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# ESTIMATIONS OF THE LAGRANGE MULTIPLIERS' NORMS IN SET-VALUED OPTIMIZATION

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**Abstract:** This paper presents some results concerning the existence of Lagrange multipliers for general vector optimization problems with set-valued maps. Two main subdifferentials are considered: the Clarke subdifferential (for which exact calculus rules can be used in Banach spaces) and the Fréchet subdifferential (with fuzzy calculus rules in Asplund spaces). In every case, estimations of the multipliers' norms are given.

**Key words:** *Fréchet subdifferential, Clarke subdifferential, set-valued maps, scalarization, optimization*

**Mathematics Subject Classification:** *46A30, 54C60, 90C26*

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## 1 Introduction

The aim of this paper is to give a more precise look on some existence results of Lagrange multipliers for vector optimization problems governed by set-valued maps. In general, such results ensure the existence (explicitly or implicitly) of a non-zero Lagrangian for Clarke subdifferential (with exact calculus rules in general Banach spaces) or for Fréchet subdifferential (with fuzzy calculus rules in Asplund spaces). For example, for problem $(P_1)$ in Section 2, [7, Theorem 3.5] states the following (see the notations below): if $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ is a local weak minimum point for $F$ then there exists $y^* \in K^*$ s.t.

$$(0, -y^*) \in \partial^C d_{\operatorname{Gr} F}(\overline{x}, \overline{y}).$$

If $A$ and $C$ are nonempty, closed subsets in $X$ and $Y$, respectively, consider the function $h : X \to \overline{\mathbb{R}}$, $h(u) := d(F(u), C)$ ($d(A, B)$ denotes the distance between the sets $A$ and $B$) and the problem: minimize $h(u)$, subject to $u \in A$. An existence result of an implicit Lagrangian for this problem is the following (see [4, Theorem 2.1] and its proof): suppose that $X, Y$ are Banach spaces, which admit $C^1-$smooth Lipschitz bump functions and $F$ is upper semicontinuous with closed graph. If $x \in A$ is a local minimum point for the above problem then for every $\varepsilon > 0$, $U^*$ and $V^*$ weak-star neighborhoods of 0 in $X^*$ and $Y^*$, either there exists $s_\varepsilon \in B(x, \varepsilon)$ s.t. $0 \in \partial^F h(s_\varepsilon)$ or there exist $s_\varepsilon \in B(x, \varepsilon)$, $u_\varepsilon \in A \cap B(x, \varepsilon)$, $x_\varepsilon \in B(x, \varepsilon)$, $y_\varepsilon \in F(x_\varepsilon)$, $z_\varepsilon \in C$ s.t.

$$0 \in D^*_{\partial^F} F(x_\varepsilon, y_\varepsilon)(N_{\partial^F}(C, z_\varepsilon) + V^*) \setminus \{0\} + N_{\partial^F}(A, u_\varepsilon) + U^*.$$

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In the same framework (i.e. working with Clarke and Fréchet subdifferentials), the recent paper of Ng and Zheng [17] presents existence results for Lagrangians for problems without constraints in the more general case of Pareto minimum.

Using a scalarization technique introduced in [9] and also used in [5], [6] (and which works only in the case when the interior of the ordering cone in nonempty) we study here some general problems in vector optimization with set-valued maps and we establish the existence of Lagrange multipliers for these problems. In contrast with the above quoted results, a calculus of the subdifferential of the scalarization functional, allows us to give estimations for the norms of the multipliers. In other words, we are able to have a better knowledge of the area where the multipliers can be found; moreover, it is shown that this area is depending on an element in the interior of the cone which can be chosen at the start. It seems to be for the first time when this scalarization method is used in order to derive optimality conditions for optimization problems with set-values maps objectives. The approach used in this paper allows to give an unifying treatment for both types of above mentioned subdifferentials. The case of the limiting (Mordukhovich) coderivative in the Asplund spaces setting is considered as well and, as usual, for obtaining necessary optimality conditions in normal (Kuhn-Tucher) form for optimization problems with constraints in these terms we need to impose some constraint qualification conditions.

## $\boxed{2}$ Main Tools

Throughout the paper $X, Y, Z$ are Banach spaces over the real field $\mathbb{R}$. If additional properties for these spaces will be needed, this will be stated explicitly. We denote by $B(x,\varepsilon)$ and $D(x,\varepsilon)$ the open ball and the closed ball, respectively, with center $x \in X$ and radius $\varepsilon > 0$; $B_X$, $S_X$ are the open unit ball and the unit sphere of $X$, respectively. $X^*$ is the topological dual space of $X$ and by $w$ and $w^*$ we mean the weak topology on $X$ and the weak star topology on $X^*$, respectively. On a product space such $X \times Y$ we consider the sum norm; note that the dual norm in $X^* \times Y^*$ is the box norm. If $x \in X$, we denote the distance from $x$ to $S$ by $d(x,S) := \inf_{y\in S} \|x-y\|$ and we write $d_S$ for the distance function with respect to $S$, $d_S(x) = d(x,S)$ for every $x \in X$ (by convention, $d(x,\emptyset) = \infty$); $I_S$ is the indicator function of $S$ ($I_S(x) = 0$ if $x \in S$ and $I_S(x) = \infty$, if $x \notin S$).

We consider a pointed closed convex cone $K \subset Y$ which introduces a partial order on $Y$ by the equivalence $y_1 \leq_K y_2$ iff $y_2 - y_1 \in K$; we also suppose that $K$ has nonempty interior (i.e. $\operatorname{int} K \neq \emptyset$). We set $K^* := \{y^* \in Y^* \mid y^*(y) \geq 0, \forall y \in K\}$ for the dual cone of $K$. A cone $Q$ with similar properties is considered in $Z$. The notion of minima with respect to the order given by $K$ which work with in this paper is the following.

**Definition 2.1** *Let $A \subset Y$ be a nonempty subset of $Y$. A point $\overline{y} \in A$ is said to be a weak minimum point of $A$ with respect to $K$ (we write $\overline{y} \in \operatorname{WMin}(A,K)$) if $(A-\overline{y}) \cap (-\operatorname{int} K) = \emptyset$.*

In the sequel we consider some set-valued maps $F : X \rightrightarrows Y$, $G : X \rightrightarrows Z$. As usual, the domain and the graph of $F$ are $\operatorname{Dom} F = \{x \in X \mid F(x) \neq \emptyset\}$ and $\operatorname{Gr} F = \{(x,y) \mid y \in F(x)\}$, respectively. If $A \subset X$, $F(A) := \bigcup_{x\in A} F(x)$ and the inverse set-valued map of $F$ is $F^{-1} : Y \rightrightarrows X$ given by $(y,x) \in \operatorname{Gr} F^{-1}$ iff $(x,y) \in \operatorname{Gr} F$.

The basic tool for our study is the next lemma which present a separating functional and its main properties (see also [5], [6]). In this result $\partial$ denotes the Fenchel subdifferential of a convex function and $\operatorname{bd}(K)$ denotes the topological boundary of $K$. For the reader's convenience we present a proof of it.

**Lemma 2.1** *Let $K \subset Y$ be a closed convex cone with nonempty interior. Then for every $e \in \operatorname{int} K$ the functional $s_e : Y \to \mathbb{R}$ given by*

$$s_e(y) = \inf\{\lambda \in \mathbb{R} \mid \lambda e \in y + K\} \tag{1}$$

*is continuous, sublinear, strictly-*$\operatorname{int} K$*-monotone and:*

*(i) $\partial s_e(0) = \{v^* \in K^* \mid v^*(e) = 1\}$;*

*(ii) for every $u \in Y$, $\partial s_e(u) \neq \emptyset$ and $\partial s_e(u) = \{v^* \in K^* \mid v^*(e) = 1, v^*(u) = s_e(u)\}$.*

*Moreover, $s_e$ is $d(e, \operatorname{bd}(K))^{-1}$–Lipschitz and for every $u \in Y$ and $v^* \in \partial s_e(u)$, $\|e\|^{-1} \leq \|v^*\| \leq d(e, \operatorname{bd}(K))^{-1}$.*

*If $A \subset Y$ is a nonempty set s.t. $0 \in \operatorname{WMin}(A, K)$ then $s_e(a) \geq 0$ for every $a \in A$.*

*Proof.* The function $s_e$ defined above is the function $\varphi$ defined in [10, Corollary 2.3.5]. Then $s_e$ is a strictly $\operatorname{int} K$-monotone continuous sublinear function. Moreover,

$$\{y \mid \ s_e(y) \leq \lambda\} = \lambda e - K.$$

Let $v^* \in \partial s_e(0)$ and take $k \in \operatorname{int} K$. From the monotonicity of $s_e$ and taking into account the definition of the subdifferential of a convex function (note that $s_e(0) = 0$), we can write $0 > s_e(-k) \geq v^*(-k)$, i.e. $v^*(k) > 0$ for every $k \in \operatorname{int} K$. This proves that $v^* \in K^*$.

Since, $v^* \in \partial s_e(0)$ means that

$$s_e(y) - s_e(0) \geq v^*(y), \forall y \in Y,$$

i.e.

$$\forall y \in Y, \forall \lambda \in \mathbb{R}, y \in \lambda e - K \Rightarrow v^*(y) \leq \lambda, \tag{2}$$

that is

$$\lambda \geq \sup\{v^*(\lambda e - k) \mid k \in K\} = \sup\{\lambda v^*(e) - v^*(k) \mid k \in K\} = \lambda v^*(e).$$

So,

$$\lambda \geq \lambda v^*(e), \forall \lambda \in \mathbb{R}.$$

It implies that $v^*(e) = 1$, hence $\partial s_e(0) \subset \{v^* \in K^* \mid v^*(e) = 1\}$. Suppose now that $v^* \in K^*$, $v^*(e) = 1$, fix $y \in Y$ and $\lambda \geq s_e(y)$. Hence $y \in \lambda e - K$ and we can write

$$\lambda = \sup\{v^*(\lambda e - k) \mid k \in K\} \geq v^*(y).$$

Since $y \in Y$ and $\lambda \geq s_e(y)$ were arbitrarily chosen we can conclude that $s_e(y) \geq v^*(y)$ for every $y$, i.e. $v^* \in \partial s_e(0)$. This proves $(i)$. Because $s_e$ is continuous, its subdifferential is nonempty at any point and since it is sublinear as well we can apply [19, Theorem 2.4.14] in order to deduce $(ii)$.

Because $e \in \operatorname{int} K$ and $K$ is closed we have $D(e, d(e, \operatorname{bd}(K))) \subset K$. Take $u \in D(0, 1)$; then $e \in d(e, \operatorname{bd}(K))u + K$, whence $d(e, \operatorname{bd}(K))^{-1}e \in u + K$; consequently, for $v^* \in \partial s_e(0)$,

$$v^*(u) \leq d(e, \operatorname{bd}(K))^{-1}.$$

Since $u$ was arbitrary chosen in $D(0, 1)$, we conclude that $\|v^*\| \leq d(e, \operatorname{bd}(K))^{-1}$. The inequality $\|e\|^{-1} \leq \|v^*\|$ follows from $v^*(e) = 1$. The functional $s_e$ is Lipschitz since every sublinear continuous functional has this property and its Lipschitz constant is $d(e, \operatorname{bd}(K))^{-1}$, because $D(0, 1) \subset \{y \mid s_e(y) \leq d(e, \operatorname{bd}(K))^{-1}\}$.

It is clear that if $a \in A$ and $s_e(a) < 0$, since $s_e(a)e \in a+K$, $a \in -\operatorname{int} K$, in contradiction with the minimality of 0. □

In the sequel, for an $e \in \operatorname{int} K$ we denote by $L_e$ the positive number $d(e, \operatorname{bd}(K))^{-1}$. The next well-known result can be found in [3] and we recall it here as another main tool in what follows.

**Lemma 2.2** *Let $X$ be a normed vector space, $x_0 \in C \subset X$, and $f : X \to \mathbb{R}$ be a function locally $L-$Lipschitz ($L > 0$) at $x_0$. If $x_0$ is a local minimum point for $f$ over $C$, then there exists a neighborhood $V$ of $x_0$ s.t. the function $x \longmapsto f(x) + Ld_C(x)$ attains its minimum over $V$ at $x_0$.*

Our aim here is to work, mainly, with the Clarke subdifferential ($\partial^C$) and the Fréchet subdifferential ($\partial^F$) as most important subdifferentials with share the exact calculus rules on Banach spaces and fuzzy calculus rules on Asplund spaces, respectively, but, of course, an axiomatic approach of the underlying subdifferential can be considered as well. If in the case of Clarke subdifferential we use only two exact calculus rules (from [3]) which will be recalled at theirs applications in the proofs, for the Fréchet subdifferential we list some notations and fuzzy calculus rules.

First, we recall the definition of the Fréchet subdifferential. If $f : X \to \mathbb{R} \cup \{\infty\}$ is a function, we denote the domain of $f$ by $\operatorname{Dom} f = \{x \in X \mid f(x) < \infty\}$.

**Definition 2.2** *Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous (lsc for short) function; we say that $x^* \in X^*$ belongs to the Fréchet subdifferential of $f$ at $x \in \operatorname{Dom} f$ (denoted $\partial^F f(x)$) if*

$$\liminf_{t \to 0} (\inf_{u \in U_X} t^{-1}(f(x+tu) - f(x)) - x^*(u)) \geq 0.$$

Using the Fréchet subdifferential we define the Fréchet normal cone to a closed set $S \subset X$ at a point $x \in S$ in the following way:

$$N_{\partial^F}(S, x) := \partial^F I_S(x) = \mathbb{R}_+ \partial^F d_S(x).$$

The Clarke normal cone to $S \subset X$ at $x \in S$ is defined similarly.

If the graph of $F$ is closed the Fréchet coderivative of $F$ at a point $(x, y) \in \operatorname{Gr} F$ is the set-valued map $D^*_{\partial^F} F(x, y) : Y^* \rightrightarrows X^*$ given by:

$$D^*_{\partial^F} F(x, y)(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N_{\partial^F}(\operatorname{Gr} F, (x, y))\}.$$

Again, the Clarke coderivative is defined similarly (see [17]). Let us observe that $x^* \in D^*_{\partial^F} F(x, y)(y^*)$ iff $y^* \in D^*_{\partial^F} F^{-1}(x, y)(x^*)$.

The following notations will be used:

- $u \xrightarrow{f} x$ means that $u \to x$ and $f(u) \to f(x)$;
- $x^* \in \|\cdot\|^* - \limsup\limits_{u \to x} \partial^F f(u)$ means that for every $\varepsilon > 0$ there exist $x_\varepsilon$ and $x^*_\varepsilon$ such that $x^*_\varepsilon \in \partial^F f(x_\varepsilon)$ and $\|x_\varepsilon - x\| < \varepsilon$, $\|x^*_\varepsilon - x^*\| < \varepsilon$; the notation $x^* \in \|\cdot\|^* - \limsup\limits_{u \xrightarrow{f} x} \partial^F f(u)$ has a similar interpretation;
- $x^* \in w^* - \limsup\limits_{u \to x} \partial^F f(u)$ means that for every $\varepsilon > 0$ and $U$ a weak-star neighborhood of 0 in $X^*$, there exists $x_{\varepsilon,U}$ and $x^*_{\varepsilon,U}$ such that $x^*_{\varepsilon,U} \in \partial^F f(x_{\varepsilon,U})$ and $\|x_{\varepsilon,U} - x\| < \varepsilon$, $x^*_{\varepsilon,U} \in x^* + U$. In order to keep the notations as simple as possible, we shall write $x_\varepsilon$ and $x^*_\varepsilon$ instead of $x_{\varepsilon,U}$ and $x^*_{\varepsilon,U}$.

We list below the main properties of the Fréchet subdifferential which we shall use in the sequel (see [2], [8], [11], [12]). All the functions considered in these properties are lsc unless stated otherwise.

(A1) If $f$ attains a local minimum at $x \in \operatorname{Dom} f$, then $0 \in \partial^F f(x)$.

(A2) If $f$ is a convex function, then $\partial^F f$ is the subdifferential $\partial f$ in the sense of convex analysis.

(A3) If $X$ is an Asplund space, $\varphi_1, \varphi_2, ..., \varphi_n : X \to \mathbb{R}$ is a family of Lipschitz functions and $x \in \operatorname{Dom} f$, then

$$\partial^F (f + \sum_{i=1}^{n} \varphi_i)(x) \subset \|\cdot\|^* - \limsup_{y \xrightarrow{f} x, z_i \to x} (\partial^F f(y) + \sum_{i=1}^{n} \partial^F \varphi_i(z_i)).$$

(A4) If $X$ is an Asplund space, then for every family $f_1, f_2, ..., f_n : X \to \mathbb{R} \cup \{\infty\}$ of lsc functions, $x \in \bigcap_{i=1}^{n} \operatorname{Dom} f_i$ one has

$$\partial^F (\sum_{i=1}^{n} f_i)(x) \subset w^* - \limsup_{x_i \xrightarrow{f_i} x} \sum_{i=1}^{n} \partial^F f_i(x_i).$$

As one can see, the Fréchet subdifferential does not possess robust calculus rules and for this reason Mordukhovich have employed more robust objects satisfying better calculus rules. We recall here such objects only in the setting we use it. If $X$ is an Asplund space, the limiting normal cone at $S$ in $x$ is

$$N_M(S, x) = \{x^* \mid \exists x_n \xrightarrow{S} x, x_n^* \xrightarrow{w^*} x^*, x_n^* \in N_{\partial^F}(S, x_n)\},$$

and the limiting coderivarive $D_M^* F$ is defined as the Fréchet coderivative, replacing the normal cone in definition (see [16] for further details).

## $\boxed{3}$ Estimating the Lagrangians' Norms

Taking into account the previously made notations, we consider the following optimization problems with set-valued maps.

$$\begin{array}{ll} (P_1) & \text{minimize } F(x), \text{ subject to } x \in X, \\ (P_2) & \text{minimize } F(x), \text{ subject to } x \in S \subset X, \\ (P_3) & \text{minimize } F(x), \text{ subject to } x \in X, 0 \in G(x) + Q. \end{array}$$

**Definition 3.1** *A point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ is called a local weak minimum point for problem $(P_1)$ if $\overline{y} \in \operatorname{WMin}(F(U) \cap V, K)$ for some neighborhoods $U$ and $V$ of $\overline{x}$ and $\overline{y}$, respectively.*

The definitions for this sort of solutions for $(P_2)$ and $(P_3)$ are obtained by replacing $U$ in the above definition by $U \cap S$ and by $U \cap \{x \mid 0 \in G(x) + Q\} = U \cap G^{-1}(-Q)$, respectively. In the sequel we assume that $\operatorname{Gr} F$ and $\operatorname{Gr} G$ are closed sets.

**Theorem 3.1** *If $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ is a local weak minimum point for $(P_1)$ then for every $e \in \operatorname{int} K$ there exists $y^* \in K^*$ s.t. $y^*(e) = L_e^{-1}$ and*

$$(0, -y^*) \in \partial^C d_{\operatorname{Gr} F}(\overline{x}, \overline{y}). \tag{3}$$

*In particular, $(L_e \|e\|)^{-1} \leq \|y^*\| \leq 1$.*

*Proof.* Fix some $e \in \operatorname{int} K$ and consider $f : X \times Y \to \mathbb{R}$, $f(x,y) = s_e(y - \overline{y})$. Since $\overline{y} \in \operatorname{WMin}(F(U) \cap V)$ for some neighborhoods $U$ of $\overline{x}$ and $V$ of $\overline{y}$, we have (from Lemma 2.1) that $f(x,y) \geq 0$ for all $(x,y) \in (U \times V) \cap \operatorname{Gr} F$. Since $f(\overline{x},\overline{y}) = 0$, we can conclude that $(\overline{x},\overline{y})$ is a local minimum for $f$ over $\operatorname{Gr} F$. Taking into account that $s_e$ is Lipschitz, it is easy to see that $f$ is Lipschitz as well, with the same constant ($L_e$). Following Lemma 2.2, $(\overline{x},\overline{y})$ is a local minimum point (without constraints) for $f + L_e d_{\operatorname{Gr} F}$. Consequently, using a well-known calculus rule for the Clarke subdifferential,

$$(0,0) \in \partial^C f(\overline{x},\overline{y}) + L_e \partial^C d_{\operatorname{Gr} F}(\overline{x},\overline{y}).$$

But, $\partial^C f(\overline{x},\overline{y}) = \{0\} \times \partial s_e(0)$, whence there exists $y^* \in \partial s_e(0)$ s.t.

$$(0, -y^*) \in L_e \partial^C d_{\operatorname{Gr} F}(\overline{x},\overline{y})$$

i.e.

$$(0, -L_e^{-1} y^*) \in \partial^C d_{\operatorname{Gr} F}(\overline{x},\overline{y}).$$

From Lemma 2.1, $L_e^{-1} y^* \in K^*$ and $\|e\|^{-1} L_e^{-1} \leq \left\| -L_e^{-1} y^* \right\| = L_e^{-1} \|y^*\| \leq 1$. The proof is complete. □

An easy consequence of the above theorem is the following result.

**Corollary 3.1** *If $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ is a local weak minimum point for $(P_1)$ then there exists $y^* \in K^*$ s.t. $\|y^*\| = 1$ and*

$$0 \in D^*_{\partial^C} F(\overline{x},\overline{y})(y^*).$$

Let us observe that Theorem 3.1 covers Theorem 3.5 from [7] (hence Theorem 3.1 from [1] as well) and, moreover, it gives an estimation of the norm of the Lagrangians. In the particular case of weak minima the conclusion of Corollary 3.1 gives a sharp result, in contrast with [17, Theorem 3.1].

For the problem $(P_2)$ we can deduce the next corollary.

**Corollary 3.2** *If $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ is a local weak minimum point for $(P_2)$ then for every $e \in \operatorname{int} K$ there exists $y^* \in K^*$ s.t. $y^* = L_e^{-1}$ and*

$$(0, -y^*) \in \partial^C d_{(S \times Y) \cap \operatorname{Gr} F}(\overline{x},\overline{y}).$$

*In particular, $(L_e \|e\|)^{-1} \leq \|y^*\| \leq 1$.*

Considering again problem $(P_1)$, and the case of the Fréchet subdifferential we are able to present a result which make use of the fuzzy calculus rules on Asplund spaces.

**Theorem 3.2** *Suppose that $X, Y$ are Asplund spaces and $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ is a local weak minimum point for $(P_1)$. Then for every $e \in \operatorname{int} K$, $a > 0$ and $\varepsilon > 0$, there exist $y^* \in K^*$ s.t. $y^*(e) = a$ and $(x_\varepsilon, y_\varepsilon) \in \operatorname{Gr} F$ with $(x_\varepsilon, y_\varepsilon) \in B((\overline{x},\overline{y}), \varepsilon)$ s.t.*

$$0 \in D^*_{\partial^F} F(x_\varepsilon, y_\varepsilon)(y^* + \varepsilon B_{Y^*}) + \varepsilon B_{X^*}. \tag{4}$$

*In particular, $a \|e\|^{-1} \leq \|y^*\| \leq a L_e$.*

*Proof.* Take again the function $f$ as in the proof of Theorem 3.1 for a fixed $e \in \operatorname{int} K$. Since $(\overline{x}, \overline{y})$ is a local minimum point for $(P_1)$, then it is a local minimum point for $f + I_{\operatorname{Gr} F}$ and hence for $af + I_{\operatorname{Gr} F}$ as well. Then, from (A1),

$$(0,0) \in \partial^F (af(\cdot,\cdot) + I_{\operatorname{Gr} F}(\cdot,\cdot))(\overline{x}, \overline{y}).$$

Because $f$ is locally Lipschitz, the hypotheses allow us to apply (A3). Thus, for every $\varepsilon > 0$, there exist $(x_\varepsilon^1, y_\varepsilon^1), (x_\varepsilon^2, y_\varepsilon^2)$ and $(x_\varepsilon^{*1}, y_\varepsilon^{*1}), (x_\varepsilon^{*2}, y_\varepsilon^{*2})$ with $(x_\varepsilon^2, y_\varepsilon^2) \in \operatorname{Gr} F$, $\left\|(x_\varepsilon^i, y_\varepsilon^i) - (\overline{x}, \overline{y})\right\| < \varepsilon$, $i = 1, 2$, $\max(\left\|x_\varepsilon^{*1} + x_\varepsilon^{*2}\right\|, \left\|y_\varepsilon^{*1} + y_\varepsilon^{*2}\right\|) < \varepsilon$ s.t.

$$(x_\varepsilon^{*1}, y_\varepsilon^{*1}) \in \partial^F f(x_\varepsilon^1, y_\varepsilon^1) = \{0\} \times a\partial s_e(y_\varepsilon^1 - \overline{y})$$

and

$$(x_\varepsilon^{*2}, y_\varepsilon^{*2}) \in \partial^F I_{\operatorname{Gr} F}(x_\varepsilon^2, y_\varepsilon^2).$$

Consequently,

$$x_\varepsilon^{*1} = 0 \in x_\varepsilon^{*2} + \varepsilon B_{X^*} \subset D_{\partial^F}^* F(x_\varepsilon^2, y_\varepsilon^2)(-y_\varepsilon^{*2}) + \varepsilon B_{X^*} \subset D_{\partial^F}^* F(x_\varepsilon^2, y_\varepsilon^2)(y_\varepsilon^{*1} + \varepsilon B_{X^*}) + \varepsilon B_{X^*}.$$

Taking $x_\varepsilon = x_\varepsilon^2, y_\varepsilon = y_\varepsilon^2, y^* = y_\varepsilon^{*1}$ and using the properties of the subdifferential of $s_e$ we have the conclusion. □

Note that for every $\varepsilon > 0$ we can choose in the above theorem $y^* \in S_{Y^*}$. Indeed, it is enough to choose $e \in S_Y$ and $a = 1$, because in this situation, $\|y^*\| \geq 1$ and we can multiply the relation in the conclusion of the theorem by $\|y^*\|^{-1}$ without loosing the radius of the involved balls. Thus, in this particular case we obtain the same conclusion as Theorem 4.1 from [17]. Of course, the quoted result has the advantage that does not require the nonemptiness of the $\operatorname{int} K$. On the other hand, our Theorem 3.2 proves the existence of possible infinitely many multipliers. Note also that Theorem 3.2 can be written for any subdifferential satisfying (A1), (A2), (A3) (see [11], [12], [18] for many examples and applications of such subdifferentials). It is clear that using the same arguments as in [17] one can obtain a robust condition for limiting Mordukhovich coderivative (see [17, Theorem 4.2 (c)]).

**Corollary 3.3** *Let $X$ and $Y$ be Asplund spaces and $(\overline{x}, \overline{y})$ be a local weak minimum point for $(P_1)$. Then there exists $y^* \in K^* \cap S_{Y^*}$ s.t.*

$$0 \in D_M^* F(\overline{x}, \overline{y})(y^*).$$

Unfortunately, it is not possible to use the same technique for $(P_2)$, because $I_{(S\times Y)\cap \operatorname{Gr} F} = I_{S\times Y} + I_{\operatorname{Gr} F}$ and we must deal with two non-Lipschitz functions. Then we must consider an additional assumption or to work with strong-weak fuzzy calculus rules as (A4). We illustrate both directions in what follows.

The metric regularity condition needed in order to obtain strong fuzzy optimality conditions is: there exists $k > 0$ s.t. for all $(x, y)$ in a neighborhood of $(\overline{x}, \overline{y})$

$$d((x,y),(S \times Y) \cap \operatorname{Gr} F) \leq k(d(x,S) + d((x,y), \operatorname{Gr} F)). \tag{5}$$

**Theorem 3.3** *Let $X, Y$ be Asplund spaces, and $(\overline{x}, \overline{y})$ be a local weak minimum point for $(P_2)$ s.t. (5) holds. Then for every $e \in \operatorname{int} K$ and for every $\varepsilon > 0$, there exists $(x_\varepsilon, y_\varepsilon) \in B((\overline{x}, \overline{y}), \varepsilon)$, $z_\varepsilon \in B(\overline{x}, \varepsilon)$, $y^* \in K^*$ s.t. $y^*(e) = (L_e k)^{-1}$ and*

$$(-\partial^F d_S(z_\varepsilon) \times \{-y^*\}) \cap (\partial^F d_{\operatorname{Gr} F}(x_\varepsilon, y_\varepsilon) + \varepsilon B_{X^* \times Y^*}) \neq \emptyset.$$

*In particular, $(L_e k \|e\|)^{-1} \leq \|y^*\| \leq k^{-1}$.*

*Proof.* Take function $f$ as in the proof of Theorem 3.1 for a fixed $e \in \operatorname{int} K$. Since $(\overline{x}, \overline{y})$ is a local minimum point for $(P_2)$, then it is a local minimum point for $f + L_e d_{(S\times Y)\cap \operatorname{Gr} F}$; from (5), $(\overline{x}, \overline{y})$ is a local minimum point for $f + kL_e d_{S\times Y} + kL_e d_{\operatorname{Gr} F}$, hence for $(kL_e)^{-1} f + d_{S\times Y} + d_{\operatorname{Gr} F}$ as well. Then, from (A1),

$$(0,0) \in \partial^F((kL_e)^{-1} f(\cdot,\cdot) + d_{S\times Y}(\cdot,\cdot) + d_{\operatorname{Gr} F}(\cdot,\cdot))(\overline{x}, \overline{y}).$$

We can apply (A3). Accordingly, for every $\varepsilon > 0$, there exist $(x_\varepsilon^i, y_\varepsilon^i)$ and $(x_\varepsilon^{*i}, y_\varepsilon^{*i})$ with $\left\|(x_\varepsilon^i, y_\varepsilon^i) - (\overline{x}, \overline{y})\right\| < \varepsilon$, $i = 1, 2, 3$, $\max(\left\|x_\varepsilon^{*1} + x_\varepsilon^{*2} + x_\varepsilon^{*3}\right\|, \left\|y_\varepsilon^{*1} + y_\varepsilon^{*2} + y_\varepsilon^{*3}\right\|) < \varepsilon$ s.t.

$$(x_\varepsilon^{*1}, y_\varepsilon^{*1}) \in \partial^F (kL_e)^{-1} f(x_\varepsilon^1, y_\varepsilon^1) = \{0\} \times (kL_e)^{-1} \partial s_e(y_\varepsilon^1 - \overline{y}),$$

$$(x_\varepsilon^{*2}, y_\varepsilon^{*2}) \in \partial^F d_{S\times Y}(x_\varepsilon^2, y_\varepsilon^2) = \partial^F d_S(x_\varepsilon^2) \times \{0\}.$$

and

$$(x_\varepsilon^{*3}, y_\varepsilon^{*3}) \in \partial^F d_{\operatorname{Gr} F}((x_\varepsilon^3, y_\varepsilon^3)).$$

We deduce that $x_\varepsilon^{*1} = 0, y_\varepsilon^{*2} = 0$ and we can write:

$$(-x_\varepsilon^{*2}, -y_\varepsilon^{*1}) \in (x_\varepsilon^{*3}, y_\varepsilon^{*3}) + \varepsilon B_{X^*\times Y^*} \subset \partial^F d_{\operatorname{Gr} F}(x_\varepsilon^3, y_\varepsilon^3) + \varepsilon B_{X^*\times Y^*}.$$

It follows that

$$(-\partial^F d_S(x_\varepsilon^2) \times \{-y_\varepsilon^{*1}\}) \cap (\partial^F d_{\operatorname{Gr} F}(x_\varepsilon^3, y_\varepsilon^3) + \varepsilon B_{X^*\times Y^*}) \neq \emptyset.$$

Taking $x_\varepsilon = x_\varepsilon^3, y_\varepsilon = y_\varepsilon^3, y^* = y_\varepsilon^{*1}, z_\varepsilon = x_\varepsilon^2$ and using the properties of the subdifferential of $s_e$ we have the conclusion. □

Using the metric regularity condition we get an exact result for the Clarke subdifferential.

**Theorem 3.4** *If $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ is a local weak minimum point for $(P_2)$ and relation (5) holds, then for every $e \in \operatorname{int} K$ there exists $y^* \in K^*$ s.t. $y^*(e) = (L_e k)^{-1}$ and*

$$(-\partial^C d_S(\overline{x}) \times \{-y^*\}) \cap \partial^C d_{\operatorname{Gr} F}(\overline{x}, \overline{y}) \neq \emptyset.$$

*In particular, $(L_e k \left\|e\right\|)^{-1} \leq \left\|y^*\right\| \leq k^{-1}$.*

*Proof.* The proof is similar to the proofs of Theorem 3.1 and Theorem 3.3. □

**Theorem 3.5** *Let $X, Y$ be Asplund spaces, and $(\overline{x}, \overline{y})$ be a local weak minimum for $(P_2)$. Then for every $e \in \operatorname{int} K$, $a > 0$, $\varepsilon > 0$, $U^*$ and $V^*$ symmetric weak$^*$ neighborhoods of 0 in $X^*$ and $Y^*$ respectively, there exists $(x_\varepsilon, y_\varepsilon) \in B((\overline{x}, \overline{y}), \varepsilon) \cap \operatorname{Gr} F$, $z_\varepsilon \in S \cap B(\overline{x}, \varepsilon)$, $y^* \in K^*$ s.t. $y^*(e) = a$ and*

$$0 \in D^*_{\partial^F} F(x_\varepsilon, y_\varepsilon)(y^* + V^*) + N_{\partial^F}(S, z_\varepsilon) + U^*.$$

*In particular, $a \left\|e\right\|^{-1} \leq \left\|y^*\right\| \leq aL_e$.*

*Proof.* Using the same arguments as above, we have

$$(0,0) \in \partial^F(af(\cdot,\cdot) + I_{S\times Y} + I_{\operatorname{Gr} F}(\cdot,\cdot))(\overline{x}, \overline{y}).$$

We can apply (A4). Accordingly, for every $\varepsilon > 0$ and for every $U^*, V^*$ symmetric weak$^*$ neighborhoods of 0 in $X^*$ and $Y^*$, respectively there exist $(x_\varepsilon^i, y_\varepsilon^i)$ and $(x_\varepsilon^{*i}, y_\varepsilon^{*i})$ with

$\left\|(x_\varepsilon^i, y_\varepsilon^i) - (\overline{x}, \overline{y})\right\| < \varepsilon,\ i = 1,2,3,\ x_\varepsilon^2 \in S,\ (x_\varepsilon^3, y_\varepsilon^3) \in \operatorname{Gr} F,\ x_\varepsilon^{*1} + x_\varepsilon^{*2} + x_\varepsilon^{*3} \in U^*, y_\varepsilon^{*1} + y_\varepsilon^{*2} + y_\varepsilon^{*3} \in V^*$ s.t.

$$(x_\varepsilon^{*1}, y_\varepsilon^{*1}) \in \partial^F af(x_\varepsilon^1, y_\varepsilon^1) = \{0\} \times a\partial s_e(y_\varepsilon^1 - \overline{y}),$$

$$(x_\varepsilon^{*2}, y_\varepsilon^{*2}) \in \partial^F I_{S\times Y}(x_\varepsilon^2, y_\varepsilon^2) = \partial^F I_S(x_\varepsilon^2) \times \{0\},$$

and

$$(x_\varepsilon^{*3}, y_\varepsilon^{*3}) \in \partial^F I_{\operatorname{Gr} F}(x_\varepsilon^3, y_\varepsilon^3).$$

We deduce that $x_\varepsilon^{*1} = 0, y_\varepsilon^{*2} = 0$ and:

$$\begin{aligned} 0 &\in x_\varepsilon^{*2} + x_\varepsilon^{*3} + U^* \\ &\subset D^*_{\partial^F} F(x_\varepsilon^3, y_\varepsilon^3)(-y_\varepsilon^{*3}) + N_{\partial^F}(S, x_\varepsilon^{*2}) + U^* \\ &\subset D^*_{\partial^F} F(x_\varepsilon^3, y_\varepsilon^3)(y_\varepsilon^{*1} + V^*) + N_{\partial^F}(S, x_\varepsilon^{*2}) + U^*. \end{aligned}$$

The conclusion follows. □

As we already said, for obtaining robust optimality conditions in terms of limiting coderivative we need some constraints qualification condition. We say that $F$ satisfies the condition $(CQ)$ at $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ if for every sequences $(x_n, y_n) \xrightarrow{\operatorname{Gr} F} (\overline{x}, \overline{y})$, $(y_n^*)$ bounded and $x_n^* \in D^*_{\partial^F} F(x_n, y_n)(y_n^*)$ imply that $(x_n^*)$ is bounded too. Following [13, Theorem 3.2] this condition is satisfied if $F$ is pseudo-Lipschitz at $(\overline{x}, \overline{y})$, which is equivalent with the openness with linear rate property of $F^{-1}$ at $(\overline{y}, \overline{x})$.

The next result is in line with [17, Corollary 4.1], but the proof is different.

**Corollary 3.4** *Suppose that $X, Y$ are Asplund spaces, $(\overline{x}, \overline{y})$ is a local weak minimum point for $(P_2)$ and $F$ satisfies $(CQ)$ at $(\overline{x}, \overline{y})$. Then there exists $y^* \in K^* \cap S_{Y^*}$ s.t.*

$$0 \in D^*_M F(\overline{x}, \overline{y})(y^*) + N_M(S, \overline{x}).$$

*Proof.* We fix in the above theorem $e \in S_X$ and $a = 1$. Then for every natural number $n$ we can find $(x_n, y_n) \xrightarrow{\operatorname{Gr} F} (\overline{x}, \overline{y})$, $z_n \xrightarrow{S} \overline{x}$, $y_n^* \in K^*$, $1 \leq \|y_n^*\| \leq L_e$, $u_n^* \in N_{\partial^F}(S, z_n)$, $p_n^* \xrightarrow{w^*} 0_{X^*}$, $v_n^* \xrightarrow{w^*} 0_{Y^*}$ s.t.

$$-p_n^* - u_n^* \in D^*_{\partial^F} F(x_n, y_n)(y_n^* + v_n^*).$$

Since $(y_n^*)$ is bounded, we can suppose, without loosing the generality, that it converges weakly$^*$ to an element $y^*$ which is not $0_{Y^*}$ (cf. [17, relation (3.9)]). Applying $(CQ)$, the sequence $(p_n^* + u_n^*)$ is bounded, so it is convergent weakly$^*$ (without relabeling) to an element $u^*$. Then, $u_n^* \xrightarrow{w^*} u^*$. This shows that $u^* \in N_M(S, \overline{x})$ and $-u^* \in D^*_M F(\overline{x}, \overline{y})(y^*)$, i.e. the conclusion, because we can multiply to get an $y^*$ with norm 1. □

In order to present optimality conditions for $(P_3)$ in terms of the Clarke subdifferential, we need another metric regularity conditions for the multifunction $G : X \rightrightarrows Z$ (see [1, Definition 3.1]): $G$ is called metrically regular at $(\overline{x}, \overline{z}) \in \operatorname{Gr} G$ with $\overline{z} \in -Q$ relative to $-Q$ if there exist a constant $m > 0$ and some neighborhoods $V$ and $W$ of $\overline{x}$ and $\overline{z}$ respectively s.t. for all $x \in V$ and $z \in W \cap G(x)$

$$d(x, G^{-1}(-Q)) \leq md(z, -Q).$$

The next theorem is along the lines of [7, Theorem 3.3], but, thanks to the scalarization technique we use, the conclusion is more precise.

**Theorem 3.6** *Let $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ and $\overline{z} \in G(\overline{x}) \cap -Q$. Suppose that $G$ is metrically regular at $(\overline{x},\overline{z})$ relative to $-Q$ and relation (5) holds for $S := G^{-1}(-Q)$. If $(\overline{x},\overline{y})$ is a local weak minimum point for $(P_3)$, then for every $e \in \operatorname{int} K$ there exist $y^* \in K^*$, $z^* \in Q^*$ with $y^*(e) = 1$, $z^*(\overline{z}) = 0$ and*

$$(0, -y^*, -z^*) \in L_e k \partial^C d_{\operatorname{Gr} F}(\overline{x},\overline{y}) \times \{0\} + (1 + L_e k + L_e km)\partial^C h(\overline{x},\overline{y},\overline{z}),$$

*where $h(x,y,z) = d((x,z), \operatorname{Gr} G)$. In particular, $0 \in D^*_{\partial^C} F(\overline{x},\overline{y})(y^*) + D^*_{\partial^C} G(\overline{x},\overline{z})(z^*)$ and $\|e\|^{-1} \leq \|y^*\| \leq L_e$.*

*Proof.* Because $(\overline{x},\overline{y})$ is a local weak minimum point for $(P_3)$, then, following Lemma 2.2, it is also local minimum point for the scalar function $f + L_e d_{(G^{-1}(-Q)\times Y)\cap \operatorname{Gr} F}$. By use of (5), $(\overline{x},\overline{y})$ is local minimum point for $f + L_e k d_{(G^{-1}(-Q)\times Y)} + L_e k d_{\operatorname{Gr} F}$. By metric regularity condition of $G$ relative to $-Q$ we deduce that $(\overline{x},\overline{y},\overline{z})$ is a local minimum point over $\operatorname{Gr} G$ for $g + L_e k d_{\operatorname{Gr} F\times Z} + L_e kmd_{X\times Y\times -Q}$, where $g(x,y,z) = f(x,y) = s_e(y - \overline{y})$. We can apply again Lemma 2.2 in order to deduce that $(\overline{x},\overline{y},\overline{z})$ is a local minimum for $g + L_e k d_{\operatorname{Gr} F\times Z} + L_e kmd_{X\times Y\times -Q} + (1 + L_e k + L_e km)d_{\operatorname{Gr} G\underline{\times} Y}$, where $(x,y,z) \in \operatorname{Gr} G\underline{\times} Y$ iff $(x,z) \in \operatorname{Gr} G$ and $y \in Y$. Then

$$\begin{aligned}(0,0,0) \in \{0\} \times \partial s_e(0) \times \{0\} &+ L_e k(\partial^C d_{\operatorname{Gr} F}(\overline{x},\overline{y}) \times \{0\}) \\ &+ L_e km(\{0\} \times \{0\} \times \partial^C d_{-Q}(\overline{z})) + (1 + L_e k + L_e km)\partial^C h(\overline{x},\overline{y},\mathrm{z}).\end{aligned}$$

Since $-Q$ is a convex set, $\partial^C d_{-Q}(\overline{z})$ is a subset of the normal cone in the sense of convex analysis to $-Q$ at $\overline{z}$. Therefore, if $z^* \in \partial^C d_{-Q}(\overline{z})$, $z^*(z - \overline{z}) \leq 0$ for every $z \in -Q$. This implies $z^*(\overline{z}) = 0$ and $z^* \in Q^*$. From Lemma 2.1, if $y^* \in \partial s_e(0)$ then $y^* \in K^*$ and $y^*(e) = 1$. Hence we can write

$$(0, -y^*, -z^*) \in L_e k \partial^C d_{\operatorname{Gr} F}(\overline{x},\overline{y}) \times \{0\} + (1 + L_e k + L_e km)\partial^C h(\overline{x},\overline{y},\overline{z}).$$

In particular,

$$(0, -y^*, -z^*) \in N_{\partial^C} \operatorname{Gr} F(\overline{x},\overline{y}) \times \{0\} + N_{\partial^C} \operatorname{Gr} G(\overline{x},\overline{z})\underline{\times}\{0\}.$$

This shows that $0 \in D^*_{\partial^C} F(\overline{x},\overline{y})(y^*) + D^*_{\partial^C} G(\overline{x},\overline{z})(z^*)$. □

For the use of strong-weak fuzzy calculus rules we do not need regularity conditions. We have the following result concerning problem $(P_3)$.

**Theorem 3.7** *Let $X, Y, Z$ be Asplund spaces, $(\overline{x},\overline{y})$ be a local weak minimum point for $(P_3)$ and $\overline{z} \in G(\overline{x}) \cap -Q$. Then for every $e \in \operatorname{int} K$, $a > 0$, $\varepsilon > 0$, $U^*, V^*$ and $W^*$ symmetric weak$^*$ neighborhoods of 0 in $X^*$, $Y^*$ and $Z^*$, respectively, there exist $(x_\varepsilon, y_\varepsilon) \in B((\overline{x},\overline{y}),\varepsilon) \cap \operatorname{Gr} F$, $(u_\varepsilon, v_\varepsilon) \in B((\overline{x},\overline{z}),\varepsilon) \cap \operatorname{Gr} G$, $y^* \in K^*$ s.t. $y^*(e) = a$, $z^* \in Q^*$ and*

$$0 \in D^*_{\partial^F} F(x_\varepsilon, y_\varepsilon)(y^* + V^*) + D^*_{\partial^F} G(u_\varepsilon, v_\varepsilon)(z^* + W^*) + U^*.$$

*In particular, $a\,\|e\|^{-1} \leq \|y^*\| \leq aL_e$ and $z^*(z_\varepsilon) = 0$ for an element $z_\varepsilon \in B(\overline{z},\varepsilon) \cap -Q$.*

*Proof.* We use the notations in the proof of the preceding result. Then $(\overline{x},\overline{y},\overline{z})$ is a local minimum point for the scalar function $g + I_{G^{-1}(-Q)\times Y\times Z} + I_{\operatorname{Gr} F\times Z}$ (in fact is minimum over $R \times T \times Z$, where $R, T$ are some neighborhoods of $\overline{x}$ and $\overline{y}$). Since $(X \times -Q) \cap \operatorname{Gr} G \subset G^{-1}(-Q) \times Z$, we have that $(\overline{x},\overline{y},\overline{z})$ is a local minimum point for $ag + I_{X\times Y\times -Q} + I_{\operatorname{Gr} F\times Z} + I_{\operatorname{Gr} G\underline{\times} Y}$. Hence,

$$(0,0,0) \in \partial^F(ag(\cdot,\cdot,\cdot) + I_{X\times Y\times -Q}(\cdot,\cdot,\cdot) + I_{\operatorname{Gr} F\times Z}(\cdot,\cdot,\cdot) + I_{\operatorname{Gr} G\underline{\times} Y}(\cdot,\cdot,\cdot))(\overline{x},\overline{y},\overline{z})$$

Then, for every $\varepsilon > 0$, every symmetric weak* neighborhoods of 0, $U^*, V^*, W^*$ there exists $(x_\varepsilon^i, y_\varepsilon^i, z_\varepsilon^i)$ and $(x_\varepsilon^{*i}, y_\varepsilon^{*i}, z_\varepsilon^{*i})$ with $\left\|(x_\varepsilon^i, y_\varepsilon^i, z_\varepsilon^i) - (\overline{x}, \overline{y}, \overline{z})\right\| < \varepsilon$, $i = 1,2,3,4$, $z_\varepsilon^2 \in -Q$, $(x_\varepsilon^3, y_\varepsilon^3) \in \operatorname{Gr} F$, $(x_\varepsilon^4, z_\varepsilon^4) \in \operatorname{Gr} G$, $x_\varepsilon^{*1} + x_\varepsilon^{*2} + x_\varepsilon^{*3} + x_\varepsilon^{*4} \in U^*$, $y_\varepsilon^{*1} + y_\varepsilon^{*2} + y_\varepsilon^{*3} + y_\varepsilon^{*4} \in V^*$, $z_\varepsilon^{*1} + z_\varepsilon^{*2} + z_\varepsilon^{*3} + z_\varepsilon^{*4} \in W^*$ s.t.

$$(x_\varepsilon^{*1}, y_\varepsilon^{*1}, z_\varepsilon^{*1}) \in \partial^F ag(x_\varepsilon^1, y_\varepsilon^1, z_\varepsilon^1) = \{0\} \times a\partial s_e(y_\varepsilon^1 - \overline{y}) \times \{0\},$$

$$(x_\varepsilon^{*2}, y_\varepsilon^{*2}, z_\varepsilon^{*2}) \in \partial^F I_{X\times Y\times -Q}(x_\varepsilon^2, y_\varepsilon^2, z_\varepsilon^2) = \{0\} \times \{0\} \times N_{\partial^C}(-Q, z_\varepsilon^2),$$

$$x_\varepsilon^{*3} \in D_{\partial^F}^* F(x_\varepsilon^3, y_\varepsilon^3)(-y_\varepsilon^{*3}); z_\varepsilon^{*3} = 0,$$

and

$$x_\varepsilon^{*4} \in D_{\partial^F}^* G(x_\varepsilon^4, z_\varepsilon^4)(-z_\varepsilon^{*4}); y_\varepsilon^{*4} = 0.$$

Therefore, we can write:

$$\begin{aligned} 0 &\in x_\varepsilon^{*3} + x_\varepsilon^{*4} + U^* \\ &\subset D_{\partial^F}^* F(x_\varepsilon^3, y_\varepsilon^3)(-y_\varepsilon^{*3}) + D_{\partial^F}^* G(x_\varepsilon^4, z_\varepsilon^4)(-z_\varepsilon^{*4}) + U^* \\ &\subset D_{\partial^F}^* F(x_\varepsilon^3, y_\varepsilon^3)(y_\varepsilon^{*1} + V^*) + D_{\partial^F}^* G(x_\varepsilon^4, z_\varepsilon^4)(z_\varepsilon^{*2} + W^*) + U^*. \end{aligned}$$

Using above considerations on the normal cone to $-Q$ we obtain the conclusion. □

For the limiting subdifferential we have the following result (compare with [17, Theorem 4.3]).

**Corollary 3.5** *Suppose that $X, Y, Z$ are Asplund spaces, $(\overline{x}, \overline{y})$ be a local weak minimum point for $(P_3)$ and $\overline{z} \in G(\overline{x}) \cap -Q$. If $F$ satisfies $(CQ)$ at $(\overline{x}, \overline{y})$ and $G^{-1}$ satisfies $(CQ)$ at $(\overline{z}, \overline{x})$, then there exists $y^* \in K^* \cap S_{Y^*}$ and $z^* \in Q^*$ s.t.*

$$0 \in D_M^* F(\overline{x}, \overline{y})(y^*) + D_M^* G(\overline{x}, \overline{z})(z^*).$$

*Proof.* Take in the above theorem $e \in S_X$ and $a = 1$. Then for every natural number $n$ we can find $(x_n, y_n) \xrightarrow{\operatorname{Gr} F} (\overline{x}, \overline{y})$, $(u_n, v_n) \xrightarrow{\operatorname{Gr} G} (\overline{x}, \overline{z})$, $y_n^* \in K^*$, $1 \le \|y_n^*\| \le L_e$, $z_n^* \in Q^*$, $p_n^* \xrightarrow{w^*} 0_{X^*}$, $r_n^* \xrightarrow{w^*} 0_{Y^*}$, $q_n^* \xrightarrow{w^*} 0_{Z^*}$, $u_n^* \in D_{\partial^F}^* F(x_n, y_n)(y_n^* + r_n^*)$, $v_n^* \in D_{\partial^F}^* G(u_n, v_n)(z_n^* + q_n^*)$ s.t.

$$u_n^* + v_n^* + p_n^* = 0$$

Since $(y_n^*)$ is bounded, we can again suppose, without loosing the generality, that it converges weakly* to an element $y^*$ which is not $0_{Y^*}$. Applying $(CQ)$ for $F$, the sequence $(u_n^*)$ is bounded, so it is convergent weakly* (again without relabeling) to an element $u^*$. Then, from the last relation, $v_n^*$ is bounded too, whence weakly* convergent to $v^*$. From $(CQ)$ applied for $G^{-1}$, it follows that $(z_n^* + q_n^*)$ is bounded and, therefore, $z_n^*$ is weakly* convergent to an element in $Q^*$. Moreover, $u^* + v^* = 0$. The conclusion follows. □

Finally, let us mention that the recent two-volume book of Mordukhovich [15] contains various new developments on multiobjective optimization based on Mordukhovich's constructions in Asplund spaces mentioned in the present paper: using some notions of multiobjective optimization (also studied in [14]) the author gives many results on necessary optimality conditions in Asplund spaces. Since these notions of multiobjective optimization are different from the notions considered here it would be interesting to compare theirs interrelations. This will be the subject of a future work.

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