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VECTOR QUASI-VARIATIONAL INEQUALITIES UNDER PERTURBATIONS

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Abstract: Painlevé-Kuratowski upper convergence of solutions to perturbed Vector Quasi-Variational Inequalities is studied. Using different partial orders, various types of solutions are proposed and convergence results are established under certain set-valued mappings properties and (pseudo-)monotonicity assumptions on the operators. Some examples show that these conditions are of minimal character.

Key words: *Vector Quasi-Variational Inequalities, set-valued mappings, Painlevé-Kuratowski upper convergence, Banach spaces*

Mathematics Subject Classification: *47J20, 49J40, 49J53, 58E35, 58K55*

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1 Introduction

Quasi-Variational Inequalities have been introduced in [5] and investigated in [1], [3] and [24]. Vector Quasi-Variational Inequalities and some generalization have been considered in [7] and in [9] for what concerning the existence of solutions and in [21] for what concerning well-posedness properties. In this paper we are interested to the behavior of the solution sets of perturbed Vector Quasi-Variational Inequalities in line with previous results concerning Variational Inequalities ([17] and [20]) and Quasi-Variational Inequalities ([18]). More precisely, we are interested in looking for conditions under which the solution sets of perturbed Vector Quasi-Variational Inequalities converge, in some sense, to the solution set of the "limit" problem. The set convergences considered here are the upper and the lower convergence in the sense of Painlevé-Kuratowski (see, for example, [2], [24]). We prove that, under suitable and reasonable assumptions, the upper convergence of the solution sets can be achieved. The lower convergence of the solution sets is not discussed in this paper being well known that it can be obtained only under very particular conditions, even in the scalar case (see, for example, [4], [8], [19], [22]).
The results are expressed with respect to a sequence of perturbations of the data but they could be written also with respect to a parameter. Employed tools are (pseudo-)monotonicity properties for operators defined on partially ordered vector spaces and a previous result concerning lower convergent sequences of sets ([16], Lemma 3.1).

2 Settings and Preliminaries

Let U be a normed space, σ be a convergence in U, and let $(H_n)_n$ be a sequence of subsets of U. Then:

- $z \in \sigma - \text{Limsup}\, H_n$ if there exists a sequence $(z_k)_k$ $\sigma-$converging to z in U such that $z_k \in H_{n_k}$, for a subsequence (H_{n_k}) of $(H_n)_n$ and for each $k \in \mathbb{N}$;
- $z \in \sigma - \text{Liminf}\, H_n$ if there exists a sequence $(z_n)_n$ $\sigma-$converging to z in U and such that $z_n \in H_n$ for n sufficiently large.

Assume now that τ and σ are two convergences in U. Let $S : U \longrightarrow 2^U$ and $S_n : U \longrightarrow 2^U$ be set-valued maps from U to itself for any $n \in \mathbb{N}$. We recall that: S is closed-valued (resp. convex-valued) if $S(u)$ is a nonempty closed (resp. convex) subset of U for every $u \in U$; S is *sequentially* (τ, σ)*-closed* if $\sigma - \text{Limsup}\, S(u_n) \subseteq S(u)$ for all $(u_n)_n$ τ-converging to u, that is for every sequence $(u_n)_n$ τ-converging to u in U and for every sequence $(z_n)_n$ σ-converging to z in U and such that $z_n \in S(u_n)$ for n sufficiently large, one has $z \in S(u)$; S is *sequentially* (τ, σ)*-lower semicontinuous* if $S(u) \subseteq \sigma - \text{Liminf}\, S(u_n)$ for all $(u_n)_n$ τ-converging to u, that is for every sequence $(u_n)_n$ τ-converging to u in U and for every $z \in S(u)$ there exists a sequence $(z'_n)_n$ σ-converging to z such that $z'_n \in S(u_n)$ for n sufficiently large. According to [14] we say that $(S_n)_n$ is:

- (τ, σ)-*sequentially upper convergent to* S if:
  $\sigma - \text{Limsup}\, S_n(u_n) \subseteq S(u)$ for every sequence $(u_n)_n$ τ-convergent to u and for every $u \in U$;
- (τ, σ)-*sequentially lower convergent to* S if:
  $S(u) \subseteq \sigma - \text{Liminf}\, S_n(u_n)$ for every sequence $(u_n)_n$ τ-convergent to u and for every $u \in U$.

In the following: we denote by $G(T)$ the graph of any map $T : U \longrightarrow 2^Z$, that is the set $\{(u, v) \in U \times Z / v \in T(u)\}$ and by $B(u, \delta)$ the open ball with center in u and ray δ; we consider reflexive real Banach spaces; we omit the term *sequentially* and we denote by w and s, respectively, the weak and strong convergences. Moreover, we write $u_n \xrightarrow{s} u$ to denote that $(u_n)_n$ strongly converges to u, $u_n \rightharpoonup u$ to denote that $(u_n)_n$ weakly converges to u and $\mathit{int}\, H$ to denote the interior of the set H. We note that when the sequence $(S_n)_n$ is (w, w)-upper converging and (w, s)-lower converging to S, then the sequence $(S_n(u_n))_n$ Mosco converges to $S(u)$ for every $u \in U$ and every sequence $(u_n)_n$ weakly converging to u (see [24]).

Let U and V be two reflexive real Banach spaces. For any $n \in \mathbb{N}$, let K be a nonempty, closed and convex subset of U, $A : K \longrightarrow \mathcal{L}(U, V)$, $S : K \longrightarrow 2^K$, $A_n : K \longrightarrow \mathcal{L}(U, V)$ and $S_n : K \longrightarrow 2^K$, where $\mathcal{L}(U, V)$ is the space of all linear and continuous maps from U to V. Assume that $\mathcal{C}$ is a pointed and closed cone in V with apex in the origin and nonempty interior. The cone $\mathcal{C}$ induces on V two strict order relations, denoted by $\leq_{\mathit{int}\,\mathcal{C}}$ and $\leq_{\mathcal{C}\setminus\{0\}}$ and defined as below:

$$u \leq_{\mathit{int}\,\mathcal{C}} v \Longleftrightarrow v - u \in \mathit{int}\,\mathcal{C}$$
$$u \leq_{\mathcal{C}\setminus\{0\}} v \Longleftrightarrow v - u \in \mathcal{C}\setminus\{0\}\ .$$

We are interested in the following Vector Quasi-Variational Inequalities, that become a classical Quasi-Variational Inequality when $V = \mathbb{R}$ and $\mathcal{C} = [0, +\infty[$:

$$(WVQ): \quad \begin{cases} \text{find } u \in K \text{ such that } u \in S(u) \text{ and} \\ \langle Au, v - u \rangle \not\leq_{\mathit{int}\,\mathcal{C}} 0 \ \ \forall\, v \in S(u) \end{cases}$$

$$(VQ): \quad \begin{cases} \text{find } u \in K \text{ such that } u \in S(u) \text{ and} \\ \langle Au, v - u \rangle \not\leq_{\mathcal{C}\setminus\{0\}} 0 \ \ \forall\, v \in S(u) \end{cases}$$

In the scalar case, the investigation of Variational or Quasi-Variational Inequalities leads to consider linearized or Minty's problems (dual problems in [3]). The linearized problems corresponding to the above Vector Quasi-Variational Inequalities are the following:

$$(WLVQ): \quad \begin{cases} \text{find } u \in K \text{ such that } u \in S(u) \text{ and} \\ \langle Av, v-u \rangle \not\leq_{int\,\mathcal{C}} 0 \ \ \forall\, v \in S(u) \end{cases}$$

$$(LVQ): \quad \begin{cases} \text{find } u \in K \text{ such that } u \in S(u) \text{ and} \\ \langle Av, v-u \rangle \not\leq_{\mathcal{C}\setminus\{0\}} 0 \ \ \forall\, v \in S(u). \end{cases}$$

All these problems have been already considered in [21] for what concerns well-posedness properties. Here, our aim is to study the behavior of the solution sets of the perturbed problems $(WVQ)_n$ (resp. $(VQ)_n, (WLVQ)_n$ and $(LVQ)_n$) defined by the maps A_n and the constraints S_n. The sets of solutions of the perturbed problems are denoted, respectively, by WQ_n, Q_n, WL_n, L_n, while the sets of solutions of the unperturbed problems are denoted, respectively, by WQ, Q, WL, L. Since $int\,\mathcal{C} \subseteq \mathcal{C}\setminus\{0\}$, we point out that

$$Q \subseteq WQ \ \text{ and } \ L \subseteq WL,$$

$$Q_n \subseteq WQ_n \ \text{ and } \ L_n \subseteq WL_n.$$

In the case where the set-valued map S is constantly equal to K, problem (VQ) (resp. (WVQ)) amounts to a Vector Variational Inequality (resp. Weak Vector Variational Inequality) and problem (LVQ) (resp. $(WLVQ)$) amounts to the Minty's Vector Variational Inequality (resp. Weak Minty's Vector Variational Inequality), introduced and investigated by Giannessi in [10] and in [11]. As in the scalar case, in order to investigate Vector Quasi-Variational or Variational Inequalities in infinite dimensional spaces, some continuity and monotonicity properties are useful. Thus, an operator A from U to $\mathcal{L}(U, V)$ is said to be:

- ▷ *hemicontinuous* if it is continuous from the segments of U to $\mathcal{L}(U, V)$ endowed with the weak topology;
- ▷ *monotone* if, for every $u, v \in U$:

$$\langle Au - Av, u - v \rangle \geq_{\mathcal{C}} 0 \ ;$$

- ▷ *pseudomonotone* if, for every $u, v \in U$:

$$\langle Av, u - v \rangle \geq_{\mathcal{C}} 0 \Longrightarrow \langle Au, u - v \rangle \geq_{\mathcal{C}} 0 \ ;$$

- ▷ *strictly pseudomonotone* if, for every $u, v \in U$ and $u \neq v$:

$$\langle Av, u - v \rangle \geq_{\mathcal{C}\setminus\{0\}} 0 \Longrightarrow \langle Au, u - v \rangle \geq_{\mathcal{C}\setminus\{0\}} 0 \ .$$

- ▷ *W-monotone* if, for every $u, v \in U$ and $u \neq v$:

$$\langle Au - Av, u - v \rangle \geq_{int\,\mathcal{C}} 0 \ ;$$

- ▷ *W-pseudomonotone* if, for every $u, v \in U$ and $u \neq v$:

$$\langle Av, u - v \rangle \geq_{int\,\mathcal{C}} 0 \Longrightarrow \langle Au, u - v \rangle \geq_{int\,\mathcal{C}} 0 \ .$$

A sequence of operators $(A_n)_n$ is $G(s^-, s^-)$-*converging* to an operator A if for every $v \in U$ there exists a sequence $(v'_n)_n$ strongly convergent to v such that $A_n v'_n \xrightarrow{s} Av$, that is $G(A) \subseteq s - \text{Liminf}\, G(A_n)$. Such convergence, weaker than pointwise convergence, has been used by Mosco in [24], in the setting of scalar Variational Inequalities, and by Lignola and Morgan in [18], in the setting of scalar Quasi-Variational Inequalities.
The next proposition recalls the links between the solution sets of the different types of Vector Quasi-Variational Inequalities.

Proposition 2.1 ([21], Prop. 3.1) *Assume that the following assumptions hold:*

i) *the set-valued map S is closed-valued and convex-valued on U;*

ii) *the operator A is pseudomonotone (resp. W-pseudomonotone) on U.*

Then $Q \subseteq L$ (resp. $WQ \subseteq WL$). If the operator A is hemicontinuous on U, then $WQ \supseteq WL$.

We point out that, in general, the Vector Quasi-Variational Inequality (VQ) defined by A and S is not equivalent to the linearized Vector Quasi-Variational Inequality (LVQ) even in finite dimensional spaces with a continuous operator A and a constant set-valued map S (see [11]).
Finally, we recall a result on the lower convergence of a sequence of convex sets, that will be widely used in the next section.

Proposition 2.2 ([16], Lemma 3.1) *Let $(H_n)_n$ be a sequence of nonempty subsets of a Banach space E such that:*

i) *H_n is convex for every $n \in N$;*

ii) $H \subseteq \text{Liminf}\, H_n$;

iii) *there exists $m \in N$ such that $int \bigcap_{n \geq m} H_n \neq \emptyset$.*

Then, for every $u \in int\, H$ there exists a positive real number δ such that: $B(u, \delta) \subseteq H_n \ \forall\, n \geq m$.
If E is a finite dimensional space, then assumption iii) *can be substituted by:* iii') *$int\, H \neq \emptyset$.*

3 Convergence of Solutions

In this section, we first consider the problem (WVQ) and we investigate the behavior of the solutions to the perturbed problems $(WVQ)_n$ $(n \in \mathbb{N})$. More precisely, when the operators are assumed to be W-pseudomonotone or monotone, we give sufficient conditions for the weak convergence of a sequence of solutions to $(WVQ)_n$ to a solution to (WVQ). Then, we investigate the upper convergence of the solution sets of the linearized problems $(WLVQ)_n$ to the solution set of $(WLVQ)$. Finally, some examples show that it is not possible to obtain neither upper convergence results for the problems (VQ) and (LVQ) nor lower convergence results for all considered problems.

In this section, we consider set-valued maps S and S_n with non-empty, convex and closed values. The first result concerns the upper convergence of the solution sets of $(WVQ)_n$ to the solution set of (WVQ) when the operators $(A_n)_n$ are W-pseudomonotone.

Theorem 3.1 *Assume that the following conditions are satisfied:*

i) *A is hemicontinuous;*

ii) *A_n is W-pseudomonotone for any $n \in \mathbb{N}$;*

iii) *$A_n v \xrightarrow{s} Av$ for any $v \in U$;*

iv) *the sequence $(S_n)_n$ is (w, w)-upper converging and (w, s)-lower converging to S;*

v) *for any $u \in U$, $int\, S(u) \neq \emptyset$ and for any sequence $(u_n)_n$ weakly convergent to u, there exists $m \in \mathbb{N}$ such that $int \bigcap_{n \geq m} S_n(u_n) \neq \emptyset$.*

Then:

$$w - \text{Limsup}\, WQ_n \subseteq WQ\ .$$

When V is finite dimensional, in v) *it is sufficient only to assume $int\, S(u) \neq \emptyset$ for any $u \in U$.*

Proof. By contradiction, assume that $u \in w - \text{Limsup}\, WQ_n$ and $u \notin WQ$. So, there exists a sequence $(u_k)_k$ weakly converging to u such that $u_k \in WQ_{n_k}$ for a subsequence (WQ_{n_k}) of $(WQ_n)_n$ and for any k. Since $u_k \in S_{n_k}(u_k)$ for any k and the sequence $(S_n)_n$ is (w, w)-upper converging to S, one gets $u \in S(u)$. Therefore, having assumed that $u \notin WQ$ and A being hemicontinuous, in light of Proposition 2.1, $u \notin WL$, that is u does not solve $(WLVQ)$. So:

$(\mathcal{H})$: *there exists $v' \in S(u)$ such that $\langle Av', v' - u \rangle \leq_{int\, \mathcal{C}} 0$, that is*

$$\langle Av', u - v' \rangle \in int\, \mathcal{C}\ . \tag{1}$$

Statement $(\mathcal{H})$ implies the following:

$(\mathcal{T})$: *there exists $\hat{v} \in int\, S(u)$ which satisfies* (1)*.*

In fact, if $(\mathcal{H})$ is satisfied and $(\mathcal{T})$ fails to be true, one has:

$$\langle Av, u - v \rangle \notin int\, \mathcal{C} \quad \forall\, v \in int\, S(u), \tag{2}$$

and the element v' satisfying $\mathcal{H}$ is such that $v' \in S(u) \backslash int\, S(u)$. Taken a sequence $(v_n)_n \subseteq int\, S(u)$ which lies on a segment and which (strongly) converges to v', A being hemicontinuous, from (2) we obtain:

$$\langle Av_n, u - v_n \rangle \xrightarrow{s} \langle Av', u - v' \rangle \notin int\, \mathcal{C}\ ,$$

which is in conflict with (1). Hence statement $(\mathcal{T})$ holds.
Now, let $\hat{v}$ be an element satisfying $(\mathcal{T})$. In light of v) and Proposition 2.2, $\hat{v}$ belongs to $S_{n_k}(u_k)$ for k sufficiently large. The assumption iii) implies

$$s - \lim_k \langle A_{n_k} \hat{v}, u_k - \hat{v} \rangle = \langle A\hat{v}, u - \hat{v} \rangle\ . \tag{3}$$

So, being $\langle A\hat{v}, u - \hat{v} \rangle \in int\ \mathcal{C}$, we get $\langle A_{n_k} \hat{v}, u_k - \hat{v} \rangle \in int\ \mathcal{C}$ and $u_k \neq \hat{v}$ for k sufficiently large. Since A_{n_k} is W-pseudomonotone for any k, we have

$$\langle A_{n_k} u_k, u_k - \hat{v} \rangle \in int\, \mathcal{C} \quad \text{for } k \text{ sufficiently large,}$$

which is in contradiction with the assumption $u_k \in WQ_{n_k}$. □

We remark that in order to obtain the weak convergence to a solution to (WVQ) of a sequence of solutions to the problems $(WVQ)_n$, we cannot drop the W-pseudomonotonicity of the operators A_n. The following example considers a continuous and non W-pseudomonotone operator A defined on an Hilbert space and such that the set of solutions of (WVQ) is not sequentially closed in the weak convergence.

Example 3.1 Let U be an infinite dimensional separable Hilbert space, $V = \mathbb{R}$, $\mathcal{C} = [0, +\infty[$, and let $Au = A_n u = -u$ and $S(u) = S_n(u) = \mathcal{B} = \{v \in U/ \parallel v \parallel \leq 1\}$ for every $u \in U$ and every $n \in \mathbb{N}$. One can check that $(A_n)_n$ is pointwise converging to A but it is not W-pseudomonotone. Set $\mathcal{S} = \{u \in U/ \parallel u \parallel = 1\}$, one has $WQ = WQ_n = \mathcal{S} \cup \{0\}$. In fact, the statement is obvious for $u = 0$ and for u such that $\parallel u \parallel = 1$ one gets $\langle u, v\rangle \leq \parallel v \parallel \leq 1$ for each $v \in \mathcal{B}$, so $\langle Au, u - v\rangle = - \parallel u \parallel^2 + \langle u, v\rangle \leq 0$ for each $v \in S(u)$. On the other hand, if $u \in \mathcal{B}$ and $0 < \parallel u \parallel < 1$, then there exists $t > 1$ such that $v' = tu \in \mathcal{B}$ and $\langle Au, u - v'\rangle = - \parallel u \parallel^2 + \langle u, v'\rangle = - \parallel u \parallel^2 + t \parallel u \parallel^2 > 0$. Hence $w - \text{Limsup}\, WQ_n = cl^w(WQ) = cl^w(\mathcal{S} \cup \{0\})$ and $cl^w(\mathcal{S} \cup \{0\}) = \mathcal{B}$ (see [6], Chapter 3). So $w - \text{Limsup}\, WQ_n \not\subseteq WQ$, that is the result of Theorem 3.1 does not hold.

We observe that Theorem 3.1 (and the same for Theorem 3.2) could be proved without any monotonicity assumptions but assuming that the sequence of operators $(A_n)_n$ is (w, s)-continuously converging to A, that is: $u_n \rightharpoonup u \Rightarrow A_n u_n \xrightarrow{s} Au$. This kind of convergence, which has been used in [13], is a very strong assumption in infinite dimensional spaces, as one can look in the above example, where a sequence $(A_n)_n$ of operators, which is not (w, s)-continuously converging, is considered.

Now, under a stronger monotonicity condition, it is possible to weaken the convergence for operators used in the above theorem. In fact, we have the following result.

Theorem 3.2 *Assume that the following conditions are satisfied:*

i) *A is hemicontinuous;*

ii) *A_n is monotone for any $n \in \mathbb{N}$;*

iii) *$(A_n)_n$ $G(s^-, s^-)$ converges to A, that is $G(A) \subseteq s - \text{Liminf}\, G(A_n)$;*

iv) *the sequence $(S_n)_n$ (w, w)-upper converges and (w, s)-lower converges to S;*

v) *for any sequence $(u_n)_n$ weakly convergent in U and any sequence $(z_n)_n$ strongly convergent to 0 in U, one has $\langle A_n u_n, z_n\rangle \xrightarrow{s} 0$.*

Then:

$$w - \text{Limsup}\, WQ_n \subseteq WQ\ .$$

Proof. Assume that: u_n solves the problem $(WVQ)_n$ for every n, that is:

$$u_n \in S_n(u_n) \quad \text{and} \quad \langle A_n u_n, z - u_n\rangle \not\leq_{int\,\mathcal{C}} 0 \quad \forall\, z \in S_n(u_n)\ ,$$

the sequence (u_n) weakly converges to u and the point u does not solve the problem (WVQ). Since, by assumption iv), $u \in S(u)$, in light of hemicontinuity of the operator A and Proposition 2.1, there exists $v \in S(u)$ such that $\langle Av, v - u\rangle \leq_{int\,\mathcal{C}} 0$, that is $\langle Av, u - v\rangle \in int\,\mathcal{C}$. Since the sequence $(A_n)_n$ is $G(s^-, s^-)$-converging to A, there exists a sequence $(v'_n)_n$, with

$v'_n \xrightarrow{s} v$, such that $A_n v'_n \xrightarrow{s} Av$. Therefore, one has $\langle A_n v'_n, u_n - v'_n \rangle \in int\,\mathcal{C}$ for n sufficiently large. Assumption iv) ensures the existence of a sequence $(v_n)_n$ such that $v_n \xrightarrow{s} v$ and $v_n \in S_n(u_n)$ for n sufficiently large. Being $u_n \in WQ_n$ for any n, one gets

$$\langle A_n u_n, u_n - v_n \rangle \notin int\,\mathcal{C} \quad \text{for } n \text{ sufficiently large}\,. \tag{4}$$

Since the operator A_n is monotone, one has

$$\langle A_n u_n - A_n v'_n, u_n - v'_n \rangle \in \mathcal{C} \quad \forall\, n \in \mathbb{N}\,.$$

Then, one has

$$z_n = \langle A_n u_n, u_n - v'_n \rangle = \langle A_n u_n - A_n v'_n, u_n - v'_n \rangle + \langle A_n v'_n, u_n - v'_n \rangle \in int\,\mathcal{C}$$

and there exists $\delta_n > 0$ such that $B(z_n, \delta_n) \subseteq int\,\mathcal{C}$ for n sufficiently large. In light of assumption v) the sequence $(\langle A_n u_n, v'_n - v_n \rangle)_n$ strongly converges to 0 in V, so there exists a strictly increasing sequence of positive integers $(n_k)_k$ such that

$$\| \langle A_{n_k} u_{n_k}, v'_{n_k} - v_{n_k} \rangle \| < \delta_{n_k} \quad \text{for } k \text{ sufficiently large}\,.$$

Therefore, being

$$\| \langle A_{n_k} u_{n_k}, u_{n_k} - v_{n_k} \rangle - \langle A_{n_k} u_{n_k}, u_{n_k} - v'_{n_k} \rangle \| = \| \langle A_{n_k} u_{n_k}, v'_{n_k} - v_{n_k} \rangle \| < \delta_{n_k}$$

one gets $\langle A_{n_k} u_{n_k}, u_{n_k} - v_{n_k} \rangle \in int\,\mathcal{C}$ and this contradicts (3). □

Similar results have been obtained in [20] for scalar Variational Inequalities. As the following example shows, it is not always possible to get the upper convergence of the solution sets when the operators A_n are W-pseudomonotone and the sequence $(A_n)_n$ $G(s^-, s^-)$-converges to A, even in the scalar case.

Example 3.2 Let $U = V = \mathbb{R}$, $A_n : u \in [0, +\infty[\longrightarrow e^{-(nu)^2} - 1 \in \mathbb{R}$, $Au = -1$ and $S(u) = S_n(u) = [u, u+1]$ for all $u \in [0, +\infty[$. One can check that the operators A_n are W-pseudomonotone and the sequence $(A_n)_n$ is $G(s^-, s^-)$-converging to A. However, one has $WQ_n = \{0\}$ for all n, but $0 \notin WQ$.

Concerning the perturbed problems $(VQ)_n$ and $(VLQ)_n$, the following example shows that, in general, it is not possible to give reasonable sufficient conditions for the upper convergence of the solution sets of the problems $(VQ)_n$ and $(LVQ)_n$.

Example 3.3 Let $U = \mathbb{R}$, $V = \mathbb{R}^2$, $A_n = A : u \in [-1, 0] \longrightarrow (1, u)$, $S(u) = S_n(u) = [-1, u]$ for any $u \in [-1, 0]$ and $\mathcal{C} = [0, +\infty[^2$. The inequality $\langle Au, v - u \rangle \not\leq_{\mathcal{C}\setminus\{0\}} (0,0)$ means that $(u - v, u^2 - uv) \notin \mathcal{C} \setminus \{0\}$. Then, for any $n \in \mathbb{N}$, the set $Q_n = Q = [-1, 0[$ (while $WQ_n = WQ = [-1, 0]$). Therefore $\text{Limsup}\, Q_n = [-1, 0] \not\subseteq Q$.

Obviously, since $Q_n \subseteq WQ_n$, the hypothesis of the previous theorems guarantee that $w-$ Limsup $Q_n \subseteq WQ$. However, the upper convergence of the sequence $(Q_n)_n$ to the set WQ can be obtained also as follows:

Proposition 3.1 *Assume that the assumption* i), iii), iv), v) *of Theorem 3.1 and the following are satisfied:*

ii)′ A_n *is strictly pseudomonotone for any* $n \in \mathbb{N}$*;*

Then:

$$w - \text{Limsup}\, Q_n \subseteq WQ\ .$$

Proof. Assume that $u \in w - \text{Limsup}\, Q_n$ and $u \notin WQ$. Proceeding as in the proof of Theorem 3.1, one obtains (3). So, for k sufficiently large, $\langle A_{n_k}\hat{v}, u_k - \hat{v}\rangle \in int\,\mathcal{C}$ and $u_k \neq \hat{v}$. Since the maps A_{n_k} are strictly pseudomonotone, one gets $\langle A_{n_k}u_k, u_k - \hat{v}\rangle \in \mathcal{C}\backslash\{0\}$ for k sufficiently large, and this is in conflict with $u_n \in Q_n$ for every n. □

For what concerning the linearized problems $(WLVQ)_n$, we have the following result.

Theorem 3.3 *Assume that the following conditions are satisfied:*

i) *A is hemicontinuous;*

ii) *$A_n v \xrightarrow{s} Av$ for any $v \in U$;*

iii) *the sequence $(S_n)_n$ (w,w)-upper converges and (w,s)-lower converges to S.*

iv) *for any $u \in U$, $int\,S(u) \neq \emptyset$ and for any sequence $(u_n)_n$ weakly convergent to u, there exists $m \in \mathbb{N}$ such that $int \bigcap_{n\geq m} S_n(u_n) \neq \emptyset$.*

Then:

$$w - \text{Limsup}\, WL_n \subseteq WL\ .$$

When V is finite dimensional, in v) *it is sufficient only to assume $int\,S(u) \neq \emptyset$ for any $u \in U$.*

Proof. Assume that $(u_n)_n$ is a sequence of solutions to the problems $(WLVQ)_n$, that is

$$u_n \in S_n(u_n) \ \text{ and } \ \langle A_n v, v - u_n\rangle \not\leq_{int\,\mathcal{C}} 0 \ \ \forall\, v \in S_n(u_n),$$

and $u_n \rightharpoonup u$. Then, by assumption iii), $u \in S(u)$. Now, let $v \in S(u)$. If $v \in int\,S(u)$, from iv) and Proposition 2.2, one gets $v \in S_n(u_n)$ for n sufficiently large. So one has $\langle A_n v, v - u_n\rangle \not\leq_{int\,\mathcal{C}} 0$, that is $\langle A_n v, u_n - v\rangle \notin int\,\mathcal{C}$, for n sufficiently large. Therefore:

$$\langle Av, u - v\rangle = s - \lim_n \langle A_n v, u_n - v\rangle \notin int\,\mathcal{C},$$

that is $\langle Av, v - u\rangle \not\leq_{int\,\mathcal{C}} 0$.
If $v \in S(u) \setminus int\,S(u)$, there exists a sequence $(v_n)_n$, which lies on a segment included in $int\,S(u)$, such that $v_n \xrightarrow{s} v$. By assumption i) one has

$$\langle Av, u - v\rangle = s - \lim_n \langle Av_n, u - v_n\rangle \notin int\,\mathcal{C}$$

and the proof is completed. □

We note that, using Proposition 2.1, Theorem 3.1 could be deduced from Theorem 3.3. Indeed, when the operators A_n are W-pseudomonotone, one has $WQ_n \subseteq WL_n$ for all n, and when the operator A is hemicontinuous, one has $WL \subseteq WQ$. However, to emphasize the different natures of the problems (WVQ) and $(WLVQ)$, we have chosen to give a direct proof of Theorem 3.1.

Remark 3.1 We point out that, as shown again by Example 3.2, it is not possible to get upper convergence results of the solution sets of the problems $(WLVQ)_n$ when the sequence of operators $(A_n)_n$ is $G(s^-, s^-)$-converging. Indeed, the operators A and A_n considered in Example 3.2 are W-pseudomonotone and continuous, so $WL = WQ$, $WL_n = WQ_n = \{0\}$ and $0 \notin WL$. Moreover, Example 3.3 shows that it is not possible to obtain upper convergence results of the solution sets of the problems $(LVQ)_n$, since the operators considered therein are monotone, continuous and $Q_n = L_n$.

Finally, we point out that Theorems 3.1, 3.2 and 3.3 can be easily reformulated for weak Vector Variational Inequalities (or Minty's type) taking $S_n(u_n) = K_n$, for every $n \in \mathbb{N}$, and $S(u) = K$.

4 Conclusions

We have investigated stability properties of the solution sets of Vector Quasi-Variational Inequalities and linearized Vector Quasi-Variational Inequalities under perturbations of the data. Summarizing the results, we have obtained that:

▷ Problems (WVQ) are "stable" with respect to upper convergence when the operators are W-pseudomonotone and pointwise converging, or when the operators are monotone and $G(s^-, s^-)$-converging.

▷ Problems $(WLVQ)$ are stable with respect to upper convergence when the operators are pointwise converging, but, in general, they are not stable when the operators are $G(s^-, s^-)$-converging.

▷ Problems (VQ) and (LVQ) are not stable, in general, with respect to upper convergence, even in finite dimensional spaces.

▷ All problems are not stable, in general, with respect to lower convergence, that is: not every solution to the limit problem can be approached by sequences of solutions to the perturbed problems.

In the scalar case (see [15]), some results of lower convergence have been obtained for suitable approximate solutions. So, in a next paper we will investigate approximate solutions for Vector Quasi-Variational Inequalities and their possible lower convergence.

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