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A GOAL INTERVAL PROGRAMMING MODEL AND ITS APPLICATION TO PORTFOLIO SELECTION

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Abstract: The aim of this paper is to deal with a goal programming problem with interval coefficients. Based on two order relations between intervals, the noninferior solution to this problem is defined. The goal interval programming is converted into a linear programming problem with a objective function composed of the linear combinations of positive and negative deviation variables. Considering the uncertain returns of assets in capital markets, a goal interval programming model for portfolio selection problem is presented with given returns and risks. Then, an approach is proposed to solve the above model and at last a numerical example is provided.

Key words: *goal programming, interval number, order relation, possible deviation, portfolio selection, absolute deviation*

Mathematics Subject Classification: *90C29, 65K05*

1 Introduction

In classical mathematical programming, some coefficients of objective functions and constraints are determined as crisp values. However, due to lack of abundance of information it may be difficult to determine the crisp coefficients. In such cases, interval numbers, stochastic variables or fuzzy variables are proper choice to represent the imprecise information. Over the last two decades, interval programming grounded on interval analysis has been developed as a useful and simple method to deal with uncertainty.

Early research on mathematical programming with interval coefficients was Charnes and Granot [6], Bitran [3], Ben-Israel and Robers [2]. Since then, some authors studied linear programming with interval coefficients, such as [2, 11, 14, 19, 25, 27]; some authors studied multiple objective linear programming with interval coefficients, such as [3, 4, 12, 26] etc. Moreover, interval programming was applied in a few practical fields, such as municipal waste management and planning [7] and portfolio selection [14].

Goal programming (GP), proposed by Charnes and Cooper [5] in 1961, has been, and still is, the most widely used multiple objective decision making technique [22]. It is necessary for decision maker (DM) to specify aspiration levels for objective functions. GP algorithms attempt to achieve as many goals as possible by minimizing deviation from aspiration levels, depending on their relative weights. Though this approach is an apt decision aid, and has been extensively used in solving decision making problems involving multiple conflicting goals, such as portfolio selection [15, 21], resource allocation, water resources planning and management [16] etc, it also has a major limitation. The aspiration level and/or priority factors (and occasionally the weights to be assigned to the goals) are imprecise in nature for the DM, so it is difficult to specify crisp aspiration level and weights. Moreover, the input data, e. g., available resources and/or technological coefficients, may not be precisely determined because of incomplete or non-obtainable information. As stated above, it is appropriate to extend the idea of interval programming into goal programming. Goal interval programming was proposed by Ignizio [8]. Inuiguchi and Kume studied goal programming problems with interval coefficients and target intervals in [9]. However, they didn't consider the above model from the point of view of order relations between two interval numbers. It is well-known that the order relations between two interval numbers is not a complete ordering, but a partial ordering. So, in this paper, we will introduce some preference relations between two interval numbers, and propose a goal programming problems with interval coefficients in objective functions and constraints (also called as goal interval programming). Moreover, we propose a goal interval programming model to portfolio selection by using absolute deviation of return of portfolio to measure risk which is different from the variance measure in Markowitz model.

In the following example, we show that it is natural to describe uncertain expected return and tolerated risk as intervals in capital market.

Example 1. *Suppose an investor decides to allocate his wealth among two risky securities and a risk-less asset. However, not only the vagueness or tolerance of the investor but also the imprecision or uncertainty in capital market make it difficult to give determined values of expected returns and tolerated risk of securities. The expected returns of securities are given by intervals and listed in Table 1. Denote w_j as the absolute deviation of return of security j $(j = 1, 2)$. We assume that w_j can be obtained from the expected return R_j. Now, in order to make the expected return and tolerated risk of the portfolio close to the given interval R and w how does the investor allocate his wealth between securities?*

This problem can be written as follows,

$$\left\{\begin{array}{rl} \text{Goals}: & R_1x_1 + R_2x_2 + R_3x_3 = R, \\ & w_1x_1 + w_2x_2 + w_3x_3 = w, \\ \text{s.t.} & x_1 + x_2 + x_3 = 1, \\ & 0 \leqslant x_j \leqslant u_j, \quad j = 1, 2, 3, \end{array}\right.$$

where R and w are the expected return of portfolio $x = (x_1, x_2, x_3)$ and the absolute deviation of return of portfolio x, respectively, x_j is the investment proportion of the risky security j, x_3 is the investment proportion of the risk-less asset, and u_1, u_2, u_3 are the upper bounds of the investment proportions, respectively.

Table 1: The Interval Returns and Risks of The Three Stocks (%)

	Security 1	Security 2	Riskless Asset	Portfolio
R_i	$R_1 = [3.1, 3.3]$	$R_2 = [4.3, 4.7]$	$R_3 = 0.4$	R=[2.2,2.7]
w_i	$w_1 = [1.8, 2.0]$	$w_2 = [3.1, 3.5]$	$w_3 = 0$	$w = [1.5, 1.8]$

We organize this paper as follows. Some basic concepts of interval numbers are reviewed, and two order relations between interval numbers are proposed in Section 2. A goal interval programming (GIP) model is proposed and the properties are considered in Section 3. In Section 4, the absolute deviation of portfolio return is adopted to measure risk and a goal interval programming model for portfolio selection problem is presented. Simple numerical example is provided in Section 5 and some concluding remarks are given in Section 6.

2 Basic Operations and Two Order Relations of Interval Numbers

The set of all real numbers is denoted by R. An ordered pair in a bracket defines an interval as

$$a = [a^{\mathrm{L}}, a^{\mathrm{U}}] = \{x \in \mathbf{R} | a^{\mathrm{L}} \leqslant x \leqslant a^{\mathrm{U}}\},$$

where a^{L} is the left limit and a^{U} is the right limit of an interval a. The center of an interval a and the width of an interval a may be calculated by

$$m[a] = (a^{\mathrm{L}} + a^{\mathrm{U}})/2 \quad \text{and} \quad w[a] = a^{\mathrm{U}} - a^{\mathrm{L}}.$$

For a detailed discussion of interval arithmetic, refer to [20, 1].

Definition 1. *[1] Let $* \in \{+, -, \times, \div\}$ be a binary operation on $\mathbf{R}$. If a and b are two closed intervals, then*

$$a(*)b = \{z | z = x * y : x \in a, y \in b\} \tag{2.1}$$

or

$$a(*)b = \{z | z = x * y : a^{\mathrm{L}} \leqslant x \leqslant a^{\mathrm{U}}, b^{\mathrm{L}} \leqslant y \leqslant b^{\mathrm{U}}\} \tag{2.2}$$

defines a binary operation on the set of all the closed intervals. In the case of division, it is assumed that $0 \notin b$.

From Definition 1, we have

$$a(+)b = [a^{\mathrm{L}}, a^{\mathrm{U}}](+)[b^{\mathrm{L}}, b^{\mathrm{U}}] = [a^{\mathrm{L}} + b^{\mathrm{L}}, a^{\mathrm{U}} + b^{\mathrm{U}}], \tag{2.3}$$
$$a(-)b = [a^{\mathrm{L}}, a^{\mathrm{U}}](-)[b^{\mathrm{L}}, b^{\mathrm{U}}] = [a^{\mathrm{L}} - b^{\mathrm{U}}, a^{\mathrm{U}} - b^{\mathrm{L}}], \tag{2.4}$$

and

$$ka = k[a^{\mathrm{L}}, a^{\mathrm{U}}] = \begin{cases} [ka^{\mathrm{L}}, ka^{\mathrm{U}}], & \text{for } k \geqslant 0, \\ [ka^{\mathrm{U}}, ka^{\mathrm{L}}], & \text{for } k < 0, \end{cases} \tag{2.5}$$

where k is a real number.

$a(*)b$ estimates the possible region of the value of $x * y$ restricted by $x \in a$ and $y \in b$. In this sense, an operator $(*)$ is called 'a possible extended operator of $*$'. $a(+)b$ and $a(-)b$ are called the possibly extended addition and the possibly extended subtraction (for simplicity, a possible addition and a possible subtraction), respectively.

The following properties hold for $a(+)b$ and $a(-)b$:

$$m[a(+)b] = m[a] + m[b], \quad m[a(-)b] = m[a] - m[b],$$
$$w[a(+)b] = w[a(-)b] = w[a] + w[b].$$

The possibly extended maximum of intervals a and b is derived from

$$a(\vee)b = [a^{\mathrm{L}} \vee b^{\mathrm{L}}, a^{\mathrm{U}} \vee b^{\mathrm{U}}],$$

where $a^{\mathrm{L}} \vee b^{\mathrm{L}} = \max\{a^{\mathrm{L}}, b^{\mathrm{L}}\}, a^{\mathrm{U}} \vee b^{\mathrm{U}} = \max\{a^{\mathrm{U}}, b^{\mathrm{U}}\}$.

The absolute value of a is defined as [20]

$$|a| = \begin{cases} [a^{\mathrm{L}}, a^{\mathrm{U}}], & \mathrm{a}^{\mathrm{L}} \geqslant 0, \\ [0, (-a^{\mathrm{L}}) \vee a^{\mathrm{U}}], & \mathrm{a}^{\mathrm{L}} < 0 < \mathrm{a}^{\mathrm{U}}, \\ [a^{\mathrm{U}}, a^{\mathrm{L}}], & \mathrm{a}^{\mathrm{U}} \leqslant 0. \end{cases}$$

Let us consider the following interval equation with a possible addition

$$a(+)x = b, \tag{2.6}$$

where $a = [a^{\mathrm{L}}, a^{\mathrm{U}}]$ and $b = [b^{\mathrm{L}}, b^{\mathrm{U}}]$. To solve equation (2.6), an interval $x = [x^{\mathrm{L}}, x^{\mathrm{U}}]$ such that the possible sum of a and x, $a(+)x$, equals to b should be determined. Because $w[a(+)x] = w[a] + w[x] = w[b]$ and $w[x] \geqslant 0$, the necessary condition for the existences of interval x is that $w[a] \leqslant w[b]$. When the condition that $w[a] \leqslant w[b]$ is fulfilled, solving the following linear equations

$$a^{\mathrm{L}} + x^{\mathrm{L}} = b^{\mathrm{L}}, \quad \text{and} \quad a^{\mathrm{U}} + x^{\mathrm{U}} = b^{\mathrm{U}},$$

the solution of (2.6) is obtained as $x = [b^{\mathrm{L}} - a^{\mathrm{L}}, b^{\mathrm{U}} - a^{\mathrm{U}}]$.

Definition 2. *Given two intervals $a = [a^{\mathrm{L}}, a^{\mathrm{U}}]$ and $b = [b^{\mathrm{L}}, b^{\mathrm{U}}]$ such that $w[a] \leqslant w[b]$, a operation difference of a from b is defined as follows,*

$$b) - (a = [b^{\mathrm{L}} - a^{\mathrm{L}}, b^{\mathrm{U}} - a^{\mathrm{U}}]. \tag{2.7}$$

From Definition 2, we know that the difference of a from b, $b) - (a$, is different from the possible subtraction $b(-)a$. Because $b(-)a = [b^{\mathrm{L}} - a^{\mathrm{U}}, b^{\mathrm{U}} - a^{\mathrm{L}}]$ and $b) - (a = [b^{\mathrm{L}} - a^{\mathrm{L}}, b^{\mathrm{U}} - a^{\mathrm{U}}]$, the following property is viable:

$$b) - (a \subseteq b(-)a.$$

Considering a linear constraint with interval coefficients

$$(+)_{j=1}^{n} a_j x_j \leqslant b,$$

where $a_j = [a_j^{\mathrm{L}}, a_j^{\mathrm{U}}]$, $b = [b^{\mathrm{L}}, b^{\mathrm{U}}]$ and $x_j \geqslant 0$, we have

$$(+)_{j=1}^{n} [a_j^{\mathrm{L}}, a_j^{\mathrm{U}}] x_j = \Big[\sum_{j=1}^{n} a_j^{\mathrm{L}} x_j, \sum_{j=1}^{n} a_j^{\mathrm{U}} x_j \Big] \leqslant [b^{\mathrm{L}}, b^{\mathrm{U}}]. \tag{2.8}$$

How to explain the meaning of (2.8)? Moore [20], Ishibuchi and Tanaka [12], Sengupta and Pal [23] presented some order relations between interval numbers. Denote intervals a and b as uncertain resources (profits) of two alternatives, respectively. The following definition of is proposed to compare interval numbers.

Definition 3. *Let $a = [a^{\mathrm{L}}, a^{\mathrm{U}}]$ and $b = [b^{\mathrm{L}}, b^{\mathrm{U}}]$ be two intervals. We define two order relations $\preceq_1$ and $\preceq_2$ between intervals a and b as*

$$\begin{array}{lllll}
(1) & a \preceq_1 b & \text{iff} & a^{\mathrm{L}} \leqslant b^{\mathrm{L}} \quad \text{and} \quad a^{\mathrm{U}} \leqslant b^{\mathrm{U}}; \\
 & a \prec_1 b & \text{iff} & a \preceq_1 b \quad \text{and} \quad a \neq b. \\
(2) & a \preceq_2 b & \text{iff} & a^{\mathrm{U}} \leqslant b^{\mathrm{U}} \quad \text{and} \quad m[a] \leqslant m[b]; \\
 & a \prec_2 b & \text{iff} & a \preceq_2 b \quad \text{and} \quad a \neq b.
\end{array}$$

The order relation $\preceq_1$ represents the decision-maker's preference for alternative b (a) with higher (lower) minimum profit (resources) by $a^{\mathrm{L}} \leqslant b^{\mathrm{L}}$ and higher (lower) maximal profit (resources) by $a^{\mathrm{U}} \leqslant b^{\mathrm{U}}$. The order relation $\preceq_2$ represents the decision-maker's preference for alternative a with lower average resources by $m[a] \leqslant m[b]$ and lower maximal resources by $a^{\mathrm{U}} \leqslant b^{\mathrm{U}}$.

Based on Definition 3, (2.8) can be explained into two normal linear constraints according to the DM's preference.

3 Problem Formulation and Solution Method

The classical multiple objective linear programming is written as

$$\begin{cases} \max & \sum_{j=1}^{n} c_{ij}x_j, \quad i=1,2,\cdots,m, \\ \text{s.t.} & \sum_{j=1}^{n} a_{kj}x_j \leqslant b_k, \quad k=1,2,\cdots,l, \\ & x_j \geqslant 0, \quad j=1,2,\cdots,n. \end{cases} \tag{3.9}$$

Assume the coefficients in (3.9) as interval numbers, e.g., $c_{ij}=[c_{ij}^{\mathrm{L}},c_{ij}^{\mathrm{U}}]$, $a_{kj}=[a_{kj}^{\mathrm{L}},a_{kj}^{\mathrm{U}}]$ and $b_k=[b_k^{\mathrm{L}},b_k^{\mathrm{U}}]$. In this paper, we use the techniques of goal programming to solve multiple objective programming with interval coefficients.

Consider the following goal programming problem with interval coefficients and target intervals (also called as goal interval programming):

$$(\text{GIP})\begin{cases} \text{Goals}: & (+)_{j=1}^{n}[c_{ij}^{\mathrm{L}},c_{ij}^{\mathrm{U}}]x_j=[t_i^{\mathrm{L}},t_i^{\mathrm{U}}], & i=1,2,\ldots,m, \\ \text{s.t.} & (+)_{j=1}^{n}[a_{kj}^{\mathrm{L}},a_{kj}^{\mathrm{U}}]x_j \preceq_2 [b_k^{\mathrm{L}},b_k^{\mathrm{U}}], & k=1,2,\ldots,l, \\ & x_j \geqslant 0, & j=1,2,\ldots,n, \end{cases} \tag{3.10}$$

where $[c_{ij}^{\mathrm{L}},c_{ij}^{\mathrm{U}}]$ denotes the uncertain return, $[t_i^{\mathrm{L}},t_i^{\mathrm{U}}]$ denotes the aspiration levels of returns, $[a_{kj}^{\mathrm{L}},a_{kj}^{\mathrm{U}}]$ denotes the uncertain cost and $[b_k^{\mathrm{L}},b_k^{\mathrm{U}}]$ denotes the uncertain total resource. Based on Definition 2, $\sum_{j=1}^{n}[a_{kj}^{\mathrm{L}},a_{kj}^{\mathrm{U}}]x_j \preceq_2 [b_k^{\mathrm{L}},b_k^{\mathrm{U}}]$ can be converted into usual constraints, $k=1,2,\ldots,l$. Actually, they denote that a feasible solution to (GIP) is a solution such that the average costs and the costs in the worst case scenario are less than or equal to the average value and the maximal possible value of the uncertain resources, respectively. Alike the traditional goal programming model, there exist deviation interval of $(+)_{j=1}^{n}c_{ij}x_j=(+)_{j=1}^{n}[c_{ij}^{\mathrm{L}},c_{ij}^{\mathrm{U}}]x_j$ from $T_i=[t_i^{\mathrm{L}},t_i^{\mathrm{U}}]$. Thus, in order to solve (GIP), we should find a solution x which minimizes the deviation interval under some uncertain constraints.

According to Definition 2, if $w[(+)_{j=1}^{n}c_{ij}x_j] \leqslant w[T_i]$, the difference of $(+)c_{ij}x_j$ from T_i can be represented as

$$T_i)-((+)_{j=1}^{n}c_{ij}x_j=[t_i^{\mathrm{L}},t_i^{\mathrm{U}}])-([\sum_{j=1}^{n}c_{ij}^{\mathrm{L}}x_j,\sum_{j=1}^{n}c_{ij}^{\mathrm{U}}x_j]=\left[t_i^{\mathrm{L}}-\sum_{j=1}^{n}c_{ij}^{\mathrm{L}}x_j,t_i^{\mathrm{U}}-\sum_{j=1}^{n}c_{ij}^{\mathrm{U}}x_j\right].$$

Using deviational variables $d_i^{\mathrm{L}-}$, $d_i^{\mathrm{L}+}$, $d_i^{\mathrm{U}-}$ and $d_i^{\mathrm{U}+}$ such that

$$\sum_{j=1}^{n} c_{ij}^{\mathrm{L}}x_j+d_i^{\mathrm{L}-}-d_i^{\mathrm{L}+}=t_i^{\mathrm{L}}, \quad i=1,2,\ldots,m, \tag{3.11}$$

$$\sum_{j=1}^{n} c_{ij}^{\mathrm{U}}x_j+d_i^{\mathrm{U}-}-d_i^{\mathrm{U}+}=t_i^{\mathrm{U}}, \quad i=1,2,\ldots,m, \tag{3.12}$$

$$d_i^{\mathrm{L}-}d_i^{\mathrm{L}+}=0, \quad d_i^{\mathrm{U}-}d_i^{\mathrm{U}+}=0, \quad i=1,2,\ldots,m, \tag{3.13}$$

we have

$$T_i)-((+)_{j=1}^{n}c_{ij}x_j=[d_i^{\mathrm{L}-}-d_i^{\mathrm{L}+},d_i^{\mathrm{U}-}-d_i^{\mathrm{U}+}], \quad i=1,2,\ldots,m. \tag{3.14}$$

The deviation interval $D_i(x)$ of $(+)_{j=1}^{n}c_{ij}x_j$ from T_i can be represented as

$$D_i(x)=[d_i^{\mathrm{L}}(x),d_i^{\mathrm{U}}(x)]=|T_i)-((+)_{j=1}^{n}c_{ij}x_j|=\left|\left[t_i^{\mathrm{L}}-\sum_{j=1}^{n}c_{ij}^{\mathrm{L}}x_j,t_i^{\mathrm{U}}-\sum_{j=1}^{n}c_{ij}^{\mathrm{U}}x_j\right]\right|. \tag{3.15}$$

Proposition 1. *Suppose $w[(+)_{j=1}^{n}c_{ij}x_j] \leqslant w[T_i] (i=1,2,\ldots,m)$. Then*

$$D_i=|T_i)-((+)_{j=1}^{n}c_{ij}x_j|=|[d_i^{\mathrm{L}-}-d_i^{\mathrm{L}+},d_i^{\mathrm{U}-}-d_i^{\mathrm{U}+}]|=[d_i^{\mathrm{L}-}+d_i^{\mathrm{U}+},d_i^{\mathrm{L}+}\vee d_i^{\mathrm{U}-}], \tag{3.16}$$

where $d_i^{\mathrm{L}-}$, $d_i^{\mathrm{L}+}$, $d_i^{\mathrm{U}-}$ and $d_i^{\mathrm{U}+}$ satisfy (3.11), (3.12) and (3.13).

Proof. From (3.13) and (3.14), the following three cases are possible:

(1) If $d_i^{\mathrm{L}-} = 0$ and $d_i^{\mathrm{U}-} = 0$, then $D_i = [d_i^{\mathrm{U}+}, d_i^{\mathrm{L}+}]$.

(2) If $d_i^{\mathrm{L}-} = 0$ and $d_i^{\mathrm{U}+} = 0$, then $D_i = [0, d_i^{\mathrm{L}+} \vee d_i^{\mathrm{U}-}]$.

(3) If $d_i^{\mathrm{L}+} = 0$ and $d_i^{\mathrm{U}+} = 0$, then $D_i = [d_i^{\mathrm{L}-}, d_i^{\mathrm{U}-}]$.

We show that the case of $d_i^{\mathrm{L}+} = 0, d_i^{\mathrm{U}-} = 0$ is impossible. Because $w[T_i] \geqslant w[(+)_{j=1}^{n} c_{ij} x_j]$, it follows that

$$t_i^{\mathrm{U}} - t_i^{\mathrm{L}} \geqslant \sum_{j=1}^{n} c_{ij}^{\mathrm{U}} x_j - \sum_{j=1}^{n} c_{ij}^{\mathrm{L}} x_j,$$

which is equivalent to $t_i^{\mathrm{U}} - \sum_{j=1}^{n} c_{ij}^{\mathrm{U}} x_j \geqslant t_i^{\mathrm{L}} - \sum_{j=1}^{n} c_{ij}^{\mathrm{L}} x_j$. From (3.11) and (3.12) we have

$$d_i^{\mathrm{U}-} - d_i^{\mathrm{U}+} \geqslant d_i^{\mathrm{L}-} - d_i^{\mathrm{L}+}.$$

If $d_i^{\mathrm{L}+} = 0, d_i^{\mathrm{U}-} = 0$, it follows that $-d_i^{\mathrm{U}+} \geqslant d_i^{\mathrm{L}-}$ which is a contradiction. □

Extending the deviation variables in the sense of (3.15), (GIP) can be stated as the following problem (BGIP):

$$\begin{cases} \min\limits_{\preceq_1} : & \{D_1(x), \ldots, D_m(x)\} = \\ & \left\{ \left| \left[t_1^{\mathrm{L}} - \sum\limits_{j=1}^{n} c_{1j}^{\mathrm{L}} x_j, t_1^{\mathrm{U}} - \sum\limits_{j=1}^{n} c_{1j}^{\mathrm{U}} x_j \right] \right|, \ldots, \left| \left[t_m^{\mathrm{L}} - \sum\limits_{j=1}^{n} c_{mj}^{\mathrm{L}} x_j, t_m^{\mathrm{U}} - \sum\limits_{j=1}^{n} c_{mj}^{\mathrm{U}} x_j \right] \right| \right\} \\ & = \{[d_1^{\mathrm{L}-} + d_1^{\mathrm{U}+}, d_1^{\mathrm{L}+} \vee d_1^{\mathrm{U}-}], \cdots, [d_m^{\mathrm{L}-} + d_m^{\mathrm{U}+}, d_m^{\mathrm{L}+} \vee d_m^{\mathrm{U}-}]\}, \\ \text{s.t.} & \sum_{j=1}^{n} c_{ij}^{\mathrm{L}} x_j + d_i^{\mathrm{L}-} - d_i^{\mathrm{L}+} = t_i^{\mathrm{L}}, \\ & \sum_{j=1}^{n} c_{ij}^{\mathrm{U}} x_j + d_i^{\mathrm{U}-} - d_i^{\mathrm{U}+} = t_i^{\mathrm{U}}, \\ & t_i^{\mathrm{U}} - t_i^{\mathrm{L}} \geqslant \sum_{j=1}^{n} c_{ij}^{\mathrm{U}} x_j - \sum_{j=1}^{n} c_{ij}^{\mathrm{L}} x_j, \\ & d_i^{\mathrm{L}-} d_i^{\mathrm{L}+} = 0, \quad d_i^{\mathrm{U}-} d_i^{\mathrm{U}+} = 0, \quad i = 1, 2, \ldots, m, \\ & \sum_{j=1}^{n} a_{kj}^{\mathrm{U}} x_j \leqslant b_k^{\mathrm{U}}, \\ & \sum_{j=1}^{n} \frac{a_{kj}^{\mathrm{L}} + a_{kj}^{\mathrm{U}}}{2} x_j \leqslant \frac{b_k^{\mathrm{L}} + b_k^{\mathrm{U}}}{2}, \quad k = 1, 2, \ldots, l, \\ & x_j \geqslant 0, d_i^{\mathrm{L}-}, d_i^{\mathrm{L}+}, d_i^{\mathrm{U}-}, d_i^{\mathrm{U}+} \geqslant 0, \quad j = 1, 2, \ldots, n. \end{cases}$$

In (BGIP), by definition of order relation $\preceq_1$, we intend to minimize the lower and upper bounds of the deviation intervals $D_i(x), i = 1, 2, \ldots, m$.

Definition 4. *Let* $x = (x_1, \ldots, x_n)$, $d^{\mathrm{L}-} = (d_1^{\mathrm{L}-}, \ldots, d_m^{\mathrm{L}-})$, $d^{\mathrm{L}+} = (d_1^{\mathrm{L}+}, \ldots, d_m^{\mathrm{L}+})$, $d^{\mathrm{U}-} = (d_1^{\mathrm{U}-}, \ldots, d_m^{\mathrm{U}-})$, $d^{\mathrm{U}+} = (d_1^{\mathrm{U}+}, \ldots, d_m^{\mathrm{U}+})$. *A feasible solution* $(x, d^{\mathrm{L}-}, d^{\mathrm{L}+}, d^{\mathrm{U}-}, d^{\mathrm{U}+})$ *is said to be a noninferior solution to (BGIP) if and only if there is no other feasible solution* $(x', d^{\mathrm{L}-'}, d^{\mathrm{L}+'}, d^{\mathrm{U}-'}, d^{\mathrm{U}+'})$

$$D_i(x') \preceq_1 D_i(x), \quad i = 1, 2, \ldots, m,$$

with $D_k(x') \prec_1 D_k(x)$ *for at least one* $k, k \in \{1, 2, \ldots, m\}$.

In conventional goal programming, regret function is composed of deviation variables from the goal levels depending on their relatives weights. Here, the deviation variables are replaced by extended deviation interval $D_i(x)$. From (3.16) and Definition 3, we replace the objective functions of (BGIP) with $(+)_{i=1}^{m} w_i D_i(x)$. Then (BGIP1) can be formulated as follows:

$$
\text{(BGIP1)}\left\{\begin{array}{ll}
\min\limits_{\preceq_1}: & (+)_{i=1}^{m} w_i D_i = (+)_{i=1}^{m} w_i [d_i^{\mathrm{L}-} + d_i^{\mathrm{U}+}, d_i^{\mathrm{L}+} \vee d_i^{\mathrm{U}-}], \\
\text{s.t.} & \sum_{j=1}^{n} c_{ij}^{\mathrm{L}} x_j + d_i^{\mathrm{L}-} - d_i^{\mathrm{L}+} = t_i^{\mathrm{L}}, \\
& \sum_{j=1}^{n} c_{ij}^{\mathrm{U}} x_j + d_i^{\mathrm{U}-} - d_i^{\mathrm{U}+} = t_i^{\mathrm{U}}, \\
& t_i^{\mathrm{U}} - t_i^{\mathrm{L}} \geqslant \sum_{j=1}^{n} c_{ij}^{\mathrm{U}} x_j - \sum_{j=1}^{n} c_{ij}^{\mathrm{L}} x_j, \\
& d_i^{\mathrm{L}-} d_i^{\mathrm{L}+} = 0, \quad d_i^{\mathrm{U}-} d_i^{\mathrm{U}+} = 0, \quad i = 1, 2, \ldots, m, \\
& \sum_{j=1}^{n} a_{kj}^{\mathrm{U}} x_j \leqslant b_k^{\mathrm{U}}, \\
& \sum_{j=1}^{n} \frac{a_{kj}^{\mathrm{L}} + a_{kj}^{\mathrm{U}}}{2} x_j \leqslant \frac{b_k^{\mathrm{L}} + b_k^{\mathrm{U}}}{2}, \quad k = 1, 2, \ldots, l, \\
& x_j \geqslant 0, d_i^{\mathrm{L}-}, d_i^{\mathrm{L}+}, d_i^{\mathrm{U}-}, d_i^{\mathrm{U}+} \geqslant 0, \quad j = 1, 2, \ldots, n,
\end{array}\right.
$$

where

$$
\begin{aligned}
\min_{\preceq_1} (+)_{i=1}^{m} w_i D_i(x) &= \min_{\preceq_1} (+)_{i=1}^{m} w_i [d_i^{\mathrm{L}-} + d_i^{\mathrm{U}+}, d_i^{\mathrm{L}+} \vee d_i^{\mathrm{U}-}] \\
&= \Big\{ \min \sum_{i=1}^{m} w_i (d_i^{\mathrm{L}-} + d_i^{\mathrm{U}+}), \min \sum_{i=1}^{m} w_i (d_i^{\mathrm{L}+} \vee d_i^{\mathrm{U}-}) \Big\}, \qquad (3.17)
\end{aligned}
$$

and weight coefficients $w_i \geqslant 0$, $\sum_{i=1}^{n} w_i = 1$ which are assigned by the decision makers to reflect the relative importance. From (3.17), we know that (BGIP1) is a bi-objective mathematical programming problem.

Definition 5. *Suppose* $(x,\ d^{\mathrm{L}-},\ d^{\mathrm{L}+},\ d^{\mathrm{U}-},\ d^{\mathrm{U}+})$ *is a feasible solution to (BGIP1). Feasible solution* $(x,\ d^{\mathrm{L}-},\ d^{\mathrm{L}+},\ d^{\mathrm{U}-},\ d^{\mathrm{U}+})$ *is a noninferior solution to (BGIP1) if and only if there exists no other feasible solution* $(x',\ d^{\mathrm{L}-'},\ d^{\mathrm{L}+'},\ d^{\mathrm{U}-'},\ d^{\mathrm{U}+'})$ *such that*

$$(+)_{i=1}^{m} w_i D_i(x') \prec_1 (+)_{i=1}^{m} w_i D_i(x).$$

Theorem 1. *Suppose* $(x,\ d^{\mathrm{L}-},\ d^{\mathrm{L}+},\ d^{\mathrm{U}-},\ d^{\mathrm{U}+})$ *is a feasible solution to (BGIP1). If it is a noninferior solution to (BGIP1), then it is also a noninferior solution to (BGIP).*

Proof. Assume $(x,\ d^{\mathrm{L}-},\ d^{\mathrm{L}+},\ d^{\mathrm{U}-},\ d^{\mathrm{U}+})$ is a noninferior solution to (BGIP1) but not a noninferior solution to (BGIP). Then, by Definition 4, there exists another feasible solution $(x',\ d^{\mathrm{L}-'},\ d^{\mathrm{L}+'},\ d^{\mathrm{U}-'},\ d^{\mathrm{U}+'})$ such that

$$D_i(x') \preceq_1 D_i(x), \quad i = 1, 2, \ldots, m,$$

with $D_k(x') \prec_1 D_k(x)$ for at least one $k, k \in \{1, 2, \ldots, m\}$. Since

$$D_i(x) = [d_i^{\mathrm{L}-} + d_i^{\mathrm{U}+}, d_i^{\mathrm{L}+} \vee d_i^{\mathrm{U}-}],\ D_i(x') = [d_i^{\mathrm{L}-'} + d_i^{\mathrm{U}+'}, d_i^{\mathrm{L}+'} \vee d_i^{\mathrm{U}-'}],$$

it follows that

$$d_i^{\mathrm{L}-'} + d_i^{\mathrm{U}+'} \leqslant d_i^{\mathrm{L}-} + d_i^{\mathrm{U}+}, \quad d_i^{\mathrm{L}+'} \vee d_i^{\mathrm{U}-'} \leqslant d_i^{\mathrm{L}-} \vee d_i^{\mathrm{U}+}, \quad i = 1, 2, \ldots, m,$$

and

$$d_k^{\mathrm{L}-'} + d_k^{\mathrm{U}+'} < d_k^{\mathrm{L}-} + d_k^{\mathrm{U}+} \quad \text{or} \quad d_k^{\mathrm{L}+'} \vee d_k^{\mathrm{U}-'} < d_k^{\mathrm{L}-} \vee d_k^{\mathrm{U}+}, \quad k \in \{1, 2, \cdots, m\}.$$

With the choice of $w_i > 0$, $i = 1, 2, \ldots, m$, we have

$$\sum_{i=1}^{m} w_i (d_i^{\mathrm{L}-'} + d_i^{\mathrm{U}+'}) < \sum_{i=1}^{m} w_i (d_i^{\mathrm{L}-} + d_i^{\mathrm{U}+}), \sum_{i=1}^{m} w_i (d_i^{\mathrm{L}+'} \vee d_i^{\mathrm{U}-'}) \leqslant \sum_{i=1}^{m} w_i (d_i^{\mathrm{L}-} \vee d_i^{\mathrm{U}+})$$

or

$$\sum_{i=1}^{m} w_i(d_i^{\mathrm{L}-'} + d_i^{\mathrm{U}+'}) \leqslant \sum_{i=1}^{m} w_i(d_i^{\mathrm{L}-} + d_i^{\mathrm{U}+}), \sum_{i=1}^{m} w_i(d_i^{\mathrm{L}+'} \vee d_i^{\mathrm{U}-'}) < \sum_{i=1}^{m} w_i(d_i^{\mathrm{L}-} \vee d_i^{\mathrm{U}+}).$$

So far we have proved that there exists a feasible solution $(x', d^{\mathrm{L}-'}, d^{\mathrm{L}+'}, d^{\mathrm{U}-'}, d^{\mathrm{U}+'})$ such that $(+)_{i=1}^{m} D_i(x') \prec_1 (+)_{i=1}^{m} D_i(x)$, which contradicts the assumption. Hence, $(x, d^{\mathrm{L}-}, d^{\mathrm{L}+}, d^{\mathrm{U}-}, d^{\mathrm{U}+})$ is a noninferior solution to (BGIP). This completes the proof. □

From Theorem 1, the noninferior solution to (BGIP) can be obtained by solving (BGIP1). Introducing a parameter $\lambda \in [0,1]$, the objective functions of (BGIP1) is converted into one objective function,

$$\begin{aligned} &\lambda \sum_{i=1}^{m} w_i(d_i^{\mathrm{L}-} + d_i^{\mathrm{U}+}) + (1-\lambda) \sum_{i=1}^{m} w_i d_i^{\mathrm{L}-} \vee d_i^{\mathrm{U}+} \\ &= \sum_{i=1}^{m} w_i[\lambda(d_i^{\mathrm{L}-} + d_i^{\mathrm{U}+}) + (1-\lambda) d_i^{\mathrm{L}+} \vee d_i^{\mathrm{U}-}]. \end{aligned} \tag{3.18}$$

Then, we have the following mathematical programming problem,

$$(\mathrm{PGIP}) \left\{ \begin{array}{ll} \min & \sum_{i=1}^{m} w_i[\lambda(d_i^{\mathrm{L}-} + d_i^{\mathrm{U}+}) + (1-\lambda)v_i], \\ \text{s.t.} & \sum_{j=1}^{n} c_{ij}^{\mathrm{L}} x_j + d_i^{\mathrm{L}-} - d_i^{\mathrm{L}+} = t_i^{\mathrm{L}}, \\ & \sum_{j=1}^{n} c_{ij}^{\mathrm{U}} x_j + d_i^{\mathrm{U}-} - d_i^{\mathrm{U}+} = t_i^{\mathrm{U}}, \\ & t_i^{\mathrm{U}} - t_i^{\mathrm{L}} \geqslant \sum_{j=1}^{n} c_{ij}^{\mathrm{U}} x_j - \sum_{j=1}^{n} c_{ij}^{\mathrm{L}} x_j, \\ & d_i^{\mathrm{L}+} \leqslant v_i, d_i^{\mathrm{U}-} \leqslant v_i, \\ & d_i^{\mathrm{L}-} d_i^{\mathrm{L}+} = 0, \quad d_i^{\mathrm{U}-} d_i^{\mathrm{U}+} = 0, \quad i = 1,2,\ldots,m, \\ & \sum_{j=1}^{n} a_{kj}^{\mathrm{L}} x_j \leqslant b_k^{\mathrm{L}}, \\ & \sum_{j=1}^{n} \frac{a_{kj}^{\mathrm{U}} + a_{kj}^{\mathrm{L}}}{2} x_j \leqslant \frac{b_k^{\mathrm{U}} + b_k^{\mathrm{L}}}{2}, \quad k = 1,2,\ldots,l, \\ & x_j \geqslant 0, d_i^{\mathrm{L}-}, d_i^{\mathrm{L}+}, d_i^{\mathrm{U}-}, d_i^{\mathrm{U}+} \geqslant 0, \quad j = 1,2,\ldots,n. \end{array} \right.$$

where $d_i^{\mathrm{L}-} \vee d_i^{\mathrm{U}+}$ are replaced by $v_i (i = 1, 2, \cdots, m)$, and λ reflects the degree of the DM's pessimism. If the decision maker is an optimist, λ should be selected as 0 and the objective function of (PGIP) is $\sum_{i=1}^{m} w_i(d_i^{\mathrm{L}+} \vee d_i^{\mathrm{U}-})$; If the decision maker is a pessimist, λ should be selected as 1 and the objective function is $\sum_{i=1}^{m} w_i(d_i^{\mathrm{L}-} + d_i^{\mathrm{U}+})$.

When the constraints

$$d_i^{\mathrm{L}-} d_i^{\mathrm{L}+} = 0 \quad \text{and} \quad d_i^{\mathrm{U}-} d_i^{\mathrm{U}+} = 0, \quad i = 1,2,\ldots,m$$

are omitted from (PGIP), the following linear programming problem (PGIP1), which can be solved by Simplex Method, is generated,

$$(\mathrm{PGIP1}) \left\{ \begin{array}{ll} \min & \sum_{i=1}^{m} w_i[\lambda(d_i^{\mathrm{L}-} + d_i^{\mathrm{U}+}) + (1-\lambda)v_i], \\ \text{s.t.} & \sum_{j=1}^{n} c_{ij}^{\mathrm{L}} x_j + d_i^{\mathrm{L}-} - d_i^{\mathrm{L}+} = t_i^{\mathrm{L}}, \\ & \sum_{j=1}^{n} c_{ij}^{\mathrm{U}} x_j + d_i^{\mathrm{U}-} - d_i^{\mathrm{U}+} = t_i^{\mathrm{U}}, \\ & t_i^{\mathrm{U}} - t_i^{\mathrm{L}} \geqslant \sum_{j=1}^{n} c_{ij}^{\mathrm{U}} x_j - \sum_{j=1}^{n} c_{ij}^{\mathrm{L}} x_j, \\ & d_i^{\mathrm{L}+} \leqslant v_i, d_i^{\mathrm{U}-} \leqslant v_i, \quad i = 1,2,\ldots,m, \\ & \sum_{j=1}^{n} a_{kj}^{\mathrm{L}} x_j \leqslant b_k^{\mathrm{L}}, \\ & \sum_{j=1}^{n} \frac{a_{kj}^{\mathrm{U}} + a_{kj}^{\mathrm{L}}}{2} x_j \leqslant \frac{b_k^{\mathrm{U}} + b_k^{\mathrm{L}}}{2}, \quad k = 1,2,\ldots,l, \\ & x_j \geqslant 0, d_i^{\mathrm{L}-}, d_i^{\mathrm{L}+}, d_i^{\mathrm{U}-}, d_i^{\mathrm{U}+} \geqslant 0, \quad j = 1,2,\ldots,n. \end{array} \right.$$

Theorem 2. *The optimal solution to (PGIP1) satisfies the complementary constraints, i.e.,* $d_i^{\mathrm{L}-*} d_i^{\mathrm{L}+*} = 0, d_i^{\mathrm{U}-*} d_i^{\mathrm{U}+*} = 0$ *for an arbitrary* $i \in \{1, 2, \ldots, m\}$.

Proof. Denote $v^* = (v_1^*, \ldots, v_m^*)$. Suppose ($x^*$, $d^{\mathrm{L}-*}$, $d^{\mathrm{L}+*}$, $d^{\mathrm{U}-*}$, $d^{\mathrm{U}+*}$) is an optimal solution to (PGIP1). For every feasible solution (x, $d^{\mathrm{L}-}$, $d^{\mathrm{L}+}$, $d^{\mathrm{U}-}$, $d^{\mathrm{U}+}$), we have

$$\sum_{i=1}^{m} w_i[\lambda(d_i^{\mathrm{L}-*} + d_i^{\mathrm{U}+*}) + (1-\lambda)v_i^*] \leqslant \sum_{i=1}^{m} w_i[\lambda(d_i^{\mathrm{L}-} + d_i^{\mathrm{U}+}) + (1-\lambda)v_i]. \tag{3.19}$$

Suppose that there exists a $p \in \{1, 2, \ldots, m\}$ such that $d_p^{\mathrm{L}-*} d_p^{\mathrm{L}+*} \neq 0$. It follows that $d_p^{\mathrm{L}-*} > 0, d_p^{\mathrm{L}+*} > 0$. Two deviational variables $d_0^{\mathrm{L}-}, d_0^{\mathrm{L}+}$ can be constructed such that

$$\begin{aligned} d_p^{\mathrm{L}-*} - d_p^{\mathrm{L}+*} &= d_0^{\mathrm{L}-} - d_0^{\mathrm{L}+}, \\ d_0^{\mathrm{L}-} &\geqslant 0, \\ d_0^{\mathrm{L}+} &\geqslant 0, \\ d_0^{\mathrm{L}-} d_0^{\mathrm{L}+} &= 0. \end{aligned}$$

A new feasible solution to (PGIP1) can be generated by replacing $d_p^{\mathrm{L}-*}$ and $d_p^{\mathrm{L}+*}$ with $d_0^{\mathrm{L}-}$ and $d_0^{\mathrm{L}+}$, respectively. For $d_0^{\mathrm{L}-} d_0^{\mathrm{L}+} = 0$, it follows that

$$d_p^{\mathrm{L}-*} > d_p^{\mathrm{L}-*} - d_p^{\mathrm{L}+*} = d_0^{\mathrm{L}-} - d_0^{\mathrm{L}+} = d_0^{\mathrm{L}-}$$

in the case that $d_0^{\mathrm{L}+} = 0$, and

$$d_p^{\mathrm{L}-*} - d_p^{\mathrm{L}+*} = d_0^{\mathrm{L}-} - d_0^{\mathrm{L}+} = -d_0^{\mathrm{L}+} > -d_p^{\mathrm{L}+*}$$

in the case that $d_0^{\mathrm{L}-} = 0$. Without loss of generality, assume that $d_p^{\mathrm{L}-*} > d_0^{\mathrm{L}-}$, it follows

$$\begin{aligned} &\sum_{i=1}^{m} w_i[\lambda(d_i^{\mathrm{L}-*} + d_i^{\mathrm{U}+*}) + (1-\lambda)v_i^*] \\ &> \sum_{i=1, i\neq p}^{m} w_i[\lambda(d_i^{\mathrm{L}-*} + d_i^{\mathrm{U}+*}) + (1-\lambda)v_i^*] + w_p[\lambda(d_0^{\mathrm{L}-} + d_p^{\mathrm{U}+*}) + (1-\lambda)v_p^*] \end{aligned}$$

which contradicts (3.19). Hence, $d_i^{\mathrm{L}-*} d_i^{\mathrm{L}+*} = 0, \forall i \in \{1, 2, \ldots, m\}$. Similarly, we may prove that $d_i^{\mathrm{U}-*} d_i^{\mathrm{U}+*} = 0$. □

From Theorem 2 we know that (PGIP) is equivalent to (PGIP1). By solving linear programming problem (PGIP1), the optimal solution to (PGIP) , which is also a noninferior solution to model (BGIP) or (GIP), can be obtained.

4 An Application to Portfolio Selection

In 1952, Markowitz [17] published his pioneering work that paved the foundation for portfolio selection analysis. In his seminal work [17, 18], Markowitz employed the standard deviation of the return as the measure of risk and formulated a mean-variance portfolio selection model. The model, contrary to its theoretic reputation, has not been used extensively from its original form to construct a large-scale portfolio [13]. The first reason is in the nature of the input data required for portfolio analysis. If the accurate expectations about future mean returns for each stock and the correlation of return between each pair of stocks could be obtained, then Markowitz model would produce optimum portfolios. The problem lies in obtaining accurate expectations of input data needed for this model. Another one is the computational difficulty associated with solving a large-scale quadratic programming problems with a dense covariance matrix. Several authors have tried to alleviate these problems by using various approximation schemes to obtain linear problems. In recent years, a absolute deviation risk function, where the measure of risk differs from the ones used by Markowitz [18], Sharpe [24] and the others, was introduced and a mean-absolute deviation portfolio selection model was formulated in [13]. It can be realized the intention of Markowitz's model by solving a linear programming instead of a quadratic programming problem.

By far, most of the existing portfolio selection models are grounded on probability theory. However, there are some uncertain factors which are different from random variables found in capital market. Some theories, such as fuzzy set theory [29], possibility theory [30], have been adopted to handle some non-stochastic factors in capital market [28, 10]. Whereas it may be not easy to specify the membership functions of possibilistic returns. So, at least in some cases, it is a good and simple idea for investors to determine the uncertain returns of assets as intervals. Some efforts to introduce interval numbers into portfolio selection is outlined in Lai et al. [14]. In the mean-absolute deviation portfolio selection model, the arithmetic means of historical returns are considered to be the expected returns of securities. In practice, we believe that the expected return of a security should be larger than the arithmetic mean obtained from the historical data if the recent historical return of a security is increasing or the corporation is operating fine. In such cases, we can assign the arithmetic mean as the lower limit of the expected return and estimate the upper limit of the expected return grounded on the financial report of the corporation. Therefore, in this paper we use the absolute deviation of portfolio return to measure risk and consider the portfolio selection problem with interval coefficients.

Assume that an investor wants to allocate his wealth among n risky securities with a tendency to rising prices based on the recent historical data or the financial report of the corporation and a risk-less asset offering a fixed rate of return. Denote x_j as the proportion of the total investment devoted to the risky security j $(j = 1, 2, \dots, n)$ and denote x_{n+1} as the proportion to a risk-less asset. In our model short selling is not permitted. Thus $\sum_{j=1}^{n+1} x_j = 1$ and $0 \leqslant x_j \leqslant u_j$ $(j = 1, 2, \dots, n+1)$, where u_j is the upper bound of the proportion of the total investment devoted to the asset j.

Suppose that the data are observed for the risky securities over T time periods. Let r_{jt} be the realization of random variable r_j during period t $(t = 1, 2, \dots, T)$. We also assume that the expected value of the random variable can be approximated by the average derived from these data. Since the n risky securities having a tendency to rising prices, the range over the expected return of security j may be represented by the following interval

$$\widetilde{R}_j = [R_j^{\mathrm{L}}, R_j^{\mathrm{U}}] = \Big[\frac{1}{T}\sum_{t=1}^{T} r_{jt}, R_j^{\mathrm{U}}\Big], \quad j = 1, 2, \ldots, n,$$

where R_j^{U} can be estimated by investors based on the corporation's financial report. Denote R_{n+1} as the rate of return of the risk-less asset $n+1$.

The interval over which the expected return of portfolio $x = (x_1, x_2, \ldots, x_{n+1})$ can be vary is represented as

$$\widetilde{R}(x) = (+)_{j=1}^{n} \widetilde{R}_j x_j + R_{n+1}x_{n+1} = \Big[\sum_{j=1}^{n} R_j^{\mathrm{L}} x_j + R_{n+1}x_{n+1}, \sum_{j=1}^{n} R_j^{\mathrm{U}} x_j + R_{n+1}x_{n+1}\Big].$$

If the expected returns of securities are accurately estimated, the absolute deviation of the return of portfolio $x = (x_1, x_2, \ldots, x_{n+1})$ can be represented as

$$w(x) = E\Big[\Big|\sum_{j=1}^{n} r_j x_j - E(\sum_{j=1}^{n} r_j x_j)\Big|\Big] = \frac{1}{T}\sum_{t=1}^{T}\Big|\sum_{j=1}^{n}(R_j - r_{jt})x_j\Big|,$$

where R_j is the expected return of security j and $R_j = \frac{1}{T}\sum_{t=1}^{T} r_{jt}$.

In practice, we may only obtain the variational interval of the security's return. So the range of the absolute deviation of the return of portfolio $x = (x_1, x_2, \ldots, x_{n+1})$ can be described as the following interval

$$\begin{aligned}\widetilde{w}(x) &= \frac{1}{T}(+)_{t=1}^{T}\Big|\sum_{j=1}^{n}\widetilde{R}_j x_j) - (\sum_{j=1}^{n} r_{jt}x_j\Big| \\ &= \frac{1}{T}(+)_{t=1}^{T}\Big|\Big[\sum_{j=1}^{n}(R_j^{\mathrm{L}} - r_{jt})x_j, \sum_{j=1}^{n}(R_j^{\mathrm{U}} - r_{jt})x_j\Big]\Big|.\end{aligned}$$

Proposition 2. *For a given portfolio $x = (x_1, x_2, \ldots, x_{n+1}) \in R^{n+1}$ and the interval of return $[R_j^{\mathrm{L}}, R_j^{\mathrm{U}}]$ of security j, the range of the absolute deviation of the return of portfolio x can be represented as follows*

$$\begin{aligned}\widetilde{w}(x) &= \frac{1}{T}(+)_{t=1}^{T}\Big|\Big[\sum_{j=1}^{n}(R_j^{\mathrm{L}} - r_{jt})x_j, \sum_{j=1}^{n}(R_j^{\mathrm{U}} - r_{jt})x_j\Big]\Big| \\ &= \frac{1}{T}(+)_{t=1}^{T}\Big[\max\Big\{0, \sum_{j=1}^{n}(R_j^{\mathrm{L}} - r_{jt})x_j\Big\} - \min\Big\{0, \sum_{j=1}^{n}(R_j^{\mathrm{U}} - r_{jt})x_j\Big\}, \\ &\qquad \max\Big\{\sum_{j=1}^{n}(R_j^{\mathrm{U}} - r_{jt})x_j, -\sum_{j=1}^{n}(R_j^{\mathrm{L}} - r_{jt})x_j\Big\}\Big].\end{aligned}$$

Proof. Denote the absolute deviation of return over the past period t as

$$[y_t^{\mathrm{L}}, y_t^{\mathrm{U}}] = \Big|\Big[\sum_{j=1}^{n}(R_j^{\mathrm{L}} - r_{jt})x_j, \sum_{j=1}^{n}(R_j^{\mathrm{U}} - r_{jt})x_j\Big]\Big|, \quad t = 1, 2, \ldots, T,$$

then the range of absolute deviation of the return of portfolio $x = (x_1, \ldots, x_{n+1})$ can be written as $\widetilde{w}(x) = \frac{1}{T}(+)_{t=1}^{T}[y_t^{\mathrm{L}}, y_t^{\mathrm{U}}]$.

For the given portfolio x and the given return interval $[R_j^{\mathrm{L}}, R_j^{\mathrm{U}}]$ of security j, the following three cases are possible:

(i) If $\sum_{j=1}^{n}(R_j^{\mathrm{U}} - r_{jt})x_j \geqslant \sum_{j=1}^{n}(R_j^{\mathrm{L}} - r_{jt})x_j \geqslant 0$, then

$$[y_t^{\mathrm{L}}, y_t^{\mathrm{U}}] = \left[\sum_{j=1}^{n}(R_j^{\mathrm{L}} - r_{jt})x_j, \sum_{j=1}^{n}(R_j^{\mathrm{U}} - r_{jt})x_j\right].$$

(ii) If $\sum_{j=1}^{n}(R_j^{\mathrm{U}} - r_{jt})x_j \geqslant 0 \geqslant \sum_{j=1}^{n}(R_j^{\mathrm{L}} - r_{jt})x_j$, then

$$[y_t^{\mathrm{L}}, y_t^{\mathrm{U}}] = \left[0, (-\sum_{j=1}^{n}(R_j^{\mathrm{L}} - r_{jt})x_j) \vee \sum_{j=1}^{n}(R_j^{\mathrm{U}} - r_{jt})x_j\right].$$

(iii) If $0 \geqslant \sum_{j=1}^{n}(R_j^{\mathrm{U}} - r_{jt})x_j \geqslant \sum_{j=1}^{n}(R_j^{\mathrm{L}} - r_{jt})x_j$, then

$$[y_t^{\mathrm{L}}, y_t^{\mathrm{U}}] = \left[-\sum_{j=1}^{n}(R_j^{\mathrm{U}} - r_{jt})x_j, -\sum_{j=1}^{n}(R_j^{\mathrm{L}} - r_{jt})x_j\right].$$

Then y_t^{L} and y_t^{U} can be denoted as follows:

$$\begin{aligned} y_t^{\mathrm{L}} &= \max\{0, \sum_{j=1}^{n}(R_j^{\mathrm{L}} - r_{jt})x_j\} - \min\{0, \sum_{j=1}^{n}(R_j^{\mathrm{U}} - r_{jt})x_j\}, \\ y_t^{\mathrm{U}} &= \max\{\sum_{j=1}^{n}(R_j^{\mathrm{U}} - r_{jt})x_j, -\sum_{j=1}^{n}(R_j^{\mathrm{L}} - r_{jt})x_j\}, \end{aligned}$$

which implies

$$\begin{aligned} \widetilde{w}(x) &= \frac{1}{T}(+)_{t=1}^{T}[y_t^{\mathrm{L}}, y_t^{\mathrm{U}}] \\ &= \frac{1}{T}(+)_{t=1}^{T}\left[\max\left\{0, \sum_{j=1}^{n}(R_j^{\mathrm{L}} - r_{jt})x_j\right\} - \min\left\{0, \sum_{j=1}^{n}(R_j^{\mathrm{U}} - r_{jt})x_j\right\},\right. \\ &\qquad\qquad \left.\max\left\{\sum_{j=1}^{n}(R_j^{\mathrm{U}} - r_{jt})x_j, -\sum_{j=1}^{n}(R_j^{\mathrm{L}} - r_{jt})x_j\right\}\right]. \end{aligned}$$

This completes the proof. □

Denote $y_t^{\mathrm{L}} = y_t^{\mathrm{L}1} + y_t^{\mathrm{L}2}$, $y_t^{\mathrm{L}1} = \max\{0, \sum_{j=1}^{n}(R_j^{\mathrm{L}} - r_{jt})x_j\}$ and $y_t^{\mathrm{L}2} = -\min\{0, \sum_{j=1}^{n}(R_j^{\mathrm{U}} - r_{jt})x_j\}$, where $y_t^{\mathrm{L}1}$, $y_t^{\mathrm{L}2}$ and y_t^{U} satisfy the following constraints:

$$\begin{aligned} &y_t^{\mathrm{L}1} \geqslant 0,\ y_t^{\mathrm{L}1} \geqslant \sum_{j=1}^{n}(R_j^{\mathrm{L}} - r_{jt})x_j;\ y_t^{\mathrm{L}2} \geqslant 0,\ y_t^{\mathrm{L}2} \geqslant -\sum_{j=1}^{n}(R_j^{\mathrm{U}} - r_{jt})x_j; \\ &y_t^{\mathrm{U}} \geqslant \sum_{j=1}^{n}(R_j^{\mathrm{U}} - r_{jt})x_j,\ y_t^{\mathrm{U}} \geqslant -\sum_{j=1}^{n}(R_j^{\mathrm{L}} - r_{jt})x_j. \end{aligned}$$

Suppose that the investor wants to maximize his or her portfolio return under some given level of portfolio return $[R^{\mathrm{L}}, R^{\mathrm{U}}]$ and minimize his or her absolute deviation under some given level of portfolio risk $[w^{\mathrm{L}}, w^{\mathrm{U}}]$, respectively. Here, w^{L} and w^{U} are two given constants. w^{L} represents the tolerated risk level when the expected returns are predicted pessimistically as R^{L}, and w^{U} represents the tolerated risk level when the expected returns are predicted optimistically as R^{U}. Then, we will simultaneously consider two objectives in the following portfolio selection model, i.e.,

$$(+)_{j=1}^{n}[R_j^{\mathrm{L}}, R_j^{\mathrm{U}}]x_j(+)R_{n+1}x_{n+1} = [R^{\mathrm{L}}, R^{\mathrm{U}}]$$

and

$$\frac{1}{T}(+)_{t=1}^{T}\left|\left[\sum_{j=1}^{n}(R_j^{\mathrm{L}} - r_{jt})x_j, \sum_{j=1}^{n}(R_j^{\mathrm{U}} - r_{jt})x_j\right]\right| = \frac{1}{T}(+)_{t=1}^{T}[y_t^{\mathrm{L}}, y_t^{\mathrm{U}}] = [w^{\mathrm{L}}, w^{\mathrm{U}}].$$

So the goal interval programming for portfolio selection can be stated as

$$\begin{cases} \text{Goals}_{(\preceq_1)}: & (+)_{j=1}^{n}[R_j^{\mathrm{L}}x_j, R_j^{\mathrm{U}}x_j](+)R_{n+1}x_{n+1} = [R^{\mathrm{L}}, R^{\mathrm{U}}], \\ & \frac{1}{T}(+)_{t=1}^{T}[y_t^{\mathrm{L}}, y_t^{\mathrm{U}}] = [w^{\mathrm{L}}, w^{\mathrm{U}}], \\ \text{s.t.} & y_t^{\mathrm{L}} = y_t^{\mathrm{L}1} + y_t^{\mathrm{L}2}, \\ & y_t^{\mathrm{L}1} \geqslant 0, \quad y_t^{\mathrm{L}1} \geqslant \sum_{j=1}^{n}(R_j^{\mathrm{L}} - r_{jt})x_j, \\ & y_t^{\mathrm{L}2} \geqslant 0, \quad y_t^{\mathrm{L}2} \geqslant -\sum_{j=1}^{n}(R_j^{\mathrm{U}} - r_{jt})x_j, \\ & y_t^{\mathrm{U}} \geqslant \sum_{j=1}^{n}(R_j^{\mathrm{U}} - r_{jt})x_j, \\ & y_t^{\mathrm{U}} \geqslant -\sum_{j=1}^{n}(R_j^{\mathrm{L}} - r_{jt})x_j, \quad t = 1, 2, \ldots, T, \\ & \sum_{j=1}^{n+1} x_j = 1, \quad 0 \leqslant x_j \leqslant u_j, \quad j = 1, 2, \ldots, m. \end{cases} \tag{4.20}$$

Denote D_1 and D_2 as the possible deviation of portfolio return and the possible deviation of portfolio risk,

$$D_1 = \left|[R^{\mathrm{L}}, R^{\mathrm{U}}]) - (\left[(+)_{j=1}^{n}[R_j^{\mathrm{L}}, R_j^{\mathrm{U}}]x_j(+)R_{n+1}x_{n+1}\right]\right|,$$
$$D_2 = \left|[w^{\mathrm{L}}, w^{\mathrm{U}}]) - (\frac{1}{T}(+)_{t=1}^{T}[y_t^{\mathrm{L}}, y_t^{\mathrm{U}}]\right|,$$

respectively.

Actually, we want to minimize these possible deviations as small as possible. From Proposition 1, if $R^{\mathrm{U}} - R^{\mathrm{L}} \geqslant \sum_{j=1}^{n}(R_j^{\mathrm{U}} - R_j^{\mathrm{L}})x_j$ and $w^{\mathrm{U}} - w^{\mathrm{L}} \geqslant \frac{1}{T}\sum_{t=1}^{T}(y_t^{\mathrm{U}} - y_t^{\mathrm{L}})$, we have

$$D_1 = [d_1^{\mathrm{L}-} + d_1^{\mathrm{U}+}, d_1^{\mathrm{L}+} \vee d_1^{\mathrm{U}-}] \quad \text{and} \quad D_2 = [d_2^{\mathrm{L}-} + d_2^{\mathrm{U}+}, d_2^{\mathrm{L}+} \vee d_2^{\mathrm{U}-}],$$

where $d_i^{\mathrm{L}-}$, $d_i^{\mathrm{L}+}$, $d_i^{\mathrm{U}-}$ and $d_i^{\mathrm{U}+}$ $(i = 1, 2)$ such that

$$\sum_{j=1}^{n} R_j^{\mathrm{L}}x_j + R_{n+1}x_{n+1} + d_1^{\mathrm{L}-} - d_1^{\mathrm{L}+} = R^{\mathrm{L}},$$
$$\sum_{j=1}^{n} R_j^{\mathrm{U}}x_j + R_{n+1}x_{n+1} + d_1^{\mathrm{U}-} - d_1^{\mathrm{U}+} = R^{\mathrm{U}},$$
$$\frac{1}{T}\sum_{t=1}^{T} y_t^{\mathrm{L}} + d_2^{\mathrm{L}-} - d_2^{\mathrm{L}+} = w^{\mathrm{L}}, \frac{1}{T}\sum_{t=1}^{T} y_t^{\mathrm{U}} + d_2^{\mathrm{U}-} - d_2^{\mathrm{U}+} = w^{\mathrm{U}},$$
$$d_i^{\mathrm{L}-}d_i^{\mathrm{L}+} = 0, \quad d_i^{\mathrm{L}-}d_i^{\mathrm{L}+} = 0, d_i^{\mathrm{L}-}, d_i^{\mathrm{L}+}, d_i^{\mathrm{U}-}, d_i^{\mathrm{U}+} \geqslant 0, \quad i = 1, 2.$$

By Theorem 1, problem (4.21) should be considered. The unwanted deviations D_1 and D_2 are assigned weights w_1 and w_2 according to their relative importance to the investors, respectively. Then, (4.20) can be formulated as

$$\left\{\begin{array}{ll}\min\limits_{\preceq_1} & w_1D_1(+)w_2D_2 = w_1[d_1^{\mathrm{L}-}+d_1^{\mathrm{U}+}, d_1^{\mathrm{L}+}\vee d_1^{\mathrm{U}-}](+)w_2[d_2^{\mathrm{L}-}+d_2^{\mathrm{U}+}, d_2^{\mathrm{L}+}\vee d_2^{\mathrm{U}-}],\\ \text{s.t.} & \sum_{j=1}^n R_j^{\mathrm{L}}x_j + R_{n+1}x_{n+1} + d_1^{\mathrm{L}-} - d_1^{\mathrm{L}+} = R^{\mathrm{L}},\\ & \sum_{j=1}^n R_j^{\mathrm{U}}x_j + R_{n+1}x_{n+1} + d_1^{\mathrm{U}-} - d_1^{\mathrm{U}+} = R^{\mathrm{U}},\\ & \frac{1}{T}\sum_{t=1}^T y_t^{\mathrm{L}} + d_2^{\mathrm{L}-} - d_2^{\mathrm{L}+} = w^{\mathrm{L}},\\ & \frac{1}{T}\sum_{t=1}^T y_t^{\mathrm{U}} + d_2^{\mathrm{U}-} - d_2^{\mathrm{U}+} = w^{\mathrm{U}},\\ & d_i^{\mathrm{L}-}d_i^{\mathrm{L}+} = 0, \quad d_i^{\mathrm{U}-}d_i^{\mathrm{U}+} = 0, \quad d_i^{\mathrm{L}-}, d_i^{\mathrm{L}+}, d_i^{\mathrm{U}-}, d_i^{\mathrm{U}+} \geqslant 0, \quad i = 1,2,\\ & R^{\mathrm{U}} - R^{\mathrm{L}} \geqslant \sum_{j=1}^n (R_j^{\mathrm{U}} - R_j^{\mathrm{L}})x_j,\\ & w^{\mathrm{U}} - w^{\mathrm{L}} \geqslant \frac{1}{T}\sum_{t=1}^T (y_t^{\mathrm{U}} - t_t^{\mathrm{L}}),\\ & y_t^{\mathrm{L}} = y_t^{\mathrm{L}1} + y_t^{\mathrm{L}2},\\ & y_t^{\mathrm{L}1} \geqslant 0, \quad y_t^{\mathrm{L}1} \geqslant \sum_{j=1}^n (R_j^{\mathrm{L}} - r_{jt})x_j,\\ & y_t^{\mathrm{L}2} \geqslant 0, \quad y_t^{\mathrm{L}2} \geqslant -\sum_{j=1}^n (R_j^{\mathrm{U}} - r_{jt})x_j,\\ & y_t^{\mathrm{U}} \geqslant \sum_{j=1}^n (R_j^{\mathrm{U}} - r_{jt})x_j,\\ & y_t^{\mathrm{U}} \geqslant -\sum_{j=1}^n (R_j^{\mathrm{L}} - r_{jt})x_j \quad t = 1,2,\ldots,T,\\ & \sum_{j=1}^{n+1} x_j = 1, \quad 0 \leqslant x_j \leqslant u_j, \quad j = 1,2,\ldots,m.\end{array}\right. \tag{4.21}$$

where parameters $w_1, w_2 \geqslant 0$ and $w_1 + w_2 = 1$.

Obviously, the objective function in (4.21) is equivalent to

$$\begin{aligned}&\min_{\preceq_1} w_1[d_1^{\mathrm{L}-}+d_1^{\mathrm{U}+}, d_1^{\mathrm{L}+}\vee d_1^{\mathrm{U}-}](+)w_2[d_2^{\mathrm{L}-}+d_2^{\mathrm{U}+}, d_2^{\mathrm{L}+}\vee d_2^{\mathrm{U}-}]\\ &= \min\left\{w_1(d_1^{\mathrm{L}-}+d_1^{\mathrm{U}+}) + w_2(d_2^{\mathrm{L}-}+d_2^{\mathrm{U}+}), w_1(d_1^{\mathrm{L}+}\vee d_1^{\mathrm{U}-}) + w_2(d_2^{\mathrm{L}+}\vee d_2^{\mathrm{U}-})\right\}.\end{aligned}$$

From (3.18), we have

$$\begin{aligned}&\lambda\big[w_1(d_1^{\mathrm{L}-}+d_1^{\mathrm{U}+}) + w_2(d_2^{\mathrm{L}-}+d_2^{\mathrm{U}+})\big] + (1-\lambda)\big[w_1(d_1^{\mathrm{L}+}\vee d_1^{\mathrm{U}-}) + w_2(d_2^{\mathrm{L}+}\vee d_2^{\mathrm{U}-})\big]\\ &= w_1[\lambda(d_1^{\mathrm{L}-}+d_1^{\mathrm{U}+}) + (1-\lambda)v_1] + w_2[\lambda(d_2^{\mathrm{L}-}+d_2^{\mathrm{U}+}) + (1-\lambda)v_2].\end{aligned}$$

The noninferior solution to (4.21) can be generated by solving the following linear programming problem:

$$\left\{\begin{array}{ll}\min & w_1[\lambda(d_1^{\mathrm{L}-}+d_1^{\mathrm{U}+}) + (1-\lambda)v_1] + w_2[\lambda(d_2^{\mathrm{L}-}+d_2^{\mathrm{U}+}) + (1-\lambda)v_2],\\ \text{s.t.} & \sum_{j=1}^n R_j^{\mathrm{L}}x_j + R_{n+1}x_{n+1} + d_1^{\mathrm{L}-} - d_1^{\mathrm{L}+} = R^{\mathrm{L}},\\ & \sum_{j=1}^n R_j^{\mathrm{U}}x_j + R_{n+1}x_{n+1} + d_1^{\mathrm{U}-} - d_1^{\mathrm{U}+} = R^{\mathrm{U}},\\ & \frac{1}{T}\sum_{t=1}^T y_t^{\mathrm{L}} + d_2^{\mathrm{L}-} - d_2^{\mathrm{L}+} = w^{\mathrm{L}},\\ & \frac{1}{T}\sum_{t=1}^T y_t^{\mathrm{U}} + d_2^{\mathrm{U}-} - d_2^{\mathrm{U}+} = w^{\mathrm{U}},\\ & d_1^{\mathrm{L}+} \geqslant v_1, d_1^{\mathrm{U}-} \geqslant v_1, \quad d_2^{\mathrm{L}+} \geqslant v_2, d_2^{\mathrm{U}-} \geqslant v_2,\\ & d_1^{\mathrm{L}-}, d_1^{\mathrm{L}+}, d_1^{\mathrm{U}-}, d_1^{\mathrm{U}+}, d_2^{\mathrm{L}-}, d_2^{\mathrm{L}+}, d_2^{\mathrm{U}-}, d_2^{\mathrm{U}+} \geqslant 0,\\ & R^{\mathrm{U}} - R^{\mathrm{L}} \geqslant \sum_{j=1}^n (R_j^{\mathrm{U}} - R_j^{\mathrm{L}})x_j,\\ & w^{\mathrm{U}} - w^{\mathrm{L}} \geqslant \frac{1}{T}\sum_{t=1}^T (y_t^{\mathrm{U}} - t_t^{\mathrm{L}}),\\ & y_t^{\mathrm{L}} = y_t^{\mathrm{L}1} + y_t^{\mathrm{L}2},\\ & y_t^{\mathrm{L}1} \geqslant 0, \quad y_t^{\mathrm{L}1} \geqslant \sum_{j=1}^n (R_j^{\mathrm{L}} - r_{jt})x_j,\\ & y_t^{\mathrm{L}2} \geqslant 0, \quad y_t^{\mathrm{L}2} \geqslant -\sum_{j=1}^n (R_j^{\mathrm{U}} - r_{jt})x_j,\\ & y_t^{\mathrm{U}} \geqslant \sum_{j=1}^n (R_j^{\mathrm{U}} - r_{jt})x_j,\\ & y_t^{\mathrm{U}} \geqslant -\sum_{j=1}^n (R_j^{\mathrm{L}} - r_{jt})x_j, \quad t = 1,2,\ldots,T,\\ & \sum_{j=1}^{n+1} x_j = 1, \quad 0 \leqslant x_j \leqslant u_j, \quad j = 1,2,\ldots,m.\end{array}\right. \tag{4.22}$$

where $v_1 = d_1^{\mathrm{L}+} \vee d_1^{\mathrm{U}-}$ and $v_2 = d_2^{\mathrm{L}+} \vee d_2^{\mathrm{U}-}$, and parameter $\lambda \in (0,1)$ denotes the degree of risk aversion. If $\lambda \to 0$, the DM is a rather pessimistic; If $\lambda \to 1$, the DM is a rather optimistic. By Theorem 2, the optimal solutions to (4.22) satisfy the complementary constraints $d_i^{\mathrm{L}-} d_i^{\mathrm{L}+} = 0$ and $d_i^{\mathrm{U}-} d_i^{\mathrm{U}+} = 0$, $i = 1, 2$.

The noninferior solution to (4.21) can be obtained by solving linear programming problem (4.22).

5 A Numerical Example

We consider the following problem: an investor decides to invest his wealth among a risk-less asset and twelve stocks in the Shanghai Stock Exchange. The twelve stocks are listed in Table 2.

Table 2: The Names of The Twelve Stocks

	Name		Name		Name
St. 1	Shanggang Jixiang	St. 2	Shenneng Gufen	St. 3	Baogang Gufen
St. 4	Dongbei Gaosu	St. 5	Shanghai Jichang	St. 6	Yun Tianhua
St. 7	Tongbao Nengyuan	St. 8	Huabei Zhiyiao	St. 9	Hayao Jituan
St. 10	Waiyun Fazhan	St. 11	Yili Gufen	St. 12	Dongfeng Qiche

Historical data of the twelve stocks from Jan. 2002 to Dec. 2003 are downloaded from the website www.stockstar.com. In this example, the return of the risk-less asset is 0.004 and two months is chosen as a basic period to obtain the historical returns. The historical returns are given in Table 3.

Table 3: The Historic Return of The Stocks (%)

Period	1	2	3	4	5	6	7	8	9	10	11	12
St. 1	0.55	6.61	4.97	−5.69	−7.39	−3.50	18.28	23.56	3.12	5.17	−0.77	8.81
St. 2	−11.96	10.27	9.60	−11.60	−11.23	−4.46	14.33	15.36	−6.03	−0.34	1.08	15.76
St. 3	7.59	−0.49	14.85	−1.32	−5.80	−3.51	22.03	5.33	−7.59	7.02	5.76	18.03
St. 4	−5.18	11.43	10.63	4.51	10.36	8.52	12.10	−3.62	−1.28	−1.51	−3.68	4.92
St. 5	4.67	7.70	13.59	−1.28	−3.11	−7.42	13.42	5.0	2.60	7.09	−4.33	2.89
St. 6	−1.87	11.29	3.98	−2.43	8.48	−12.38	15.39	10.09	−8.58	−6.80	5.61	5.18
St. 7	−6.45	14.86	0.71	4.87	−6.57	7.77	25.69	26.19	5.27	−1.04	1.18	2.59
St. 8	−5.48	7.83	9.81	1.86	−12.79	−9.25	17.97	19.96	−12.91	−10.48	15.37	11.80
St. 9	−10.41	32.89	4.43	14.22	10.44	15.50	−1.27	16.99	−3.79	−12.98	−4.12	−14.68
St. 10	1.38	0.31	4.90	−11.59	−1.51	−0.50	15.82	13.64	−3.20	3.40	−1.13	0.17
St. 11	1.89	3.48	6.56	−2.68	9.48	−1.39	10.29	9.34	−1.99	−5.12	4.73	4.96
St. 12	6.98	4.30	2.80	1.92	3.41	16.24	−1.85	−1.71	−4.06	−10.29	−6.66	12.94

Because the Shanghai Stock Exchange is a very young, the arithmetic means may not be good estimation of the actual returns which will come true in the future. Based on the corporations' financial reports, the interval of expected returns of the twelve stocks can be estimated. The upper and lower bounds of the intervals are listed in Table 4.

Here we assume that there is no limits on the investment proportions for the twelve stocks and the risk-less asset, i.e. $u_j = 1, j = 1, 2, \ldots, 13$. Then from (4.22), a linear programming problem is obtained which can be solved by LINDO software. For the given risk and return levels investment strategies are obtained and these alternatives are listed in Table 5. From Table 5 we know that (0, 0.000254, 0.108931, 0.035099, 0, 0.002693, 0.012611, 0.012962, 0,

Table 4: The Intervals of Expected Returns of The Twelve Stocks (%)

	St. 1	St. 2	St. 3	St. 4	St. 5	St. 6	St. 7	St. 8	St. 9	St. 10	St. 11	St. 12
R_j^{L}	4.48	1.79	5.16	3.93	3.40	2.33	6.26	2.81	3.94	1.81	3.30	2.00
R_j^{U}	4.51	1.82	5.18	3.95	3.44	2.36	6.28	2.84	3.97	1.84	3.33	2.05

Table 5: Some Alternatives

w^{L}	w^{U}	R^{L}	R^{U}	λ	St. 1	St. 2	St. 3	St. 4	St. 5	St. 6	St. 7
						St. 8	St. 9	St. 10	St. 11	St. 12	St. 13
1	2	1	2	0.1	0	0.000254	0.108931	0.035099	0	0.002693	0.012611
						0.012962	0	0	0	0.067789	0.759662
2	3	1.5	2.5	0.2	0	0.000508	0.217862	0.070198	0	0.005386	0.025221
						0.025924	0	0	0	0.135577	0.519324
3	4	2.5	3.5	0.5	0	0.000762	0.326793	0.105296	0	0.008079	0.037832
						0.038886	0	0	0	0.203366	0.278986
4	5	2.5	4	0.8	0	0.001016	0.435723	0.140395	0	0.010772	0.050442
						0.051849	0	0	0	0.271155	0.038648

0, 0, 0.067789, 0.759662) is a better investment strategy for an investor with a conservative and pessimistic mind, and (0, 0.001016, 0.435723, 0.140395, 0, 0.010772, 0.050442, 0.051849, 0, 0, 0, 0.271155, 0.038648) is a better investment strategy for an investor with an aggressive and optimistic mind. If the investor is not satisfied with any of the current alternatives, more alternatives can be generated by varying the values of $w^{\mathrm{L}}, w^{\mathrm{U}}, R^{\mathrm{L}}, R^{\mathrm{U}}$ and λ.

6 Conclusion

In this paper, a goal programming problem with interval coefficients in objective functions and constraints is discussed. Based on two order relations between interval numbers, the noninferior solutions to this problem can be generated by solving a linear programming problem. When the uncertain returns of assets in a capital market are assumed as intervals, the absolute deviation of the risk of portfolio is extended to the interval case. Then, a portfolio selection model is presented and can be transformed into a linear programming problem with interval coefficients. Finally, we should notice that the proposed method in this paper is only a tentative attempt to handle portfolio selection problem with interval coefficients. Extensions to the model are also possible. Moreover, it is reasonable to assume that the 'preemptive' weights or the combinations of 'preemptive' and 'relative' weights are interval numbers and the authors intends to investigate these problems in another papers.

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