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ASYMPTOTIC SUMS OF MONOTONE OPERATORS

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Abstract: We establish a sequential description of the subdifferential sum formula for proper lower semicontinuous convex functions. Motivated by this description, we introduce the notion of an asymptotic sum of two maximal monotone operators, generated by enlargements of the operators, and examine the relationships with the pointwise sum of the maximal monotone operators. We also present various sufficient conditions, including a new dual condition, for the equality of the asymptotic sum and the pointwise sum of two maximal monotone operators.

Key words: *maximal monotone operators, dual conditions, asymptotic sums, pointwise sums*

Mathematics Subject Classification: *90C25, 49J52, 49J53*

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1 Introduction

The pointwise sum of two maximal monotone operators is not necessarily a maximal monotone operator without a regularity condition. For instance, the pointwise sum of the subdifferential operators of two lower semicontinuous proper convex functions is in general not a maximal monotone operator. This prompted the development of various notions of extended sums [6, 11] of maximal monotone operators, which are maximal monotone in several important cases. These extended sums were generated by certain enlargements of the operators [1, 2, 3].

The purpose of this paper is to introduce a notion of asymptotic sum of two monotone operators, generated by enlargements of the operators, and to examine the relationships with the extended sum [11] and the pointwise sum of maximal monotone operators. We present conditions which ensure that the asymptotic sum of two maximal monotone operators equals the pointwise sum of the two operators. In particular, we show that both sums coincide under a new dual closure condition [9], which is weaker than the popularly known interior-point type conditions [12].

2 Preliminaries

We begin by fixing some definitions and notations. We assume throughout that X and Y are Banach spaces unless stated otherwise. The continuous dual space of X will be denoted by X' and will be endowed with the weak* topology. For the set $D \subset$ X, the **closure** of D will be denoted by $\bar{D}$. If a set $A \subset X'$, the expression $\bar{A}$ will stand for the weak* closure. The **indicator function** δ_D is defined as $\delta_D(x) = 0$ if $x \in D$ and $\delta_D(x) = +\infty$ if $x \notin D$. The

support function σ_D is defined by $\sigma_D(u) = \sup_{x\in D} u(x)$. The **normal cone** of D is given by $N_D(x) = \{v \in X' \ : \ v(y-x) \leq 0, \forall y \in D\}$ when $x \in D$, and $N_D(x) = \emptyset$ when $x \notin D$. Let $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$. Then, the **conjugate** function of f, $f^* : X' \to \mathbb{R} \cup \{+\infty\}$, is defined by

$$f^*(v) = \sup\{v(x) - f(x) \mid x \in \ \operatorname{dom} f\},$$

where the *domain* of f, $\operatorname{dom} f$, is given by $\operatorname{dom} f = \{x \in X \mid f(x) < +\infty\}$. The function f is said to be *proper* if f does not take on the value $-\infty$ and $\operatorname{dom} f \neq \emptyset$. The *epigraph of* f, $\operatorname{Epi} f$, is defined by

$$\operatorname{Epi} f = \{(x, r) \in X \times \mathbb{R} \mid x \in \operatorname{dom} f, \ f(x) \leq r\}.$$

The function f is lower semicontinuous if and only if $\operatorname{Epi} f$ is a closed subset of $X \times \mathbb{R}$. The *lower semicontinuous regularization*, $\operatorname{cl} f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$, is the function whose epigraph is equal to the closure of the epigraph of f in $X \times \mathbb{R}$:

$$\operatorname{Epi}(\operatorname{cl} f) := \operatorname{cl}(\operatorname{Epi} f).$$

The *subdifferential of* f, $\partial f : X \rightrightarrows X'$ is defined as

$$\partial f(x) = \{v \in X' \mid f(y) \geq f(x) + v(y-x), \ \forall\, y \in X\}.$$

Note also that $\partial \delta_D = N_D$. If $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semi-continuous *sublinear* function, i.e. f is convex and positively homogeneous ($f(0) = 0$, and $f(\lambda x) = \lambda f(x)$, $\forall x \in X, \forall \lambda \in (0, \infty)$), then $\partial f(0)$ is non-empty and for each $x \in \operatorname{dom} f$,

$$\partial f(x) = \{v \in \partial f(0) \mid v(x) = f(x)\}.$$

For the functions $f, g : X \to \mathbb{R} \cup \{-\infty, +\infty\}$, the *infimal convolution* of f with g, denoted by $f \oplus g : X \to \mathbb{R} \cup \{-\infty, +\infty\}$, is defined by

$$f \oplus g(z) := \inf_{z_1+z_2=z} \{f(z_1) + g(z_2)\}.$$

The infimal convolution of f with g is said to be *exact* provided the infimum above is achieved for every $z \in X$. The following basic lemmas will be used later in the paper.

Lemma 2.1. *Let* $f, g : X \to \mathbb{R} \cup \{+\infty\}$ *be proper lower semi-continuous convex functions such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *Then* $\operatorname{Epi}(f+g)^* = \operatorname{cl}(\operatorname{Epi} f^* + \operatorname{Epi} g^*)$.

Proof. It follows from Theorem 3.2 and Theorem 2.2(e) in [14]. □

Lemma 2.2. *[4] Let* $f, g : X \to \mathbb{R} \cup \{+\infty\}$ *be proper lower semi-continuous convex functions such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *If* $\operatorname{Epi} f^* + \operatorname{Epi} g^*$ *is weak* closed, then*

$$\partial(f+g)(x) = \partial f(x) + \partial g(x)).$$

Lemma 2.3. *(Hiriart-Urruty and Phelps [7]) Let* $f, g : X \to \mathbb{R} \cup \{+\infty\}$ *be two proper lower semicontinuous convex functions. Then for every* $x \in \operatorname{dom} f \cap \operatorname{dom} g$

$$\partial(f+g)(x) = \cap_{\epsilon>0} \overline{\partial_\epsilon f(x) + \partial_\epsilon g(x)}. \tag{2.1}$$

Lemma 2.4. *[13] Let $A : X \times X' \to R \cup \{+\infty\}$ be proper and convex,*

$$((x, x^*) \in X \times X') \Rightarrow A(x, x^*) \geq (x, x^*)$$

and

$$((x, x^*) \in X \times X') \Rightarrow A^*(x^*, x) \geq (x, x^*).$$

Then $G := \{(x, x^*) \in X \times X' \mid A^*(x^*, x) = (x, x^*)\}$ is a maximal monotone subset of $X \times X'$.

Let $\sigma, \tau : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions and let

$$P_X \mathrm{dom}\, \sigma \quad = \quad \{x \in X \mid \exists u \in Y \text{ such that } (x, u) \in \mathrm{dom}\, \sigma\}.$$

Let

$$\begin{aligned} (\mathrm{Epi}\sigma^*)_{X' \times Y' \times \{0\} \times \mathbb{R}} &:= \{(s^*, u^*, 0, c) \mid (s^*, u^*, c) \in \mathrm{Epi}\, \sigma^*\}; \\ (\mathrm{Epi}\sigma^*)_{X' \times \{0\} \times Y' \times \mathbb{R}} &:= \{(s^*, 0, u^*, c) \mid (s^*, u^*, c) \in \mathrm{Epi}\, \sigma^*\}. \end{aligned}$$

For $(x, y) \in X \times Y$, let

$$\rho(x, y) := \inf\{\sigma(x, u) + \tau(x, v) : u, v \in Y, u + v = y\}.$$

Then, ρ is a convex function.

Theorem 2.1. *(see [9]) Let $\sigma, \tau : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions such that $P_X(\mathrm{dom}\, \sigma) \cap P_X(\mathrm{dom}\, \tau) \neq \emptyset$. If $(\mathrm{Epi}\sigma^*)_{X' \times Y' \times \{0\} \times \mathbb{R}} + (\mathrm{Epi}\tau^*)_{X' \times \{0\} \times Y' \times \mathbb{R}}$ is weak* closed, then for each $(x^*, y^*) \in X' \times Y'$,*

$$\rho^*(x^*, y^*) = \min\{\sigma^*(s^*, y^*) + \tau^*(t^*, y^*) : s^*, t^* \in X', s^* + t^* = x^*\}. \tag{2.2}$$

For other sufficient conditions for the bivariate infimal convolution formula (2.2), see [13].

3 Asymptotic Sums of Monotone Operators

Given two monotone operators $S, T : X \to X'$, define the operator $S + T$ as the usual pointwise sum of S and T:

$$(S + T)(x) = S(x) + T(x), \quad x \in X.$$

Then the sum is always monotone. However, the sum is not necessarily a maximal monotone operator without a regularity condition. The regularity condition requires an interior-point type condition which frequently fails. This motivated researchers to look for a general notion of the sum of monotone operators. An *extended sum* of two monotone operators $A, B : X \to X'$ at the point $x \in X$ as follows:

$$A +_{\mathrm{ext}} B(x) = \cap_{\epsilon > 0} \overline{A^\epsilon x + B^\epsilon x},$$

where the operator $A^\epsilon : X \to X'$ defined by

$$A^\epsilon(x) := \{x^* \in X' \mid (y^* - x^*, y - x) \geq -\epsilon \text{ for any } (y, y^*) \in G(A)\}, \; x \in X.$$

This concept was first considered in [10], and formally introduced as a definition in [1]. It is based on the fact that the sum of two proper lower semicontinuous convex functions has the property (2.1).

The sequential description of the subdifferential of the sum of two proper lower semicontinuous functions in the following lemma prompts us to introduce a new notion, called an asymptotic sum. Note that $\lim_{\alpha\in I} x^*_\alpha$ for a net $\{x^*_\alpha\} \subset X', \forall\alpha \in I$ is the limit in the weak* topology, where I is the directed set.

Theorem 3.1. *Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be two proper lower semicontinuous convex functions such that* $\mathrm{dom} f \cap \mathrm{dom} g \neq \emptyset$. *Then for every $x \in \mathrm{dom} f \cap \mathrm{dom} g$,*

$$\partial(f+g)(x) = \{\lim_{\alpha\in I} x^*_\alpha : \{x^*_\alpha\}_{\alpha\in I} \subset X',\ \{\epsilon_\alpha\}_{\alpha\in I} \subset R_+,\ \lim_{\alpha\in I} \epsilon_\alpha = 0,$$
$$x^*_\alpha \in \partial_{\epsilon_\alpha} f(x) + \partial_{\epsilon_\alpha} g(x), \forall\alpha \in I\}.$$

Proof. [⇒]. Let $x^* \in \partial(f+g)(x)$. Then, for each $y \in X$,

$$(y - x, x^*) - ((f+g)(y) - (f+g)(x)) \leq 0.$$

So, $(x^*, (x, x^*) - (f+g)(x)) \in \mathrm{Epi}(f+g)^*$. By Lemma 2.1, $\mathrm{Epi}(f+g)^* = cl(\mathrm{Epi} f^* + \mathrm{Epi} g^*)$ and so, there exist nets $\{(y^*_\alpha, \gamma_\alpha)\} \subset \mathrm{Epi} f^*$ and $\{(z^*_\alpha, \beta_\alpha)\} \subset \mathrm{Epi} g^*$ such that $\lim_\alpha y^*_\alpha + z^*_\alpha = x^*$ and $\lim_\alpha \gamma_\alpha + \beta_\alpha = (x, x^*) - (f+g)(x)$. Since

$$\mathrm{Epi}\ f^* = \bigcup_{\epsilon\geq 0} \{(u, (u, x) + \epsilon - f(x)) \mid u \in \partial_\epsilon f(x)\}$$

and

$$\mathrm{Epi}\ g^* = \bigcup_{\eta\geq 0} \{(u, (u, x) + \eta - g(x)) \mid u \in \partial_\eta g(x)\},$$

there exist nets $\{\eta_\alpha\}, \{\zeta_\alpha\} \subset R_+$ such that $y^*_\alpha \in \partial_{\eta_\alpha} f(x), z^*_\alpha \in \partial_{\zeta_\alpha} g(x)$,

$$\gamma_\alpha = (x, y^*_\alpha) - f(x) + \eta_\alpha \text{ and } \beta_\alpha = (x, z^*_\alpha) - g(x) + \zeta_\alpha.$$

Adding the two equalities, we obtain

$$\gamma_\alpha + \beta_\alpha = (x, y^*_\alpha + z^*_\alpha) - (f+g)(x) + \eta_\alpha + \zeta_\alpha.$$

Passing to limit as $\alpha \to \infty$, we get that $\lim_\alpha \eta_\alpha + \zeta_\alpha = 0$. This gives us that $\lim_\alpha \eta_\alpha = 0$ and $\lim_\alpha \zeta_\alpha = 0$, since $\{\eta_\alpha\}, \{\zeta_\alpha\} \subset R_+$. Letting $\epsilon_\alpha = \max\{\eta_\alpha, \zeta_\alpha\}$, we see that $y^*_\alpha \in \partial_{\epsilon_\alpha} f(x), z^*_\alpha \in \partial_{\epsilon_\alpha} g(x)$ with $\lim_\alpha y^*_\alpha + z^*_\alpha = x^*$and $\lim_\alpha \epsilon_\alpha = 0$.

[⇐]. Suppose that there exist $\{\epsilon_\alpha\} \subset R_+$ with $\lim_\alpha \epsilon_\alpha = 0$, and $\{y^*_\alpha\}, \{z^*_\alpha\} \subset X'$ such that $y^*_\alpha \in \partial_{\epsilon_\alpha} f(x), z^*_\alpha \in \partial_{\epsilon_\alpha} g(x)$ and $\lim_\alpha y^*_\alpha + z^*_\alpha = x^*$. Then, for each $y \in X$,

$$(y - x, y^*_\alpha) \leq f(y) - f(x) + \epsilon_\alpha \text{ and } (y - x, z^*_\alpha) \leq g(y) - g(x) + \epsilon_\alpha.$$

Thus, for each $y \in X$,

$$(y - x, y^*_\alpha + z^*_\alpha) \leq (f+g)(y) - (f+g)(x) + 2\epsilon_\alpha.$$

Passing to the limit as $\alpha \to \infty$, we obtain that for each $y \in X$, $(y - x, x^*) \leq (f+g)(y) - (f+g)(x)$ for any $y \in X$, i.e., $x^* \in \partial(f+g)(x)$. □

We now see that Theorem 3.1 yields a sequential condition characterizing optimality of a convex function over a closed convex set. For related details, see [8].

Corollary 3.1. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. Let C be a closed convex subset of X and let $a \in C \cap \operatorname{dom} f$. Then the point a is a minimizer of f on C if and only if there exist nets $\{u_\alpha\}, \{v_\alpha\} \subset X', \{\epsilon_\alpha\} \subset R_+$ with $u_\alpha \in \partial_{\epsilon_\alpha} f(a)$, $v_\alpha \in N_C^{\epsilon_\alpha}(a)$ and $\lim_\alpha \epsilon_\alpha = 0$ such that $\lim_\alpha u_\alpha + v_\alpha = 0$.*

Proof. Let $g(x) := \delta_C(x)$. Then $a \in C$ is a minimizer of f on C if and only if $0 \in \partial(f+g)(a)$. The conclusion now follows from Theorem 3.1. □

We now introduce the notion of an asymptotic sum of two monotone operators S and T.

Definition 3.1. (Asymptotic Sum) Let $S, T : X \to X'$ be two monotone operators and $x \in X$. Then, the asymptotic sum of S and T is given by

$$S +_{\lim} T(x) := \{\lim_{\alpha \in I} x_\alpha^* : \{x_\alpha^*\}_{\alpha \in I} \subset X',\ \{\epsilon_\alpha\}_{\alpha \in I} \subset R_+,\ \lim_{\alpha \in I} \epsilon_\alpha = 0,$$
$$x_\alpha^* \in S^{\epsilon_\alpha}(x) + T^{\epsilon_\alpha}(x), \forall \alpha \in I\}.$$

Note from the definition that $\partial_{\epsilon_\alpha} f(x) \subset (\partial f)^{\epsilon_\alpha}(x)$. Links among the asymptotic sum, the pointwise sum and the extended sum are illustrated by the following Proposition.

Proposition 3.1. *Let S and T be two monotone operators. For any $x \in X$,*
1°. $(S+T)(x) \subset \overline{S+T}(x) \subset S +_{\lim} T(x)$;
2°. $S +_{\lim} T(x) = S +_{\text{ext}} T(x)$.

Proof. 1°. Clearly, for each $x \in X$, $(S+T)(x) \subset \overline{S+T}(x)$ and it is easy to see that $\overline{S+T}(x) \subset S +_{\lim} T(x)$. Indeed, for any $x^* \in \overline{S+T}(x)$, there exists a net $\{x_\alpha^*\} \subset (S+T)(x)$ such that $\lim_\alpha x_\alpha^* = x$. Let $\epsilon_\alpha = 0$ for each α. Then $\lim_\alpha \epsilon_\alpha = 0$ and $x_\alpha^* \in S^{\epsilon_\alpha}(x) + T^{\epsilon_\alpha}(x)$. Thus, $x^* \in S +_{\lim} T(x)$.

2°. Firstly, we show that $S +_{\lim} T(x) \subset S +_{\text{ext}} T(x)$, let $x^* \in S +_{\lim} T(x)$. Then, by the definition there exists a directed set I such that

$$\begin{aligned} &(i) && \{\epsilon_\alpha\}_{\alpha \in I} \subset R_+ \text{ with } \lim_{\alpha \in I} \epsilon_\alpha = 0 \\ &(ii) && x_\alpha^* \in S^{\epsilon_\alpha}(x) + T^{\epsilon_\alpha}(x), \forall \alpha \in I \\ &(iii) && \lim_{\alpha \in I} x_\alpha^* = x^*. \end{aligned}$$

Let $\epsilon > 0$. Then, there exists a terminal set* such that $\epsilon_\alpha \le \epsilon$, for all $\alpha \in J$. So, $S^\epsilon(x) + T^\epsilon(x) \supset S^{\epsilon_\alpha}(x) + T^{\epsilon_\alpha}(x)$ for all $\alpha \in J$. Since $\lim_{\alpha \in I} x_\alpha^* = x^*$, $x^* \in \overline{S^\epsilon x + T^\epsilon x}$ and hence,

$$x^* \in \cap_{\epsilon > 0} \overline{S^\epsilon x + T^\epsilon x} = S +_{\text{ext}} T(x).$$

We now show that $S +_{\text{ext}} T(x) \subset S +_{\lim} T(x)$. Let $\Lambda = \{\alpha = (\epsilon, V) : \epsilon > 0 \text{ and } V \in \mathcal{N}(x^*)\}$, where $\mathcal{N}(x^*)$ stands for the family of weak* neighborhood of x^*. We say that

$$\alpha' = (\epsilon', V') \le \alpha'' = (\epsilon'', V'') \iff \epsilon'' \le \epsilon' \text{ and } V'' \subset V'.$$

Then Λ is a directed set, that is,

$$\forall \alpha', \alpha'' \in \Lambda, \exists \alpha_0 \in \Lambda \text{ such that } \alpha_0 \ge \alpha' \text{ and } \alpha_0 \ge \alpha''.$$

Let $x^* \in S +_{\text{ext}} T(x)$. Then, for each $\epsilon > 0$ and each neighborhood V of x^* in the weak* topology, there exists $x_\alpha^* \in S^\epsilon x + T^\epsilon x$. ($x_\alpha^*$ depends on ϵ and V.) Set $\epsilon_\alpha = \epsilon$, then the preceding inclusion reads

$$x_\alpha^* \in S^{\epsilon_\alpha} x + T^{\epsilon_\alpha} x.$$

*A subset J of I is a terminal set if there exists $j \in I$ such that $k \in J$ for all $k \ge j$

In order to prove that $x^* \in S +_{\lim} T(x)$, it remains to prove that the nets (x^*_α) and (ϵ_α) respectively converge to x^*(in the weak* topology) and to 0.

Let $V_0 \in \mathcal{N}(x^*)$ be arbitrary and $\delta > 0$. Set $\alpha_0 = (\epsilon_0, V_0)$, with $\epsilon_0 < \delta$. If $\alpha = (\epsilon, V) \geq \alpha_0$, then by definition $V \subset V_0$ and $\epsilon \leq \epsilon_0$ and therefore $x^*_\alpha \in V_0$ and $\epsilon_\alpha = \epsilon \leq \epsilon_0 \leq \delta$.

□

Corollary 3.2. *Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions such that* $\mathrm{dom} f \cap \mathrm{dom} g \neq \emptyset$. *Then for each $x \in \mathrm{dom} f \cap \mathrm{dom} g$,*

$$\partial(f+g)(x) = \partial f +_{\lim} \partial g(x).$$

Proof. By Theorem 4.1 in [11], we see that

$$\partial(f+g)(x) = \partial f +_{\mathrm{ext}} \partial g(x).$$

The conclusion now follows from Proposition 3.1. □

4 Asymptotic Sums in Reflexive Spaces

In this section, we assume that X is a reflexive Banach space. We give a dual condition, which ensures that the asymptotic sum of two maximal monotone operators coincides with the point-wise sum.

Let $A : X \to X'$ be maximal monotone with graph $G(A) := \{(x, x^*) \in X \times X' : x^* \in A(x)\}$. We define $\psi_A : X \times X' \to \mathbb{R} \cup \{+\infty\}$ by

$$\psi_A(x, x^*) = \sup_{(s,s^*) \in G(A)} (x - s, s^* - x^*),$$

and define the Fitzpatrick function $\varphi_A : X \times X' \to \mathbb{R} \cup \{+\infty\}$ associated with A by

$$\varphi_A(x, x^*) = \sup_{(s,s^*) \in G(A)} [(s, x^*) + (x, s^*) - (s, s^*)] = \psi_A(x, x^*) + (x, x^*).$$

Theorem 4.1. *Let $S, T : X \to X'$ be maximal monotone such that $P_X \mathrm{dom}\, \varphi_S \cap P_X \mathrm{dom}\, \varphi_T \neq \emptyset$. If $(\mathrm{Epi}\varphi^*_S)_{X' \times X \times \{0\} \times \mathbb{R}} + (\mathrm{Epi}\varphi^*_T)_{X' \times \{0\} \times X \times \mathbb{R}}$ is closed then $S+T$ is a maximal monotone operator and*

$$S +_{\lim} T(x) = (S+T)(x).$$

Proof. Since, for each $u^*, v^* \in X'$ such that $u^* + v^* = x^*$,

$$\rho(x, x^*) = \inf_{u^*+v^*=x^*} (\varphi_S(x, u^*) + \varphi_T(x, v^*)) \geq (x, x^*) > -\infty,$$

it follows from Theorem 2.1 and the hypothesis that, for each $(x^*, x) \in X' \times X$,

$$\rho^*(x^*, x) = \min\{\varphi^*_S(u^*, x) + \varphi^*_T(v^*, x) : u^*, v^* \in X', u^* + v^* = x^*\}.$$

Thus,

$$\rho^*(x^*, x) \geq \min_{u^*+v^*=x^*} [(x, u^*) + (x, v^*)] = (x, x^*).$$

By Lemma 2.4, the set

$$\{(x, x^*) \mid \rho^*(x^*, x) = (x, x^*)\}$$

is a maximal monotone subset of $X \times X'$. We now show that

$$G(S+T) = \{(x, x^*) \mid \rho^*(x^*, x) = (x, x^*)\}.$$

Let $\Omega = \{(x, x^*) \in X \times X' \mid \rho^*(x^*, x) = (x, x^*)\}$. For each $(x, x^*) \in \Omega$, $\rho^*(x^*, x) = (x, x^*)$. By $\rho^*(x^*, x) = \min\{\varphi_S^*(s^*, x) + \varphi_T^*(t^*, x) \mid s^*, t^* \in X', s^* + t^* = x^*\}$, there exist $s^*, t^* \in X'$ such that $s^* + t^* = x^*$ and $\rho^*(x^*, x) = \varphi_S^*(s^*, x) + \varphi_T^*(t^*, x) = (x, x^*)$. Since $\varphi_S^*(s^*, x) \geq (x, s^*)$ and $\varphi_T^*(t^*, x) \geq (x, t^*)$,then $\varphi_S^*(x, s^*) = (s^*, x)$ and $\varphi_T^*(t^*, x) = (x, t^*)$. So, $(x, s^*) \in G(S)$ and $(x, t^*) \in G(T)$. Hence $(x, x^*) \in G(S+T)$.

Conversely, let $(x, x^*) \in G(S+T)$, then there exist $s^*, t^* \in X'$ such that $(x, s^*) \in G(S)$ and $(x, t^*) \in G(T)$. This gives us that

$$\varphi_S^*(s^*, x) = (x, s^*) \text{ and } \varphi_T^*(t^*, x) = (x, t^*).$$

So, $\rho^*(x^*, x) = \min\{\varphi_S^*(s^*, x) + \varphi_T^*(t^*, x) \mid s^*, t^* \in X', s^* + t^* = x^*\} \leq (x, x^*)$. Moreover,

$$\rho^*(x^*, x) \geq \min_{u^*+v^*=x^*} [(x, u^*) + (x, v^*)] = (x, x^*).$$

Thus, $\rho^*(x^*, x) = (x, x^*)$, i.e., $(x, x^*) \in \Omega$. Hence, $G(S+T) = \Omega$. Hence, $S+T$ is maximal monotone. Now, it follows from Proposition 3.1 that $S +_{\text{lim}} T(x) = S +_{\text{ext}} T(x)$ and by Corollary 4.2 in [11], we know that $S +_{\text{ext}} T(x) = (S+T)(x)$. Thus, $S +_{\text{lim}} T(x) = (S+T)(x)$. □

Remark 4.1. *Theorem 4.1 may also be derived from the results of [9]. However, for self containment of the paper, we have provided a direct proof here. It is worth noting from Lemma 5.1 [13] that if $S, T : X \to X'$ are maximal monotone operators such that $\cup_{\lambda>0} \lambda(pr_X \operatorname{dom} \varphi_S - pr_X \operatorname{dom} \varphi_T)$ is a closed subspace of X then $S+T$ is maximal monotone, and so, $S +_{\text{lim}} T(x) = (S+T)(x)$, where*

$$pr_X \operatorname{dom} \varphi_S := \{x \in X \mid \exists\, x^* \in X' \text{ such that } (x, x^*) \in \operatorname{dom} \varphi_S\}.$$

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