

ABOUT THE SECOND THEOREM OF WELFARE ECONOMICS WITH STOCK MARKETS

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Abstract: This paper discusses necessary optimality conditions for multi-objective optimization problems with application to the Second Theorem of Welfare Economics. We use the extremal principle, since we consider non-convex sets and non-smooth functions. Particularly, we develop a slight generalization of the main result of Jofré–Rivera [9], which allows more flexibility in a stochastic economy with production and stock market. Formally, we define a stock market equilibrium through the necessary optimality conditions at a constrained Pareto optimal allocation. We show that the Second Theorem of Welfare Economics holds in a two-period framework. But, by mean of an example, we show that this later result is no longer true for multi-period economies.

Key words: *multi-objective optimization, extremal principle, non-smooth analysis, non-convex programming, first-order necessary conditions, second theorem of welfare economics*

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1 Introduction

From a mathematical point of view, this paper is mainly concerned with multi-objective optimization problem and the analysis of the first-order necessary conditions. Typically, we face non-convex, non-smooth problem. That is why we use the most recent tools of Variational Analysis. (See Mordukovich [11, 12] and Rockafellar–Wets [13] and references therein.)

Our aims in this paper consist of: (1) defining an equilibrium in a stochastic production economy with stock markets, then (2) checking whether the Second Theorem of Welfare Economics can be extended to this framework.

We use the extremal principle, which is closely related to the works of Mordukovich. (See [11, 12].) It goes back to the contribution of Cornet–Rockafellar [5] (see also Aliprantis et al [1]) and extended in Jofré–Rivera [9]. We provide a slight improvement by considering a product of closed sets, which does not lie necessarily in the same linear space. Moreover, we obtain sharper results than those of Bonnisseau–Lachiri [2], since we use limiting normal cones as a refinement of Clarke’s normal cones. With respect to previous works on that subjects (see Drèze [7], Grossman–Hart [8], moreover Magill–Quinzii [10] Chapter 6 for a survey) the generality of the maximal principle allows to have a global approach dealing with all variables simultaneously.

Let’s now come to the presentation of the economic problem. We consider a production economy with several periods, uncertainty and stock markets.

To accomplish objective (1) (*i.e.*, defining an equilibrium in this framework), we must introduce a decision criterion for the firms, since, contrary to the standard approach of an Arrow-Debreu economy, the maximization of profit is ill-defined. This is a consequence of the fact that there are several possibilities to compute the actualized value of a production allocation.

Intuitively, a decision criterion for the producers can be obtained with the first-order necessary conditions at a Pareto optimal allocation. But, in our framework, it is hopeless to obtain first best Pareto optimal allocations when the markets are incomplete. So, the concept of constrained Pareto optima was introduced by Diamond [6] and Drèze [7] to take into account the limitations on the possible transfers induced by the financial structure: it is assumed that any transfer is feasible at the present time but, for the future periods, the transfer must take place through the stock markets.

Formally, we follow Definition 31.7 of Magill–Quinzii [10] to define the constrained attainability. Thus, the set of constrained attainable allocations is independent of the preferences of the agents as in the standard case of an economy à la Arrow-Debreu. This set is typically non-convex. From the concept of constrained attainability, one immediately define the constrained Pareto optimality.

Using the extremal principle (Theorem 2.4) with a constraint qualification condition, a condition introduced by Cornet in [4], we recover the first-order necessary conditions at a constrained Pareto optimal allocation. The one concerning the firms is the extension of the Drèze’s Criterion. The firms satisfy a first-order necessary condition for profit maximization with respect to the Drèze’s prices, which are weighted sums of the state-price vectors of the stockholders. But, since there are several periods, the weights change state by state because the shares are not the same due to the trading on the stock market.

A difficulty appears when we would like to compute the Drèze’s prices, since the state prices of the stockholders enter in its formulation. Indeed, the state prices of stockholders are not uniquely defined due to the non smoothness of the stockholders’ preferences. To overcome this difficulty, we proceed in two steps. We start by characterizing the preferences’ maximization problem of an individual consumer. This characterization (Lemma 4.2) consists of the first-order necessary conditions using a corollary of Theorem 2.4 (Corollary 2.5). Moreover, we provide conditions under which those conditions are sufficient. Then, taking a stockholders’ state prices that satisfy those first-order conditions, we define a stock market equilibrium where producers follow the extended Drèze’s Criterion.

Our objective (2) (*i.e.* proving that a Pareto optimal allocation is an equilibrium allocation if the initial endowments are suitably redistributed among the consumers) is achieved without significant difficulties in the two-period setting. In other words, the Second Theorem of Welfare Economics hold when stock trade is restricted to the initial date.

Surprisingly, in standard economy with three-period, we provide a particular constrained Pareto optimal allocation, which is not an allocation of a stock market equilibrium, even with income transfers at the first period. This result contradicts the usual conclusion of the Second Theorem of Welfare Economics. Indeed, the stock prices, which allow to finance the allocation, exhibits arbitrage opportunities for the state-price vectors of the consumers. Consequently, the allocation is optimal, but if we open the stock market, there is no equilibrium for these stock prices. To decentralize a constrained Pareto optimal allocation, a Social Planner must use not only the budget constraints but also additional constraints on the net trade on the stock markets.

The remainder of the paper is arranged as follows. Necessary preliminary results in Non-Smooth Analysis and the extremal principle are presented in Section 2. In Section 3, we describe the economic model and we define constrained feasible and constrained Pareto opti-

mal allocations. Then, we characterize constrained Pareto optimal allocations. In Section 4, we give a definition of a pre-equilibrium. Then, through the first-order necessary conditions at an individually optimal consumption allocation, (Lemma 4.2), we define a stock market equilibrium. We show that the Second Theorem of Welfare Economics holds in two-period setting, but not with more periods. We conclude by Section 5 within which we collect the proofs of Theorem 3.4 and Corollary 2.5, respectively.

2 The Extremal Principle

We present in this section the tools of Non-Smooth Analysis, which allow us to state our result under weak assumptions, and, in particular, to avoid convexity and differentiability assumptions on the fundamentals of the economy. In particular, we introduce the extremal principle, which is the fundamental step of the proof of our main result. We refer to the book of Rockafellar and Wets [13] for a detailed presentation. We do not use the same approach but we follow the same notations.*

Let's denote f a function defined from \mathbb{R}^n to $\mathbb{R} \cup \{+\infty\}$ and $x \in \mathbb{R}^n$ such that $f(x) \in \mathbb{R}$.

Definition 2.1. An element $y \in \mathbb{R}^n$ is a *proximal subgradient* of f at x if there exists $r > 0$ and $k > 0$, such that for all $x' \in B(x, r)$,

$$f(x') \geq f(x) + y \cdot (x' - x) - k\|x' - x\|^2$$

The set of proximal subgradients of f at x is denoted $\partial_P f(x)$.

Definition 2.2. An element $y \in \mathbb{R}^n$ is a *subgradient* of f at x if there exists a sequence $(x^\nu, y^\nu)_\nu$ in $\mathbb{R}^n \times \mathbb{R}^n$ that converges to (x, y) , such that for all integer ν , $f(x^\nu) \in \mathbb{R}$ and $y^\nu \in \partial_P f(x^\nu)$.

The set of subgradients (called often in the literature limiting subgradients) of f at x is denoted $\partial f(x)$. The set of subgradients is smaller than the set of subgradients in the sense of Clarke [3]. This allows us to obtain sharper results than the one in Bonnisseau–Lachiri [2].

Definition 2.3. Let X a subset of \mathbb{R}^n and x an element of X .

(a) The proximal normal cone to X at x , denoted $N_X^P(x)$, is defined by:

$$N_X^P(x) = \{y \in \mathbb{R}^n \mid \exists \alpha > 0, B(x + \alpha y, \alpha\|y\|) \cap X = \emptyset\}$$

(b) The normal cone to X at x , denoted $N_X(x)$, (called often in the literature limiting normal cone) is defined by:

$$N_X(x) = \{y \in \mathbb{R}^n \mid \exists (x^\nu, y^\nu) \subset X \times \mathbb{R}^n, (x^\nu, y^\nu) \rightarrow (x, y), y^\nu \in N_X^P(x^\nu), \forall \nu \in \mathbb{N}\}$$

The next proposition gives a formula that links the notions of subgradients and normal vectors. (See Theorem 8.9 and Exercise 8.14 in [13].) This formula requires the notions of

*Notations: if x and y are two vectors of \mathbb{R}^n , we denote by $x \cdot y = \sum_{i=1}^n x_i y_i$ the usual inner product, by $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$ the Euclidean norm. For all real number $r > 0$, $B(x, r)$ (resp. $\bar{B}(x, r)$) denotes the open (resp. closed) ball of center x and radius r . If X is a subset of \mathbb{R}^n , $\text{bdry} X$ denotes the boundary of X and \bar{X} the closure of X .

the epigraph of a function and of the indicator function of a set, therefore we recall their definitions.

The epigraph of f , denoted $\text{epi} f$, is the set defined by

$$\text{epi} f = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq t\}.$$

The indicator function of X , a subset of \mathbb{R}^n , is defined by

$$\delta_X(x) = \begin{cases} 0 & \text{if } x \in X \\ +\infty & \text{if } x \notin X \end{cases}$$

Proposition 2.1. (a) Let f be a function from \mathbb{R}^n to $\mathbb{R} \cup \{+\infty\}$ and $x \in \mathbb{R}^n$ such that $f(x) \in \mathbb{R}$. Then, $\partial f(x) = \{y \in \mathbb{R}^n \mid (y, -1) \in N_{\text{epi} f}(x, f(x))\}$.

(b) Let X a subset of \mathbb{R}^n and x an element of X . Then $N_X(x) = \partial \delta_X(x)$.

The concepts of the subgradients and normal vectors generalize the usual notions coming from convex analysis. Indeed, if f is convex (see Proposition 8.12 in [13]), then

$$\partial f(x) = \{y \in \mathbb{R}^n \mid f(x') \geq f(x) + y \cdot (x' - x), \forall x' \in \mathbb{R}^n\}$$

and, if X is convex (see Theorem 6.9 in [13]), then

$$N_X(x) = \{y \in \mathbb{R}^n \mid y \cdot x' \leq y \cdot x, \forall x' \in X\}$$

We now state the properties of the subgradients and normal cones, which will be used in the remaining of this paper.

Proposition 2.2. Let X be a subset of \mathbb{R}^n and f be a function from \mathbb{R}^n to $\mathbb{R} \cup \{+\infty\}$.

- (a) If $x \in X \cap U$, with U open subset of \mathbb{R}^n , then $N_{X \cap U}(x) = N_X(x)$;
- (b) For $k = 1, \dots, h$, let n^k be a positive integer and X^k be a subset of \mathbb{R}^{n^k} . Then, for all $x = (x^1, \dots, x^h) \in X = \prod_{k=1}^h X^k$, $N_X(x) = \prod_{k=1}^h N_{X^k}(x^k)$;
- (c) For $k = 1, \dots, h$, let n^k be a positive integer number and (f^k) be a lower semi-continuous function from \mathbb{R}^{n^k} to $\mathbb{R} \cup \{+\infty\}$. For all $(x^1, \dots, x^h) \in \prod_{k=1}^h \mathbb{R}^{n^k}$ such that $f^k(x^k)$ is finite for every k , $\partial(\sum_{k=1}^h f^k)(x^1, \dots, x^h) = \prod_{k=1}^h \partial f^k(x^k)$;
- (d) Let (x^ν, y^ν) be a sequence of $X \times \mathbb{R}^n$ converging to $(x, y) \in X \times \mathbb{R}^n$. If $y^\nu \in N_X(x^\nu)$ for all ν , then $y \in N_X(x)$.
- (e) Let (x^ν, y^ν) be a sequence of $\mathbb{R}^n \times \mathbb{R}^n$ converging to (x, y) such that $f(x^\nu)$ is finite and $y^\nu \in \partial f(x^\nu)$ for all ν . If the sequence $(f(x^\nu))$ converges to $f(x)$ and $f(x)$ is finite, then $y \in \partial f(x)$.
- (f) If f is locally Lipschitz continuous on a neighborhood of x of rank h , then $\partial f(x) \subset \bar{B}(0, h)$;
- (g) If f is continuously differentiable on a neighborhood of x , then $\partial f(x) = \{\nabla f(x)\}$;
- (h) If f is locally Lipschitz continuous on a neighborhood of x and g is a lower semicontinuous function from \mathbb{R}^n to $\mathbb{R} \cup \{+\infty\}$ such that $g(x)$ is finite, then $\partial(f + g)(x) \subset \partial f(x) + \partial g(x)$;

- (i) For all $\lambda > 0$ and x such that $f(x)$ is finite, $\partial(\lambda f)(x) = \lambda \partial f(x)$, $\lambda > 0$;
- (j) Let g be a locally Lipschitz continuous function from \mathbb{R}^m to $\mathbb{R} \cup \{+\infty\}$ and F be a locally Lipschitz continuous mapping from \mathbb{R}^n to \mathbb{R}^m . Let $f = g \circ F$. Then, for all $x \in \mathbb{R}^n$ such that $f(x)$ is finite, $\partial f(x) \subset \cup_{y \in \partial g(F(x))} \partial(y \cdot F)(x)$;
- (k) If f is locally Lipschitz continuous on a neighborhood of x and X is closed in \mathbb{R}^n , then, if x is a local minimizer of f on X , $0 \in \partial f(x) + N_X(x)$.
- (l) Let U an open subset of \mathbb{R}^n . Let $(f^i)_{i=1}^\ell$ and $(g^k)_{k=1}^m$ $\ell + m$ continuously differentiable functions from U to \mathbb{R} . Let

$$X = \{x \in U \mid \forall i = 1, \dots, \ell, f^i(x) = 0, \forall k = 1, \dots, m, g^k(x) \leq 0\}$$

Let $x \in X$ such that the gradient vectors $(\nabla f^i(x))_{i=1}^\ell$ are linearly independent and there exists $v \in \mathbb{R}^n$ such that for all $i = 1, \dots, \ell$, $\nabla f^i(x) \cdot v = 0$ and for all $k = 1, \dots, m$ such that $g^k(x) = 0$, $\nabla g^k(x) \cdot v < 0$. Then,

$$N_X(x) = \left\{ \sum_{i=1}^\ell \lambda^i \nabla f^i(x) + \sum_{k=1}^m \mu^k \nabla g^k(x) \mid \begin{array}{l} (\lambda^i) \in \mathbb{R}^\ell, (\mu^k) \in \mathbb{R}_+^m \\ \mu^k g^k(x) = 0, \forall k = 1, \dots, m \end{array} \right\}$$

Proof. The proofs of the above properties of the normal cone and of the subgradient set are given in the book of Rockafellar and Wets [13]. We give now the precise references. First note that Theorem 9.13 shows that $\partial^\infty f(x) = \{0\}$ if f is locally Lipschitz continuous on a neighborhood of x . Then, some qualification condition are obviously satisfied with locally Lipschitz continuous function. Assertion (a) comes from the definition of the normal cone as the upper limit of the proximal normal cone and from the fact that for all $x' \in X \cap U$, $N_{X \cap U}^P(x') = N_X^P(x')$. Assertion (b) is given in Proposition 6.41. Assertion (c) is given in Proposition 10.5. Assertion (d) is given in Proposition 6.6. Assertion (e) is a direct consequence of Assertion (c) and Assertion (a) of Proposition 2.1. Assertion (f) is given in Theorem 9.13. Assertion (g) is given in Exercise 8.8. Assertion (h) is given in Exercise 10.10. Assertion (i) is given in Proposition 10.19. Assertion (j) is given in Theorem 10.49. Assertion (k) is given in Theorem 8.15. Assertion (l) is given in Example 6.40. \square

We now come to the main result of this section, called extremal principle in [13], which is the fundamental tool in the proofs of our paper. A first version of this result by Cornet and Rockafellar dates back to the beginning of the 90's. (See, Aliprantis, Cornet and Tourky [1].) It was then generalized by Jofré and Rivera Cayupi in [9]. It is closely related to numerous works of Mordukhovich. (See [11, 12].)

We choose the formulation of Rockafellar and Wets [13], which is simpler than the one of Jofré-Rivera Cayupi, and we show how one deduces the result of Jofré and Rivera Cayupi as well as the first formulation of Cornet and Rockafellar. We give a direct demonstration that mimic the proof given in [9], since the one given in [13] uses sophisticated results on the derivatives of set-valued mappings. The basic idea of using a sequence of perturbed optimization problems dates back to Cornet–Rockafellar [5].

Theorem 2.4. *Let X be a closed subset of \mathbb{R}^n and F a mapping from \mathbb{R}^n to \mathbb{R}^m . Let $x \in X$ such that F is locally Lipschitz continuous at x and $F(x) \in \text{bdry} F(X)$. Then, there exists $\pi \in \mathbb{R}^m \setminus \{0\}$ such that:*

$$0 \in \partial(\pi \cdot F)(x) + N_X(x).$$

As a corollary of Theorem 2.4, we give hereafter in (c) the initial result of Cornet and Rockafellar, in (b) the particular case where F is continuously differentiable and in (a) a general result, which in contrast to the one of Jofré and Rivera Cayupi applies to a system of sets that may not lie in linear spaces with the same dimensionality.

Corollary 2.5. (a) For $k = 1, \dots, h$, let n^k be a positive integer number and X^k be a subset of \mathbb{R}^{n^k} . Let F be a mapping from $\mathbb{R}^{\sum_{k=1}^h n^k}$ to \mathbb{R}^m . Let $x = (x^1, \dots, x^h) \in \prod_{k=1}^h X^k$ such that F is locally Lipschitz continuous at x and $x \in \text{bdry} F(\prod_{k=1}^h X^k)$. Then, there exists $\pi \in \mathbb{R}^m \setminus \{0\}$ such that:

$$0 \in \partial(\pi \cdot F)(x) + \prod_{k=1}^h N_{X^k}(x^k)$$

(b) If F is continuously differentiable in a neighborhood of x , then, there exists $\pi \in \mathbb{R}^m \setminus \{0\}$ such that:

$$0 \in DF^t(x)(\pi) + \prod_{k=1}^h N_{X^k}(x^k)$$

where $DF^t(x)$ is the transpose of $DF(x)$ the differential of F at x .

(c) If $n^1 = n^2 = \dots = n^h = m$ and $\sum_{k=1}^h x^k \in \text{bdry} \sum_{k=1}^h X^k$, then there exists $\pi \in \mathbb{R}^m \setminus \{0\}$ such that for all k ,

$$0 \in \pi + N_{X^k}(x^k)$$

Proof. Corollary 2.5 is easily derived from Theorem 2.4 using Assertion (b) of Proposition 2.2 to prove (a), Assertion (f) to prove (b) and applying the result to $F(x^1, \dots, x^h) = \sum_{k=1}^h x^k$ to prove (c). \square

Remark 2.6. Remark that one can actually get sharper results than those of Theorem 2.4 and Corollary 2.4. It suffices to consider the subgradient $\partial d_X(x)$ of the distance function d_X to the set X at a point x instead of the normal cone $N_X(x)$ to the same set at the same point. (See Jofré-Rivera-Cayupi [9] for a discussion about this point.)

Now, we turn back to Theorem 2.4 and we present its proof.

Proof. Let $r > 0$ such that F is locally Lipschitz continuous of rank h on $\bar{B}(x, r)$. Since $\bar{z} = F(x) \in \text{bdry} F(X)$, there exists a sequence $(z^\nu) \in \mathbb{R}^m$, which converges to \bar{z} and such that $z^\nu \notin F(X)$ for all ν . For all ν , we consider the following minimization problem:

$$(\text{Pb}^\nu) \begin{cases} \text{Minimize } g^\nu(x) = \|F(x) - z^\nu\| + \|x - x\|^2 \\ x \in X \cap \bar{B}(x, r) \end{cases}$$

For all ν , the problem (Pb^ν) has at least one solution (x^ν) since the objective function is continuous and the set $X \cap \bar{B}(x, r)$ is compact. Since $z^\nu \notin F(X)$, one has $F(x^\nu) \neq z^\nu$. One also remarks that x satisfies the constraints, then, $g^\nu(x^\nu) \leq g^\nu(x) = \|F(x) - z^\nu\|$. Consequently, $\|x^\nu - x\|^2 \leq \|F(x) - z^\nu\|$. Since the sequence (z^ν) converges to $\bar{z} = F(x)$, one deduces that (x^ν) converges to x . Consequently, there exists $\underline{\nu}$ such that for all $\nu \geq \underline{\nu}$, $x^\nu \in B(x, r)$. The sequence $\left(\pi^\nu = \frac{F(x^\nu) - z^\nu}{\|F(x^\nu) - z^\nu\|}\right)$ has a cluster point π satisfying $\|\pi\| = 1$. Up to a subsequence, we can assume that $x^\nu \in B(x, r)$ and (π^ν) converges to π .

From Proposition 2.2(k), for all ν , $0 \in \partial g^\nu(x^\nu) + N_{X \cap \bar{B}(x, r)}(x^\nu)$. From Proposition 2.2(a), since $x^\nu \in B(x, r)$, $N_{X \cap \bar{B}(x, r)}(x^\nu) = N_X(x^\nu)$. From Proposition 2.2(g,h,j), since

$z \rightarrow \|z - z^\nu\|$ is continuously differentiable in a neighborhood of $F(x^\nu)$ and its gradient at $F(x^\nu)$ is equal to π^ν ,

$$\partial g^\nu(x^\nu) \subset \partial(\pi^\nu \cdot F)(x^\nu) + \{2(x^\nu - x)\}$$

From Proposition 2.2(h), since $\pi^\nu \cdot F = \pi \cdot F + (\pi^\nu - \pi) \cdot F$, one has $\partial(\pi^\nu \cdot F)(x^\nu) \subset \partial(\pi \cdot F)(x^\nu) + \partial(\pi^\nu - \pi) \cdot F(x^\nu)$. Since $(\pi^\nu - \pi) \cdot F$ is Lipschitz continuous of rank $h\|\pi^\nu - \pi\|$ on $\bar{B}(x, r)$, from Proposition 2.2 (f), $\partial(\pi^\nu - \pi) \cdot F(x^\nu) \subset \bar{B}(0, h\|\pi^\nu - \pi\|)$. Gathering all these elements, one gets:

$$0 \in \partial(\pi \cdot F)(x^\nu) + \{2(x^\nu - x)\} + \bar{B}(0, h\|\pi^\nu - \pi\|) + N_X(x^\nu)$$

Hence, there exists $\xi^\nu \in \partial(\pi \cdot F)(x^\nu)$ and $\zeta^\nu \in N_X(x^\nu)$ such that $\|\xi^\nu + \zeta^\nu\| \leq h\|\pi^\nu - \pi\| + 2\|x^\nu - x\|$. Since $\pi \cdot F$ is Lipschitz continuous of rank h on $\bar{B}(x, r)$, from Proposition 2.2 (f), $\|\xi^\nu\| \leq h$. Hence the sequence (ξ^ν) has a subsequence converging to ξ . Since (x^ν, π^ν) converges to (x, π) , the same subsequence of (ζ^ν) converges to $-\xi$. From Proposition 2.2 (d,e), $\xi \in \partial(\pi \cdot F)(x)$ and $-\xi \in N_X(x)$, which ends the proof. \square

We end this section by recalling a condition introduced in Cornet [4], which plays the role of a constraint qualification condition when the sets are not convex.

Definition 2.7. Let X be a subset of \mathbb{R}^n and $x \in \bar{X}$. X satisfies the Condition (D) at x if there exist $v \in \mathbb{R}^n$ and $\underline{t} > 0$ such that for all $t \in]0, \underline{t}[$,

$$tv + (\bar{X} \cap \bar{B}(x, t)) \subset X.$$

The direction v is a locally inward direction at x with respect to X . In the following, it can be interpreted as a desirability direction. In Cornet [4], the following sufficient conditions are proved. We refer to [13] for the definition of epilipschitzianity.

Proposition 2.3. Let X be a subset of \mathbb{R}^n and $x \in \bar{X}$. X satisfies the Condition (D) at x if one of the following conditions holds true:

- (a) X is convex;
- (b) X is closed;
- (c) $X + \text{int}\mathbb{R}_+^n \subset X$;
- (d) X is epilipschitzian at x .

3 Constrained Pareto Optimal Allocations

In this section, we will start by presenting the economic model. Next, we recall the definition of constrained feasible allocations, then the one of constrained Pareto optimal allocations. Then, we close by a characterization of constrained Pareto optimal allocations.

Time, uncertainty and the information revelation over time are modeled by an event-tree. Time is finite and discrete, denoted by $t = 0, \dots, T$, where time $t = 0$ and $t = 1, \dots, T$ represent respectively the present and the future. Uncertainty is about the accomplishment of a finite set $\Omega = \{1, \dots, S\}$ of states of nature in the future. We assume that uncertainty is solved through a finite sequence of partitions $\mathcal{F} = (\mathcal{F}_t)_{t=0}^T$ of the set Ω that increases over time, where $\mathcal{F}_0 = \Omega$ and $\mathcal{F}_T = \{\{1\}, \dots, \{S\}\}$. Therefore, given (Ω, \mathcal{F}) , the event tree,

denoted by \mathfrak{D} , is the finite set of nodes $\xi = (t, \sigma)$ provided that $\sigma \in \mathcal{F}_t$, for all $t = 1, \dots, T$.[†] For needs, we borrow from Magill–Quinzii [10] a set of notations. We denote ξ_0 the unique node that occur with certainty at date 0. The unique predecessor of a non initial node $\xi \in \mathfrak{D}^+ = \mathfrak{D} \setminus \{\xi_0\}$ is denoted ξ^- . The set of immediate successors of a non terminal node $\xi \in \mathfrak{D}_- = \mathfrak{D} \setminus \mathfrak{D}_T$ is denoted ξ^+ . Finally, we denote $\mathfrak{D}_-^+ = \mathfrak{D}_- \setminus \{\xi_0\}$ the set of non initial and non terminal nodes.

We consider a simple model with a unique commodity at each node since this is enough to obtain our main conclusion. Thus, the commodity space is $\mathbb{R}^{\mathfrak{D}}$. At each node, the spot commodity market takes place and the spot price is normalized to 1.

There are I consumers represented by the superscript $i \in \mathcal{I} = \{1, \dots, I\}$ and J producers represented by the superscript $j \in \mathcal{J} = \{1, \dots, J\}$.

Each consumer i has a consumption set $X^i \subset \mathbb{R}^{\mathfrak{D}}$. We denote by X the Cartesian product of the individual consumption sets. The preferences of the consumer i are represented by a preference relation \mathcal{P}^i from X to X^i . For all $x = (x^1, \dots, x^I) \in X$, $\mathcal{P}^i(x)$ is the set of consumption plan strictly preferred to x^i by consumer i taken into account the consumption plans $(x^k)_{k \neq i}$ of the other consumers. The initial endowment is denoted by $e^i \in \mathbb{R}^{\mathfrak{D}}$.

The producer j has a production sets, denoted Y_j , which is a subset of $\mathbb{R}^{\mathfrak{D}}$. At date 0, the i th consumer has a initial share $a_j^i(\xi_0^-) \geq 0$ on the profit or losses of firm j . The notation is chosen to simplify forthcoming equations. This share is actually an initial endowment like e^i for the commodities. As usual, these shares satisfy the constraints $\sum_{i \in \mathcal{I}} a_j^i(\xi_0^-) = 1$ for all $j \in \mathcal{J}$.

In addition to spot commodity markets, there are stock markets for J assets that correspond to the stocks of the firms, but only for each nonterminal node $\xi \in \mathfrak{D}_-$. A portfolio of shares of stocks of a consumer i across nodes is denoted $a^i = (a_j^i(\xi))_{j \in \mathcal{J}}^{\xi \in \mathfrak{D}_-} \in (\mathbb{R}_+^{\mathfrak{D}_-})^J$. Note that some times we will allow consumer i to go short in the stock markets, that is each share $a_j^i(\xi)$ belongs to $\mathbb{R}^{\mathfrak{D}_-}$ instead of $\mathbb{R}_+^{\mathfrak{D}_-}$. We denote the stock price of each stock j at node ξ by $q^j(\xi)$.

As usual, we assume that each consumer i has a budget set $\mathcal{B}^i(q, y)$ given a production plan $y = (y_j) \in \prod_{j \in \mathcal{J}} Y_j$ and stock prices $q \in (\mathbb{R}^{\mathfrak{D}})^J$. To simplify the formulation of the budget set $\mathcal{B}^i(q, y)$, we let $q^j(\xi) = 0$ and $a^i(\xi^-) = a^i(\xi)$, for all $\xi \in \mathfrak{D}_T$. Thus $(x^i, a^i) \in X^i \times (\mathbb{R}^{\mathfrak{D}_-})^J$ is in $\mathcal{B}^i(q, y)$ if for all $\xi \in \mathfrak{D}$

$$x^i(\xi) - e^i(\xi) \leq y(\xi) \cdot a^i(\xi^-) + q(\xi) \cdot (a^i(\xi^-) - a^i(\xi)), \quad (3.1)$$

where \cdot denotes, throughout the paper, the usual inner product in \mathbb{R}^J .

The inequality (3.1) means that at a generic node ξ , the net trade $(x^i(\xi) - e^i(\xi))$ on the spot commodity market must be lower or equal to the payoff coming from the dividends $a^i(\xi^-) \cdot y(\xi)$ (profits or losses) of the firms distributed according to the portfolio of shares $a^i(\xi^-)$ held at the immediate predecessor node ξ^- , plus the expense or the income coming from the transaction on the stock market, $q(\xi) \cdot (a^i(\xi^-) - a^i(\xi))$.

We now come to the notion of constrained feasibility. As for the usual concept of feasibility, the constraints on the allocations are the usual physical constraints saying that the sum of the consumptions at each node must be equal to the sum of the initial endowments plus the productions. Due to the limited number of financial instruments, all physically

[†]We refer to Magill–Quinzii [10] pages 215-218 for a detailed presentation of the event tree approach to model uncertainty.

feasible allocations are not attainable through the financial markets. Thus, constrained feasible allocations must satisfy the additional condition that the net trade must be financed by trades on the stock market, except at the initial node. (See Magill–Quinzii [10] pages for a lengthy discussion about this concept of feasibility.)

Definition 3.1. An element $(x, a, y) \in \prod_{i \in \mathcal{I}} X^i \times ((\mathbb{R}_+^{\mathfrak{D}})^J)^I \times \prod_{j \in \mathcal{J}} Y_j$ is said to be constrained feasible provided that:

- (a) there exists a stock price $q \in (\mathbb{R}^{\mathfrak{D}^+})^J$, with $q(\xi) = 0$ for all $\xi \in \mathfrak{D}_T$ such that for all $i \in \mathcal{I}$,

$$x^i(\xi) - e^i(\xi) = y(\xi) \cdot a^i(\xi^-) + q(\xi) \cdot (a^i(\xi^-) - a^i(\xi)), \forall \xi \in \mathfrak{D}^+, \quad (3.2)$$

- (b) $\sum_{i \in \mathcal{I}} (x^i(\xi_0) - e^i(\xi_0)) = \sum_{j \in \mathcal{J}} y^j(\xi_0)$, and

- (c) $\sum_{i \in \mathcal{I}} a_j^i(\xi) = 1$, for all $j \in \mathcal{J}$ and for all $\xi \in \mathfrak{D}_+$.

Note first that, the previous definition can be clearly extended to include no short-sales constraints on the shares. It is close to Definition 31.7 of Magill–Quinzii [10] since we do not assume that the consumers are at an optimal consumption with respect to the prices. Indeed, we limit the constraints to physical and financial ones. This may be justified by two arguments. First, we try to obtain a simple definition. Second, in the usual setting of an Arrow–Debreu economy with complete markets, the feasible allocations are defined independently of the consumers' preferences. Our definition satisfies this property, which does not hold for the one of Magill–Quinzii [10].

Remark 3.2. Remark that, the concept of constrained feasibility of Definition 3.1 is equivalent to Definition 2 of Bonnisseau–Lachiri [2] as soon as we consider a unique commodity per node with the next fact. Aggregating over \mathcal{I} the financial constraint (a) for each node $\xi \in \mathfrak{D}^+$ and using the feasibility constraints (c), one obtains the market clearing condition (b) at each node $\xi \in \mathfrak{D}^+$. Then, considering in addition condition (b), we recover the spot market clearing conditions (b) of Definition 2 of Bonnisseau–Lachiri [2]. Accordingly, this means that a constrained feasible allocation of Definition 3.1 is also physically feasible.

Now, we state a definition of constrained Pareto optimal allocations induced by the concept of feasibility of Definition 3.1.

Definition 3.3. An element $(x, a, y) \in \prod_{i \in \mathcal{I}} X^i \times ((\mathbb{R}_+^{\mathfrak{D}^-})^J)^I \times \prod_{j \in \mathcal{J}} Y_j$ is said to be constrained Pareto optimal provided that it is constrained feasible and it does not exist a constrained feasible element (x', a', y') such that $x'^i \in \mathcal{P}^i(x)$ for all $i \in \mathcal{I}$.

In the next, we present the first result of this paper, that is the necessary conditions at a constrained Pareto optimal allocation. In particular, we recover a multi-period Drèze Criterion satisfied by each producer under weaker assumptions than the corresponding result in [2]. Moreover, this result is sharper since we consider the normal cone instead of the Clarke's normal cone. This result will provide, later on, the production criterion in the definition of a stock market equilibrium (Definition 4.5).

Theorem 3.4. Let $(\bar{x}, \bar{a}, \bar{y})$ be a constrained Pareto optimal allocation. Since it is constrained feasible, there exists a stock price vector $\bar{q} \in \mathbb{R}^{\mathfrak{D}^+}$, with $\bar{q}(\xi) = 0$ for all $\xi \in \mathfrak{D}_T$, which finances the allocation $(\bar{x}, \bar{a}, \bar{y})$, that is, Equations 3.2 are satisfied. We assume that for each $i \in \mathcal{I}$, $\bar{x}^i \in \mathcal{P}^i(\bar{x}^i)$, and $\mathcal{P}^i(\bar{x}^i)$ satisfies Condition (D) at \bar{x}^i , and, for all $j \in \mathcal{J}$, Y^j is closed. Then, there exists a common price $\pi(\xi_0)$ for the initial node, personal prices $(\pi^i(\xi))_{\xi \in \mathfrak{D}^+}$ for each consumer $i \in \mathcal{I}$, and short-term values $(v^j(\xi))_{\xi \in \mathfrak{D}_-}$ for each firm $j \in \mathcal{J}$ such that $(\pi(\xi_0), (\pi^i)_{i \in \mathcal{I}}) \neq 0$, which satisfies the following properties,

- (i) $\forall i \in \mathcal{I}, \bar{\pi}^i = \left(\pi(\xi_0), (\pi^i(\xi))_{\xi \in \mathfrak{D}^+} \right) \in -N_{\bar{\mathcal{P}}^i(\bar{x})}(\bar{x}^i).$
- (ii) $\forall j \in \mathcal{J}, \bar{\pi}^j = \left(\pi(\xi_0), (\sum_{i \in \mathcal{I}} \pi^i(\xi) \bar{a}_j^i(\xi^-))_{\xi \in \mathfrak{D}^+} \right) \in N_{Y^j}(\bar{y}^j).$
- (iii) $\forall j \in \mathcal{J}, \forall \xi \in \mathfrak{D}_-^+, \sum_{i \in \mathcal{I}} \pi^i(\xi) (\bar{a}_j^i(\xi) - \bar{a}_j^i(\xi^-)) = 0.$
- (iv) $\forall i \in \mathcal{I}, \text{ for all } \xi \in \mathfrak{D}_-,$

$$v(\xi) + \pi^i(\xi) \bar{q}(\xi) - \sum_{\eta \in \xi^+} \pi^i(\eta) (\bar{q}(\eta) + \bar{y}(\eta)) \geq 0 \quad (3.3a)$$

$$\bar{a}^i(\xi) \cdot (v(\xi) + \pi^i(\xi) \bar{q}(\xi) - \sum_{\eta \in \xi^+} \pi^i(\eta) (\bar{q}(\eta) + \bar{y}(\eta))) = 0 \quad (3.3b)$$

with $\bar{q}(\xi_0) = 0.$

The proof of this theorem is given in Section 5.

For an economic interpretation of the above conditions with the consequence on the evaluation of the production plans by the consumers and the managers of the firms, we refer to [2]. We just comments conditions (ii), which is the key points in Definition 4.5.

Condition (ii) means that each producer satisfies the first-order necessary condition for profit maximization with respect to the price $\bar{\pi}^j$. This price is computed at each node as a convex combination of the personal state prices of the shareholders at this node. The weights at a given node in the convex combination are given by the shares at the previous node, since the profit of the firms are distributed according to these shares. This condition is the extension of Drèze's Criterion to a multi-period economy.

4 Equilibrium Allocations and the Second Theorem of Welfare Economics

For needs, we start this section by recalling the usual equilibrium conditions on the consumer's behavior and on the market clearing when productions are exogenously fixed. We call such collection a pre-equilibrium since we do not impose any condition on the producers. Later, we give the definition of a stock market equilibrium, which is a pre-equilibrium with the production allocations satisfying the condition (ii) of Theorem 3.4.

Definition 4.1. An element $(q, x, a, y) \in \mathbb{R}^{\mathfrak{D}-} \times \prod_{i \in \mathcal{I}} X^i \times (\mathbb{R}_+^{\mathfrak{D}-})^I \times \prod_{j \in \mathcal{J}} Y_j$ is said to be a pre-equilibrium provided that:

- (a) for each i , $(x^i, a^i) \in \mathcal{B}^i(q, y)$, and $\left(\mathcal{P}^i(x) \times (\mathbb{R}_+^{\mathfrak{D}-})^J \right) \cap \mathcal{B}^i(q, y) = \emptyset,$
- (b) $\sum_{i \in \mathcal{I}} (x^i - e^i) = \sum_{j \in \mathcal{J}} y^j$, and
- (c) $\sum_{i \in \mathcal{I}} a_j^i(\xi) = 1$, for all $j \in \mathcal{J}$ and for all $\xi \in \mathfrak{D}.$

In the forthcoming definition of a stock market equilibrium, we need to know the first-order necessary conditions at a maximal element for the preferences in the budget set, which are given in the next lemma. This result constitutes an application of Corollary 2.5 under some weak conditions on consumers' preferences. We postpone the proof of this lemma to Section 5.

Lemma 4.2. Let $\bar{x} \in X$ and $\bar{a}^i \in (\mathbb{R}_+^{\mathfrak{D}-})^J$.

- (a) Let $(\bar{x}^i, \bar{a}^i) \in X^i \times (\mathbb{R}_+^{\mathfrak{D}-})^J$ a maximal element of \mathcal{P}^i in the budget set $\mathcal{B}^i(q, y)$ with $q(\xi) = 0$ for all $\xi \in \mathfrak{D}_T$. Then, if in addition $\bar{x}^i \in \mathcal{P}^i(\bar{x})$ and $\mathcal{P}^i(\bar{x})$ satisfies Condition (D) at \bar{x}^i , there exists $\pi^i \in \mathbb{R}_+^{\mathfrak{D}} \setminus \{0\}$ such that

$$-\pi^i \in N_{\bar{\mathcal{P}}^i(\bar{x})}(\bar{x}^i) \quad (4.1)$$

for all $j \in \mathcal{J}$, $\forall \xi \in \mathfrak{D}_-$,

$$\pi^i(\xi)q^j(\xi) - \sum_{\eta \in \xi^+} \pi^i(\eta)(q^j(\eta) + y^j(\eta)) \geq 0 \quad (4.2a)$$

$$\bar{a}_j^i(\xi) \left(\pi^i(\xi)q^j(\xi) - \sum_{\eta \in \xi^+} \pi^i(\eta)(q^j(\eta) + y^j(\eta)) \right) = 0, \quad (4.2b)$$

- (b) Conversely, if $\mathcal{P}^i(\bar{x})$ is open and convex, $\bar{x}^i \in \bar{\mathcal{P}}^i(\bar{x})$, all inequalities (3.1) are binding at (\bar{x}^i, \bar{a}^i) and Equations (4.1) and (4.2) are satisfied, then (\bar{x}^i, \bar{a}^i) is a maximal element of \mathcal{P}^i in the budget set $\mathcal{B}^i(q, y)$.

Remark 4.3. If the preferences are represented by a differentiable utility function, the vector π^i is given by the marginal rate of substitution and we get standard necessary conditions for a mathematical programming problem. Conditions (4.2) are then the standard complementary slackness conditions.

Remark 4.4. We remark that, in contrast to Equations 4.2 of Lemma 4.2, the short term values $v_j(\xi)$ in Equations 3.3 of Theorem 3.4 may be different from 0. Thus, clearly at constrained Pareto optimal allocations, the first-order necessary conditions for shareholders' preferences maximization are not necessarily satisfied at a non terminal node. This fact constitute the key point of the main economic result in the conclusion of this section.

We can now state the definition of a stock market equilibrium allocation, that is a consequence of Definition 4.1, Lemma 4.2 and condition (ii) of Theorem 3.4.

Definition 4.5. An element $(\bar{q}, \bar{x}, \bar{a}, \bar{y}) \in (\mathbb{R}^{\mathfrak{D}-})^J \times \prod_{i \in \mathcal{I}} X^i \times ((\mathbb{R}_+^{\mathfrak{D}-})^J)^I \times \prod_{j \in \mathcal{J}} Y_j$ is said to be a stock market equilibrium provided that it is a pre-equilibrium in the sense of Definition 4.1 and

- (d) there exist a state price $\bar{\pi}(\xi_0)$ and for each $i \in \mathcal{I}$, a state price vector $(\bar{\pi}^i(\xi))_{\xi \in \mathfrak{D}^+}$, such that $\bar{\pi}^i = (\bar{\pi}(\xi_0), (\bar{\pi}^i(\xi))_{\xi \in \mathfrak{D}^+})$ satisfies the conclusion of Lemma 4.2, and for all $j \in \mathcal{J}$,

$$\bar{\pi}^j = \left(\bar{\pi}(\xi_0), \left(\sum_{i \in \mathcal{I}} \bar{\pi}^i(\xi) \bar{a}_j^i(\xi^-) \right)_{\xi \in \mathfrak{D}^+} \right) \in N_{Y_j}(\bar{y}^j). \quad (4.3)$$

In order to understand the previous concept of equilibrium, assume, for instance, that all consumptions \bar{x}^i are positive and the agents' preferences are represented standard utility functions u^i , that are differentiable with gradient $\nabla u^i(\bar{x}^i) \in \mathbb{R}_{++}^{\mathfrak{D}}$. In this case, every state price $\bar{\pi}^i$ is the normalized normal vector, that is the vector defined by

$$\left(\frac{1}{\frac{\partial u^i}{\partial x^i}(\bar{x}^i)} \right) \nabla u^i(\bar{x}^i)$$

with the first coordinate equals to 1. Then, the rule can be reformulated by only substituting $\bar{\pi}(\xi_0)$ by 1 in (4.3).

Additionally, if the production set Y_j is convex, then the formula (4.3) means that the firm maximizes its profit with respect to a state price $\bar{\pi}^j$, which is computed node by node as a convex combination of the marginal rates of substitution of its shareholders.

We can now state the Second Theorem of Welfare Economics. We start with a two-period economy. In this framework, the definition of a constrained feasible allocation does not require the existence of a stock price since the second period is also the terminal one. Thus, the financial feasibility constraint is simply $x^i(\xi) - e^i(\xi) = y(\xi) \cdot a^i(\xi_0)$ for all $\xi \in \mathfrak{D}_1$.

Theorem 4.6. *We consider a two-period economy. Let $(\bar{x}, \bar{a}, \bar{y})$ be a constrained Pareto optimal allocation satisfying the assumptions of Theorem 3.4. We also assume that $\mathcal{P}^i(\bar{x}^i)$ is open and convex and for some $i_0 \in \mathcal{I}$, there exists $\tilde{x}^{i_0} \in \mathcal{P}^i(\bar{x})$, with $\tilde{x}^{i_0}(\xi_0) > \bar{x}^{i_0}(\xi_0)$ and $\tilde{x}^{i_0}(\xi) = \bar{x}^{i_0}(\xi)$ for all $\xi \in \mathfrak{D}_1$. Then there exists a stock price $\bar{q} \in \mathbb{R}^J$ and a transfer $t \in \mathbb{R}^I$ such that $\sum_{i \in \mathcal{I}} t^i = 0$ and $(\bar{q}, \bar{x}, \bar{a}, \bar{y})$ is a stock market equilibrium of the economy where the initial endowments are $\tilde{e}^i = (e^i(\xi_0) + t^i, (e^i(\xi))_{\xi \in \mathfrak{D}_1})$.*

Remark 4.7. The convexity and openness assumption on the preferred sets are necessary to obtain the fact that the consumptions are optimal. Without this assumption, we obtain a weaker condition saying that only the first-order necessary conditions are satisfied. The additional assumption on the preference means that the agent i_0 can obtain a strictly better allocation by only increasing consumption at the first period. This property is weaker than assuming that the preferences are strictly increasing. A stronger version of this non-satiation assumption is often used in the incomplete market equilibrium models.

Now, we present the proof of Theorem 4.6.

Proof. By virtue of Theorem 3.4, there exists $(\pi(\xi_0), (\pi(\xi))_{\xi \in \mathfrak{D}_1}, (v^j(\xi_0)))$ satisfying the properties (i) to (iv). We first prove that $\pi(\xi_0) > 0$. Indeed, since $\mathcal{P}^{i_0}(\bar{x})$ is convex and open, since $\tilde{x}^{i_0} \in \mathcal{P}^i(\bar{x})$, with $\tilde{x}^{i_0}(\xi_0) > \bar{x}^{i_0}(\xi_0)$ and $\tilde{x}^{i_0}(\xi) = \bar{x}^{i_0}(\xi)$ for all $\xi \in \mathfrak{D}_1$ and since $(\pi(\xi_0), (\pi^{i_0}(\xi))_{\xi \in \mathfrak{D}_1}) \in -N_{\bar{\mathcal{P}}^i(\bar{x})}(\bar{x}^i)$, one has $\pi(\xi_0)\bar{x}^{i_0}(\xi_0) + \sum_{\xi \in \mathfrak{D}_1} \pi^{i_0}(\xi)\bar{x}^{i_0}(\xi) < \pi(\xi_0)\tilde{x}^{i_0}(\xi_0) + \sum_{\xi \in \mathfrak{D}_1} \pi^{i_0}(\xi)\tilde{x}^{i_0}(\xi)$. This clearly implies $\pi(\xi_0) > 0$.

Now, we define the stock prices by $\bar{q}^j(\xi_0) = v^j(\xi_0)/\pi(\xi_0)$. We also define the transfer $t^i = \bar{x}^i(\xi_0) - e^i(\xi_0) - \bar{y}(\xi_0) \cdot a^i(\xi_0^-) - \bar{q}(\xi_0) \cdot (a^i(\xi_0^-) - \bar{a}^i(\xi_0))$. Since $\sum_{i \in \mathcal{I}} (\bar{x}^i(\xi_0) - e^i(\xi_0)) = \sum_{j \in \mathcal{J}} \bar{y}^j(\xi_0)$ and for all $j \in \mathcal{J}$, $\sum_{i \in \mathcal{I}} a^i(\xi_0^-) = \sum_{i \in \mathcal{I}} \bar{a}^i(\xi_0) = 1$, one easily shows that $\sum_{i \in \mathcal{I}} t^i = 0$.

Since $(\bar{x}, \bar{a}, \bar{y})$ is constrained feasible, from the definition of the transfers (t^i) , one deduces that for all i , (\bar{x}, \bar{a}) belongs to the budget set $\mathcal{B}^i(\bar{q}, \bar{y})$ associated to the initial endowments (\tilde{e}^i) and all inequalities are binding. From Lemma 4.2 (b) and Properties (i) and (iv) of Theorem 3.4, one deduces that (\bar{x}^i, \bar{a}^i) is a maximal element of \mathcal{P}^i in the budget set. Since $(\bar{x}, \bar{a}, \bar{y})$ is constrained feasible, one then deduces that $(\bar{q}, \bar{x}, \bar{a}, \bar{y})$ is a pre-equilibrium of the economy with initial endowments (\tilde{e}^i) . Finally, Property (ii) of Theorem 3.4 implies that Condition (d) of Definition 4.5 is also satisfied, which means that $(\bar{q}, \bar{x}, \bar{a}, \bar{y})$ is a stock market equilibrium of the economy with initial endowments (\tilde{e}^i) . \square

We conclude by a surprising result concerning the Second Theorem of Welfare Economics in a multi-period setting. We will provide a simple standard economy in which it is not possible to decentralize a fixed constrained Pareto optimal allocation.

Consider an economy with 3 periods. The initial node is 0 and it has two successors 1 and 2. Node 1 (resp. 2) has a unique successor 3 (resp. 4). One has two consumers

with the preferences represented by the same utility function $u(x) = \prod_{\xi=0}^4 x(\xi)$. The initial endowments are $e^1 = (2, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2})$ and $e^2 = (2, \frac{3}{4}, \frac{3}{4}, \frac{1}{2}, \frac{1}{2})$. The unique producer has a unique production plan $y = (-2, 1, 1, 1, 1)$ and the initial shares are $a^1(0^-) = 1$ and $a^2(0^-) = 0$. One easily checks that the allocation $\bar{x}^1 = \bar{x}^2 = (1, 1, 1, 1, 1)$, $\bar{a}^1 = (1, \frac{1}{2}, \frac{1}{2})$, $\bar{a}^2 = (0, \frac{1}{2}, \frac{1}{2})$ associated to the stock price $q = (-\frac{1}{2}, -\frac{1}{2})$ is constrained feasible. It is constrained Pareto optimal since $(\bar{x}^1, \bar{x}^2, y)$ is Pareto optimal in the economy with complete markets, that is on the unconstrained attainable set.

This allocation is not a stock market equilibrium allocation. The feasibility constraints at nodes 1 and 2 implies that the stock prices are equal to $-\frac{1}{2}$. Thus, the optimality condition of equations 4.2 in Lemma 4.2 implies that $\pi^1(3) = -\frac{1}{2}\pi^1(1)$. The state prices $(\pi^1(0), \pi^1(1), \pi^1(2), \pi^1(3), \pi^1(4))$ belongs to \mathbb{R}_{++}^5 since the preferences are strictly increasing in each state, and thus, one obtains a contradiction.

For simplicity, the above allocation is actually a first best Pareto optimum but this does not mean that a planner can always implement a first best optimum by using the financial market. This is really an exceptional case.

We interpret this phenomenon as follows. A constrained Pareto optimal allocation can be financed by stock prices that lead to arbitrage opportunities. In contrast with the individual choice of a consumer, a constrained Pareto optimal allocation is chosen by a Social Planner, who takes into account the feasibility constraints. These constraints can avoid the implementation of a Pareto improving trade due to the presence of arbitrage opportunities. For example, the stock price may be low and all consumers would like to buy more stocks, but the feasibility constraints are binding and the Social Planner is thus able to observe that the new allocation is not attainable.

5 Proofs

5.1 Proof of Theorem 3.4

What follows mimics some arguments of the demonstration of Theorem 3.1 of Bonnisseau–Lachiri [2]. Rather than repeating those complicated arguments that deal with heavy notations, we will limit ourselves with presenting some simplifications due to Condition (D), Theorem 2.4 and Corollary 2.5.

The proof takes eight short steps:

1. Define the function F .
2. Specify the set X .
3. Choose an element $x \in X$.
4. Verify that $F(x) \in \text{bdry} F(X)$.
5. Apply Theorem 2.4 to the subset X and the mapping F at the point x .
6. Compute the normal cone $N_X(x)$ to X at x .
7. Compute the subgradient $\partial(\pi \cdot F)$ of $(\pi \cdot F)$ at x .
8. Recover Conditions (i), (ii), (iii) and (iv) of Theorem 3.4.

STEP 1: Let's define $\mathbb{R}^n \equiv (\mathbb{R}^{\mathfrak{D}})^I \times (\mathbb{R}^{\mathfrak{D}})^J \times (\mathbb{R}^{\mathfrak{D}^+})^J \times (\mathbb{R}^{\mathfrak{D}-})^{JI}$, $\mathbb{R}^m \equiv (\mathbb{R}^{\mathfrak{D}-})^J \times \mathbb{R} \times (\mathbb{R}^{\mathfrak{D}^+})^I$ and a generic element of \mathbb{R}^n denoted by $x = ((x^i)_{i \in \mathcal{I}}, (y^j)_{j \in \mathcal{J}}, (q^j)_{j \in \mathcal{J}}, (a_j^i)_{i \in \mathcal{I}, j \in \mathcal{J}})$. Then, consider the function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $x \longmapsto F(x) = ((f_\xi(z))_{\xi \in \mathfrak{D}-}, g_{\xi_0}(x), (h_\xi^i(x))_{i \in \mathcal{I}, \xi \in \mathfrak{D}^+})$ such that: $f_\xi(x) = \sum_{i \in \mathcal{I}} a^i(\xi)$, $\forall \xi \in \mathfrak{D}-$; $g_{\xi_0}(x) = \sum_{i \in \mathcal{I}} x^i(\xi_0) - \sum_{j \in \mathcal{J}} y^j(\xi_0)$; $\forall i \in \mathcal{I}$, $h_\xi^i(x) = x^i(\xi) - y(\xi) \cdot a^i(\xi^-) - q(\xi) \cdot (a^i(\xi^-) - a^i(\xi))$ if $\xi \in \mathfrak{D}^+$ and $h_\xi^i(z) = x^i(\xi) - y(\xi) \cdot a^i(\xi^-)$ if $\xi \in \mathfrak{D}_T$. Note that F is continuously differentiable.

STEP 2: Since, for each $i \in \mathcal{I}$, $\mathcal{P}^i(\bar{x}^i)$ satisfies Condition (D) at \bar{x}^i , there exists a vector $v^i \in \mathbb{R}^{\mathfrak{D}}$ and $\underline{t}^i > 0$ such that for all $t \in]0, \underline{t}^i[$, $tv^i + (\bar{\mathcal{P}}^i(\bar{x}^i) \cap \bar{B}(\bar{x}^i, \underline{t}^i)) \subset \mathcal{P}^i(\bar{x}^i)$. Then, let's take $\underline{t} = \min \{\underline{t}^i \mid i \in \mathcal{I}\}$ and define $v^i \in (\mathbb{R}^{\mathfrak{D}-})^J \times \mathbb{R} \times (\mathbb{R}^{\mathfrak{D}^+})^I$ for every $i \in \mathcal{I}$ by: $v^{i,j}(\xi) = 0$, for all $j \in \mathcal{J}$ and for all $\xi \in \mathfrak{D}-$, $v^i(\xi_0) = v^i(\xi_0)$ and for all $\xi \in \mathfrak{D}^+$, $v^{i,k}(\xi) = 0$ if $k \neq i$ and $v^{i,i}(\xi) = v^i(\xi)$.

Consider the set $X = \prod_{k=1}^4 X^k \subset \mathbb{R}^n$, in which $X^1 = \prod_{i \in \mathcal{I}} (\bar{\mathcal{P}}^i(\bar{x}) \cap \bar{B}(\bar{x}^i, \underline{t}))$, $X^2 = \prod_{j \in \mathcal{J}} Y_j$, $X^3 = (\mathbb{R}^{\mathfrak{D}^+})^J$ and $X^4 = (\mathbb{R}_+^{\mathfrak{D}-})^{JI}$. Note that X is a closed but not convex subset of \mathbb{R}^n because X^1, X^2 are closed subsets, and X^3 and X^4 are closed convex subsets.

STEP 3: Let's consider $(\bar{x}, \bar{a}, \bar{y})$ a constrained Pareto optimal allocation that satisfies all assumptions of Theorem 3.4 and denote $\bar{q} \in \mathbb{R}^{\mathfrak{D}^+}$ the stock price vector that finances $(\bar{x}, \bar{a}, \bar{y})$. This define $x = ((\bar{x}^i)_{i \in \mathcal{I}}, (\bar{y}^j)_{j \in \mathcal{J}}, (\bar{q}^j)_{j \in \mathcal{J}}, (\bar{a}_j^i)_{i \in \mathcal{I}, j \in \mathcal{J}}) \in X$. Note that, if we denote $w = (1_{(\mathfrak{D}-)^J}, \sum_{i=1}^I e^i(\xi_0), (e^i(\xi))_{i \in \mathcal{I}, \xi \in \mathfrak{D}^+})$, then by virtue of Definitions 3.3 and 3.1 $w = F(x) \in F(X)$.

STEP 4: To prove the assertion: $F(x) \in \text{bdry } F(X)$, it is enough to demonstrate that $(w - t \sum_{i=1}^I v^i) \notin F(X)$ for all $t \in]0, \underline{t}[$. We proceed by contraposition. If there exists $t \in]0, \underline{t}[$ and $x = ((x^i)_{i \in \mathcal{I}}, (y^j)_{j \in \mathcal{J}}, (q^j)_{j \in \mathcal{J}}, (a_j^i)_{i \in \mathcal{I}, j \in \mathcal{J}}) \in X = \sum_{k=1}^4 X^k$ such that $F(x) = w - t \sum_{i \in \mathcal{I}} v^i$, meaning that: $\sum_{i \in \mathcal{I}} a_j^i(\xi) = 1$, $\forall \xi \in \mathfrak{D}-$, $\forall j \in \mathcal{J}$; $\sum_{i \in \mathcal{I}} e^i(\xi_0) - t \sum_{i \in \mathcal{I}} v^i(\xi_0) = \sum_{i \in \mathcal{I}} x^i(\xi_0) - \sum_{j \in \mathcal{J}} y^j(\xi_0)$; $e^i(\xi) - tv^i(\xi) = x^i(\xi) - y(\xi) \cdot a^i(\xi^-) - q(\xi) \cdot (a^i(\xi^-) - a^i(\xi))$, $\forall \xi \in \mathfrak{D}^+$, $\forall i \in \mathcal{I}$; and finally, $e^i(\xi) - tv^i(\xi) = x^i(\xi) - y(\xi) \cdot a^i(\xi^-)$, $\forall \xi \in \mathfrak{D}_T$, $\forall i \in \mathcal{I}$.

Now, let's denote $\underline{x}^i = x^i + tv^i$ for every $i \in \mathcal{I}$, $\underline{y}^j = y^j$ and $\underline{q}^j = q^j$ for every $j \in \mathcal{J}$, and, $\underline{a}_j^i = a_j^i$ for every $(i, j) \in \mathcal{I} \times \mathcal{J}$. An immediate implication is that (x, a, y) is constrained feasible allocation. Furthermore, our choice of \underline{t} implies, with the virtue of Condition (D), that $\underline{x}^i \in \mathcal{P}^i(\bar{x})$ for all $i \in \mathcal{I}$. This contradict the constrained Pareto optimality of the original allocation $(\bar{x}, \bar{a}, \bar{y})$.

STEP 5: We are now ready to apply the extremal principle. Assertions in STEPS 1, STEPS 2, STEPS 3 and STEP 4 imply, by virtue of Theorem 2.4, that there exists $\pi = ((v^j(\xi))_{j \in \mathcal{J}}^{\xi \in \mathfrak{D}-}, \pi(\xi_0), (\pi^i(\xi))_{i \in \mathcal{I}}^{\xi \in \mathfrak{D}^+}) \in \mathbb{R}^m \setminus \{0\}$ such that:

$$0 \in \partial(\pi \cdot F)(x) + N_X(x), \quad (5.1)$$

where F , X and x are respectively defined in STEPS 1, STEPS 2 and STEPS 3.

STEP 6: From Step 2, we recall that $X^1 = \prod_{i \in \mathcal{I}} (\bar{\mathcal{P}}^i(\bar{x}) \cap \bar{B}(\bar{x}^i, \underline{t}))$, $X^2 = \prod_{j \in \mathcal{J}} Y_j$, $X^3 = (\mathbb{R}^{\mathfrak{D}^+})^J$, $X^4 = (\mathbb{R}_+^{\mathfrak{D}-})^{JI}$; $x^1 = (\bar{x}^i)_{i \in \mathcal{I}}$, $x^2 = (\bar{y}^j)_{j \in \mathcal{J}}$, $x^3 = (\bar{q}^j)_{j \in \mathcal{J}}$, $x^4 = (\bar{a}_j^i)_{i \in \mathcal{I}, j \in \mathcal{J}}$; and $X = \prod_{k=1}^4 X^k$. Thus, by virtue of Assertions (b) in Proposition 2.2, $N_X(x) = \prod_{k=1}^4 N_{X^k}(x^k)$. By virtue of Assertions (b) and (l) in Proposition 2.2 and a simple algebra, we get $N_{X^3}(x^3) = \prod_{j \in \mathcal{J}} N_{\mathbb{R}^{\mathfrak{D}^+}}(\bar{q}^j)$ with $N_{\mathbb{R}^{\mathfrak{D}^+}}(\bar{q}^j) = \{0_{\mathfrak{D}-}\}$ and $N_{X^4}(x^4) = \prod_{j \in \mathcal{J}}$

$(\prod_{i \in \mathcal{J}} N_{\mathbb{R}_+^{\mathfrak{D}-}}(\bar{a}_j^i))$ with $N_{\mathbb{R}_+^{\mathfrak{D}-}}(\bar{a}_j^i) = \{\nu_j^i = (\nu_j^i(\xi))_{\xi \in \mathfrak{D}-} \in -\mathbb{R}_+^{\mathfrak{D}-} \mid \nu_j^i(\xi) \bar{a}_j^i(\xi) = 0, \forall \xi \in \mathfrak{D}-\}$. Since for every $i \in \mathcal{J}$, $\bar{x}^i \in \text{int } \bar{B}(\bar{x}^i, \eta)$, by Assertions (a) and (b) in Proposition 2.2, $N_{X^1}(x^1) = \prod_{i \in \mathcal{J}} N_{\mathcal{P}^i(\bar{x})}(\bar{x}^i)$.

STEP 7: STEP 1 implies, by virtue of Assertions (g) and (j) in Proposition 2.2, that $\partial(\pi \cdot F)(x) = \{DF^t(x)(\pi)\}$. Let's take $\gamma = (\alpha, \zeta, \kappa, \theta) \in \mathbb{R}^m$. Due to STEP 1, $DF(x)(\gamma)$ is an $(I+2)$ blocks of vectors, whose entries are respectively: $(Df_\xi(x)(\gamma))_{\xi \in \mathfrak{D}-}$, $(Dg_{\xi_0}(x)(\gamma))$, $(Dh_\xi^1(x)(\gamma))_{\xi \in \mathfrak{D}^+}$, \dots , $(Dh_\xi^I(x)(\gamma))_{\xi \in \mathfrak{D}^+}$. Where, $Df_\xi(x)(\gamma) = \sum_{i \in \mathcal{J}} \theta^i(\xi)$, $\forall \xi \in \mathfrak{D}-$; $Dg_{\xi_0}(x)(\gamma) = \sum_{i \in \mathcal{J}} \alpha^i(\xi_0) - \sum_{j \in \mathcal{J}} \zeta^j(\xi_0)$; and, for every $i \in \mathcal{J}$, $Dh_\xi^i(x)(\gamma) = \alpha^i(\xi) - (\kappa(\xi) + \zeta(\xi)) \cdot \bar{a}^i(\xi^-) - (\bar{q}(\xi) + \bar{y}(\xi)) \cdot \theta^i(\xi^-) + \bar{q}(\xi) \cdot \theta^i(\xi) + \kappa(\xi) \cdot \bar{a}^i(\xi)$ if $\xi \in \mathfrak{D}_+^+$ and $Dh_\xi^i(x)(\gamma) = \alpha^i(\xi) - \zeta(\xi) \cdot \bar{a}^i(\xi^-) - \bar{y}(\xi) \cdot \theta^i(\xi^-)$ if $\xi \in \mathfrak{D}_T$.

Using simple algebra and the formula: $\pi \cdot DF(x)(\gamma) = \gamma \cdot DF^t(x)(\pi)$, we obtain

$$DF^t(x)(\pi) = \begin{pmatrix} A & B & C & D \end{pmatrix},$$

where,

$$A = \left(\pi(\xi_0), ((\pi^i(\xi))_{\xi \in \mathfrak{D}^+})_{i \in \mathcal{J}} \right); B = - \left(\pi(\xi_0), \left(\sum_{i \in \mathcal{J}} \pi^i(\xi) \bar{a}_j^i(\xi^-) \right)_{\xi \in \mathfrak{D}^+} \right)_{j \in \mathcal{J}};$$

$$C = \left(\sum_{i \in \mathcal{J}} \pi^i(\xi) (\bar{a}_j^i(\xi) - \bar{a}_j^i(\xi^-)) \right)_{j \in \mathcal{J}}^{\xi \in \mathfrak{D}_+^+};$$

and, $D = (D(\xi_0), (D(\xi))_{\xi \in \mathfrak{D}_+^+ \setminus \mathfrak{D}_{T-1}}, (D(\xi))_{\xi \in \{\mathfrak{D}_{T-1}\}})$ with $D(\xi_0) = (v^j(\xi_0) - \sum_{\eta \in \xi_0^+} \pi^i(\eta)(\bar{q}^j(\eta) + \bar{y}^j(\eta)))_{j \in \mathcal{J}}^{i \in \mathcal{J}}$ and $D(\xi) = (v^j(\xi) + \pi^i(\xi) \bar{q}^j(\xi) - \sum_{\eta \in \xi^+} \pi^i(\eta)(\bar{q}^j(\eta) + \bar{y}^j(\eta)))_{j \in \mathcal{J}}^{i \in \mathcal{J}}$ if $\xi \in \mathfrak{D}_+^+ \setminus \mathfrak{D}_{T-1}$, $D(\xi) = (v^j(\xi) + \pi^i(\xi) \bar{q}^j(\xi) - \sum_{\eta \in \xi^+} \pi^i(\eta) \bar{y}^j(\eta))_{j \in \mathcal{J}}^{i \in \mathcal{J}}$ if $\xi \in \mathfrak{D}_{T-1}$.

STEP 8: Conditions from (i) to (iv) of Theorem 3.4, follow from STEP 5, STEP 6 and STEP 7 together with the convention $\bar{q}(\xi) = 0$ for every $\xi \in \mathfrak{D}_T$. Note that the equality in condition (iv) is obtained since each term of the inner product of the vectors is null by their non negativity. We close totally this proof by claiming that $(\pi(\xi_0), (\pi^i(\xi))) \neq 0$. Indeed, if not, then $\forall i \in \mathcal{J}, \forall j \in \mathcal{J}, \forall \xi \in \mathfrak{D}_-, \bar{a}_j^i(\xi) v^j(\xi) = 0$. Since $\sum_{i \in \mathcal{J}} \bar{a}_j^i(\xi) = 1$, there exists $i_0 \in \mathcal{J}$ such that $\bar{a}_j^{i_0}(\xi) \neq 0$, which implies that $v^j(\xi) = 0$. Thus, $\pi = 0$, which contradicts the fact that $\pi \neq 0$. Consequently, one gets the conclusion of Theorem 3.4.

5.2 Proof of Lemma 4.2

• We start by demonstrating the first assertion (a) of Lemma 4.2: Recall that the budget set is defined by formula (3.1). Let's take a maximal element (\bar{x}^i, \bar{a}^i) of \mathcal{P}^i in the budget set $\mathcal{B}^i(q, y)$. Then, by definition $(\bar{x}^i, \bar{a}^i) \in \tilde{\mathcal{P}}^i(\bar{x}) = (\mathcal{P}^i(\bar{x}) \times (\mathbb{R}_+^{\mathfrak{D}-})^J)$ and $(\bar{x}^i, \bar{a}^i) \in \mathcal{B}^i(q, y)$.

Take $X^1 = (\mathcal{P}(\bar{x}) \cap \bar{B}(\bar{x}^i, \underline{t})) \times (\mathbb{R}_+^{\mathfrak{D}-})^J$ and $X^2 = -\mathcal{B}^i(q, y)$ and $x = (x^1, x^2)$ with $x^1 = -x^2 = (\bar{x}^i, \bar{a}^i)$. We claim that $0 = x^1 + x^2 \in \text{bdry}(X^1 + X^2)$. Note that $\mathcal{P}^i(\bar{x})$ satisfies Condition (D) at \bar{x}^i . Then, there exists $v^i \in \mathbb{R}^{\mathfrak{D}}$ and $\underline{t} > 0$ such that for all $t \in]0, \underline{t}[$, $tv^i + (\mathcal{P}^i(\bar{x}) \cap \bar{B}(\bar{x}^i, \underline{t})) \subset \mathcal{P}^i(\bar{x})$. Let's define \mathbf{v}^i a vector of the form $(v^i, 0) \in \mathbb{R}^{\mathfrak{D}} \times (\mathbb{R}^{\mathfrak{D}-})^J$. Therefore, what is required to end the proof of our last claim is to demonstrate that, $t\mathbf{v}^i \notin (X^1 + X^2)$ for all $t \in]0, \underline{t}[$. By contraposition, there exist $t \in]0, \underline{t}[$, $(\tilde{x}^i, \tilde{a}^i) \in X^1$ and $-(x^i, a^i) \in X^2$ such that $x = \tilde{x}^i + tv^i$ and $a^i = \tilde{a}^i$. So $x^i \in \mathcal{P}^i(\bar{x})$, since $\mathcal{P}^i(\bar{x})$ satisfies

Condition (D) at \bar{x}^i . Thus, $(x^i, a^i) \in \tilde{\mathcal{P}}^i(\bar{x})$. But, this contradicts the fact that (\bar{x}^i, \bar{a}^i) is a maximal element of \mathcal{P}^i in the budget set, because $(x^i, a^i) \in \mathcal{B}^i(q, y)$. This ends the claim.

Now, assertion (d) of Corollary 2.5 can apply. Thus, for $x^1 \in X^1 = (\bar{\mathcal{P}}^i(\bar{x}) \cap \bar{B}(\bar{x}^i, \underline{t})) \times (\mathbb{R}_+^{\mathcal{D}-})^J$, $x^2 \in X^2 = \mathcal{B}^i(q, y)$ and $x^1 = -x^2 = (\bar{x}^i, \bar{a}^i)$ there exists a vector $(\alpha^i, \varpi^i) \in (\mathbb{R}^{\mathcal{D}} \times (\mathbb{R}^{\mathcal{D}-})^J) \setminus \{0\}$ such that

$$(\alpha^i, \varpi^i) \in N_{X^1}(x^1) \text{ and } (\alpha^i, \varpi^i) \in N_{X^2}(x^2).$$

On the one hand, by virtue of (a), (b) of Proposition 2.1, the left side of the the previous formula can be written as $\alpha^i \in N_{\bar{\mathcal{P}}^i(\bar{x})}(\bar{x}^i)$ and $\varpi^i \in N_{(\mathbb{R}_+^{\mathcal{D}-})^J}(\bar{a}^i)$. On the other hand, since $\mathcal{B}^i(q, y)$ is defined through a system of linear inequalities, there exists $\pi^i \in \mathbb{R}_+^{\mathcal{D}}$ such that $\alpha^i = -\pi^i$ and for all $j \in \mathcal{J}$, for all $\xi \in \mathcal{D}_-$, $\varpi_j^i(\xi) = -\pi^i(\xi)q^j(\xi) + \sum_{\eta \in \xi^+} \pi^i(\eta)(q^j(\eta) + y^j(\eta))$. Note that, $\pi^i \neq 0$ otherwise $(\alpha^i, \varpi^i) = 0$. So accordingly $-\pi^i \in N_{\bar{\mathcal{P}}^i(\bar{x})}(\bar{x}^i)$. Then, by the computation of $N_{(\mathbb{R}_+^{\mathcal{D}-})^J}(\bar{a}^i)$, we obtain that, conditions (4.2a) and (4.2b). This closes the proof of assertion (a).

• To demonstrate assertion (b) of Lemma 4.2 we proceed by contraposition. Let's assume that (\bar{x}^i, \bar{a}^i) is not a maximal element of $\mathcal{P}^i(\bar{x})$ in the budget set $\mathcal{B}^i(q, y)$; therefore, there exists $(x^i, a^i) \in \mathcal{B}^i(q, y)$ such that $x^i \in \mathcal{P}^i(\bar{x})$.

Now, we claim that $\pi^i \cdot x^i > \pi^i \cdot \bar{x}^i$. This follows immediately from Equation (4.1) by virtue of the convexity and the openness of $\mathcal{P}^i(\bar{x})$, and $\pi^i \in \mathbb{R}_+^{\mathcal{D}} \setminus \{0\}$. Note that, we totally close this proof by demonstrating that $\pi^i \cdot x^i \leq \pi^i \cdot \bar{x}^i$, since it contradicts the first claim. To prove that we proceed as follows. Recall from above that $(x^i, a^i) \in \mathcal{B}^i(q, y)$, which means that $x^i(\xi) - e^i(\xi) \leq y(\xi) \cdot a^i(\xi^-) + q(\xi) \cdot (a^i(\xi^-) - a^i(\xi))$ for all ξ . Multiplying this inequality by π^i and after rearranging it, we obtain

$$\pi^i \cdot x^i \leq \pi^i \cdot e^i + \sum_{\xi \in \mathcal{D}_-} a^i(\xi) \cdot \left(\sum_{\eta \in \xi^+} \pi^i(\eta)(q^j(\eta) + y^j(\eta)) - \pi^i(\xi)q(\xi) \right).$$

Considering Equation (4.2a) and $a_j^i(\xi) \geq 0$, we obtain that the second term of the right side of the last inequality is negative too. Then, $\pi^i \cdot x^i \leq \pi^i \cdot e^i$. Similar computation for \bar{x}^i using Equation (4.2b) and the fact that all inequalities of (3.1) are binding at (\bar{x}^i, \bar{a}^i) shows that $\pi^i \cdot \bar{x}^i = \pi^i \cdot e^i$. Finally, we get $\pi^i \cdot x^i \leq \pi^i \cdot \bar{x}^i$.

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