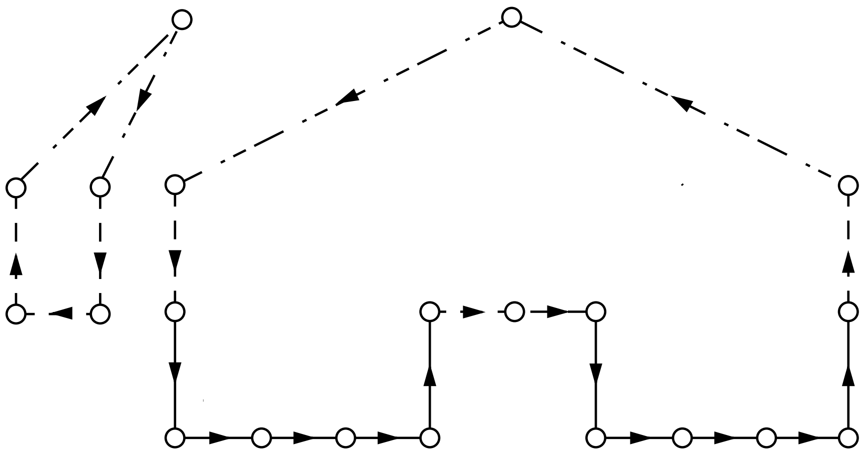


induced slice subgraph $G^{\downarrow 12}$

Figure 7: Induced slice subgraphs of G in Figure 6(b).



(**a**) a partition of $G^{\downarrow 21}$ into z-directed cycles



(**b**) another partition of $G^{\downarrow 21}$ into z-directed cycles

Figure 8: Induced slice subgraph $G^{\downarrow 21}$ in Figure 7 admits two distinct partitions, shown in (a) and (b) above.

In order to prove this result, it is useful first to introduce a few definitions. Consider an assembly circulation z of a $(\ell+1)$-level assembly system. If $\ell = 0$, a *level* 0 *contraction of* z is the operation of setting to zero the variables associated with the end (unique) facility (setting, in this case, all variables to zero). If $\ell \geq 1$, a *level* 0 *contraction of* z is the operation of adding to the inventories of the level 1 facilities the inventories of the level 0 facility associated with the same periods and then setting to zero the variables associated with the assembly facility. In any case, the vector $\tilde{z}$ thus obtained is an assembly circulation. If $\ell = 0$, this is trivially true, and if $\ell \geq 1$, it can be seen by checking the equations associated with the level 1 intermediate nodes, since the equations associated with level 0 nodes are now trivially satisfied and the values of the variables that show up in the remaining equations stay unchanged. Evidently,

$$x_i^{1k} + \tilde{y}_{i_-}^{1k} - \tilde{y}_i^{1k} = (x_i^{1k} + y_{i_-}^{1k} - y_i^{1k} - x_i^{01}) + (x_i^{01} + y_{i_-}^{01} - y_i^{01}) = 0,$$

which proves the desired result.

Consider the f_1 predecessor subtrees of the facility tree for the facilities at level 1 and the respective subvectors $\tilde{z}^{\uparrow 11}, \dots, \tilde{z}^{\uparrow 1f_1}$ of $\tilde{z}$ for each subtree. Since $(\tilde{x}^{01}, \tilde{y}^{01}) = 0$, $\tilde{z}^{\uparrow 1k}$ is an assembly circulation in the respective assembly subsystem, for $k = 1, \dots, f_1$. Now consider an assembly circulation z such that $x^{01} = y^{01} = 0$. Define a *level* 1 *contraction of* z as the operation of performing a level 0 contraction on $z^{\uparrow 1k}$ for $k = 1, \dots, f_1$. Applying the argument above to each subvector $\tilde{z}^{\uparrow 1k}$ and using the fact that $x^{01} = y^{01} = 0$ shows that the vector $\tilde{z}$ obtained is another assembly circulation and satisfies $\tilde{x}^{01} = \tilde{y}^{01} = \tilde{x}^{11} = \tilde{y}^{11} = \cdots = \tilde{x}^{1f_1} = \tilde{y}^{1f_1} = 0$. Next consider an assembly circulation z such that $x^{ik} = y^{ik} = 0$ for all appropriate k and $i < m$. Then we may view z as the collection of f_m assembly circulations, $z^{\uparrow 1k}$, for $k = 1, \dots, f_m$. Note that facility $1k$ is the level 0 facility with respect to $z^{\uparrow 1k}$. A *level* m *contraction of* z consists of performing a level 0 contraction on $z^{\uparrow mk}$ for $k = 1, \dots, f_m$.

The concept of a $\sqcup$-path of a subgraph of $\mathcal{G}$ will be used frequently in the sequel. A $\sqcup$*-path* P is an undirected path in a subgraph of $\mathcal{G}$ such that all the arcs in P are associated with the assembly facility, the first and last arcs of P are production arcs associated with distinct periods and the remaining arcs of P are inventory arcs. If the assembly system has but one level, such a path will be closed, its shape resembling a triangle if we adopt a planar drawing of the graph analogous to the one in Figure 2. For assembly systems with two or more levels, the $\sqcup$-paths will resemble those in Figure 9. The $\sqcup$-paths are partially ordered by the following *contained-in* relation. The $\sqcup$-path P is *contained in* the $\sqcup$-path P' if the set of inventory arcs of P is a subset of the corresponding set of P' as illustrated in Figure 9. Given a subgraph of $\mathcal{G}$, a path P in the subgraph is a maximal $\sqcup$-path if it is a $\sqcup$-path and it is maximal with respect to the contained-in relation just defined. By construction of $\mathcal{G}$ and by definition of the contained-in relation, maximal $\sqcup$-paths are disjoint.

A *level* m *expansion of* z is defined for assembly circulations z such that $x^{ik} = y^{ik} = 0$ for all $i \leq m$ and suitable k's. This operation requires two parameters: a scalar α and a set of maximal $\sqcup$-paths $\mathcal{P}^k$ for each $\mathcal{G}^{\uparrow mk}$, for $k = 1, \dots, f_m$ (since m is fixed it is not included in the superscript of $\mathcal{P}$ to simplify notation). First construct an assembly circulation z' as follows. If mk is a parts facility send flow α counterclockwise along the $\sqcup$-paths (cycles) in $\mathcal{P}^k$. Otherwise, for each maximal $\sqcup$-path $P = \{x_i^{mk}, y_i^{mk}, \dots, y_{j_-}^{mk}, x_j^{mk}\}$ in $\mathcal{P}^k$, send flow α counterclockwise along the cycles $mk \overset{m_-r}{\underset{i \qquad j}{\boxed{}}}$, for each predecessor m_-r of facility mk. The level m expansion of z using the scalar α and the sets $\mathcal{P}^k$ for all k is defined as the operation of adding the assembly circulation z' thus constructed to z. The resulting vector $\tilde{z} = z + z'$

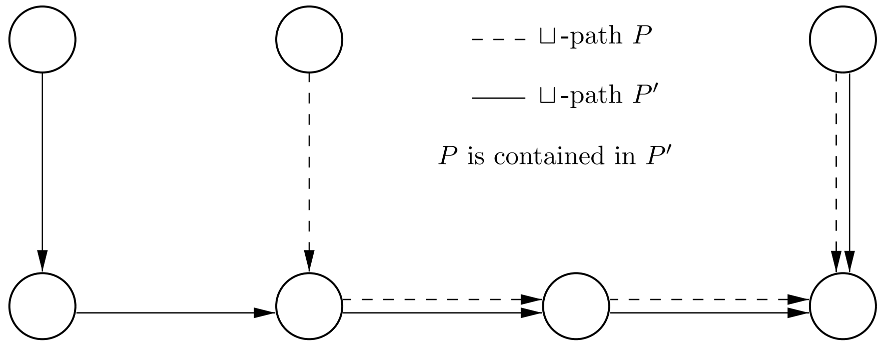


Figure 9: Contained-in relation for ⊔-paths.

is clearly another assembly circulation and the set of maximal ⊔-paths of $\tilde{G}^{\uparrow mk}$ induced by $\tilde{z}^{\uparrow mk}$ is precisely the set $\mathcal{P}^k$ used in the expansion.

The proof of Theorem 1 and its corollaries make use of a result from [11] which we include below for completeness. We kept the original notation and definitions in the statement of the theorem. Two vectors are "in harmony" if and only if they are conformal. The dimension of the space that contains the linear subspace bears no relation to the set containing the period numbers introduced in this article.

Theorem 2 (Rockafellar). *Let K be a subspace of $\mathbb{R}^N$ and $\boldsymbol{X}$ be any non-zero vector in K. Then there exist elementary vectors $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_r$ of K, such that $\boldsymbol{X} = \boldsymbol{X}_1 + \cdots + \boldsymbol{X}_r$. These elementary vectors may be chosen such that each is in harmony with $\boldsymbol{X}$ and has its support contained in the support of $\boldsymbol{X}$, but none has its support contained in the union of the support of the others, and such that r does not exceed the dimension of K or the number of elements in the support of $\boldsymbol{X}$.*

Let $\operatorname{supp} z$ denote the support of vector z and $|\operatorname{supp} z|$ the cardinality thereof. The proof of Theorem 1 may now be presented.

Proof of Theorem 1. First we show that any graph G induced by an assembly circulation z such that all nonnull variables have the same absolute value, say α, admits at least one slice-partition. Thus the statement about G being sequentially strongly biconnected cannot be vacuously satisfied. Fix the subgraph $G^{\downarrow mk}$ induced by the slice vector $z^{\downarrow mk}$. Since $z^{\downarrow mk}$ is an ordinary circulation in $G^{\downarrow mk}$, it can be expressed, see [1] and Theorem 2, as a sum of conformal elementary circulations, each consisting of flow along a z-directed cycle, such that the support of each elementary circulation is not contained in the union of the supports of the others. Thus the magnitude of flow along each elementary circulation must be α and their supports (z-directed cycles) must in fact constitute a partition of $G^{\downarrow mk}$.

Let $\ell + 1$ be the number of levels in the assembly system. First we prove by induction that given a subgraph G induced by an assembly circulation $z \neq 0$, there exists a subgraph $\tilde{G}$ induced by an assembly circulation $\tilde{z} \neq 0$ such that $\operatorname{supp} \tilde{z} \subseteq \operatorname{supp} z$; the absolute flow in each arc of $\tilde{G}$ is α; the ⊔-paths of $\tilde{G}$ are precisely the maximal ⊔-paths of G and the flow along each inventory arc of the end facility in $\tilde{G}$ is α. The claim is easily seen to be true when $\ell = 0$ since in this case the maximal ⊔-paths are cycles and $\tilde{z}$ may be obtained by sending flow α counterclockwise along each maximal ⊔-path. Assume by induction that the claim is true for assembly systems with ℓ levels or less and consider one with $\ell + 1$ levels. The following diagram illustrates the steps in the construction of $\tilde{z}$.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\bar{z},\, \bar{G} \arrow[r, "\text{induction hypothesis}", "\substack{\text{preserve } \bar{G}^{\uparrow 1k}\text{'s max } \sqcup\text{-paths, } \alpha\text{-flows} \\ \operatorname{supp} \tilde{\bar{z}} \subseteq \operatorname{supp} \tilde{z}}"'] & \tilde{\bar{z}},\, \tilde{\bar{G}} \arrow[d, "\substack{\text{expand} \\ \text{using } \alpha}"] \\
z,\, G \arrow[u, "\text{contract}"] \arrow[r, dashed, "\text{preserve } G\text{'s max } \sqcup\text{-paths}"] & \tilde{z},\, \tilde{G}
\end{tikzcd}$$

First perform a level 0 contraction of z obtaining another assembly circulation $\bar{z}$. Since $(\bar{x}^{01}, \bar{y}^{01}) = 0$, $\bar{z}^{\uparrow 1k}$ is an assembly circulation and induces the subgraph $\bar{G}^{\uparrow 1k}$, with at most ℓ levels, for $k = 1, \ldots, f_1$. Let $\tilde{\bar{z}}^{\uparrow 1k}$ be the assembly circulation satisfying the induction hypotheses, for $k = 1, \ldots, f_1$. Combine $(\tilde{\bar{x}}^{01}, \tilde{\bar{y}}^{01}) = 0$ with the assembly circulations $\{\tilde{\bar{z}}^{1k}\}_k$ to form an assembly circulation $\tilde{\bar{z}}$ of the original system.

Now let $\tilde{z}$ be obtained by performing a level 0 expansion of $\tilde{\bar{z}}$ using the maximal $\sqcup$-paths of G for the set of maximal $\sqcup$-paths and α for the scalar. We claim that $\operatorname{supp} \tilde{z} \subseteq \operatorname{supp} z$. By construction, we only need to check variables $\tilde{y}_i^{1k}$, for $k \in F_1$ and $i \in N$. Since $\tilde{\bar{y}}_i^{1k} = 0$ or α and an expansion may add either zero or $-\alpha$ to the flow along the corresponding arc, we have that $\tilde{y}_i^{1k} = 0, \pm\alpha$. Of course, only values $\pm\alpha$ need to be checked. If $\tilde{y}_i^{1k} = -\alpha$, then $\tilde{\bar{y}}_i^{1k} = 0$ and $y_i^{01} \neq 0$, since there is a maximal $\sqcup$-path in the expansion (and therefore in G) containing this arc. By the induction hypothesis, $\tilde{\bar{y}}_i^{1k} = 0$ if and only if $\bar{y}_i^{1k} = 0$ since $\bar{G}^{\uparrow 1k}$ and $\tilde{\bar{G}}^{\uparrow 1k}$ have the same maximal $\sqcup$-paths. But then $0 = \bar{y}_i^{1k} = y_i^{1k} + y_i^{01}$ and $y_i^{01} \neq 0$ imply $y_i^{1k} \neq 0$. Suppose, on the other hand, that $\tilde{y}_i^{1k} = \alpha$. This means that the level 0 expansion, and therefore G, did not include a $\sqcup$-path containing the y_i^{01} and also implies that $\tilde{\bar{y}}_i^{1k} = \alpha$. Thus $\tilde{y}_i^{01} = y_i^{01} = 0$ and $\bar{y}_i^{1k} \neq 0$, implying $0 \neq \bar{y}_i^{1k} = y_i^{1k} + y_i^{01} = y_i^{1k}$.

Consider the "only if" part. If z is elementary and $\operatorname{supp} \tilde{z} \subseteq \operatorname{supp} z$ then z and $\tilde{z}$ must have the same support (thus inducing the same G) and be multiples. Suppose not and let $i \in \operatorname{supp} \tilde{z} = \operatorname{supp} z$. Then $z - (z_i/\tilde{z}_i)\tilde{z}$ is yet another assembly circulation with support strictly contained in the support of z, contradicting the fact that z is elementary.

Therefore the nonnull elements of z have the same absolute value, since the nonnull elements of $\tilde{z}$ are $\pm\alpha$. Now suppose there is a slice-partition with respect to which G is **not** sequentially strongly biconnected and let E be an equivalence class of sequentially strongly biconnected arcs with respect to this (fixed) slice-partition, so E^c is nonempty. If a distinguished arc of a slice subgraph $G^{\downarrow mk}$ belongs to E, then so do all the arcs in the (unique with respect to the fixed slice-partition) z-directed cycle in $G^{\downarrow mk}$ that contain the distinguished arc. Thus setting the flows in the arcs in E to zero yields the nonnull assembly circulation $(\mathrm{z}_{E^c}, 0)$ whose support is a proper subset of that of z, contradicting the fact that z is elementary.

Finally, for the "if" part, we assume z is **not** elementary. If the nonnull elements of z assume distinct absolute values we are done. Now suppose all nonnull elements of z have absolute value α. If z is not elementary Theorem 2 implies that there exists an elementary assembly circulation z' conformal with z but with strictly smaller support. From the "only if" part we know the nonnull components of z' have common absolute value. Without loss of generality let α be this common value. Then $z'' = z - z'$ is another nonnull assembly circulation, and the pair z', z'' is conformal and orthogonal. The subgraphs G' and G'' induced by z' and z'' respectively have slice-partitions. Moreover, from any two such slice-partition we may build a slice-partition for G by taking the partition for each slice of z to be the union of the corresponding partitions for the same slices of z' and z''. But since z' and z'' are conformal and orthogonal, arcs in G' are not sequentially strongly biconnected in G to arcs in G'' with respect to the constructed slice-partition of G, so G is not sequentially strongly biconnected with respect to that slice-partition. $\square$

The following corollary can also be derived from the unimodularity of the constraint matrix (shown in [9]) and Theorem 7 of [7].

Corollary 3 *All elementary assembly circulations z are multiples of $0, \pm 1$ vectors.*

Proof. Set the absolute value of the flows to one in Theorem 1. $\square$

The following corollary sharpens Rockafellar's Theorem 2 regarding the maximum number (either cardinality of the vector or the dimension of the subspace) of elementary vectors in a decomposition of a vector z of a subspace.

Corollary 4 *Each (integer) nonnull assembly circulation is the sum of at most $|supp\, z| - 2 \geq 1$ elementary (integer) assembly circulations.*

Proof. It follows from Theorem 2 and Corollary 3, that there is an $0, \pm 1$ elementary vector $\tilde{z}$ conformal with z, with support contained in the support of $z \equiv z^0$. Let $\alpha = \min\{|z_i| \mid i \in \operatorname{supp} \tilde{z}\}$. Then $z' = \alpha \tilde{z}$ (resp., $z^1 \equiv z - z'$) is an (integer) elementary assembly circulation (resp., (integer) assembly circulation) conformal with z^0 and with support strictly contained in the support of z^0. Iterating this procedure we must reach an elementary (integer) assembly circulation z^i and obtain a decomposition of the (integer) assembly circulation z in $i + 1$ (integer) elementary assembly circulations. By Theorem 1, z^i must contain at least one z^i-directed cycle, which, by the structure of $\mathcal{G}$, must contain at least three arcs. Since at each iteration step the support of z decreases by at least one element and the last elementary vector's support has at least 3 elements we conclude that $3 + i \leq |\operatorname{supp} z|$, or, equivalently, $i + 1 \leq |\operatorname{supp} z| - 2$. $\square$

What is the effort involved in establishing whether a vector $\tilde{z}$ is an elementary assembly circulation? Checking whether $\tilde{z} \in \mathcal{N}$ and whether the nonnull elements have common absolute value can be done in linear time. Suppose $\tilde{z}$ satisfies both conditions. Fix an element, say r in the support of $\tilde{z}$. We claim that $\tilde{z}$ is elementary if and only if the following system has no solution:

$$\begin{aligned} Az &= 0 \\ z_i &= 0 \quad \text{for } i \in \{r\} \cup (\operatorname{supp} \tilde{z})^{\mathrm{c}} \\ \textstyle\sum_i (\text{sign } \tilde{z}_i) z_i &= 1 \\ (\text{sign } \tilde{z}_i) z_i &\geq 0 \quad \text{for all } i \end{aligned}$$

If $\tilde{z}$ is elementary, there is no nonnull assembly circulation whose support is strictly contained in that of $\tilde{z}$, thus the above system has no solution. If $\tilde{z}$ is not elementary then [11] there is z' elementary conformal with $\tilde{z}$ whose support contains r. Without loss of generality both $\tilde{z}$ and z' are $0, \pm 1$ vectors. But then the vector $(\tilde{z} - z')/|\operatorname{supp}(\tilde{z} - z')|$ satisfies the system. Finally, the above system of linear inequalities can be solved in polynomial time.

Following Whitney [18], two arcs in a graph are *biconnected* if there is a cycle containing them or they coincide. The biconnectedness relation is transitive, symmetric and reflexive, and so is an equivalence relation. A graph is *biconnected* if each pair of arcs is biconnected and there are no isolated nodes. A *biconnected component* of a graph is a maximal subgraph among those that consist of a single node or are biconnected. The biconnected components of a connected graph form a tree. These concepts are used in the characterization of the elementary assembly circulations of star assembly systems below.

Theorem 5 (Characterization of Elementary Assembly Circulations of Star Assembly Systems). *In star assembly systems, a nonnull assembly circulation z is elementary if and only if its induced subgraph G has a unique slice-partition with respect to which it is sequentially strongly biconnected, each z-directed cycle of which constitutes a biconnected component of the corresponding induced slice subgraph and the absolute flow in each arc of G is the same. Furthermore, if G contains $\sqcup$-paths, then each z-directed cycle contains at least one $\sqcup$-path.*

Proof. From Theorem 1 it only remains to show the uniqueness of the slice-partition and the biconnected component characterization. The result if trivial if there is no flow along the assembly arcs since in this case the elementary assembly circulation z must reduce to an ordinary elementary circulation in one of the slice subgraphs. In this case the subgraph G induced by z reduces to a single cycle. Assume then that there is at least one maximal $\sqcup$-path in G.

Since there are only two levels we will label the assembly facility 0 and the k^{th} first level (parts) facility k. Apply a level 0 contraction on z, obtaining $\bar{z}$. The subvector $\bar{z}^{\uparrow k}$ is a circulation in the subgraph it induces, $\bar{G}^{\uparrow k}$. Construct $\tilde{\bar{z}}^{\uparrow k}$ by sending unit flow counterclockwise along each maximal $\sqcup$-path (cycle) of $\bar{G}^{\uparrow k}$. Thus the subgraphs induced by $\tilde{\bar{z}}^{\uparrow k}$ are either empty or the union of cycles which are precisely their biconnected components. Figure 10 shows a typical nonempty instance of such a subgraph.

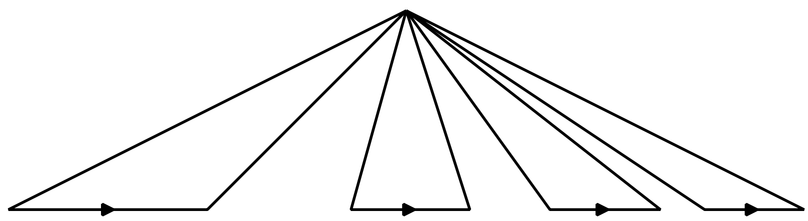


Figure 10: Subgraph $\tilde{\bar{G}}^{\uparrow k}$ induced by $\tilde{\bar{z}}^{\uparrow k}$.

Let $\tilde{z}$ be obtained by a level 0 expansion of $\tilde{\bar{z}}$ using the set of maximal $\sqcup$-paths of G and unit flow. As shown in the proof of Theorem 1, $\operatorname{supp} \tilde{z} \subseteq \operatorname{supp} z$, so $\tilde{z}$ must also be elementary and a multiple of z. Therefore $\tilde{G} = G$, and it is enough to study the possible effects of expansion on $\tilde{\bar{G}}$. Assuming the expansion operation is done in steps, one for each maximal $\sqcup$-path, it suffices to examine what happens with a generic slice in one step. Let z' be the assembly circulation added corresponding to a maximal $\sqcup$-path.

It is easy to rule out the possibility shown on Figure 11. In that case the addition of z' creates an isolated cycle C, with unit flow, in the slice subgraph, all of whose arcs are associated with one of the parts facility. Such configuration would allow two distinct partitions. This cycle, and the flow along it, won't be changed in further expansion steps since the maximal $\sqcup$-paths are disjoint. Thus C can be eliminated, i.e., the flow along its arcs may be set to zero, without disrupting conservation of flow in this or other slices, producing an assembly circulation with strictly smaller support, which contradicts the hypothesis that z is elementary. Therefore the cycle induced by z' in the slice subgraph cannot intersect two or more cycles of the slice subgraph at each step.

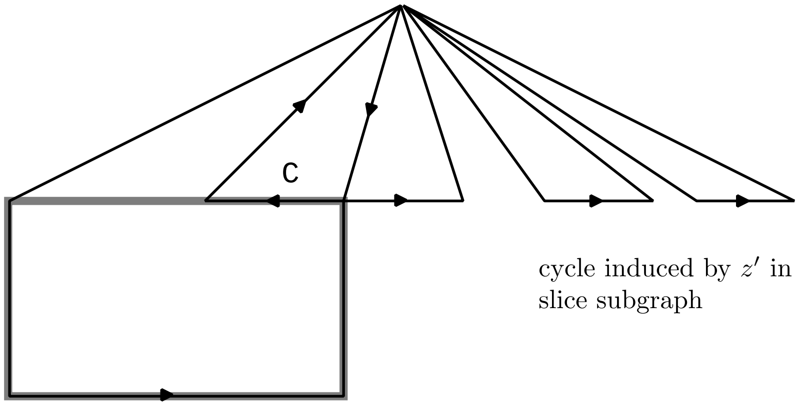


Figure 11: First case considered when performing expansion in steps contradicts hypothesis.

The admissible possibilities are that the cycle induced by z' in the slice subgraph intersect at most one cycle. If the intersection contains arcs, as illustrated by the gray cycle in Figure 12, then the addition of z' will transform the two intersecting cycles in one new larger cycle. If the intersection is empty or contains a single node, adding z' will result in adding a cycle to the slice, as illustrated by the dashed gray cycle in Figure 12.

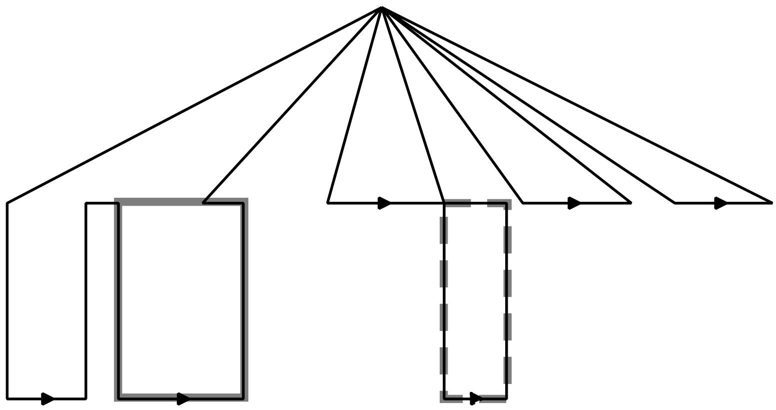


Figure 12: Some possible effects of expansion.

The expansion process begins with a subgraph $\tilde{\tilde{G}}^{\uparrow k}$ whose biconnected components are cycles and at each step this property is preserved, so it must hold for the final slice subgraph. Finally, all cycles in $\tilde{\tilde{G}}^{\uparrow k}$ will eventually be enlarged to encompass some $\sqcup$-path. The resulting subgraph G admits a unique slice-partition, each z-directed cycle of which contains a $\sqcup$-path. $\square$

It is interesting now to investigate the amount of effort involved in determining whether a given vector z is indeed an elementary assembly circulation of a star assembly system. This can be done in linear time by following the series of steps described below (where one proceeds to the next step if the outcome of the current one is positive). Notice that this complexity upperbound is better than the polynomial complexity of the method for assembly systems with two or more levels (see page 642). In the following we refer to consecutive maximal $\sqcup$-paths. Since maximal $\sqcup$-paths do not overlap, they can be labeled as $P_1, P_2, \ldots, P_p$ so that $i < j$ if P_i is entirely to the left of P_j in the graph (the embedding was fixed to be the analogous to that of Figure 3).

- Verify whether z is an assembly circulation with nonnull elements of common magnitude (accomplished in linear time).
- For each induced slice subgraph, use Tarjan's [12] algorithm to find the biconnected components therefrom and verify whether each biconnected component is indeed a cycle (linear time).
- If G contains no $\sqcup$-paths then verify whether it reduces to a unique cycle in some induced slice subgraph. If it contains exactly one $\sqcup$-path, then verify whether each slice induced subgraph consists of exactly one z-directed cycle containing this $\sqcup$-path. In both cases G is trivially sequentially strongly biconnected. Otherwise verify whether each pair of consecutive maximal $\sqcup$-paths belongs to a common z-directed cycle in some induced slice subgraph and each z-directed cycle contains bottom nodes (linear time).

We still need to argue that for z satisfying the first two steps, its induced graph G is sequentially strongly biconnected if and only if the third step is true.

Corollary 6 *Let z be a $0, \pm 1$ assembly circulation of a star assembly system such that its induced subgraph G hast at least two $\sqcup$-paths and the biconnected components of each induced slice subgraph are z-directed cycles each containing bottom nodes. Then G is sequentially strongly biconnected with respect to the (unique) slice-partition whose elements are the biconnected components of the induced slice subgraphs if and only if every pair of consecutive $\sqcup$-paths belong to a common z-directed cycle in some induced slice subgraph.*

Proof. The fact that z is $0, \pm 1$ implies that each node in each slice subgraph of G is adjacent to an even number of arcs. Thus all $\sqcup$-paths must be maximal. Suppose a z-directed cycle contains two nonadjacent $\sqcup$-paths. This cycle must contain a path connecting the last assembly arc of the left $\sqcup$-path with the first assembly arc of the $\sqcup$-path on the right. This path may contain only assembly arcs and/or level 1 inventory arcs. Therefore this path will intercept the intermediate $\sqcup$-path(s) in at least two intermediate nodes (the extremes of the $\sqcup$-path(s)). But then the intermediate $\sqcup$-path(s) must belong to the same biconnected component as the cycle, since the intersection of any two distinct biconnected component may contain at most one node [12]. Therefore the intermediate $\sqcup$-path(s) must also be contained in the cycle.

Suppose G is sequentially strongly biconnected and consider any two consecutive $\sqcup$-paths, say P_i and P_{i+1}. Consider the sequence of cycles in the unique slice-partition of G that connects an arc of P_i with an arc of P_{i+1}. Since there are only two levels, the cycles in the sequence must share assembly arcs only. Consider the first cycle in the sequence that contains assembly arcs both in or to the left of P_i and in or to the right of P_{i+1}. Such a cycle must exist if the two arcs are to be sequentially connected. But then, from the discussion in the previous paragraphs, this cycle must contain both P_i and P_{i+1}.

Now suppose every two consecutive $\sqcup$-paths belong to some common cycle in a slice subgraph. Then all $\sqcup$-paths are easily seen to be sequentially strongly biconnected. Finally, each parts facility arc in G is trivially sequentially biconnected to the $\sqcup$-path that belongs to the same cycle (biconnected component) that contains the arc. $\square$

3 Conformality

The determination of the conformal pairs of variables in assembly systems is of fundamental importance since the monotonicity results extending those in [8] depend crucially on this concept. Theorem 7 gives a complete characterization of the conformal pairs of variables for the assembly system.

In the sequel we shall frequently be given an elementary circulation $z^{\downarrow mk}$ in the slice subgraph $\mathcal{G}^{\downarrow mk}$ associated with a slice $S(mk) = (S_0, ..., S_m)$ and desire to form a related elementary circulation, called a "clone" thereof for a different slice subgraph.

Clone of an Elementary Circulation. Let rp be the lowest level facility with arcs in the induced slice subgraph $G^{\downarrow mk}$ induced by the elementary circulation $z^{\downarrow mk}$. First, for any slice $T(tj) = (T_0, ..., T_t)$ containing rp (recall $S_0 = T_0 = 01$), define the *clone* $z^{\downarrow tj}$ of $z^{\downarrow mk}$ in the slice subgraph $\mathcal{G}^{\downarrow tj}$ as follows: (i) set $(x^{T_i}, y^{T_i}) = (x^{S_i}, y^{S_i})$ for $i = 0, \ldots, m \wedge t$; (ii) if $m < t$, set $(x^{T_i}, y^{T_i}) = (x^{S_m}, 0)$ for $i = m+1, ..., t$; (iii) set all remaining variables of $z^{\downarrow tj}$ to zero. Notice that the clone of an elementary circulation is itself an elementary circulation, albeit in a different slice subgraph.

A Cloned Elementary Assembly Circulation. If one clones the elementary circulation $z^{\downarrow mk}$ in each slice subgraph whose slice contains rp, the result is a cloned assembly circulation z. Moreover, z is elementary. To see this, observe that, by construction, for each slice containing rp the arcs in the respective induced subgraph form a unique z-directed cycle and the absolute flow is the same in each arc of the cycle. Furthermore, each such z-directed cycle contains the same set of facility rp arcs. Thus the graph G induced by z is sequentially strongly biconnected with respect to this (unique) slice-partition.

It is useful to observe that if two variables belonging to a common slice subgraph are *not* conformal for the associated slice network-flow problem, then they also are not conformal for the assembly system. To see this, observe that since the two variables are not conformal in the slice subgraph, there are two elementary circulations in the slice subgraph containing the two variables, one in which the product of the two variables is positive and the other in which that product is negative. Then the two cloned elementary assembly circulations inherit this property, which establishes that the two variables are not conformal in the assembly system. This result is helpful in the elimination of potential conformal pairs of variables since the conformal pairs for a general slice network-flow problem with at least three periods and two facilities are known to be precisely the variables whose arcs belong to the boundary of a common face or are incident to a common node.

In order to avoid special cases, assume that there are at least four periods, each (sub)assembly facility has at least two predecessors and there are at least three facilities in the system. The notation for cycles summarized in Figure 4 is extensively used in the proof.

Theorem 7 (Conformality in Assembly Systems). *Two distinct variables of an assembly system are conformal if and only if they either* (*i*) *lie on the boundary of a common face in some slice subgraph,* (*ii*) *are inventories associated with the same period of distinct facilities that have a common successor or* (*iii*) *are production levels of distinct facilities other than the assembly facility in the first or last period. If they satisfy* (*i*) *and are oriented* *in the same (resp., opposite) way in the cycle that forms the boundary of the face, they are complements (resp., substitutes). If they satisfy* (*ii*)*, they are complements. And if they satisfy* (*iii*) *and belong to the same period (resp., distinct periods), they are complements (resp., substitutes).*

Proof. We use the fact that, with the possible exception of the top node, all nodes in any slice subgraph induced by a $0, \pm 1$ assembly circulation have degree 2 or 4. This follows from the topology of the slice subgraphs (the degree of any node, except the top node, is less than or equal to 4) and the conservation-of-flow equation. For nodes other than the top node, this equation amounts to a sum, with up to four ± 1's, that must equal zero. Thus there must be an even number of summands.

Given a $0, \pm 1$ elementary assembly circulation z of a $\ell + 1$ assembly system, consider the following sequence of contractions and expansions using **unit** flow:

$$
\begin{array}{c}
\underset{\mathcal{P}^{01}}{z \equiv z^0} \xrightarrow[\text{contraction}]{\text{level } 0} \underset{\{\mathcal{P}^{1k}\}_k}{z^1} \xrightarrow[\text{contraction}]{\text{level } 1} \underset{\{\mathcal{P}^{2k}\}_k}{z^2} \longrightarrow \cdots \longrightarrow \underset{\{\mathcal{P}^{\ell k}\}_k}{z^\ell} \xrightarrow[\text{contraction}]{\text{level } \ell} z^{\ell_+} \equiv 0 \\
\tilde{z} \equiv \tilde{z}^0 \xleftarrow[\text{using } \mathcal{P}^{01}]{\text{level 0 exp.}} \tilde{z}^1 \xleftarrow[\text{using } \{\mathcal{P}^{1k}\}_k]{\text{level 1 exp.}} \tilde{z}^2 \longleftarrow \cdots \longleftarrow \tilde{z}^\ell \xleftarrow[\text{using } \{\mathcal{P}^{\ell k}\}_k]{\text{level } \ell \text{ expansion}} z^{\ell_+} \equiv 0
\end{array}
$$

Using induction we show the following properties hold for the assembly circulations $\tilde{z}^m$, for each $m \in \{0, \ldots, \ell\}$:

(a) $\tilde{z}^m$ is $0, \pm 1$, $\operatorname{supp} \tilde{z}^m \subseteq \operatorname{supp} z^m$ and the $\sqcup$-paths induced by $\tilde{z}^m$ are precisely $\{\mathcal{P}^{mk}\}_k$;

(b) the sequence (in increasing order of period) of nonzero productions levels at each fixed facility alternates in sign, starting with a positive production and has an even number of elements;

(c) the sequence (in increasing order of level) of nonzero inventories of facilities on a common slice associated with a common time period alternates in sign, starting with a positive inventory, for all slices and time periods.

The elementarity of z and (a) will then imply that $\tilde{z} = \pm z$, so that $\tilde{z}^m = \pm z^m$, for all m.

Notice that performing a level ℓ expansion on the null circulation z^{ℓ_+} using the sets of maximal $\sqcup$-paths $\{\mathcal{P}^{\ell k}\}_k$ produces a $0, \pm 1$ assembly circulation $\tilde{z}^\ell$ satisfying (a) (since $\operatorname{supp} \tilde{z}^\ell = \cup_k \{\mathcal{P}^{\ell k}\}_k = \operatorname{supp} z^\ell$), (b) and (c) (since expansion is achieved by sending unit flow counterclockwise flow along arc-disjoint cycles—$\sqcup$-paths—sharing a common node, the top node).

Assume by induction that $\tilde{z}^{m_+}$ satisfies (a), (b), and (c). Let $\tilde{z}^m$ be the assembly circulation obtained by performing an expansion using $\{\mathcal{P}^{mk}\}_k$ and unit flow. Then $\tilde{z}^m$ satisfies (b). Clearly all inventory variables, in levels other than m_+, and all production variables in $\tilde{z}^m$ are $0, \pm 1$. All inventory variables, in levels other than m_+, and all production variables in the support of $\tilde{z}^m$ are easily seen to be contained in z^m, by construction and the induction hypothesis. Also, by construction, the $\sqcup$-paths induced by $\tilde{z}^m$ are those in $\{\mathcal{P}^{mk}\}_k$.

Now consider a generic level m_+ inventory variable in $\tilde{z}^{m_+}$, say $y^{m_+}{}^{m_+k}$. Denote by $\mathbf{I}(\cdot)$ the indicator function that is one when the argument is nonzero and zero otherwise. Then

$\mathbf{I}(\cdot)$ satisfies

$$\mathbf{I}(a+b) \leq \mathbf{I}(a) + \mathbf{I}(b). \tag{2}$$

Let $s(m_+k) = mj$. Then (a) and (c) imply the first equality below, and the definition of contraction gives the second:

$$\tilde{y}_i^{m_+{}^{m_+k}} = \mathbf{I}(y_i^{m_+{}^{m_+k}}) = \mathbf{I}(y_i^{m^{m_+k}} + y_i^{m^{mj}}). \tag{3}$$

The rules for expansion imply that

$$\begin{aligned} \tilde{y}_i^{m^{mj}} &= \mathbf{I}(y_i^{m^{mj}}) && \text{(4a)}\\ \tilde{y}_i^{m^{m_+k}} &= \tilde{y}_i^{m_+{}^{m_+k}} - \tilde{y}_i^{m^{mj}} && \text{(4b)} \end{aligned}$$

Substituting (4a) and (3) into (4b) and then using (2) of $\mathbf{I}(\cdot)$ we obtain

$$\tilde{y}_i^{m^{m_+k}} = \mathbf{I}(y_i^{m^{m_+k}} + y_i^{m^{mj}}) - \mathbf{I}(y_i^{m^{mj}}) \leq \mathbf{I}(y_i^{m^{m_+k}}),$$

completing the proof that $\operatorname{supp} \tilde{z}^m \subseteq \operatorname{supp} z^m$. Thus $\tilde{z}^m$ satisfies (a).

Finally, property (c) need only be checked for those periods i such that $\tilde{y}_i^{m^{mj}} = 1$. Suppose this is the case and $s(m_+k) = mk$. If $\tilde{y}_i^{m_+{}^{m_+k}} = 0$, then, by (4b), $\tilde{y}_i^{m^{m_+k}} = -1$, so an alternating pair of inventories is added, beginning with a positive inventory. If $\tilde{y}_i^{m_+{}^{m_+k}} = 1$, then $\tilde{y}_i^{m^{m_+k}} = 0$, so the alternating sign structure is maintained, with the first positive inventory being moved to a lower level. In either case (c) is satisfied by $\tilde{z}^m$.

We now establish (i) of the Theorem. Without loss of generality we may assume the elementary assembly circulations considered satisfy (a), (b) and (c), since the signs of products of pairs is unchanged if both elements in the pair are multiplied by -1. There are two types of faces, internal and external. The variables that lie on the boundary of an internal face are $x_i^{mj}, y_i^{mj}, x_{i_+}^{mj}$ and $y_i^{m_+k}$, where $s(m_+k) = mj$. If the production levels (resp., inventories) $x_i^{mj}, x_{i_+}^{mj}$ (resp., $y_i^{mj}, y_i^{m_+k}$) are nonzero in an elementary assembly circulation, then, by (b) (resp., (c)), their product is negative, that is, they are substitutes.

Now suppose x_i^{mj} and y_i^{mj} are nonzero in an elementary assembly circulation z. Consider the assembly circulation z^m. If $y_i^{mj} = 1$, (c) and the definition of contraction imply that $y_i^{m^{mj}} = 1$. Since level m_- variables are zero in z^m, the degree of the tail node of y_i^{mj} must be 2. Thus, by conservation of flow, we must have $x_i^{mj} = x_i^{m^{mj}} = 1$. Analogously, if $y_i^{mj} = -1$, then $y_i^{m^{mj}} = 0$. But $0 \neq x_i^{mj} = x_i^{m^{mj}}$, so the degree of the tail node of y_i^{mj} must be 2. Thus $y_{i_-}^{m^{mj}} \neq 0$ and, in fact, must equal 1 by (c). Finally, conservation of flow associated with this node implies $x_i^{mj} = x_i^{m^{mj}} = -1$. Thus the product $x_i^{mj} y_i^{mj}$ is positive, and the variables are complements. The fact that $y_i^{mj} x_{i_+}^{mj} \leq 0$, that is, the variables are substitutes, is shown in an analogous fashion.

Now suppose $x_{i_+}^{mj}$ and $y_i^{m_+k}$ are nonzero in a $0, \pm 1$ elementary assembly circulation z. Consider the assembly circulation $z^m (= \tilde{z}^m)$. If $y_i^{m_+k} = 1$, (c) implies that $y_i^{m^{mj}} = 0$. This and the fact that $0 \neq x_{i_+}^{mj} = x_{i_+}^{m^{mj}}$ imply that the degree of the head node of $x_{i_+}^{mj}$ must be 2. Then, by (c), $y_{i_+}^{m^{mj}} = 1$, and thus conservation of flow implies $x_{i_+}^{mj} = x_{i_+}^{m^{mj}} = 1$. The case $y_i^{m_+k} = -1$ is handled in a similar fashion, establishing that the variables are complements. Finally, the same technique may be used to show that x_i^{mj} and $y_i^{m_+k}$ are substitutes.

Now consider the external face. By (c) we conclude that y_i^{01} and y_j^{01} are complements. By (b) and (c) we conclude that the product $x_i^{mj}y_p^{01}$ is nonnegative (the variables are complements) if $i = 1$ and nonpositive (the variables are substitutes) if $i = n$.

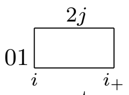
Now suppose two production variables, say x_i^{mj} and x_q^{rk} where $i, q \in \{1, n\}$, are nonzero in an elementary assembly circulation. Then (b) implies their product is positive if $i = q$ and negative otherwise. This finishes the proof of (i) and also shows (iii), since the argument does not depend on mj and rk belonging to the same slice.

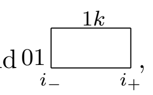
Next we establish (ii). Suppose $y_i^{m_+k}$ and $y_i^{m_+r}$ such that $s(m_+k) = s(m_+r) = mj$ are nonzero in an elementary assembly circulation z. Consider z^m. If $y_i^{m^{mj}} = 0$, then, by (c), $y_i^{m^{m_+k}} = y_i^{m_+k} = 1 = y_i^{m^{m_+r}} = y_i^{m_+r}$. If $y_i^{m^{mj}} = 1$, then, by (c), $y_i^{m^{m_+k}} = y_i^{m_+k} = -1 = y_i^{m^{m_+r}} = y_i^{m_+r}$. Thus, in any case, the two variables have the same sign and thus are complements.

It remains to show that the pairs of variables that do not satisfy (i), (ii) or (iii) are not conformal. In order to show that two variables are not conformal we build two elementary assembly circulations in which the products of the variables have opposite signs. These assembly circulations are most often cloned from simpler circulations in slices. All the circulations are built by sending unit flow counterclockwise along a specified cycle. Thus in order to describe a circulation it suffices to specify its induced cycle.

If a pair of variables belongs to the same slice, then use the fact that they are not conformal for the assembly problem if they are not for the network-flow problem whose graph is the slice subgraph. Thus two variables in a slice that do not belong to the boundary of a common face or are not incident to a common node are not conformal. By (i) it remains to show that a pair of variables whose arcs are adjacent but do not lie on the boundary of a common face are not conformal. These can be of two types: production at a facility and its successor in the same period (but not the first or last), or inventories at a facility (other than the assembly facility) in successive periods.

Consider the pair $x_i^{m_+k}, x_i^{mj}$ such that $s(m_+k) = mj$ and the pair $y_{i_-}^{m_+k}, y_i^{m_+k}$. Without loss of generality take $mj = 01$ and, consequently, $m_+k = 1k$. Suppose $i \geq 3$ (the remaining possibility, $i = 2$, is analogously treated). Consider the assembly circulation that induces (1) the two cycles in Figure 13 (a) below in all slices containing facility $1k$ (in case $1k$ is a parts facility, the three top nodes should coalesce) and (2) the unique cycle in Figure 13 (b) in all remaining slices. The assembly circulation is elementary since all arcs are sequentially strongly biconnected. Notice that $x_i^{1k}x_i^{01} < 0$ and $y_{i_-}^{1k}y_i^{1k} < 0$ in this elementary assembly circulation. On the other hand, these products are positive in the cloned elementary assembly circulations obtained by cloning the elementary circulations that induce the cycles



(where $s(2j) = 1k$) and


, respectively. Therefore these pairs of variables are not conformal.

(**a**) Induced slice subgraphs in slices containing facility $1k$

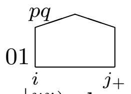
(**b**) Induced slice subgraphs in remaining slices

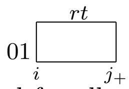
Figure 13: Adjacent production arcs that do not belong to same face are not conformal.

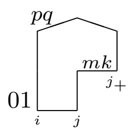
The remaining pairs of variables can be classified in the following three possible cases:

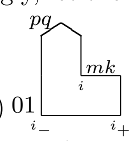
(I) x_i^{mk}, y_j^{rt} where the facilities mk and rt do not belong to the same slice.

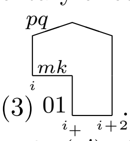
Without loss of generality the first common (possibly nonimmediate) successor of mk and rt is facility 01. Let 11 (resp., 12) be the facility on level 1 that belongs to the path from mk (resp., rt) to the assembly facility in the facility tree. Pick a slice containing mk (resp., rt), say $S(pq)$ (resp., $S(uv)$).

Suppose $i \leq j$. Consider the elementary circulation on the slice $S(pq)$ (resp., $S(uv)$) that induces the cycle 

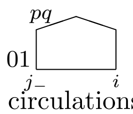
(resp., 

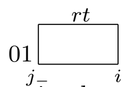
). Construct clones of the elementary circulation $z^{\downarrow pq}$ (resp., $z^{\downarrow uv}$) thus obtained for all other slices containing facility 11 (resp., all level 1 facilities other than 11). Together, these elementary circulations form a $0, \pm 1$ assembly circulation that is elementary since each induced slice subgraph contains a unique cycle that in turn contains x_i^{01}, so the induced subgraph is sequentially strongly biconnected with respect to this unique slice-partition. Notice that the product $x_i^{mk} y_j^{rt}$ is negative in this elementary assembly circulation. Next we build another elementary assembly circulation in which this product is positive, so the two variables are not conformal. Consider the cases (1) $i < j$, (2) $i = j \geq 2$ or (3) $i = j \leq n-2$. Accordingly, let the elementary circulation $z^{\downarrow pq}$ in the slice $S(pq)$ induce the cycle (1) 

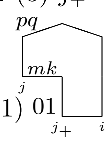
, (2) 

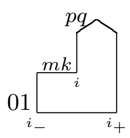
or (3) 

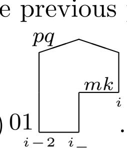
. Construct an elementary circulation $z^{\downarrow uv}$ in the slice $S(uv)$ that induces the cycle $(1')$, $(2')$, or $(3')$ obtained by replacing pq, mk with uv, rt respectively in the cycles (1), (2) and (3) above. Construct clones of the elementary circulation $z^{\downarrow pq}$ (resp., $z^{\downarrow uv}$) thus obtained for all other slices containing facility 11 (resp., all level 1 facilities other than 11). Together, these elementary circulations form an assembly circulation that is clearly elementary. In each case, the product $x_i^{mk} y_j^{rt}$ is positive as claimed.

Now suppose $i > j$. Consider the elementary circulation in the slice $S(pq)$ (resp., $S(rt)$) that induces the cycle 

(resp., 

). Build a $0, \pm 1$ elementary assembly circulation using these circulations as done in the previous paragraph. In this elementary assembly circulation $x_i^{mk}, y_j^{rt} < 0$, thus $x_i^{mk} y_j^{rt} > 0$. Now consider the cases (1) $i > j_+$, (2) $j_+ = i < n$ or (3) $j_+ = i > 2$. Following the previous paragraph, the cycle in $S(pq)$ will be respectively (1) 

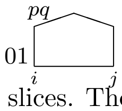
, (2) 

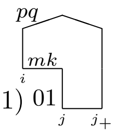
or (3) 

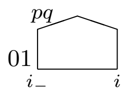
. The cycle in $S(uv)$ is again obtained by replacing pq, mk with uv, rt respectively in the cycle built for $S(pq)$. The product $x_i^{mk} y_j^{rt}$ is negative in the elementary assembly circulation obtained by repeating the procedure of the previous paragraph, which shows the pair is not conformal.

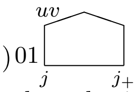
(II) x_i^{mk}, x_j^{rt}, where facilities mk and rt do not belong to the same slice and $\{i, j\} \not\subseteq \{1, n\}$.

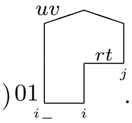
Without loss of generality $i \leq j$ and 01 is the first common (possibly nonimmediate) successor of mk and rt. Let $S(pq)$ (resp., $S(uv)$) be a slice containing mk (resp., rt). Suppose $i < j$. Construct an elementary circulation in the slice $S(pq)$ inducing the cycle



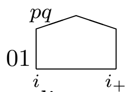
. Construct clones of the elementary circulation $z^{\downarrow pq}$ thus obtained for all other slices. The resulting $0, \pm 1$ assembly circulation is clearly elementary since each induced slice subgraph contains a unique cycle which in turn contains x_i^{01}, so the induced subgraph is sequentially strongly biconnected with respect to this unique slice-partition. Notice that the product $x_i^{mk} x_j^{rt}$ is negative in this elementary assembly circulation. To build the next one consider the cases (1) $i \leq j_- < n_-$ and (2) $1 < i \leq j_-$. Accordingly, construct an elementary circulation in $S(pq)$ inducing the cycle (1) 

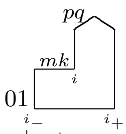
or (2) 

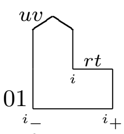
. Similarly, construct an elementary circulation in $S(rt)$ inducing the cycle (1) 

or (2) 

. Construct clones of the elementary circulation $z^{\downarrow pq}$ (resp., $z^{\downarrow uv}$) thus obtained for all other slices containing facility 11 (resp., all level 1 facilities other than 11). This leads to an elementary assembly circulation in which the product x_i^{mk}, x_j^{rt} is positive as desired.

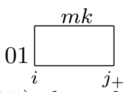
Now suppose $i = j$. Then $1 \neq i \neq n$. Construct an elementary circulation in the slice $S(pq)$ inducing the cycle 

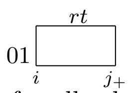
. Construct clones of the elementary circulation $z^{\downarrow pq}$ thus obtained for all other slices, which leads to an elementary assembly circulation such that $x_i^{mk} x_i^{rt} > 0$. Next build an elementary circulation in which the product is negative. Construct an elementary circulation in the slice $S(pq)$ (resp., $S(uv)$) inducing the cycle 

(resp., 

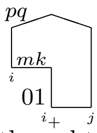
). Construct clones of the elementary circulation $z^{\downarrow pq}$ (resp., $z^{\downarrow uv}$) thus obtained for all other slices containing facility 11 (resp., all level 1 facilities other than 11). The resulting assembly circulation is clearly elementary since each induced slice subgraph contains a unique cycle that in turn contains $x_{i_-}^{01}$, so the induced subgraph is sequentially strongly biconnected with respect to this unique slice-partition.

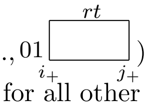
(III) y_i^{mk}, y_j^{rt} where facilities mk and rt do not belong to the same slice and either $i \neq j$ or $s(mk) \neq s(rt)$.

Without loss of generality $i \leq j$ and 01 is the first common successor of mk and rt. Let $S(pq)$ (resp., $S(uv)$) be a slice containing mk (resp., rt). Without loss of generality facility 11 (resp., 12) belongs to slice $S(pq)$ (resp., $S(uv)$). Suppose $i < j$. Construct an elementary circulation in the slice $S(pq)$ (resp., $S(uv)$) inducing the cycle 

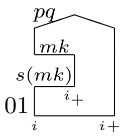
(resp., 

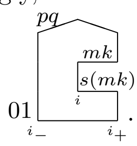
). Construct clones of the elementary circulation $z^{\downarrow pq}$ (resp., $z^{\downarrow uv}$) thus obtained for all other slices containing facility 11 (resp., all level 1 facilities other than 11). This results in an elementary assembly circulation in which $y_i^{mk} y_j^{rt} > 0$. Next build another one in which $y_i^{mk} y_j^{rt} < 0$. Construct an elementary circulation in $S(pq)$ (resp., $S(uv)$) inducing the cycle

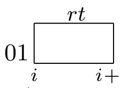


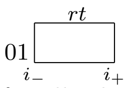
(resp., 

). Construct clones of the elementary circulation $z^{\downarrow pq}$ (resp., $z^{\downarrow uv}$) thus obtained for all other slices containing facility 11 (resp., all level 1 facilities other than 11), which leads to the desired elementary assembly circulation.

Now suppose $i = j$. The first elementary assembly circulation of the previous paragraph also applies to this case and so it suffices to build another one in which the product is negative. Since the facilities mk and rt do not have a common successor, either $m > 1$ or $r > 1$. Without loss of generality suppose $m > 1$. The possible cases are: (1) $i < n_-$ or (2) $i > 1$. Accordingly, construct an elementary circulation in slice $S(pq)$ inducing the cycle (1) 

or (2) 

. Construct an elementary circulation in $S(uv)$ inducing the cycle (1) 

or (2) 

. Construct clones of the elementary circulation $z^{\downarrow pq}$ (resp., $z^{\downarrow uv}$) thus obtained for all other slices containing facility 11 (resp., all level 1 facilities other than 11). This results in the desired $0, \pm 1$ elementary assembly circulation. $\square$

Table 1 below summarizes the complement (**C**) and substitute (**S**) pairs of the problem. Symmetry makes the completion of the lower left corner unnecessary.

	x_j^{mk}	y_w^{qu}
$x_i^{\ell r}$	**S** if $\ell r = mk, \lvert i-j\rvert = 1$ **C** if $i = j = 0$ or n **S** if $\{i, j\} = \{0, n\}$	**C** if $\ell r = qu, i = w$ **S** if $\ell r = qu, i = w_-$ **S** if $s(qu) = \ell r, i = w$ **C** if $s(qu) = \ell r, i = w_-$ **C** if $i = 0, qu = 01$ **S** if $i = n, qu = 01$
y_v^{pt}		**S** if $s(qu) = pt, w = v$ **C** if $s(qu) = s(pt), w = v$ **C** if $pt = qu = 01$

Table 1: Substitutes and complements in the assembly system.

The characterization of the elementary assembly circulations and the conformal pair of variables enables one to easily extend several results obtained for network-flow problems in [8]. In the sequel we give some examples of the application of these new results.

Example. *Increasing the safety stock of an item.* Requiring that the stock of an item be kept above a certain level is a common practice to prevent shortages due to underestimates in demand forecasts or delays in the production, for instance. Such a requirement is easily incorporated in the objective function by adding to the cost of inventory in a given period, say y, the function $\delta_+(y-\ell)$, which is the indicator function of the set $\{y \mid y \geq \ell\}$. The resulting cost function is both convex and subadditive, provided the original cost was. Suppose we want to predict the effect on the optimal solution of increasing the lower bound on y_i^{mk}, for instance. Assume the costs on the other variables are convex and lower semicontinuous.

From Theorems 7 and the Monotone Optimal-Flow Selection Theorem of [8] we conclude that there is an optimal selection such that y_i^{mk}, y_i^{mr}, for all facilities mr such that $s(mr) = s(mk)$, x_i^{mk} and $x_{i_+}^{s(mk)}$ are increasing in ℓ and $x_{i_+}^{mk}$, $x_i^{s(mk)}$, y^{m_+r}, for all facilities m_+r such that $s(m_+r) = mk$, and $y_i^{s(mk)}$ are decreasing in ℓ. The effect on the remaining variables is not predictable, unless $mk \equiv 01$. In this case we can further say that all the inventories of the end item will increase, all the productions in the first period of all products will increase and the productions in the last period of all products will decrease when ℓ is increased.

The following two examples are taken from [16].

Example. *Production cost decrease due to technological improvements.* Suppose technological improvements at facility mk are anticipated on period i that will reduce production costs. That is, production cost of x_i^{mk} may be modeled as

$$c(x_i^{mk}, t) = \begin{cases} (1/t)c(x_i^{mk}), & x_i^{mk} \geq 0 \\ c(x_i^{mk}), & x_i^{mk} < 0 \end{cases}$$

where $c(\cdot)$ is convex, nondecreasing on $[0, \infty)$, $c(0) = 0$ and $t \geq 1$. Then $c(\cdot, \cdot)$ is subadditive. Assume the costs on other variables are convex and lower semicontinuous. Technological improvements have the effect of increasing t thus reducing the production cost associated with x_i^{mk}. From Theorems 7 and the Monotone Optimal-Flow Selection Theorem of [8] we may predict that there is an optimal selection such that x_i^{mk}, y_i^{mk} and $y_{i_-}^{m_+r}$, for all m_+r predecessors of mk, are increasing in t and $y_{i_-}^{mk}$, $y_i^{m_+r}$, for all m_+r predecessors of mk, $x_{i_-}^{mk}$ and $x_{i_+}^{mk}$ are decreasing in t.

Example. *Lower bound on production cost.* Consider the situation where cost associated with production is such that up to a certain level the cost is fixed. This is the case, for instance, when there is a fixed labor cost so that up to a certain production level the cost is constant and for higher production levels overtime labor must be used, which results in a increase in cost. Thus let the cost associated with production x_i^{mk} be $c(x_i^{km}, t) = c(x_i^{mk} \vee t)$ where $c(\cdot)$ is nondecreasing and convex. This implies that $c(\cdot, \cdot)$ is doubly subadditive. If the other cost functions are convex and lower semicontinuous, the Smoothing Theorem of [8] and 7 imply that there is an optimal selection such that x_i^{mk}, $t - x_i^{mk}$, y_i^{mk}, $-y_{i_-}^{mk}$, $y_{i_-}^{m_+r}$ and $-y_i^{m_+r}$, for all m_+r predecessors of mk, are increasing in t, so that, in particular, an increase the lower bound on the production cost will result in an increase of the production level but by less than the increase in the parameter. Finally by the Ripple Selection Theorem of [8] the changes in y_i^{mk},$y_{i_-}^{mk}$, $y_{i_-}^{m_+r}$ and $y_i^{m_+r}$, for all m_+r predecessors of mk, will also be smaller in absolute value than the change in t, since they are bounded by the change in x_i^{mk}.

Example. *Comparison between investment returns in different setups.* Let t_i and t_{i_+} be the cost parameters associated with x_i^{mk} and $x_{i_+}^{mk}$. Suppose a given amount of investiment on production of facility mk on period i will decrease t_i to $\tilde{t}_i$ and result in a reduction of the minimum cost $C(t_i, t_{i_+})$. Assuming the costs $c(x_i^{mk}, t_i)$ and $d(x_{i_+}^{mk}, t_{i_+}^{mk})$ are subadditive and convex functions of the first variable, and the remaining costs are convex and lower semicontinuous, the Superadditivity of Minimum Cost in Pair of Substitutes Corollary of [8] implies that $\underline{C}(t_i, t_{i)_+})$ is superadditive. Therefore the decrease in cost $\underline{C}(t_i, t_{i_+}) - \underline{C}(\tilde{t}_i, t_{i_+})$ is increasing in t_{i_+}, that is, the same investment will produce bigger savings the greater t_{i_+} is.

Acknowledgment

I am indebted to Professor Arthur F. Veinott, Jr. for posing the problems addressed herein; for improving the formulations of several of the concepts and results including especially Theorems 1 and 7; for improvements in the proof of Theorem 1; for suggesting definitions including especially clones of elementary circulations in slice subgraphs and cloned elementary assembly circulations; for pointing out some lacunae; and for improving the exposition throughout.

References

[1] C. Berge, Les problèmes de flots et de tension, *Cahiers Centre Etudes Rech. Opér.* 3 (1961) 69–83.

[2] I. Ciurria-Stavropoulos, *Substitutes and complements in multicommodity flows and inventories*, PhD thesis, Department of Operations Research, Stanford University, Stanford, California, March 1989.

[3] R.J. Duffin, Topology of series-parallel networks, *J. Math. Anal. Appl.* 10 (1965) 303–318.

[4] D. Gale and T. Politof, Substitutes and complements in network flow problems, *Discrete Appl. Math.* 3 (1981) 175–186.

[5] A. Gautier and F. Granot, Riples, complements and substitutes in singly constrained monotropic parametric network flow problems, *Networks* 24 (1994) 285–296.

[6] A. Gautier and F. Granot, Riples, complements and substitutes in generalized networks, *Naval Res. Logist.* 43 (1996) 1–21.

[7] A. Gautier, F. Granot and H. Zheng, Qualitative sensitivity analysis in monotropic programming, *Math. Oper. Res.* 23 (1998) 695–707.

[8] F. Granot and A.F. Veinott, Jr, Substitutes, complements and ripples in network flows, *Math. Oper. Res.* 10:3 (1985) 471–497.

[9] M.P. Mello, *Generalized Leontief substitution systems and lattice programming*, PhD thesis, Department of Operations Research, Stanford University, Stanford, California, March 1993.

[10] J.S. Provan, Substitutes and complements in constrained linear models, *SIAM J. Alg. Disc. Meth.* 8 (1987) 585–603.

[11] R.T. Rockafellar, The elementary vectors of a subspace of $\mathbb{R}^n$, in *Combinatorial Mathematics and its Applications*, R.C. Bose and T.A. Dowling (eds.), University of North Carolina Press, Chapel Hill, N.C., 1969, pp. 104–127.

[12] R.E. Tarjan, Depth-first search and linear graph algorithms, *SIAM J. Comput.* 1 (1972) 146–160.

[13] A.F. Veinott, Jr, Extreme points of Leontief substitution systems, *Linear Algebra Appl.* 1 (1968) 181–194.

[14] A.F. Veinott, Jr, Substitutes and complements in network flows, *Presentation to TIMS/ORSA Joint Meeting*. San Francisco, CA, 1968.

[15] A.F. Veinott, Jr, Minimum concave-cost solution of Leontief substitution models of multi-facility inventory systems, *Oper. Res.*, 17:2 (1969) 262–291.

[16] A.F. Veinott, Jr, *Lecture Notes on Inventory Theory*, Department of Operations Research, Stanford University, 1985.

[17] A.F. Veinott, Jr. and G.B. Dantzig. Integral extreme points. *SIAM Rev.* 10 (1968) 371–372.

[18] H. Whitney, Congruent graphs and the connectivity of graphs *Amer. J. Math.* 54 (1932) 150–168.

[19] H. Whitney, Non-separable and planar graphs, *Trans. Amer. Math. Soc.* 34 (1932) 339–362.

[20] W.I. Zangwill, A backlogging model and a multi-echelon model of a dynamic lot size production system—a network approach, *Management Sci.* 15 (1969) 506–527.

Manuscript received 27 January 2006
revised 25 April 2006
accepted for publication 2 May 2006

Margarida P. Mello
Department of Applied Mathematics, IMECC, Unicamp. CP 6065 Campinas, SP, Brasil 13083-970
E-mail address: `margarid@ime.unicamp.br`