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CUBIC L_1 SPLINES ON TRIANGULATED IRREGULAR NETWORKS*

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Abstract: Bivariate cubic L_1 interpolating splines, which have previously been implemented with Sibson elements on rectangular grids, are implemented with reduced Hsieh-Clough-Tocher (rHCT) elements on triangulated irregular networks (TINs). The calculation of coefficients of a bivariate cubic L_1 spline, which minimizes the L_1 norm of its second derivatives, turns out to be a nonsmooth convex programming problem. In a generalized geometric programming framework, the dual problem has a linear objective function and convex cubic constraints. The coefficients of an L_1 spline can be obtained by solving the dual problem and then converting to a primal solution using a linear programming transformation. Our preliminary computational results show that cubic L_1 splines on TINs are flexible and capable of providing interpolation with excellent shape preservation.

Key words: *bivariate cubic L_1 spline, geometric programming, rHCT element, triangulated irregular networks*

Mathematics Subject Classification: *65D07, 65D05, 65K05, 90C30*

1 Introduction

A fundamental requirement of modern geometric modeling is "shape preservation" or, more precisely, preservation of the "shape" of discrete data by a surface passing through or near the data. In some contexts, shape preservation is interpreted as preservation of "real" characteristics of the data such as monotonicity, convexity, linearity/planarity and positivity. In other contexts, shape preservation is interpreted as preservation of characteristics—smoothness, curvature, variation of curvature—of a surface on which the data are presumed or imagined to lie [13]. For many specific applications including but not limited to modeling of airfoils, automobile bodies and ship hulls, shape preservation is at least partially understood, in large part because there is wide agreement about the structure of the surface on which the data are presumed to lie. However, shape preservation for general geometric modeling of irregular data by irregular surfaces that are not necessarily very smooth is not understood. It is precisely such modeling that is most important for most of the applications of modern interest, including representation of natural and urban terrain (for geolocation and city/regional planning), human, biological and inanimate objects, geological regions such as oil and coal reserves, economic and financial processes, sociological processes and images in general. For geometric modeling of smooth data by smooth functions, splines have been quite successful. However, most splines do not preserve the shape of irregular data well. Recently, Lavery

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proposed cubic L_1 splines [9] that do preserve the shape of irregular data well. Cubic L_1 splines are calculated by minimizing the L_1 norm of the second partial derivatives of a C^1-smooth piecewise cubic polynomial. Computational experiments on regular (tensor-product) rectangular grids have shown that cubic L_1 splines on such grids preserve shape well, even when the magnitude and spacing of the data vary abruptly [2, 9, 10]. Motivated by the shape-preserving capability of cubic L_1 splines on regular rectangular grids, we propose to investigate cubic L_1 splines on irregular triangular grids.

In 2002, Cheng *et al.* [1] proposed a geometric programming (GP) framework [5, 14, 15] for univariate cubic L_1 interpolating splines. Subsequently, Wang *et al.* [17] extended this framework to bivariate cubic L_1 interpolating splines on regular rectangular grids. (In the remainder of this paper, bivariate cubic L_1 splines will be called L_1 splines for short.) Regular rectangular grids are computationally inexpensive but are rigid and may be unsuitable for many applications involving irregular objects. In this paper, we propose to create a geometric programming framework for L_1 splines on irregular triangular grids, or, as they are more commonly called, triangulated irregular networks (TINs), which are much more flexible than regular rectangular grids. The cubic elements that we will use as the basis for L_1 splines on the triangulation are the so-called reduced Hsieh-Clough-Tocher (rHCT) elements [11].

In Section 2, we introduce the rHCT elements and phrase the L_1 spline minimization principle in terms of them. In Section 3, a generalized geometric programming framework for rHCT-element-based L_1 splines is developed. The primal problem, dual problem and primal-to-dual transformation that compose the framework are derived for a single triangle and for sets of triangles. In Section 4, computational results for small-size problems on regular and irregular triangulations are given. This section also includes comparisons of rHCT-element-based L_1 splines on TINs with Sibson-element-based L_1 splines on regular rectangular grids. Section 5 provides some concluding remarks.

2 rHCT-Element L_1 Splines on TINs

Let there be an irregular triangulation $\triangle$ covering a domain Ω in $\mathbf{R}^2$. Assume the vertices of the triangles are at the locations (x_i, y_i), $i = 0, 1, \ldots, I$. Calculating an L_1 spline on the given triangulation $\triangle$ consists of minimizing the integral of

$$\iint_{(x,y)\in\Omega} \left(\left|\frac{\partial^2 z}{\partial x^2}\right| + 2\left|\frac{\partial^2 z}{\partial x \partial y}\right| + \left|\frac{\partial^2 z}{\partial y^2}\right| \right) \mathrm{d}x\mathrm{d}y \tag{1}$$

over all surfaces $z = z(x, y)$ that consist of reduced Hsieh-Clough-Tocher elements inside each triangle and that interpolate the data (x_i, y_i, z_i), $i = 0, 1, \ldots, I$. As will be seen later in this paper, calculating an L_1 spline in this manner is equivalent to determining the values of the two first partial derivatives of the surface, namely,

$$z_i^x = \left.\frac{\partial z}{\partial x}\right|_{\substack{x=x_i\\ y=y_i}}, \qquad z_i^y = \left.\frac{\partial z}{\partial y}\right|_{\substack{x=x_i\\ y=y_i}} \tag{2}$$

at the vertices (x_i, y_i), $i = 0, 1, \ldots, I$.

2.1 Definition of rHCT Elements

An rHCT element (see [11], from which much of the following description is derived) on a given triangle is a piecewise cubic surface over that triangle. One divides the triangle into three subtriangles that have a joint vertex at the barycenter. On each of the three subtriangles, the rHCT element is represented by a bivariate cubic function. These cubic functions join in a C^1-smooth fashion on the joint boundaries of the subtriangles. Furthermore, on the joint boundaries of the given triangle and neighboring triangles, the rHCT element of the given triangle joins in a C^1-smooth manner with those of the neighboring triangles.

An rHCT element $z(x,y)$ on a given triangle depends only on the values of the function and its first derivatives at the vertices of the triangle. Such an element is depicted in Figure 1. Let the vertices of the triangle be denoted, as in Figure 1, by (x_1,y_1), (x_2,y_2) and (x_3,y_3) and let them be ordered in a counterclockwise manner (when viewed from the positive z direction, that is, when looking at the xy plane from "above"). Denote the function values and first derivatives of z at the vertices by z_i, z_i^x and z_i^y:

$$z_i = z(x_i,y_i), \quad z_i^x = \left.\frac{\partial z\,(x,y)}{\partial x}\right|_{\substack{x=x_i\\ y=y_i}}, \quad z_i^y = \left.\frac{\partial z\,(x,y)}{\partial y}\right|_{\substack{x=x_i\\ y=y_i}}, \quad i=1,2,3. \tag{3}$$

It is convenient to express an rHCT element in the barycentric coordinates λ_1, λ_2 and λ_3, which are related to the xy coordinates by the equations

$$\begin{aligned} \lambda_1+\lambda_2+\lambda_3 &= 1\\ \lambda_1x_1+\lambda_2x_2+\lambda_3x_3 &= x\\ \lambda_1y_1+\lambda_2y_2+\lambda_3y_3 &= y. \end{aligned} \tag{4}$$

The barycenter or centroid of the triangle is characterized by $\lambda_1=\lambda_2=\lambda_3=1/3$ or, equivalently,

$$(x_0,y_0) := \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

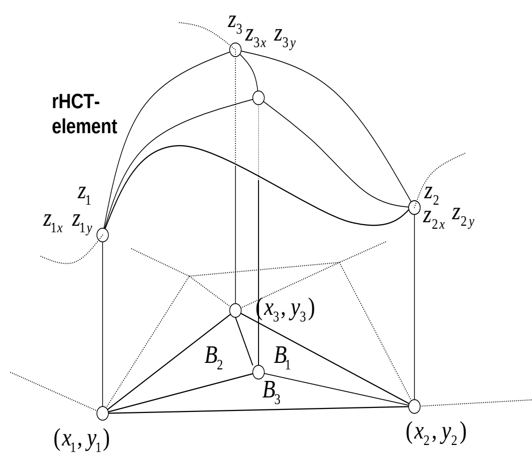


Figure 1: Reduced Hsieh-Clough-Tocher Element (rHCT)

In the definition of the rHCT element, we will use functions ρ_1, ρ_2 and ρ_3 defined on the

the three subtriangles B_j, $j = 1,2,3$ and the barycenter B_0

$$\begin{aligned} B_0 &:= \Big\{ (\lambda_1,\lambda_2,\lambda_3) \,|\quad \lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3} \Big\}, \\ B_1 &:= \{(\lambda_1,\lambda_2,\lambda_3) \,|\quad 0 \le \lambda_1 < \lambda_2, \quad \lambda_1 \le \lambda_3\}, \\ B_2 &:= \{(\lambda_1,\lambda_2,\lambda_3) \,|\quad 0 \le \lambda_2 < \lambda_3, \quad \lambda_2 \le \lambda_1\}, \\ B_3 &:= \{(\lambda_1,\lambda_2,\lambda_3) \,|\quad 0 \le \lambda_3 < \lambda_1, \quad \lambda_3 \le \lambda_2\}. \end{aligned} \tag{5}$$

We define, in particular,

$$\rho_1 := \begin{cases} \frac{1}{81} & for \quad (\lambda_1,\lambda_2,\lambda_3) \in B_0 \\ \lambda_1\lambda_2\lambda_3 + \frac{5}{6}\lambda_1^3 - \frac{1}{2}\lambda_1^2 & for \quad (\lambda_1,\lambda_2,\lambda_3) \in B_1 \\ -\frac{1}{6}\lambda_2^3 + \frac{1}{2}\lambda_2^2\lambda_3 & for \quad (\lambda_1,\lambda_2,\lambda_3) \in B_2 \\ -\frac{1}{6}\lambda_3^3 + \frac{1}{2}\lambda_3^2\lambda_2 & for \quad (\lambda_1,\lambda_2,\lambda_3) \in B_3. \end{cases} \tag{6}$$

To compress the number of formulae needed to express the ρ_i and other functions in an rHCT element, we will use the following cyclic substitution rule for indices:

$$\text{cyclic substitution :} \quad 1 \to 2 \to 3 \to 1 \tag{7}$$

where the arrows indicate replacement of an index value by its cyclic successor. Formulae for functions ρ_2 and ρ_3 are derived from formula (6) using cyclic substitution as defined in (7).

Define $x_{ij} := x_i - x_j$, $y_{ij} := y_i - y_j$, $z_{ij} := z_i - z_j$, $i,j = 1,2,3, i \neq j$. The rHCT element can be expressed as

$$z(x,y) = z_1\lambda_1 + z_2\lambda_2 + z_3\lambda_3 + M_1 + M_2 + M_3 + N_1 + N_2 + N_3 \tag{8}$$

where

$$M_1 = \left(z_2^x x_{32} + z_2^y y_{32} + z_3^x x_{23} + z_3^y y_{23}\right)\lambda_2\lambda_3\,, \tag{9}$$

$$\begin{aligned} N_1 =& \left[\left(z_2^x x_{32} + z_2^y y_{32} - z_3^x x_{23} - z_3^y y_{23}\right)/2 - z_{32}\right] \\ & \left[\lambda_2\lambda_3\left(\lambda_2 - \lambda_3\right) + 3\left(x_{13}^2 + y_{13}^2 - x_{12}^2 - y_{12}^2\right)/\left(x_{23}^2 + y_{23}^2\right)\rho_1 - \rho_2 - \rho_3\right] \end{aligned} \tag{10}$$

and M_2, M_3, N_2 and N_3 are calculated by formulae obtained from (9) and (10) using cyclic substitution.

The rHCT element has the advantageous property that two adjacent elements can be pieced together smoothly along the common boundary if they are smooth at their common vertices.

2.2 The L_1 Spline Minimization Principle in Cartesian and Barycentric Coordinates

On a given triangulation, our objective is to find

$$\begin{aligned} \arg\min_{z(x,y)} \Bigg\{ \iint\limits_{(x,y)\in\Omega} &\left[\left|\frac{\partial^2 z(x,y)}{\partial x^2}\right| + 2\left|\frac{\partial^2 z(x,y)}{\partial x \partial y}\right| + \left|\frac{\partial^2 z(x,y)}{\partial y^2}\right|\right] \mathrm{d}x\,\mathrm{d}y \\ & \Bigg| z \text{ consists of rHCT elements that cover } \Omega \text{ and} \\ & z \text{ interpolates given values at the vertices of triangles} \Bigg\} \end{aligned} \tag{11}$$

where Ω is the domain of the triangulation. The free parameters in this minimization are the values of the first derivatives of z at the nodes of the triangulation.

In each of the three subtriangles of each triangle in the triangulation, the second partial derivatives of z are linear functions of x and y and therefore also linear functions of the barycentric coordinates λ_1, λ_2 and λ_3. Recalling that $\lambda_1 + \lambda_2 + \lambda_3 = 1$, we can replace the variable λ_3 in the formulae for the second derivatives by λ_1 and λ_2. As a result, we obtain three new subtriangles in the (λ_1, λ_2) space, as shown in Figure 2, with

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= A^j_{xx}\lambda_1 + B^j_{xx}\lambda_2 + C^j_{xx},\\ \frac{\partial^2 z}{\partial x \partial y} &= A^j_{xy}\lambda_1 + B^j_{xy}\lambda_2 + C^j_{xy},\\ \frac{\partial^2 z}{\partial y^2} &= A^j_{yy}\lambda_1 + B^j_{yy}\lambda_2 + C^j_{yy},\end{aligned} \tag{12}$$

on subtriangle TB_j, $j = 1, 2, 3$, where A^j_{xx}, B^j_{xx}, C^j_{xx}, A^j_{xy}, B^j_{xy}, C^j_{xy}, A^j_{yy}, B^j_{yy}, C^j_{yy} are formulae expressed in terms of x_i, y_i, z_i, z^x_i and z^y_i at each vertex which are given in the Appendix.

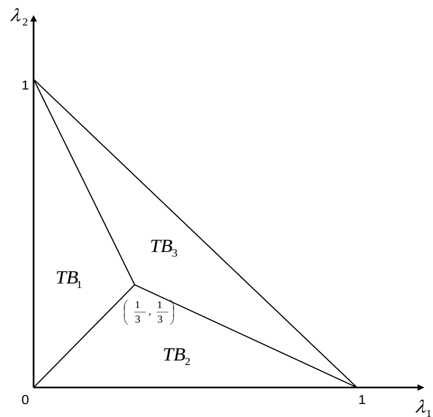


Figure 2: The new subtriangles

In the analysis presented later in this paper, we will use barycentric coordinates. The differential $\mathrm{d}x\,\mathrm{d}y$ and the differential $\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2$ are related by

$$\mathrm{d}x\,\mathrm{d}y = (x_{13}y_{23} - x_{23}y_{13})\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2. \tag{13}$$

Hence the integrals of each term of (12) on a triangle T in (x, y) space can be performed on TB_1, TB_2 and TB_3 in the (λ_1, λ_2) space (as illustrated in Figure 2). The objective function on each triangle T now becomes

$$\begin{aligned}\sum_{j=1}^{3}\int\int_{\mathrm{TB}_j}\left|f^j_{xx}(\lambda_1,\lambda_2)\right|\mathrm{d}\lambda_1\mathrm{d}\lambda_2 + 2\sum_{j=1}^{3}\int\int_{\mathrm{TB}_j}\left|f^j_{xy}(\lambda_1,\lambda_2)\right|\mathrm{d}\lambda_1\mathrm{d}\lambda_2\\ +\sum_{j=1}^{3}\int\int_{\mathrm{TB}_j}\left|f^j_{yy}(\lambda_1,\lambda_2)\right|\mathrm{d}\lambda_1\mathrm{d}\lambda_2\end{aligned} \tag{14}$$

where each of $f^j_{xx}(\lambda_1, \lambda_2)$, $f^j_{xy}(\lambda_1, \lambda_2)$ and $f^j_{yy}(\lambda_1, \lambda_2)$, $j = 1, 2, 3$, denotes a linear function of λ_1 and λ_2. Minimizing this nonsmooth functional defines the cubic L_1 spline with rHCT elements on one single triangle. The objective function on the whole space can be derived by summing functionals (14) for all of the triangles.

3 Geometric Programming Approach for Bivariate Cubic L_1 Spline on rHCT elements on TIN

In this section, we develop a generalized geometric programming framework. We first describe the theories of generalized geometric programming. Then we formulate the minimization principle for an L_1 spline on a single triangle and on multiple triangles. A primal problem and a geometric dual problem are developed for generating the C^1 smooth cubic L_1 splines on TINs.

3.1 Generalized Geometric Programming

Generalized geometric programming [14, 15] is a widely used optimization theory. In this section, we introduce the basic theory of generalized geometric programming on which our model is based. The $\mathbf{x}$ and $\mathbf{z}$ in this section are not related to the x and z used elsewhere in this paper.

3.1.1 Primal Problem

In generalized geometric programming, the *primal problem* is to find the minimizer of a real-valued convex function $\mathfrak{g}(x)$ over the intersection of the function domain $\mathfrak{C} \subseteq R^n$ and a cone $\mathfrak{X} \subseteq R^n$, that is,

$$(\textbf{Primal}) \qquad \begin{cases} \min \mathfrak{g}(\mathbf{x}) \\ \mathbf{x} \in \mathfrak{C} \cap \mathfrak{X} \end{cases} \tag{15}$$

3.1.2 Conjugate Transform

Definition 3.1 (Conjugate Transform) *Given a function $w(\mathbf{z})$ with domain $W \subseteq R^n$, the conjugate transform of $w(\mathbf{z})$ is a function $\omega(\zeta)$ with domain $\Omega \subseteq R^n$, where*

$$\Omega = \left\{ \zeta \in R^n \,\middle|\, \sup_{\mathbf{z} \in W} [\langle \zeta, \mathbf{z} \rangle - w(\mathbf{z})] < +\infty \right\}$$

and

$$\omega(\zeta) = \sup_{\mathbf{z} \in W} [\langle \zeta, \mathbf{z} \rangle - w(\mathbf{z})], \quad \forall \zeta \in \Omega$$

For a given function w, if the domain of its conjugate transform is empty, we say that its conjugate transform *does not exist.*

Theorem 3.1 *[14, 15] If a function $w(\mathbf{z})$ with domain $W \subseteq R^n$ is a convex function and W is a nonempty convex set, then there exists a conjugate transform of $w(\mathbf{z})$.*

It is known that the conjugate transform of a convex function is convex. Theorem 3.1 and the definition of the conjugate transform give us the following important inequality:

Theorem 3.2 (Conjugate Inequality) *[15] For each $\mathbf{z} \in W$ and $\zeta \in \Omega$,*

$$\langle \zeta, \mathbf{z} \rangle \leq w(\mathbf{z}) + \omega(\zeta) \tag{16}$$

with equality holding if and only if $\zeta \in \partial w(\mathbf{z})$, the set of subgradients of w at $\mathbf{z}$.

3.1.3 Dual Program

Given a convex function $\mathfrak{g}(x)$ over the domain $\mathfrak{C}$, denoted by $\mathfrak{g}:\mathfrak{C}$, the conjugate transform of $\mathfrak{g}:\mathfrak{C}$ is $\mathfrak{h}$ with domain $\mathfrak{D}$, denoted by $\mathfrak{h}:\mathfrak{D}$, where

$$\mathfrak{D} = \left\{ \mathbf{y} \in R^n \,\middle|\, \sup_{\mathbf{x}\in\mathfrak{C}}[\langle \mathbf{y},\mathbf{x}\rangle - \mathfrak{g}(\mathbf{x})] < +\infty \right\}$$

and

$$\mathfrak{h}(\mathbf{y}) = \sup_{\mathbf{x}\in\mathfrak{C}}[\langle \mathbf{y},\mathbf{x}\rangle - \mathfrak{g}(\mathbf{x})], \quad \forall \mathbf{y} \in \mathfrak{D}.$$

Let $\mathfrak{Y}$ be the dual cone of a given cone $\mathfrak{X}$, defined by

$$\mathfrak{Y} = \{\mathbf{y} \in R^n \,|\langle \mathbf{y},\mathbf{x}\rangle \geq 0, \quad \forall x \in \mathfrak{X}\}.$$

Then the dual problem of a given primal problem (15) becomes

$$(\mathbf{Dual}) \qquad \begin{cases} \min \mathfrak{h}(\mathbf{y}) \\ \mathbf{y} \in \mathfrak{D} \cap \mathfrak{Y} \end{cases} . \tag{17}$$

3.1.4 Optimality Conditions

Theorem 3.3 (Optimality Conditions) *[15]* $\mathbf{x}^*$ *and* $\mathbf{y}^*$ *are optimal solutions of the primal problem (15) and the dual problem (17), respectively, if and only if*

$$\begin{array}{ll} \text{(I)} & \mathbf{x}^* \in \mathfrak{C} \cap \mathfrak{X}, \mathbf{y}^* \in \mathfrak{D} \cap \mathfrak{Y} \\ \text{(II)} & \langle \mathbf{x}^*, \mathbf{y}^* \rangle = 0 \\ \text{(III)} & \mathbf{y}^* \in \partial \mathfrak{g}(\mathbf{x}^*) \stackrel{\triangle}{=} \{\mathbf{y} \in R^n \,|\mathfrak{g}(\mathbf{x}^*) + \langle \mathbf{y}, \mathbf{x} - \mathbf{x}^* \rangle \leq \mathfrak{g}(\mathbf{x}), \quad \forall \mathbf{x} \in \mathfrak{C}\} \end{array}$$

Optimality condition (I) indicates primal and dual feasibility. Optimality condition (II) is called the “orthogonality condition”. If the primal cone $\mathfrak{X}$ is a vector space, then its dual cone $\mathfrak{Y} = \mathfrak{X}^{\perp}$. Hence, the orthogonality condition is automatically satisfied and can be omitted. Optimality condition (III) is called the “subgradient condition”. When both function $\mathfrak{g}:\mathfrak{C}$ and cone $\mathfrak{X}$ are convex and closed, the primal problem (15) and the dual problem (17) are symmetric and the optimality condition (III) can be restated as

$$\text{(IIIa)} \;\; \mathbf{x}^* \in \partial \mathfrak{h}(\mathbf{y}^*) \text{ and } \mathbf{y}^* \in \partial \mathfrak{g}(\mathbf{x}^*)$$

Theorem 3.4 *[15] If* $\mathbf{x}$ *and* $\mathbf{y}$ *are feasible solutions of the primal problem (15) and the dual problem (17), respectively, then*

$$0 \leq \mathfrak{g}(\mathbf{x}) + \mathfrak{h}(\mathbf{y}),$$

with equality holding if and only if the optimality conditions (II) *and* (III) *are satisfied. In this case,* $\mathbf{x}$ *and* $\mathbf{y}$ *are optimal solutions of the primal problem (15) and dual problem (17), respectively.*

We denote the *relative interior* of a convex set $\mathfrak{D}$ by $ri(\mathfrak{D}) \equiv \{x \in \text{aff}(\mathfrak{D}) \,|\, \exists\, \epsilon > 0, (x + \epsilon \text{B}) \cap \text{aff}(\mathfrak{D}) \subset \mathfrak{D}\}$, where $\text{aff}(\mathfrak{D})$ is the affine hull of $\mathfrak{D}$ and B is the Euclidean unit ball in R^n, i.e., $\text{B} = \{x \,|\, \|x\|_2 \leq 1,\; x \in R^n\}$.

Theorem 3.5 *If the dual problem (17) has a feasible solution* $\mathbf{y}^* \in ri(\mathfrak{D})$ *and* $\inf_{y\in\mathfrak{D}\cap\mathfrak{Y}} \mathfrak{h}(\mathbf{y}) < +\infty$*, then the primal problem (15) has a nonempty solution set and*

$$0 = \inf_{x\in\mathfrak{C}\cap\mathfrak{X}} \mathfrak{g}(\mathbf{x}) + \inf_{y\in\mathfrak{D}\cap\mathfrak{Y}} \mathfrak{h}(\mathbf{y}).$$

Lemma 3.1 (Properties of Conjugate Transform) *Given that* $\mathfrak{g}(\boldsymbol{x}) : \boldsymbol{x} \in \mathfrak{C} \subset R^n$ *has a known conjugate transform* $\mathfrak{h}(\boldsymbol{y}) : \boldsymbol{y} \in \mathfrak{D} \subset R^n$*, then for a given vector* $\boldsymbol{u} \in R^n$*, the function* $\mathfrak{g}(\boldsymbol{x}+\boldsymbol{u}) : \boldsymbol{x}+\boldsymbol{u} \in \mathfrak{C}$ *has a conjugate transform* $\mathfrak{h}(\boldsymbol{y}) - \langle \boldsymbol{u}, \boldsymbol{y} \rangle : \boldsymbol{y} \in \mathfrak{D}$.

3.2 Geometric Programming Approach for Cubic L_1 Spline on TIN

3.2.1 Single Triangle Problem

Primal Problem:

For a single triangle problem, the objective function we want to minimize is functional (14). Since no C^1 smooth condition is necessary for a single triangle problem, the primal problem is an unconstrained optimization problem over one triangle, and the primal variables are z_i^x and z_i^y at the three vertices $i = 1, 2, 3$. For convenience, in the following derivation, we represent each function inside the absolute value signs of the right hand side of (14) as $f(\lambda_1, \lambda_2) = A\lambda_1 + B\lambda_2 + C$. This is, as mentioned, a linear function of λ_i, $i = 1, 2$. A, B and C are linear combinations of primal variables z_i^x and z_i^y that contain no λ_i.

Dual Problem:

Now we focus on how to derive the dual problem. From previous results [17], we know that the integration on a rectangular triangle T_r can be transformed into the integration on a standard triangle T_s whose three vertices are $(0,0)$, $(\frac{1}{2}, -\frac{1}{2})$ and $(-\frac{1}{2}, -\frac{1}{2})$, as shown in Figure 3. To distinguish the representation of on T_s from that on the triangles in the TINs, we use (t, s) to represent the coordinates on T_s. The objective function becomes

$$F(C_{12}, C_{03}, C_{02}) = h_i^x h_j^y \int\int_{\mathrm{T}_s} \left|2C_{12}h_i^x t + 6C_{03}h_j^y s + C_{12}h_i^x + 3C_{03}h_j^y + 2C_{02}\right| \mathrm{d}t\mathrm{d}s. \quad (18)$$

where h_i^x is the width of T_r and h_j^y is the height of the rectangle containing T_r. C_{12}, C_{03} and C_{02} are linear combinations of the primal variables, i.e., the unknown derivatives z_i^x and z_i^y.

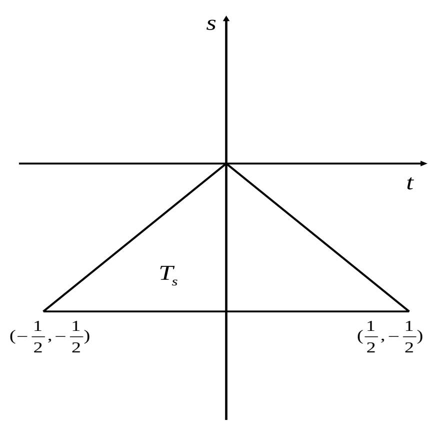


Figure 3: The standard triangle T_s

The conjugate transform of $F(C_{12}, C_{03}, C_{02})$ on T_s is

$$G(\xi, \eta, \gamma) = \sup_{(C_{12}, C_{03}, C_{02})} \{\xi C_{12} + \eta C_{03} + \gamma C_{02} - F(C_{12}, C_{03}, C_{02})\}, \quad (19)$$

defined on

$$\begin{aligned}\{(x,y,z)|x &\geq -y+z-3w+3\sqrt[3]{(-2w)(y+w)(z-w)},\\ x &\leq -y+z+3w-3\sqrt[3]{(-2w)(y-w)(z+w)},\\ y &\geq -x+z-3w+3\sqrt[3]{(-2w)(x+w)(z-w)},\\ y &\leq -x+z+3w-3\sqrt[3]{(-2w)(x-w)(z+w)},\\ z &\geq x+y-3w+3\sqrt[3]{2w(x-w)(y-w)},\\ z &\leq x+y+3w-3\sqrt[3]{2w(x+w)(y+w)} \quad \},\end{aligned} \qquad (D^*) \qquad (20)$$

where $x=\frac{\eta}{3h_j^y}$, $y=\frac{\xi}{2h_i^x}-\frac{\eta}{6h_j^y}$, $z=\frac{\xi}{2h_i^x}+\frac{\eta}{6h_j^y}-\frac{\gamma}{2}$ and $w=\frac{1}{12}h_i^x h_j^y$. Notice that x, y and z are in terms of the dual variables and w is the parameter.

Notice two things here: (i) the function $G(\xi,\eta,\gamma)$ defined in (19) will later be shown to be linear in the variables ξ, η and γ; (ii) the domain (D*) defined by (20) can actually be described in terms of cubic functions in the variables ξ, η and γ. Following [17], it can be proven that (D*) is a convex set in (ξ,η,γ) space.

To make use of this result, first we consider the special case of T_r being T_s. In this case $h_i^x=h_j^y=1$, the primal objective function becomes

$$F(C_{12},C_{03},C_{02})=\int\int_{\mathrm{T}_s}|2C_{12}t+6C_{03}s+C_{12}+3C_{03}+2C_{02}|\,\mathrm{d}t\mathrm{d}s, \qquad (21)$$

and the conjugate transform of function (21) is function (19), defined on $(x,y,z)\in D^*$, where $x=\frac{\eta}{3}$, $y=\frac{\xi}{2}-\frac{\eta}{6}$, $z=\frac{\xi}{2}+\frac{\eta}{6}-\frac{\gamma}{2}$, and $w=\frac{1}{12}$. For the present problem, each small triangle, TB_1, TB_2 and TB_3, can be transformed to triangle T_s. The previous primal variables are C_{12}, C_{03} and C_{02} and the present primal variables are z_i^x and z_i^y, $i=1,2,3$, in the representation of the coefficients A, B and C. To obtain our desired conjugate transform, we need to find a linear transformation linking the previous primal variables and the present primal variables. First we focus on finding a linear transformation mapping the old coefficient (C_{12},C_{03},C_{02}) space to the new coefficient (A,B,C) space. What we have now in TB_1 is

$$f(t,s)=At+\Big(\frac{2}{3}B-\frac{1}{3}A\Big)s+\Big(\frac{1}{3}A+\frac{1}{3}B+C\Big). \qquad (22)$$

Denote the function inside the integral in (21) by

$$g(t,s)=2C_{12}t+6C_{03}s+C_{12}+3C_{03}+2C_{02}, \qquad (23)$$

a linear function of s and t. There is a one-to-one correspondence between the coefficients in (22) and (23): $(A,\frac{2}{3}B-\frac{1}{3}A,\frac{1}{3}A+\frac{1}{3}B+C)=(2C_{12},6C_{03},C_{12}+3C_{03}+2C_{02})$. Now we consider a linear transformation that maps (C_{12},C_{03},C_{02}) space into (A,B,C) space. The linear relationship between the dual variables corresponding to (C_{12},C_{03},C_{02}) and the dual variables corresponding to (A,B,C) can be discovered in an analogous manner.

When a convex function $\mathfrak{g}(x):\mathfrak{C}$ and its conjugate transform $\mathfrak{h}(y):\mathfrak{D}$ are known, we are interested in finding the conjugate transform of $\overline{\mathfrak{g}}(x):\mathfrak{C}$, where $\overline{\mathfrak{g}}(x)=\mathfrak{g}(Tx)$ and T is a linear transformation. Suppose the conjugate transform is $\overline{\mathfrak{h}}(y)$, then

$$\overline{\mathfrak{h}}(y)=\sup_x\{yx-\mathfrak{g}(Tx)\}=\sup_x\{\big(yT^{-1}\big)\,(Tx)-\mathfrak{g}\,(Tx)\}. \qquad (24)$$

Therefore the desired conjugate transform is $\overline{\mathfrak{h}}(y)=\mathfrak{h}\left(yT^{-1}\right)$, defined on $\mathfrak{D}$.

For our problem in matrix form,

$$\left(A, \frac{2}{3}B - \frac{1}{3}A, \frac{1}{3}A + \frac{1}{3}B + C\right) = (A, B, C)\begin{pmatrix} 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}, \tag{25}$$

and

$$(2C_{12}, 6C_{03}, C_{12} + 3C_{03} + 2C_{02}) = (C_{12}, C_{03}, C_{02})\begin{pmatrix} 2 & 0 & 1 \\ 0 & 6 & 3 \\ 0 & 0 & 2 \end{pmatrix}. \tag{26}$$

The linear transformation T we look for can be presented as a matrix satisfying

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix} T = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 1 & 3 & 2 \end{pmatrix}. \tag{27}$$

Then

$$T = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{3}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 9 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{28}$$

Denote the original variables and the new variables by $Y = (\xi, \eta, \gamma)$ and $\overline{Y} = (\overline{\xi}, \overline{\eta}, \overline{\gamma})$, respectively. Then,

$$\overline{Y} = YT^{-1} = (\xi, \eta, \gamma)\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{18} & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{2}. \end{pmatrix} \tag{29}$$

Thus $(\overline{\xi}, \overline{\eta}, \overline{\gamma}) = (\frac{1}{2}\xi - \frac{1}{18}\eta, \frac{1}{9}\eta, \frac{1}{2}\gamma)$. Or, equivalently,

$$(\xi, \eta, \gamma) = (2\overline{\xi} + \overline{\eta}, 9\overline{\eta}, 2\overline{\gamma}). \tag{30}$$

Consequently, in subtriangle TB_1, the conjugate transform of $\left\|\frac{\partial^2 z}{\partial y^2}\right\|_1$, $\left\|\frac{\partial^2 z}{\partial x^2}\right\|_1$ and $\left\|\frac{\partial^2 z}{\partial x \partial y}\right\|_1$ are defined on $(x, y, z) \in D^*$ specified by $x = 3\overline{\eta}$, $y = \overline{\xi} - \overline{\eta}$, $z = \overline{\xi} + 2\overline{\eta} - \overline{\gamma}$ and $w = \frac{1}{12}$. The normal vectors along the dual cone have the same expressions as the ones in the previous results in terms of x, y and z defined here. The conjugate transform of $\left\|\frac{\partial^2 z}{\partial y^2}\right\|_1$, $\left\|\frac{\partial^2 z}{\partial x^2}\right\|_1$ and $\left\|\frac{\partial^2 z}{\partial x \partial y}\right\|_1$ on the other two subtriangles TB_2 and TB_3 can be obtained using similar linear transformations. In fact, in TB_2,

$$T = \begin{pmatrix} -1 & 9 & 0 \\ -2 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix}, \tag{31}$$

and $(\xi, \eta, \gamma) = (-\overline{\xi} - 2\overline{\eta} + \overline{\gamma}, 9\overline{\xi}, 2\overline{\gamma})$. The region on which the conjugate transform of the three terms is defined is (D^*) with $x = 3\overline{\eta}$, $y = -2\overline{\xi} - \overline{\eta} - \overline{\gamma}$, $z = \overline{\eta} - \overline{\gamma}$ and $w = \frac{1}{12}$. The normal vectors along the specified dual cone have the same expressions as the ones in the previous results in terms of x, y and z as defined here.

In TB_3,

$$T = \begin{pmatrix} -1 & 9 & 0 \\ -1 & -9 & 0 \\ 1 & 9 & 2 \end{pmatrix}, \tag{32}$$

and $(\xi,\eta,\gamma)=(-\overline{\xi}-\overline{\eta}+\overline{\gamma},-9\overline{\xi}-9\overline{\eta}+9\overline{\gamma},2\overline{\gamma})$. The region on which the conjugate transform of the three terms is defined is (D^*) with $x=-3\overline{\xi}-3\overline{\eta}+3\overline{\gamma}$, $y=-2\overline{\xi}+\overline{\eta}-\overline{\gamma}$, $z=-\overline{\xi}-2\overline{\eta}-\overline{\gamma}$ and $w=\frac{1}{12}$. The formulae of the normal vectors do not change except that x, y and z are defined differently here. Finally, to find the conjugate transform of the real primal variables z_i^x and z_i^y, $i=1,2,3$, we notice that there is a part in each expression of A, B and C which has nothing to do with the primal variables. We call this part "term constant with respect to z_i^x and z_i^y" and denote it by V_a, V_b and V_c, respectively, such that

$$
\begin{aligned}
A &= \hat{A}+V_a,\\
B &= \hat{B}+V_b,\\
C &= \hat{C}+V_c,
\end{aligned}
\tag{33}
$$

where $\hat{A}$, $\hat{B}$ and $\hat{C}$ are linear combinations of the primal variables z_i^x and z_i^y. According to Lemma 3.1, the conjugate transforms of the terms $\left\|\frac{\partial^2 z}{\partial y^2}\right\|_1$, $\left\|\frac{\partial^2 z}{\partial x^2}\right\|_1$ and $\left\|\frac{\partial^2 z}{\partial x\partial y}\right\|_1$ on TB_j, $j=1,2,3$, have the same expression which is

$$
G(\xi,\eta,\gamma)=\sup_{(A,B,C)}\{\xi A+\eta B+\gamma C-F(A,B,C)\}+V_a\xi+V_b\eta+V_c\gamma,
\tag{34}
$$

defined on the regions specified above. Each region defines a dual cone θ of the conjugate transform. Wang *et al.* [17] proved that G of (19) always equals 0, which implies that the first term on the right side of equation (34) is equal to 0. Therefore the conjugate transforms become

$$
G(\xi,\eta,\gamma)=V_a\xi+V_b\eta+V_c\gamma.
\tag{35}
$$

In function (34), $F(A,B,C)$ represents the primal functions of λ_1 and λ_2. They are the same functions as functions (12). The different representation allows us to treat the coefficients A, B and C as the variables for our primal problem. The A, B and C for each of the three terms in each subtriangle are different and are specified in the previous section. The linear transformations linking the known dual variables and our desired dual variables, are the same for the three terms in one subtriangle TB_j but are different in the other two. Finally, the dual problem becomes

$$
\begin{aligned}
\min\quad & \mathfrak{h}(\mathbf{y})=G(\xi,\eta,\gamma)\\
s.t.\quad & \mathbf{y}\in\theta
\end{aligned}
\tag{36}
$$

where θ is the domain constructed by the convex cubic constraints given in (20).

Dual to Primal Transformation:

According to Theorem 3.3, a primal optimal solution should be a linear combination of the linear independent normal vectors of the dual domain (D^*). The expressions $\hat{A},\hat{B}$ and $\hat{C}$ inside the A,B and C terms in the primal function $F(A,B,C)$ in the model, are linear combinations of the primal variables z_i^x and z_i^y. Then $\hat{A},\hat{B}$ and $\hat{C}$ should also be linear combinations of the linear independent normal vectors. Hence in the constraints of the dual to primal transformation, we set the linear combinations of the normal vectors equal to $\hat{A},\hat{B}$ and $\hat{C}$, respectively. Once we formulate the dual to primal transformation as a linear programming problem, any feasible solution to this problem is an optimal solution of the primal problem.

For a one-triangle problem with data points at (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3), a primal optimal solution can be obtained by solving the following linear optimization problem:

$$\min_{z_i^x, z_i^y, \gamma_{jyy}, \mu_{jyy}, \nu_{jyy}, \cdots} \sum_{j=1}^{3} [(|A_{yy}^j| + |B_{yy}^j| + |C_{yy}^j|) + (|A_{xx}^j| + |B_{xx}^j| + |C_{xx}^j|) + (|A_{xy}^j| + |B_{xy}^j| + |C_{xy}^j|)]$$

s.t.

$$\begin{aligned}
A_{yy}^1 - V_{a_{yy}}^1 &= \mu_{1yy}c_{11}^1 + \mu_{2yy}c_{12}^1 + \mu_{3yy}c_{13}^1 + \mu_{4yy}c_{14}^1 + \mu_{5yy}c_{15}^1 + \mu_{6yy}c_{16}^1 \\
B_{yy}^1 - V_{b_{yy}}^1 &= \mu_{1yy}c_{21}^1 + \mu_{2yy}c_{22}^1 + \mu_{3yy}c_{23}^1 + \mu_{4yy}c_{24}^1 + \mu_{5yy}c_{25}^1 + \mu_{6yy}c_{26}^1 \\
C_{yy}^1 - V_{c_{yy}}^1 &= \mu_{1yy}c_{31}^1 + \mu_{2yy}c_{32}^1 + \mu_{3yy}c_{33}^1 + \mu_{4yy}c_{34}^1 + \mu_{5yy}c_{35}^1 + \mu_{6yy}c_{36}^1 \\
A_{yy}^2 - V_{a_{yy}}^2 &= \gamma_{1yy}c_{11}^2 + \gamma_{2yy}c_{12}^2 + \gamma_{3yy}c_{13}^2 + \gamma_{4yy}c_{14}^2 + \gamma_{5yy}c_{15}^2 + \gamma_{6yy}c_{16}^2 \\
B_{yy}^2 - V_{b_{yy}}^2 &= \gamma_{1yy}c_{21}^2 + \gamma_{2yy}c_{22}^2 + \gamma_{3yy}c_{23}^2 + \gamma_{4yy}c_{24}^2 + \gamma_{5yy}c_{25}^2 + \gamma_{6yy}c_{26}^2 \\
C_{yy}^2 - V_{c_{yy}}^2 &= \gamma_{1yy}c_{31}^2 + \gamma_{2yy}c_{32}^2 + \gamma_{3yy}c_{33}^2 + \gamma_{4yy}c_{34}^2 + \gamma_{5yy}c_{35}^2 + \gamma_{6yy}c_{36}^2 \\
A_{yy}^3 - V_{a_{yy}}^3 &= \nu_{1yy}c_{11}^3 + \nu_{2yy}c_{12}^3 + \nu_{3yy}c_{13}^3 + \nu_{4yy}c_{14}^3 + \nu_{5yy}c_{15}^3 + \nu_{6yy}c_{16}^3 \\
B_{yy}^3 - V_{b_{yy}}^3 &= \nu_{1yy}c_{21}^3 + \nu_{2yy}c_{22}^3 + \nu_{3yy}c_{23}^3 + \nu_{4yy}c_{24}^3 + \nu_{5yy}c_{25}^3 + \nu_{6yy}c_{26}^3 \\
C_{yy}^3 - V_{c_{yy}}^3 &= \nu_{1yy}c_{31}^3 + \nu_{2yy}c_{32}^3 + \nu_{3yy}c_{33}^3 + \nu_{4yy}c_{34}^3 + \nu_{5yy}c_{35}^3 + \nu_{6yy}c_{36}^3 \\
A_{yy}^j &= f_{1yy}^j (z_1^x, z_1^y, z_2^x, z_2^y, z_3^x, z_3^y) \\
B_{yy}^j &= f_{2yy}^j (z_1^x, z_1^y, z_2^x, z_2^y, z_3^x, z_3^y) \\
C_{yy}^j &= f_{3yy}^j (z_1^x, z_1^y, z_2^x, z_2^y, z_3^x, z_3^y) \qquad (j = 1, 2, 3)
\end{aligned}$$

$$\begin{aligned}
A_{xx}^1 - V_{a_{xx}}^1 &= \mu_{1xx}c_{11}^1 + \mu_{2xx}c_{12}^1 + \mu_{3xx}c_{13}^1 + \mu_{4xx}c_{14}^1 + \mu_{5xx}c_{15}^1 + \mu_{6xx}c_{16}^1 \\
B_{xx}^1 - V_{b_{xx}}^1 &= \mu_{1xx}c_{21}^1 + \mu_{2xx}c_{22}^1 + \mu_{3xx}c_{23}^1 + \mu_{4xx}c_{24}^1 + \mu_{5xx}c_{25}^1 + \mu_{6xx}c_{26}^1 \\
C_{xx}^1 - V_{a_{xx}}^1 &= \mu_{1xx}c_{31}^1 + \mu_{2xx}c_{32}^1 + \mu_{3xx}c_{33}^1 + \mu_{4xx}c_{34}^1 + \mu_{5xx}c_{35}^1 + \mu_{6xx}c_{36}^1 \\
A_{xx}^2 - V_{a_{xx}}^2 &= \gamma_{1xx}c_{11}^2 + \gamma_{2xx}c_{12}^2 + \gamma_{3xx}c_{13}^2 + \gamma_{4xx}c_{14}^2 + \gamma_{5xx}c_{15}^2 + \gamma_{6xx}c_{16}^2 \\
B_{xx}^2 - V_{b_{xx}}^2 &= \gamma_{1xx}c_{21}^2 + \gamma_{2xx}c_{22}^2 + \gamma_{3xx}c_{23}^2 + \gamma_{4xx}c_{24}^2 + \gamma_{5xx}c_{25}^2 + \gamma_{6xx}c_{26}^2 \\
C_{xx}^2 - V_{c_{xx}}^2 &= \gamma_{1xx}c_{31}^2 + \gamma_{2xx}c_{32}^2 + \gamma_{3xx}c_{33}^2 + \gamma_{4xx}c_{34}^2 + \gamma_{5xx}c_{35}^2 + \gamma_{6xx}c_{36}^2 \\
A_{xx}^3 - V_{a_{xx}}^3 &= \nu_{1xx}c_{11}^3 + \nu_{2xx}c_{12}^3 + \nu_{3xx}c_{13}^3 + \nu_{4xx}c_{14}^3 + \nu_{5xx}c_{15}^3 + \nu_{6xx}c_{16}^3 \\
B_{xx}^3 - V_{b_{xx}}^3 &= \nu_{1xx}c_{21}^3 + \nu_{2xx}c_{22}^3 + \nu_{3xx}c_{23}^3 + \nu_{4xx}c_{24}^3 + \nu_{5xx}c_{25}^3 + \nu_{6xx}c_{26}^3 \\
C_{xx}^3 - V_{c_{xx}}^3 &= \nu_{1xx}c_{31}^3 + \nu_{2xx}c_{32}^3 + \nu_{3xx}c_{33}^3 + \nu_{4xx}c_{34}^3 + \nu_{5xx}c_{35}^3 + \nu_{6xx}c_{36}^3 \\
A_{xx}^j &= f_{1xx}^j (z_1^x, z_1^y, z_2^x, z_2^y, z_3^x, z_3^y) \\
B_{xx}^j &= f_{2xx}^j (z_1^x, z_1^y, z_2^x, z_2^y, z_3^x, z_3^y) \\
C_{xx}^j &= f_{3xx}^j (z_1^x, z_1^y, z_2^x, z_2^y, z_3^x, z_3^y) \qquad (j = 1, 2, 3)
\end{aligned}$$

$$
\begin{aligned}
A^1_{xy} - V^1_{a_{xy}} &= \mu_{1xy}c^1_{11} + \mu_{2xy}c^1_{12} + \mu_{3xy}c^1_{13} + \mu_{4xy}c^1_{14} + \mu_{5xy}c^1_{15} + \mu_{6xy}c^1_{16} \\
B^1_{xy} - V^2_{b_{xy}} &= \mu_{1xy}c^1_{21} + \mu_{2xy}c^1_{22} + \mu_{3xy}c^1_{23} + \mu_{4xy}c^1_{24} + \mu_{5xy}c^1_{25} + \mu_{6xy}c^1_{26} \\
C^1_{xy} - V^3_{c_{xy}} &= \mu_{1xy}c^1_{31} + \mu_{2xy}c^1_{32} + \mu_{3xy}c^1_{33} + \mu_{4xy}c^1_{34} + \mu_{5xy}c^1_{35} + \mu_{6xy}c^1_{36} \\
A^2_{xy} - V^1_{a_{xy}} &= \gamma_{1xy}c^2_{11} + \gamma_{2xy}c^2_{12} + \gamma_{3xy}c^2_{13} + \gamma_{4xy}c^2_{14} + \gamma_{5xy}c^2_{15} + \gamma_{6xy}c^2_{16} \\
B^2_{xy} - V^2_{b_{xy}} &= \gamma_{1xy}c^2_{21} + \gamma_{2xy}c^2_{22} + \gamma_{3xy}c^2_{23} + \gamma_{4xy}c^2_{24} + \gamma_{5xy}c^2_{25} + \gamma_{6xy}c^2_{26} \\
C^2_{xy} - V^3_{c_{xy}} &= \gamma_{1xy}c^2_{31} + \gamma_{2xy}c^2_{32} + \gamma_{3xy}c^2_{33} + \gamma_{4xy}c^2_{34} + \gamma_{5xy}c^2_{35} + \gamma_{6xy}c^2_{36} \\
A^3_{xy} - V^1_{a_{xy}} &= \nu_{1xy}c^3_{11} + \nu_{2xy}c^3_{12} + \nu_{3xy}c^3_{13} + \nu_{4xy}c^3_{14} + \nu_{5xy}c^3_{15} + \nu_{6xy}c^3_{16} \\
B^3_{xy} - V^2_{b_{xy}} &= \nu_{1xy}c^3_{21} + \nu_{2xy}c^3_{22} + \nu_{3xy}c^3_{23} + \nu_{4xy}c^3_{24} + \nu_{5xy}c^3_{25} + \nu_{6xy}c^3_{26} \\
C^3_{xy} - V^3_{c_{xy}} &= \nu_{1xy}c^3_{31} + \nu_{2xy}c^3_{32} + \nu_{3xy}c^3_{33} + \nu_{4xy}c^3_{34} + \nu_{5xy}c^3_{35} + \nu_{6xy}c^3_{36} \\
A^j_{xy} &= f^j_{1xy}\left(z^x_1, z^y_1, z^x_2, z^y_2, z^x_3, z^y_3\right) \\
B^j_{xy} &= f^j_{2xy}\left(z^x_1, z^y_1, z^x_2, z^y_2, z^x_3, z^y_3\right) \\
C^j_{xy} &= f^j_{3xy}\left(z^x_1, z^y_1, z^x_2, z^y_2, z^x_3, z^y_3\right) \qquad (j = 1, 2, 3)
\end{aligned}
$$

Here, the superscripts $j = 1, 2$ and 3 mean that the term is generated in triangles TB_1, TB_2 and TB_3, respectively. The subscripts xx, yy and xy indicate that the term is generated by the terms $\left\|\frac{\partial^2 z}{\partial x^2}\right\|_1$, $\left\|\frac{\partial^2 z}{\partial y^2}\right\|_1$ and $2\left\|\frac{\partial^2 z}{\partial x \partial y}\right\|_1$, respectively. The μ, γ and ν terms are the unknown coefficients in the linear combination of the normal vectors, which are also variables. The V_a, V_b and V_c terms are the parts in A, B and C that do not contain the variables z^x_i and z^y_i .

There are three groups of constraints that come from the three terms $\left\|\frac{\partial^2 z}{\partial x^2}\right\|_1$, $\left\|\frac{\partial^2 z}{\partial y^2}\right\|_1$ and $2\left\|\frac{\partial^2 z}{\partial x \partial y}\right\|_1$. In each group, the first nine constraints indicate that each of the A, B C terms is a linear function of the normal vectors. The last three constraints in each group indicate that each of the A, B, C terms is a linear function of the primal variables z^x_i and z^y_i, $i = 1, 2, 3$.

The linear programming problem for one triangle contains $27 + 6 + 18 \times 3 = 87$ variables and $(9 + 9) \times 3 = 54$ constraints. The variables can be divided into three groups. The first group contains A^j_{xy}, B^j_{xy}, C^j_{xy}, A^j_{xx}, B^j_{xx}, C^j_{xx}, A^j_{yy}, B^j_{yy}, C^j_{yy}, $j = 1, 2, 3$, for the three terms $\left\|\frac{\partial^2 z}{\partial y^2}\right\|_1$, $\left\|\frac{\partial^2 z}{\partial x^2}\right\|_1$ and $\left\|\frac{\partial^2 z}{\partial x \partial y}\right\|_1$ in the three subtriangles TB_j, respectively. The second group contains z^x_1, z^y_1, z^x_2, z^y_2, z^x_3, z^y_3, the derivatives along axis x and axis y in each (x, y) location. The third group contains all the unknown coefficients before the normal vectors which define the linear combination. We represent the objective function in terms of the absolute values of the A, B and C terms. This function is somewhat arbitrary since we seek only one feasible solution to this linear programming problem.

3.3 Solving Multi-triangle Problems

In this section we extend the geometric programming model of the one-triangle problem to multi-triangle problems.

First, we consider the problem in the original (x, y) space with objective functional (1). The interpolating function generated by the triangular patch should be C^1 smooth along the boundaries of each triangle. According to the property of the rHCT element, to obtain the smoothness, we merely need to add the constraints in the primal model such that the partial derivatives with respect to x and with respect to y are consistent at the common vertices shared by different triangles. Recall that the partial derivatives at each vertex of each triangle are the decision variables. Let Z_x be the vector containing all the partial derivatives with respect to x and Z_y be the the vector containing all the partial derivatives with respect to y.

Then a general representation of the optimization problem becomes

$$\min \iint_{(x,y)\in\Omega} \left(\left|\frac{\partial^2 z}{\partial x^2}\right| + 2\left|\frac{\partial^2 z}{\partial x \partial y}\right| + \left|\frac{\partial^2 z}{\partial y^2}\right| \right) \mathrm{d}x\mathrm{d}y$$

s.t.

$$\begin{array}{c} Consistency \quad of \quad Z_x \\ Consistency \quad of \quad Z_y \end{array} \tag{37}$$

Writing down the constraints in a general matrix form, we have the problem

$$\min \iint_{(x,y)\in\Omega} \left(\left|\frac{\partial^2 z}{\partial x^2}\right| + 2\left|\frac{\partial^2 z}{\partial x \partial y}\right| + \left|\frac{\partial^2 z}{\partial y^2}\right| \right) \mathrm{d}x\mathrm{d}y$$

s.t.

$$M \begin{pmatrix} Z_x \\ Z_y \end{pmatrix} = 0 \tag{38}$$

where M is the matrix whose elements are 0, -1 and 1. It represents that the equations making the partial derivatives at a common point equal to each other. Hence a problem containing n triangles with $3n$ data points in the (x, y)-plane requires solving an optimization problem with $6n$ variables and $\sum_{h=1}^{3n} \left[2\left(k_h - 1\right)\right]$ constraints, where k_h denotes the number of triangles sharing point h.

Based on the results from the one-triangle problem, we apply the geometric programming method to an n-triangle-problem. On each triangle l, $l = 1, 2, 3, \cdots, n$, we group the A, B, C terms according to the triangle index l by

$$\begin{aligned} A_l &:= \left(A^1_{xx}, A^1_{xy}, A^1_{yy}, A^2_{xx}, A^2_{xy}, A^2_{yy}, A^3_{xx}, A^3_{xy}, A^3_{yy}\right), \\ B_l &:= \left(B^1_{xx}, B^1_{xy}, B^1_{yy}, B^2_{xx}, B^2_{xy}, B^2_{yy}, B^3_{xx}, B^3_{xy}, B^3_{yy}\right), \\ C_l &:= \left(C^1_{xx}, C^1_{xy}, C^1_{yy}, C^2_{xx}, C^2_{xy}, C^2_{yy}, C^3_{xx}, C^3_{xy}, C^3_{yy}\right). \end{aligned} \tag{39}$$

Notice that each A, B and C term inside A_l, B_l and C_l is a linear function of the partial derivatives at the three vertices of that triangle. We define the following notation to represent this relation

$$\begin{aligned} A_l &:= f_{lA}\left(z^x_{l_1}, z^y_{l_1}, z^x_{l_2}, z^y_{l_2}, z^x_{l_3}, z^y_{l_3}\right), \\ B_l &:= f_{lB}\left(z^x_{l_1}, z^y_{l_1}, z^x_{l_2}, z^y_{l_2}, z^x_{l_3}, z^y_{l_3}\right), \\ C_l &:= f_{lC}\left(z^x_{l_1}, z^y_{l_1}, z^x_{l_2}, z^y_{l_2}, z^x_{l_3}, z^y_{l_3}\right). \end{aligned} \tag{40}$$

Now we can write down the primal problem for the n-triangle-problem as

$$\min \quad F\left(A_1, B_1, C_1, \cdots, A_l, B_l, C_l, \cdots, A_n, B_n, C_n\right)$$

s.t.

$$M\begin{pmatrix} Z_x \\ Z_y \end{pmatrix} = 0$$

$$\begin{aligned} A_l &= f_{lA}\left(z_{l_1}^x, z_{l_1}^y, z_{l_2}^x, z_{l_2}^y, z_{l_3}^x, z_{l_3}^y\right) \\ B_l &= f_{lB}\left(z_{l_1}^x, z_{l_1}^y, z_{l_2}^x, z_{l_2}^y, z_{l_3}^x, z_{l_3}^y\right) \\ C_l &= f_{lC}\left(z_{l_1}^x, z_{l_1}^y, z_{l_2}^x, z_{l_2}^y, z_{l_3}^x, z_{l_3}^y\right), \quad l = 1, 2, \cdots, n. \end{aligned} \tag{41}$$

The constraints form a primal cone $\mathfrak{X}$. We will have an individual dual problem corresponding to each one-triangle primal problem, which we denote as (TRI_l) for triangle l. Let the domain of (TRI_l) be θ_l. Also let $h_l(y)$ represent the dual objective function for each individual triangle l. Then the dual problem becomes

$$\min \quad h(y) = \sum_{l=1}^{n} h_l(y)$$

s.t.

$$y \in (\theta_1 \times \cdots \times \theta_l \cdots \times \theta_n) \bigcap \mathfrak{Y} \tag{42}$$

where $\mathfrak{Y}$ is the dual cone of $\mathfrak{X}$.

Now we are ready to present a general form for the dual to primal transform of a multi-triangle problem. Using the notation (PD_l) to denote the dual to primal transform for triangle l, the final linear programming problem we face has the following structure

$$\min \sum_{l=1}^{n} \{Objective \quad function \quad in \quad (PD_l)\}$$

s.t

$$M\begin{pmatrix} Z_x \\ Z_y \end{pmatrix} = 0 \tag{43}$$

$$Constraints \quad in \quad (PD_l), \quad l = 1, 2, 3, \cdots, n$$

4 Computational Results and Conclusion

In order to test the generalized geometric programming framework, an AMPL program [6] for the dual problem was written and submitted to the nonlinear solver MOSEK 3 [3, 4, 7], which yields a dual solution. We plug the dual solution into the dual-to-primal transformation to get corresponding primal solutions Z_x and Z_y using MOSEK 3. First, we construct an L_1 spline on a single triangle. Since there are only three data points for a single triangle, and the objective function is to minimize the second derivative of the function, the result is a plane passing through the given three data points. The L_1 spline generated by the proposed GP model is shown below with data points $(0, 0)$, $(1, 3)$ and $(3, 0)$. The elevations are 4, 12 and 8, respectively.

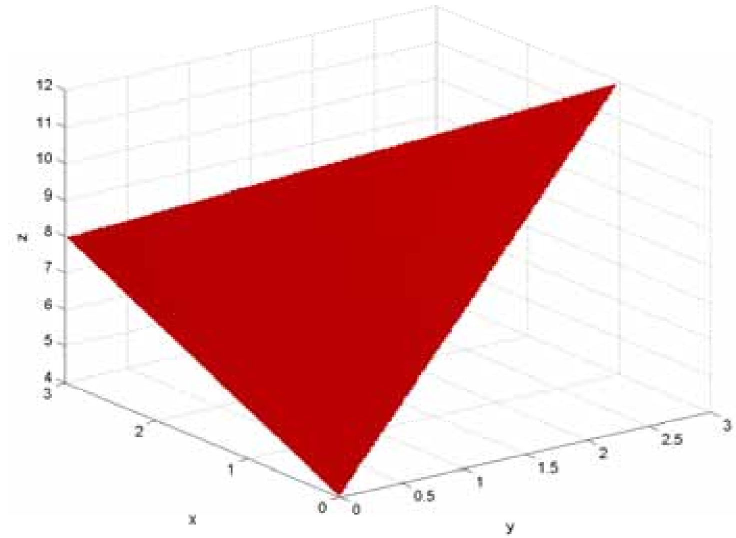
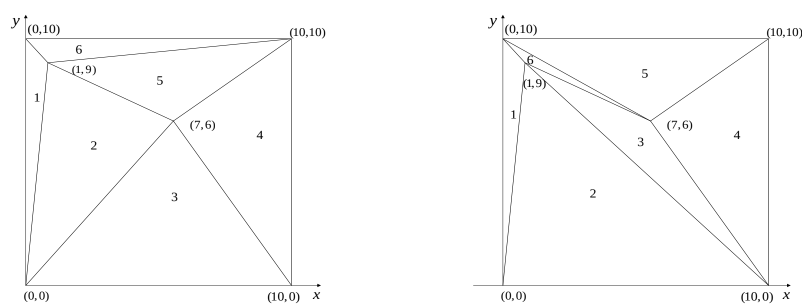


Figure 4: L_1 spline on one-triangle TIN

4.1 Experiments on Triangulated Irregular Networks

Next we build examples on triangulated irregular networks. Given six data points $(0,0)$, $(10,0)$, $(7,6)$, $(1,9)$, $(0,10)$ and $(10,10)$, we consider two potential triangulations TIN 1 and TIN 2 as shown in Figure 5.



(a) TIN 1 (b) TIN 2

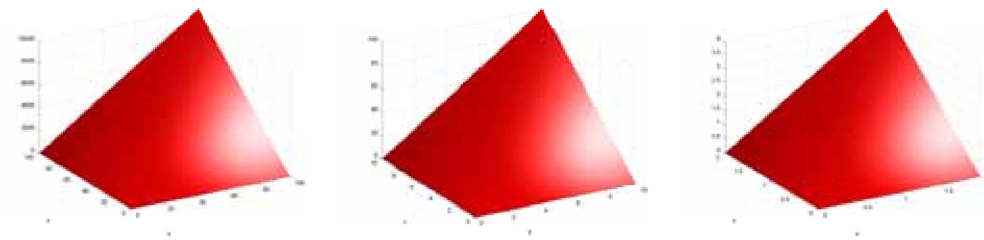
Figure 5: Two potential irregular triangulations

The elevation value at location (x_i, y_i) is assigned according to a given function of x_i and y_i to see whether the interpolating surface can preserve the shape of the original function. The following two examples are executed on both TIN 1 and TIN 2.

Example 4.1 *The elevations are given by* $z_i = x_i y_i$, $i = 1, 2, \cdots, 6$.

Example 4.2 *The elevations are given by* $z_i = x_i^2 y_i$, $i = 1, 2, \cdots, 6$.

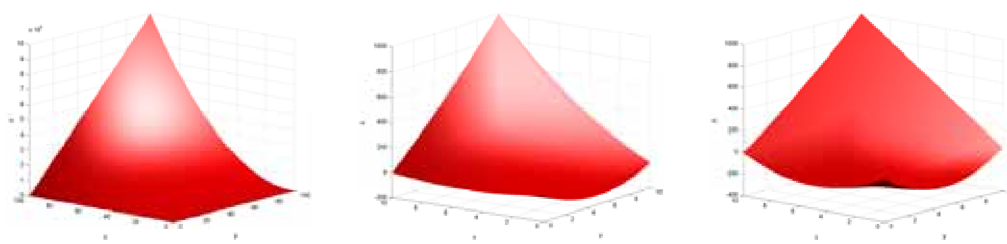
In Figures 6 and 7, plot (a) shows the shape of the original function, plot (b) the L_1 spline generated by the GP model on TIN 1, and plot (c) the L_1 spline generated by the GP model on TIN 2.



(a) $z(x, y) = xy$ (b) L_1 spline on TIN 1 (c) L_1 spline on TIN 2

Figure 6: L_1 spline for Example 4.1

For Example 4.1, Figure 6 shows that the splines generated using the rHCT elements visually preserve the shape of quadratic function very well, as one expects from previous experience on rectangular grids. Moreover, there is no visual difference between the two L_1 splines using different triangulations.



(a) $z(x, y) = x^2y$ (b) L_1 spline on TIN 1 (c) L_1 spline on TIN 2

Figure 7: L_1 splines for Example 4.2

For Example 4.2, Figure 7 also shows that the resulting L_1 splines preserve the shape of the cubic function well. However, the difference between plots (b) and (c) in Figure 7 indicates that different triangulations may result in different splines. We may notice that on the margin of the domain some undesired oscillations occur in plots (b) and (c). The reason is that the online solver quits before it achieves the optimal solution of the dual problem.

4.2 Experiments on TIN with Regular Grids

In this section, we would like to further investigate how splines are affected by different triangulations on the same data set. We are also interested in knowing whether the TIN model can produce better results than models on regular rectangular grids. For these purposes, we conduct experiments on data set at locations (x_i, y_j), $x_i = i$, $i = 0, 1, 2, 3$, and $y_j = j$,

$j = 0, 1, 2, 3$. The 3 by 3 block is formed by 9 squares (refer to Figures 8 and 10). These 9 squares are the grids on which the Sibson elements will be based (Figures 8(d) and 10(d)). For the TIN model, we consider the triangulations derived by dividing each square into two triangles. Therefore, there are $2^9 = 512$ potential triangulations in total. We execute the experiments on each of these 512 TINs and record the energy values (i.e., the objective values) of each case.

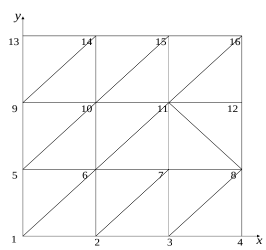
The following two examples are used in our experiments:

Example 4.3 *The elevations are given by*

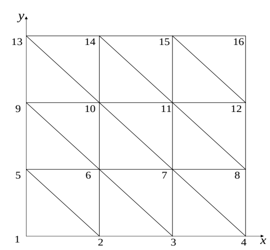
$$z = (z_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 10 \\ 0 & 0 & 10 & 0 \\ 0 & 10 & 0 & 0 \\ 10 & 0 & 0 & 0 \end{pmatrix}$$

Example 4.4 *The elevations are given by*

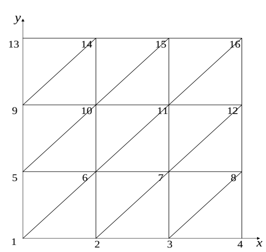
$$z = (z_{ij}) = \begin{pmatrix} 0 & 0 & 10 & 0 \\ 0 & 10 & 10 & 10 \\ 10 & 10 & 10 & 0 \\ 0 & 10 & 0 & 0 \end{pmatrix}$$



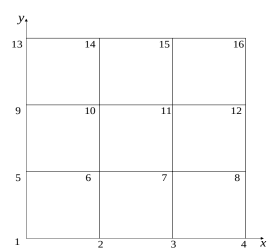
(a) TIN 3



(b) TIN 4



(c) TIN 5

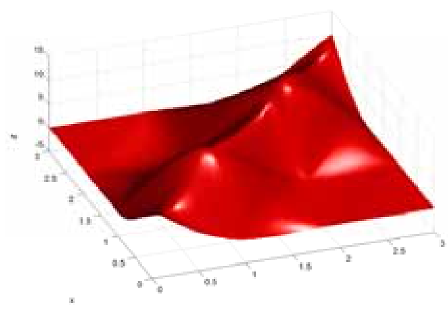


(d) Grid for Sibson elements

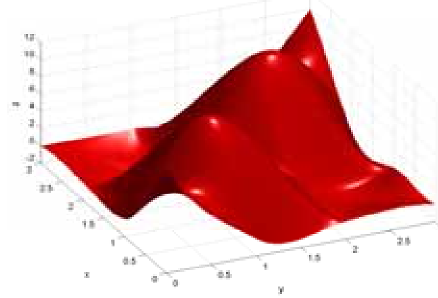
Figure 8: Grid and Three TINs for Example 4.3

To avoid early termination of the online general purpose solver, we generate the L_1 splines on both rHCT and Sibson elements using a discretized model with a primal-dual based solver [18]. The model discretizes the integral in the objective function so that the nonsmooth convex problem can be transformed into an equivalent overdetermined linear system [9] and solved using a primal-dual interior point algorithm [18]. The corresponding energy value (i.e., the objective function value) is denoted by *Obj* in the captions of the figures.

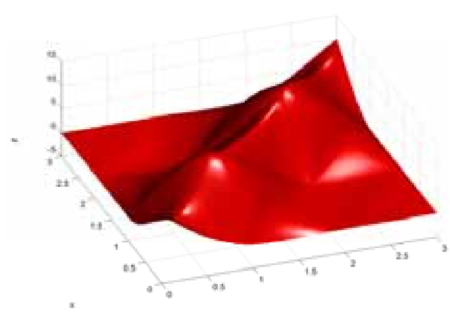
For Example 4.3, we highlight the triangulations and grid shown in Figure 8, because TIN 3 achieves the lowest energy value, TIN 4 achieves the highest energy value, and TIN 5 achieves most visually appealing shape-preserving result. The resulting L_1 splines are shown in Figure 9.



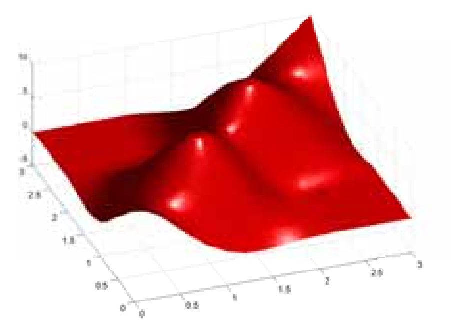
(a) L_1 spline on TIN 3: *Obj*=5.8992745



(b) L_1 spline on TIN 4: *Obj*=6.7883895



(c) L_1 spline on TIN 5: *Obj*=5.9930085



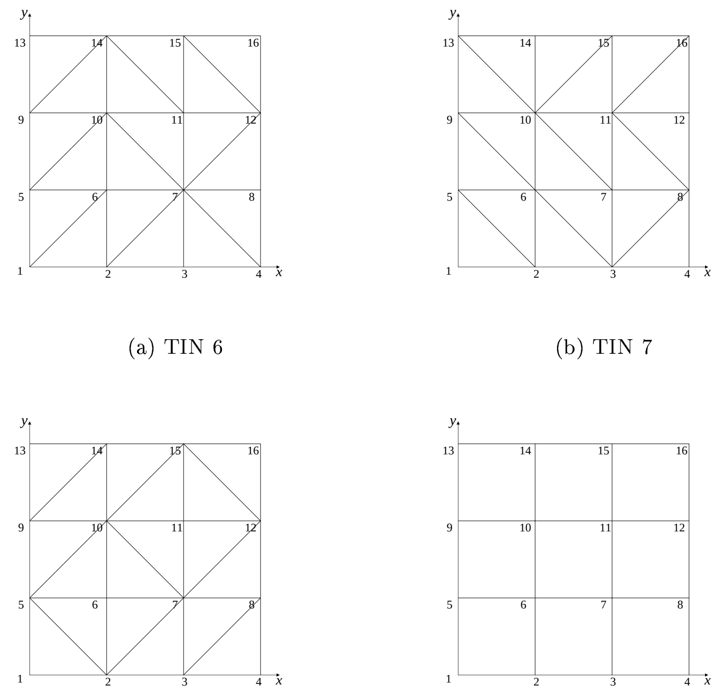
(d) L_1 spline on Sibson: *Obj*=12.412200

Figure 9: L_1 splines on rHCT and Sibson elements for Example 4.3

Figure 9 clearly shows that different triangulations lead to different interpolating surfaces. It also shows that the rHCT elements on TINs have the flexibility to "line up" the triangulation with the data to produce more visual-appealing shape-preserving cubic L_1 splines with a lower energy level than the Sibson elements on the grids.

For Example 4.4, we highlight the following triangulations and grid shown in Figure 10, because TIN 6 achieves the lowest energy value, TIN 7 achieves the highest energy value, and TIN 8 achieves most visual-appealing shape-preserving result.

The resulting L_1 splines for Example 4.4 on selected TINs and the rectangular grid are



(a) TIN 6 (b) TIN 7

(c) TIN 8 (d) Grid for Sibson elements

Figure 10: Grid and Three TINs for Example 4.4

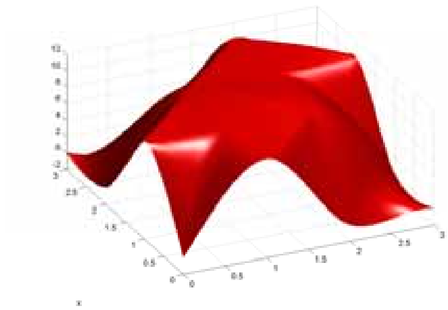
shown as Figure 11 .

Plot (a) of Figure 11 shows the spline with the lowest energy value and plot (b) shows the spline with the highest energy value. While the lower-energy plot (a) shows a much more visually appealing shape-preserving surface than the higher-energy plot (b), the most visually appealing shape-preserving surface is given by plot (c) on TIN 8. It suggests that a triangulation that "matches" the "data trend" may produce the most visually appealing shape-preserving L_1 spline, although it may not achieve the lowest objective value. Figure 11 again shows that compared to the Sibson elements, rHCT elements on TINs are more flexible and are capable of "lining up" the triangulation with the data to produce more visually appealing shape-preserving cubic L_1 splines with a lower energy level.

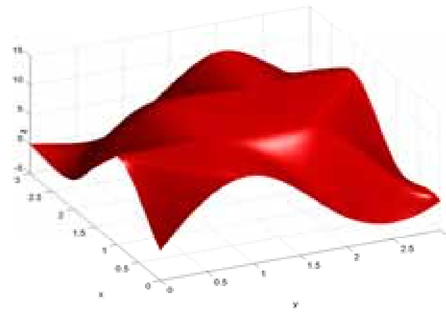
5 Concluding Remarks

In this paper, we have proposed a generalized geometric programming framework for constructing bivariate cubic L_1 splines on TINs using rHCT elements. The required primal problem, dual problem and dual-to-primal transformation have been derived. Our computational experiments not only validate this approach but also show its flexibility and superiority over the approach using Sibson elements on regular rectangular grids.

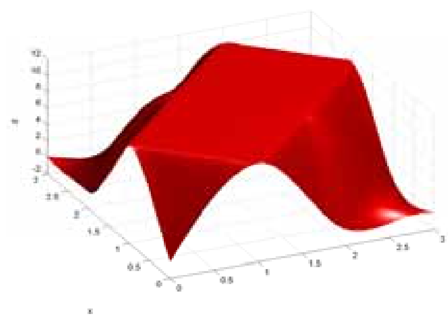
Our examples have shown that different triangulations may lead to very different L_1



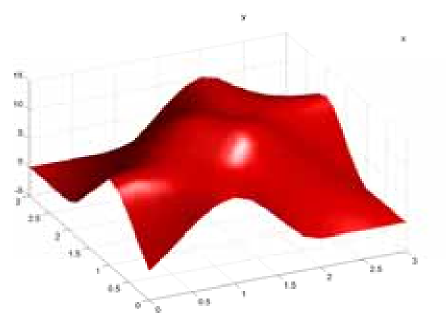
(a) L_1 spline on TIN 6: Obj=3.966516



(b) L_1 spline on TIN 7: Obj=4.786172



(c) L_1 spline on TIN 8: Obj=4.166171



(d) L_1 spline on Sibson: Obj=12.636660

Figure 11: L_1 splines on rHCT and Sibson elements for Example 4.4

splines. Most traditional triangulation procedures take account only of the locations of given data. Our experiments suggest that the elevations of the data points should also be considered in generating visually appealing shape-preserving interpolants. Our results also show that the most visually appealing shape-preserving spline does not always correspond to the spline with the lowest objective value. The results presented here may be relevant for future development of improved triangulation procedures.

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A Appendix: Formulae of A, B and Cs

We will use the abbreviations

$$
\begin{aligned}
D_{xy} &= -D_{yx} = x_1 y_{23} + x_2 y_{31} + x_3 y_{12},\\
Q_{ij} &= \left(z_{ix}x_{ji} + z_{iy}y_{ji} + z_{jx}x_{ij} + z_{jy}y_{ji}\right)/2, \quad i,j=1,2,3, \quad i \neq j,\\
C_{ij} &= \left(z_{ix}x_{ji} + z_{iy}y_{ji} - z_{jx}x_{ij} - z_{jy}y_{ji}\right)/2, \quad i,j=1,2,3, \quad i \neq j,\\
K_1 &= 3\left(x_{13}^2 + y_{13}^2 - x_{12}^2 - y_{12}^2\right) / \left(x_{23}^2 + y_{23}^2\right),\\
K_2 &= 3\left(x_{21}^2 + y_{21}^2 - x_{23}^2 - y_{23}^2\right) / \left(x_{31}^2 + y_{31}^2\right),\\
K_3 &= 3\left(x_{32}^2 + y_{32}^2 - x_{31}^2 - y_{31}^2\right) / \left(x_{12}^2 + y_{12}^2\right).
\end{aligned}
$$

The term $\frac{\partial^2 z}{\partial x^2}$ in the triangle TB_1 can be written as

$$\frac{\partial^2 z}{\partial x^2} = A_{xx}^1 \lambda_1 + B_{xx}^1 \lambda_2 + C_{xx}^1$$

where,

$$
\begin{aligned}
A_{xx}^1 = {} & C_{23}\Big[\frac{K_1}{D_{xy}^2}(5y_{23}^2 + 2y_{31}y_{12} - 2y_{23}y_{31}) + \frac{1}{D_{xy}^2}(y_{23}^2 - 2y_{31}^2 + 2y_{12}y_{31} - 2y_{23}y_{31})\Big]\\
& + C_{31}\Big[\frac{K_2}{D_{xy}^2}(-2y_{23}^2 + 2y_{12}y_{23}) + \frac{1}{D_{xy}^2}(2y_{12}^2 - 4y_{23}y_{12} + 8y_{23}^2 - 4y_{23}y_{31} + 2y_{12}y_{23})\Big]\\
& + C_{12}\Big[\frac{K_3}{D_{xy}^2}(-y_{23}^2 + 2y_{23}y_{31}) + \frac{1}{D_{xy}^2}(4y_{31}y_{23} - 2y_{31}^2 - 7y_{23}^2 - 2y_{31}y_{12} + 2y_{12}y_{23})\Big]
\end{aligned}
$$

$$\begin{aligned}
B^1_{xx} &= C_{23}\Big[\frac{K_1}{D^2_{xy}}(2y_{12}y_{23} - 2y_{23}y_{31}) + \frac{1}{D^2_{xy}}(4y_{12}y_{31} - 2y^2_{12} - 2y^2_{31} + 2y^2_{23})\Big] \\
&\quad + C_{31}\Big[\frac{K_2}{D^2_{xy}}(y^2_{23}) + \frac{1}{D^2_{xy}}(y^2_{23} - 2y_{23}y_{31})\Big] \\
&\quad + C_{12}\Big[\frac{K_3}{D^2_{xy}}(y^2_{23}) + \frac{1}{D^2_{xy}}(y^2_{23} - 2y_{12}y_{23})\Big] \\
C^1_{xx} &= \frac{2}{D^2_{xy}}\Big[Q_{23}y_{12}y_{31} + Q_{31}y_{23}y_{12} + Q_{12}y_{31}y_{23}\Big] \\
&\quad + C_{23}\Big[\frac{2}{D^2_{xy}}(y^2_{31} - y_{12}y_{31}) + \frac{K_1}{D^2_{xy}}\Big(2y_{23}y_{31} - y^2_{23} - \frac{1}{D^2_{xy}}(y^2_{23})\Big)\Big] \\
&\quad + C_{31}\Big[\frac{2}{D^2_{xy}}(y_{12}y_{23} - y^2_{23}) + \frac{K_1}{D^2_{xy}}(y^2_{23}) + \frac{1}{D^2_{xy}}(2y_{31}y_{23} - y^2_{23})\Big] \\
&\quad + C_{12}\Big[-\frac{1}{D^2_{xy}}(2y_{31}y_{23} - 2y^2_{23})\Big]
\end{aligned}$$

In triangle TB_2, we have

$$\frac{\partial^2 z}{\partial x^2} = A^2_{xx}\lambda_1 + B^2_{xx}\lambda_2 + C^2_{xx}$$

where,

$$\begin{aligned}
A^2_{xx} &= C_{23}\Big[\frac{1}{D^2_{xy}}(-y^2_{31} + 2y_{23}y_{31})\Big] \\
&\quad + C_{31}\Big[\frac{K_2}{D^2_{xy}}(2y2_{31}y_{12} - 2y_{23}y_{31}) + \frac{1}{D^2_{xy}}(2y^2_{12} - 2y^2_{31} + 2y^2_{23} - 4y_{12}y_{23})\Big] \\
&\quad + C_{12}\Big[\frac{K_3}{D^2_{xy}}(y^2_{31}) + \frac{1}{D^2_{xy}}(2y_{31}y_{12})\Big]
\end{aligned}$$

$$\begin{aligned}
B^2_{xx} &= C_{23}\Big[\frac{K_1}{D^2_{xy}}(-2y^2_{31} + 2y_{12}y_{31})\Big] \\
&= C_{31}\Big[\frac{K_2}{D^2_{xy}}(5y^2_{31} + 2y_{12}y_{23} - 2y_{31}y_{23}) \\
&\quad + \frac{1}{D^2_{xy}}(2y^2_{23} - y^2_{31} - 2y_{23}y_{12} + 2y_{31}y_{12} - 2y_{23}y_{31})\Big] \\
&\quad + C_{12}\Big[\frac{K_3}{D^2_{xy}}(-2y^2_{31} + 2y_{23}y_{31}) + \frac{1}{D^2_{xy}}(2y^2_{23} + 7y^2_{31} - 4y_{23}y_{31} - 2y_{31}y_{12}) + 2y_{23}y_{12}\Big]
\end{aligned}$$

$$\begin{aligned}
C^2_{xx} &= \frac{2}{D^2_{xy}}\Big[Q_{23}y_{12}y_{31} + Q_{31}y_{23}y_{12} + Q_{12}y_{31}y_{23}\Big] \\
&\quad + C_{23}\Big[\frac{K_1}{D^2_{xy}}(y^2_{31}) + \frac{1}{D^2_{xy}}(3y^2_{31} - 2y_{12}y_{31} - 2y_{23}y_{31})\Big] \\
&\quad + C_{31}\Big[\frac{K_2}{D^2_{xy}}(-y^2_{31} + 2y_{23}y_{31}) + \frac{1}{D^2_{xy}}(-2y^2_{23} + y^2_{31} + 2y_{23}y_{12})\Big] \\
&\quad + C_{12}\Big[\frac{1}{D^2_{xy}}(-2y^2_{31} + 2y_{23}y_{31})\Big]
\end{aligned}$$

In triangle TB_3, we have

$$\frac{\partial^2 z}{\partial x^2} = A^3_{xx}\lambda_1 + B^3_{xx}\lambda_2 + C^3_{xx}$$

where,

$$\begin{aligned}A^3_{xx} &= C_{23}\Big[\frac{1}{D^2_{xy}}(-7y^2_{12}+4y_{31}y_{12}+2y_{12}y_{23}-2y_{23}y_{31}-2y^2_{31})+\frac{K_1}{D^2_{xy}}(y^2_{12})\Big]\\ &+C_{31}\Big[\frac{1}{D^2_{xy}}(8y^2_{12}-4y_{23}y_{12}+2y_{23}y_{31}-4y_{12}y_{31})+\frac{K_2}{D^2_{xy}}(y^2_{12})\Big]\\ &+C_{12}\Big[\frac{1}{D^2_{xy}}(y^2_{12}-2y^2_{31}-2y_{23}y_{12}+2y_{23}y_{31}+2y_{12}y_{31})+\frac{K_3}{D^2_{xy}}(2y_{31}y_{12}-2y_{23}y_{31}-5y^2_{12})\Big]\end{aligned}$$

$$\begin{aligned}B^3_{xx} &= C_{23}\Big[\frac{1}{D^2_{xy}}(-8y^2_{12}+4y_{31}y_{12}+4y_{12}y_{23}-2y_{23}y_{31}-2y^2_{31})+\frac{K_1}{D^2_{xy}}(2y^2_{12}-2y_{31}y_{12})\Big]\\ &+C_{31}\Big[\frac{1}{D^2_{xy}}(7y^2_{12}+2y^2_{23}-4y_{23}y_{12}+2y_{23}y_{31}-2y_{12}y_{31})+\frac{K_2}{D^2_{xy}}(y^2_{12}-2y_{12}y_{23})\Big]\\ &+C_{12}\Big[\frac{1}{D^2_{xy}}(-y^2_{12}+2y^2_{23}+2y_{31}y_{12}-2y_{23}y_{31})+\frac{K_3}{D^2_{xy}}(2y_{31}y_{23}+2y_{23}y_{12}+5y^2_{12})\Big]\end{aligned}$$

$$\begin{aligned}C^3_{xx} &= \frac{2}{D^2_{xy}}\Big[Q_{23}y_{12}y_{31}+Q_{31}y_{23}y_{12}+Q_{12}y_{31}y_{23}\Big]\\ &+C_{23}\Big[\frac{1}{D^2_{xy}}(5y^2_{12}-2y_{31}y_{12}-2y_{12}y_{23}+2y_{23}y_{31}+2y^2_{31})+\frac{K_1}{D^2_{xy}}(-y^2_{12}+2y_{31}y_{12})\Big]\\ &+C_{31}\Big[\frac{1}{D^2_{xy}}(-5y^2_{12}-2y^2_{23}+2y_{23}y_{12}-2y_{23}y_{31}+2y_{12}y_{31})+\frac{K_2}{D^2_{xy}}(4y^2_{12}+2y_{31}y_{23})\Big]\\ &+C_{12}\Big[\frac{1}{D^2_{xy}}(-2y_{31}y_{12}+2y_{23}y_{12})+\frac{K_3}{D^2_{xy}}(2y_{31}y_{23}+4y^2_{12})\Big]\end{aligned}$$

The term $\frac{\partial^2 z}{\partial y^2}$ in the triangle TB_1 can be written as

$$\frac{\partial^2 z}{\partial y^2} = A^1_{yy}\lambda_1 + B^1_{yy}\lambda_2 + C^1_{yy}$$

where,

$$\begin{aligned}A^1_{yy} &= C_{23}\Big[\frac{K-1}{D^2_{yx}}(5y^2_{23}+2y_{31}y_{12}-2y_{23}y_{31})+\frac{1}{D^2_{yx}}(y^2_{23}-2y^2_{31}+2y_{12}y_{31}-2y_{23}y_{31})\Big]\\ &+C_{31}\Big[\frac{K_2}{D^2_{yx}}(-2y^2_{23}+2y_{12}y_{23})+\frac{1}{D^2_{yx}}(2y^2_{12})-4y_{23}y_{12}+8y^2_{23}-4y_{23}y_{31}+2y_{12}y_{23}\Big]\\ &+C_{12}\Big[\frac{K_3}{D^2_{yx}}(-y^2_{23}+2y_{23}y_{31})+\frac{1}{D^2_{yx}}(4y_{31}y_{23}-2y^2_{31}-7y^2_{23}-2y_{31}y_{12}+2y_{12}y_{23})\Big]\end{aligned}$$

$$\begin{aligned}B^1_{yy} &= C_{23}\Big[\frac{K_2}{D^2_{yx}}(2y_{12}y_{23}-2y_{23}y_{31})+\frac{1}{D^2_{yx}}(4y_{12}y_{31}-2y^2_{12}-2y^2_{31}+2y^2_{23})\Big]\\ &+C_{31}\Big[\frac{K_2}{D^2_{yx}}(y^2_{23})+\frac{1}{D^2_{yx}}(y^2_{23}-2y_{23}y_{31})\Big]\\ &+C_{12}\Big[\frac{K_3}{D^2_{yx}}(y^2_{23})+\frac{1}{D^2_{yx}}(y^2_{23}-2y_{12}y_{23})\Big]\end{aligned}$$

$$
\begin{aligned}
C^1_{yy} = \frac{2}{D^2_{yx}}\Big[Q_{23}y_{12}y_{31} + Q_{31}y_{23}y_{12} + Q_{12}y_{31}y_{23}\Big] \\
+ C_{23}\Big[\frac{2}{D^2_{yx}}(y^2_{31} - y_{12}y_{31}) + \frac{K_1}{D^2_{yx}}(2y_{23}y_{31} - y^2_{23}) - \frac{1}{D^2_{yx}}(y^2_{23})\Big] \\
+ C_{31}\Big[\frac{2}{D^2_{yx}}(y_{12}y_{23} - y^2_{23}) + \frac{K_1}{D^2_{yx}}(y^2_{23}) + \frac{1}{D^2_{yx}}(2y_{31}y_{23} - y^2_{23})\Big] \\
+ C_{12}\Big[-\frac{1}{D^2_{yx}}(2y_{31}y_{23} - 2y^2_{23})\Big]
\end{aligned}
$$

In triangle TB_2, we have

$$
\frac{\partial^2 z}{\partial y^2} = A^2_{yy}\lambda_1 + B^2_{yy}\lambda_2 + C^2_{yy}
$$

where,

$$
\begin{aligned}
A^2_{yy} &= C_{23}\Big[\frac{1}{D^2_{yx}}(-y^2_{31} + 2y_{23}y_{31})\Big] \\
&+ C_{31}\Big[\frac{K_2}{D^2_{yx}}(2y2_{31}y_{12} - 2y_{23}y_{31}) + \frac{1}{D^2_{yx}}(2y^2_{12} - 2y^2_{31} + 2y^2_{23} - 4y_{12}y_{23})\Big] \\
&+ C_{12}\Big[\frac{K_3}{D^2_{yx}}(y^2_{31}) + \frac{1}{D^2_{yx}}(2y_{31}y_{12})\Big] \\
B^2_{yy} &= C_{23}\Big[\frac{K_1}{D^2_{yx}}(-2y^2_{31} + 2y_{12}y_{31})\Big] \\
&= C_{31}\Big[\frac{K_2}{D^2_{yx}}(5y^2_{31} + 2y_{12}y_{23} - 2y_{31}y_{23}) + \frac{1}{D^2_{yx}}(2y^2_{23} - y^2_{31} - 2y_{23}y_{12} + 2y_{31}y_{12} - 2y_{23}y_{31})\Big] \\
&+ C_{12}\Big[\frac{K_3}{D^2_{yx}}(-2y^2_{31} + 2y_{23}y_{31}) + \frac{1}{D^2_{yx}}(2y^2_{23} + 7y^2_{31} - 4y_{23}y_{31} - 2y_{31}y_{12} + 2y_{23}y_{12})\Big] \\
C^2_{yy} &= \frac{2}{D^2_{yx}}\Big[Q_{23}y_{12}y_{31} + Q_{31}y_{23}y_{12} + Q_{12}y_{31}y_{23}\Big] \\
&+ C_{23}\Big[\frac{K_1}{D^2_{yx}}(y^2_{31}) + \frac{1}{D^2_{yx}}(3y^2_{31} - 2y_{12}y_{31} - 2y_{23}y_{31})\Big] \\
&+ C_{31}\Big[\frac{K_2}{D^2_{yx}}(-y^2_{31} + 2y_{23}y_{31}) + \frac{1}{D^2_{yx}}(-2y^2_{23} + y^2_{31} + 2y_{23}y_{12})\Big] \\
&+ C_{12}\Big[\frac{1}{D^2_{yx}}(-2y^2_{31} + 2y_{23}y_{31})\Big]
\end{aligned}
$$

In triangle TB_3, we have

$$
\frac{\partial^2 z}{\partial y^2} = A^3_{yy}\lambda_1 + B^3_{yy}\lambda_2 + C^3_{yy}
$$

where,

$$
\begin{aligned}
A^3_{yy} &= C_{23}\Big[\frac{1}{D^2_{yx}}(-7y^2_{12} + 4y_{31}y_{12} + 2y_{12}y_{23} - 2y_{23}y_{31} - 2y^2_{31}) + \frac{K_1}{D^2_{yx}}(y^2_{12})\Big] \\
&+ C_{31}\Big[\frac{1}{D^2_{yx}}(8y^2_{12} - 4y_{23}y_{12} + 2y_{23}y_{31} - 4y_{12}y_{31}) + \frac{K_2}{D^2_{yx}}(y^2_{12})\Big] \\
&+ C_{12}\Big[\frac{1}{D^2_{yx}}(y^2_{12} - 2y^2_{31} - 2y_{23}y_{12} + 2y_{23}y_{31} + 2y_{12}y_{31}) + \frac{K_3}{D^2_{yx}}(2y_{31}y_{12} - 2y_{23}y_{31} - 5y^2_{12})\Big]
\end{aligned}
$$

$$\begin{aligned}
B^3_{yy} &= C_{23}\Big[\frac{1}{D^2_{yx}}(-8y^2_{12}+4y_{31}y_{12}+4y_{12}y_{23}-2y_{23}y_{31}-2y^2_{31})+\frac{K_1}{D^2_{yx}}(2y^2_{12}-2y_{31}y_{12})\Big]\\
&+C_{31}\Big[\frac{1}{D^2_{yx}}(7y^2_{12}+2y^2_{23}-4y_{23}y_{12}+2y_{23}y_{31}-2y_{12}y_{31})+\frac{K_2}{D^2_{yx}}(y^2_{12}-2y_{12}y_{23})\Big]\\
&+C_{12}\Big[\frac{1}{D^2_{yx}}(-y^2_{12}+2y^2_{23}+2y_{31}y_{12}-2y_{23}y_{31})+\frac{K_3}{D^2_{yx}}(2y_{31}y_{23}+2y_{23}y_{12}+5y^2_{12})\Big]\\
C^3_{yy} &= \frac{2}{D^2_{yx}}\Big[Q_{23}y_{12}y_{31}+Q_{31}y_{23}y_{12}+Q_{12}y_{31}y_{23}\Big]\\
&+C_{23}\Big[\frac{1}{D^2_{yx}}(5y^2_{12}-2y_{31}y_{12}-2y_{12}y_{23}+2y_{23}y_{31}+2y^2_{31})+\frac{K_1}{D^2_{yx}}(-y^2_{12}+2y_{31}y_{12})\Big]\\
&+C_{31}\Big[\frac{1}{D^2_{yx}}(-5y^2_{12}-2y^2_{23}+2y_{23}y_{12}-2y_{23}y_{31}+2y_{12}y_{31})+\frac{K_2}{D^2_{yx}}(4y^2_{12}+2y_{31}y_{23})\Big]\\
&+C_{12}\Big[\frac{1}{D^2_{yx}}(-2y_{31}y_{12}+2y_{23}y_{12})+\frac{K_3}{D^2_{yx}}(2y_{31}y_{23}+4y^2_{12})\Big]
\end{aligned}$$

The term $\frac{\partial^2 z}{\partial x \partial y}$ in the triangle TB_1 can be written as

$$\frac{\partial^2 z}{\partial x\partial y} = A^1_{xy}\lambda_1 + B^1_{xy}\lambda_2 + C^1_{xy}$$

where,

$$\begin{aligned}
A^1_{xy} &= C_{23}\Big[\frac{K_1}{D_{xy}D_{yx}}(5x_{23}y_{23}+y_{31}x_{12}+y_{12}x_{31}-y_{23}x_{31}-x_{23}y_{31})\Big]\\
&+C_{23}\Big[\frac{1}{D_{xy}D_{yx}}(x_{23}y_{23}-x_{23}y_{12}-x_{12}y_{23}+x_{31}y_{23})\Big]\\
&+C_{23}\Big[\frac{1}{D_{xy}D_{yx}}(-2x_{31}y_{13}+2x_{13}y_{12}+2x_{12}y_{31})\Big]\\
&+C_{31}\Big[\frac{K_2}{D_{xy}D_{yx}}(x_{23}y_{12}-2x_{23}y_{23}+x_{23}y_{12})\Big]\\
&+C_{31}\Big[\frac{1}{D_{xy}D_{yx}}(7x_{23}y_{23}-x_{23}y_{12}-x_{12}y_{23})\Big]\\
&+C_{31}\Big[\frac{1}{D_{xy}D_{yx}}(x_{31}y_{23}+x_{23}y_{31}-2x_{31}y_{13}+2x_{13}y_{12}+2x_{12}y_{31})\Big]\\
&+C_{12}\Big[\frac{K_3}{D_{xy}D_{yx}}(-x_{23}y_{23}+x_{31}y_{23}+x_{23}y_{31})\Big]\\
&+C_{12}\Big[\frac{1}{D_{xy}D_{yx}}(-7x_{23}y_{23}-x_{12}y_{31}-x_{31}y_{12}+x_{23}y_{12})\\
&+C_{12}\Big[\frac{1}{D_{xy}D_{yx}}(x_{12}y_{23}+2x_{31}y_{23}-2x_{31}y_{13}+3x_{23}y_{31}+x_{31}y_{23})\Big]\\
B^1_{xy} &= C_{23}\Big[\frac{K_1}{D_{xy}D_{yx}}(x_{23}y_{12}+x_{12}y_{23}-x_{31}y_{23}-x_{23}y_{31})\Big]\\
&+C_{23}\Big[\frac{1}{D_{xy}D_{yx}}(2x_{23}y_{23}+4x_{31}y_{12}+4x_{12}y_{31}-2x_{12}y_{21}-2x_{31}y_{13})\Big]\\
&+C_{31}\Big[\frac{K_2}{D_{xy}D_{yx}}(-x_{23}y_{23})+\frac{1}{D_{xy}D_{yx}}(-x_{12}y_{23}+2x_{23}y_{32}-2x_{23}y_{12})\Big]\\
&+C_{12}\Big[\frac{K_3}{D_{xy}D_{yx}}(x_{23}y_{23})+\frac{1}{D_{xy}D_{yx}}(-x_{23}y_{12}-x_{12}y_{23}+x_{23}y_{32}-x_{23}y_{31}-x_{31}y_{23})\Big]
\end{aligned}$$

$$C_{xy}^{1} = \frac{1}{D_{xy}D_{yx}}\Big[Q_{23}(x_{31}y_{12}+x_{12}y_{31})+Q_{31}(x_{12}y_{23}+x_{23}y_{12})+Q_{12}(x_{23}y_{31}+x_{31}y_{23})\Big]$$
$$+C_{23}\Big[\frac{1}{D_{xy}D_{yx}}(-x_{23}y_{23}+2x_{31}y_{13}-2x_{31}y_{12}-2x_{12}y_{31})-\frac{K_1}{D_{xy}D_{yx}}(x_{23}y_{23}-x_{31}y_{23}+x_{23}y_{31})\Big]$$
$$+C_{31}\Big[\frac{1}{D_{xy}D_{yx}}(-2x_{23}y_{23}+x_{23}y_{12}+2x_{12}y_{23}-2x_{23}y_{32}+2x_{23}y_{12})+\frac{K_2}{D_{xy}D_{yx}}(x_{23}y_{23})\Big]$$
$$+C_{12}\Big[\frac{1}{D_{xy}D_{yx}}(2x_{23}y_{23}-x_{31}y_{23}-x_{23}y_{31})\Big]$$

In triangle TB_2, we have

$$\frac{\partial^2 z}{\partial x\partial y} = A_{xy}^{2}\lambda_1 + B_{xy}^{2}\lambda_2 + C_{xy}^{2}$$

where,

$$A_{xy}^{2} = C_{23}\Big[\frac{1}{D_{xy}D_{yx}}(x_{12}y_{31}+x_{31}y_{12}-x_{31}y_{31}+x_{31}y_{23}+x_{23}y_{31})+\frac{K_1}{D_{xy}D_{yx}}(-x_{31}y_{23})\Big]$$
$$+C_{31}\Big[\frac{1}{D_{xy}D_{yx}}(-2x_{31}y_{31}+2x_{12}y_{21}-4x_{23}y_{12}-4x_{12}y_{23}+2x_{23}y_{32})\Big]$$
$$+C_{31}\Big[\frac{K_2}{D_{xy}D_{yx}}(x_{12}y_{31}+x_{31}y_{12}-x_{31}y_{23}-x_{23}y_{31})\Big]$$
$$+C_{12}\Big[\frac{1}{D_{xy}D_{yx}}(x_{12}y_{31}+x_{31}y_{12}+x_{23}y_{31}+x_{31}y_{23}-2x_{31}y_{13}+x_{31}y_{31})+\frac{K_3}{D_{xy}D_{yx}}(x_{31}y_{31})\Big]$$

$$B_{xy}^{2} = C_{23}\Big[\frac{1}{D_{xy}D_{yx}}(-6x_{31}y_{31}-x_{23}y_{12}-x_{12}y_{23}+2x_{31}y_{23}+2x_{23}y_{31}+4x_{31}y_{12}+4x_{12}y_{31})\Big]$$
$$+C_{23}\Big[\frac{1}{D_{xy}D_{yx}}(-2x_{12}y_{21}-2x_{31}y_{13})+\frac{K_1}{D_{xy}D_{yx}}(x_{23}y_{31}-2x_{31}y_{13}+x_{31}y_{12})\Big]$$
$$+C_{31}\Big[\frac{1}{D_{xy}D_{yx}}(-x_{31}y_{23}-x_{23}y_{31}+x_{12}y_{31}+x_{31}y_{12}-x_{31}y_{31}-2x_{12}y_{23}+2x_{23}y_{32}-2x_{23}y_{12})\Big]$$
$$+C_{31}\Big[\frac{K_2}{D_{xy}D_{yx}}(5x_{31}y_{31}+x_{23}y_{12}+x_{12}y_{23}-x_{31}y_{23}-x_{23}y_{31})\Big]$$
$$+C_{12}\Big[\frac{1}{D_{xy}D_{yx}}(7x_{31}y_{31}-x_{12}y_{31}-x_{31}y_{12}+x_{23}y_{12}+x_{12}y_{23}+2x_{23}y_{32}-3x_{23}y_{31}-3x_{31}y_{23})\Big]$$
$$+C_{12}\Big[\frac{K_3}{D_{xy}D_{yx}}(-x_{31}y_{31}+x_{31}y_{23}+x_{23}y_{31})\Big]$$

$$C_{xy}^{2} = \frac{1}{D_{xy}D_{yx}}\Big[Q_{23}(x_{31}y_{12}+x_{12}y_{31})+Q_{31}(x_{12}y_{23}+x_{23}y_{12})+Q_{12}(x_{23}y_{31}+x_{31}y_{23})\Big]$$
$$+C_{23}\Big[\frac{1}{D_{xy}D_{yx}}(x_{31}y_{31}-x_{31}y_{23}-x_{23}y_{31}+2x_{31}y_{13}-2x_{31}y_{12}-2x_{12}y_{31})-\frac{K_1}{D_{xy}D_{yx}}(x_{31}y_{31})\Big]$$
$$+C_{31}\Big[\frac{1}{D_{xy}D_{yx}}(x_{31}y_{31}+2x_{12}y_{23}-2x_{23}y_{32}+2x_{23}y_{12})+\frac{K_2}{D_{xy}D_{yx}}(x_{31}y_{23}+x_{23}y_{31}-x_{31}y_{31})\Big]$$
$$+C_{12}\Big[\frac{1}{D_{xy}D_{yx}}(-2x_{31}y_{31}+x_{31}y_{23}+x_{23}y_{31})\Big]$$

In triangle TB_3, we have

$$\frac{\partial^2 z}{\partial x\partial y} = A_{xy}^{3}\lambda_1 + B_{xy}^{3}\lambda_2 + C_{xy}^{3}$$

where,

$$
\begin{aligned}
A^3_{xy} &= C_{23}\Big[\frac{1}{D_{xy}D_{yx}}(7x_{12}y_{12} + x_{12}y_{31} + x_{31}y_{12} + x_{23}y_{12} + x_{12}y_{23} - x_{31}y_{23} - x_{23}y_{31} - 2x_{31}y_{31})\Big]\\
&+ C_{23}\Big[\frac{1}{D_{xy}D_{yx}}(2x_{31}y_{12} + 2x_{12}y_{31}) + \frac{K_1}{D_{xy}D_{yx}}(x_{12}y_{12} - x_{12}y_{31} - x_{31}y_{12})\Big]\\
&+ C_{31}\Big[\frac{1}{D_{xy}D_{yx}}(-2x_{12}y_{31} - 2x_{31}y_{12} + 2x_{12}y_{21} - 4x_{23}y_{12} - 4x_{12}y_{23} + 6x_{12}y_{12} + x_{31}y_{23})\Big]\\
&+ C_{31}\Big[\frac{1}{D_{xy}D_{yx}}(x_{23}y_{31} + 2x_{23}y_{23}) + \frac{K_2}{D_{xy}D_{yx}}(2x_{12}y_{12} - x_{23}y_{12} - x_{12}y_{23})\Big]\\
&+ C_{12}\Big[\frac{1}{D_{xy}D_{yx}}(x_{12}y_{12} + 2x_{23}y_{31} + 2x_{31}y_{23} - 2x_{31}y_{13} + x_{12}y_{31} + x_{31}y_{12} - x_{23}y_{12} - x_{12}y_{23})\Big]\\
&+ C_{12}\Big[\frac{K_3}{D_{xy}D_{yx}}(x_{12}y_{31} + x_{31}y_{12} - x_{31}y_{23} - x_{23}y_{31})\Big]
\end{aligned}
$$

$$
\begin{aligned}
B^3_{xy} &= C_{23}\Big[\frac{1}{D_{xy}D_{yx}}(x_{23}y_{12} + x_{12}y_{23} + 4x_{31}y_{12} + 4x_{12}y_{31} - 2x_{12}y_{21} - 6x_{12}y_{12} + x_{23}y_{12})\Big]\\
&+ C_{23}\Big[\frac{1}{D_{xy}D_{yx}}(x_{12}y_{23} - x_{31}y_{23} - x_{23}y_{31} - 2x_{31}y_{31}) + \frac{K_1}{D_{xy}D_{yx}}(2x_{12}y_{12} - x_{12}y_{31} - x_{31}y_{12})\Big]\\
&+ C_{31}\Big[\frac{1}{D_{xy}D_{yx}}(-x_{23}y_{12} - x_{12}y_{23} + 7x_{12}y_{12} + x_{31}y_{23} + x_{23}y_{31} - x_{12}y_{31} - x_{31}y_{12} - 2x_{12}y_{23})\Big]\\
&+ C_{31}\Big[\frac{1}{D_{xy}D_{yx}}(2x_{23}y_{32} - 2x_{23}y_{12} + C_{31}) + \frac{K_2}{D_{xy}D_{yx}}(x_{12}y_{12} - x_{23}y_{12} - x_{12}y_{23})\Big]\\
&+ C_{12}\Big[\frac{1}{D_{xy}D_{yx}}(-x_{12}y_{12} + 2x_{23}y_{23} - 2x_{23}y_{31} - 2x_{31}y_{23} + x_{12}y_{31} + x_{31}y_{12} - x_{23}y_{12} - x_{12}y_{31})\Big]\\
&+ C_{12}\Big[\frac{K_3}{D_{xy}D_{yx}}(x_{23}y_{12} + x_{12}y_{23} - x_{31}y_{23} - x_{23}y_{31})\Big]
\end{aligned}
$$

$$
\begin{aligned}
C^3_{xy} &= \frac{1}{D_{xy}D_{yx}}\Big[Q_{23}(x_{31}y_{12} + x_{12}y_{31}) + Q_{31}(x_{12}y_{23} + x_{23}y_{12}) + Q_{12}(x_{23}y_{31} + x_{31}y_{23})\Big]\\
&+ C_{23}\Big[\frac{1}{D_{xy}D_{yx}}(5x_{12}y_{12} - x_{23}y_{12} - x_{12}y_{23} + x_{31}y_{23} + x_{23}y_{31} + 2x_{31}y_{31} - 2x_{31}y_{12} - 2x_{12}y_{31})\Big]\\
&+ C_{23}\Big[\frac{K_1}{D_{xy}D_{yx}}(-x_{12}y_{12} + x_{12}y_{31} + x_{31}y_{12})\Big]\\
&+ C_{31}\Big[\frac{1}{D_{xy}D_{yx}}(5x_{12}y_{12} - x_{31}y_{23} - x_{23}y_{31} + x_{12}y_{31} + x_{31}y_{12} + 2x_{12}y_{23} - 2x_{23}y_{31} + 2x_{23}y_{12})\Big]\\
&+ C_{31}\Big[\frac{K_2}{D_{xy}D_{yx}}(-x_{12}y_{12} + x_{23}y_{12} + x_{12}y_{23})\Big]\\
&+ C_{12}\Big[\frac{1}{D_{xy}D_{yx}}(-x_{12}y_{31} - x_{31}y_{12} + x_{23}y_{12} + x_{12}y_{23}) + \frac{K_3}{D_{xy}D_{yx}}(-x_{12}y_{12} + x_{31}y_{23} + x_{23}y_{31})\Big]
\end{aligned}
$$