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FIRST-ORDER OPTIMALITY CONDITIONS IN CONSTRAINED SET-VALUED OPTIMIZATION

Giovanni P. Crespi, Ivan Ginchev and Matteo Rocca

Abstract: A constrained optimization problem $\min_C F(x)$, $G(x) \cap (-K) \neq \emptyset$ is considered, where X, Y, Z are normed spaces, $F : X \rightsquigarrow Y$, $G : X \rightsquigarrow Z$ are set-valued functions and C and K are closed convex (not necessarily pointed) cones. The solutions of the set-valued problem are called minimizers. The notions of w-minimizers (weakly efficient points), p-minimizers (properly efficient points) and i-minimizers (isolated minimizers) are introduced. These notions are investigated and characterized by first order necessary and sufficient conditions given by means of a Dini derivative for set-valued maps. The case of convex-along-rays data is considered to have sufficient optimality conditions for weak minimizers. This paper generalizes [4] where an unconstrained set-valued optimization problem was considered.

Key words: *vector optimization, set-valued optimization, first-order optimality conditions*

Mathematics Subject Classification: *90C29, 90C30, 90C46, 49J52*

1 Introduction

We consider the set-valued optimization problem (for short, svp)

$$\min_C F(x)\,, \quad x \in G(x) \cap (-K) \neq \emptyset\,, \tag{1}$$

where $F : X \rightsquigarrow Y$ and $G : X \rightsquigarrow Z$ are nonempty-valued set-valued functions (svf), X, Y and Z are normed spaces, $C \subseteq Y$ and $K \subseteq Z$ are closed convex cones. We do not assume pointedness of C, since, as we will see, this assumption is too restrictive when constrained problems are considered. Following [1] we use the squiggled arrow $\rightsquigarrow$ to denote a set-valued function and the usual arrow $\to$ for a single-valued one. When we say a point $x \in X$ is feasible we mean, throughout the paper, that $G(x) \cap (-K) \neq \emptyset$. Clearly, the unconstrained problem

$$\min_C F(x)\,, \quad x \in X\,, \tag{2}$$

is a particular case of problem (1).

If instead of svf F and G we consider vector-valued functions (vvf) $f : X \to Y$ and $g : X \to Z$, we come to the vector-valued problem (vvp)

$$\min_C f(x)\,, \quad g(x) \in -K\,. \tag{3}$$

The aim of the paper is to derive first order optimality conditions for problem (1) in terms of a suitable first order Dini derivative of the involved svf, proving results analogous to those stated in [4] for the unconstrained problem (2). Another difference with the results in [4] is given by non-pointedness of the cone C. When we consider a single-valued function we name it with a small letter, say f, while for a set-valued one we apply a capital letter, say F. The concept of isolated minimizer is put into the center of the investigations in [6] and the results there concern finite-dimensional spaces, an assumption which is also important for some of the considerations in the present paper.

The solutions of svp (1) are defined as pairs (x^0, y^0), where x^0 is feasible and $y^0 \in F(x^0)$. With the exception of Section 4, we consider solutions in a local sense. Similarities with vector optimization problems allow us to use in set-valued optimization notions from vector optimization. In particular, the solutions of svp (1) can be called efficient points. We prefer, like in scalar optimization, to call them minimizers. In Section 2 we define different types of minimizers and recall their characterizations in terms of the so called oriented distance. Among them the notions of w-minimizer (weakly efficient point), p-minimizer (properly efficient point) and i-minimizer (isolated minimizer) play an important role. Also the concept of locally Lipschitz svf is recalled. It is shown that when F is locally Lipschitz each i-minimizer is a p-minimizer. In Section 3 we give necessary conditions for w-minimizers and sufficient conditions for i-minimizers of (1). The reversal in case of a i-minimizer is also obtained. Section 4 discusses the reversal of the necessary conditions for w-minimizers and establishes such a possibility under convexity type conditions.

2 Concepts of Optimality and Preliminaries

Here $\mathbb{R}$ is the set of the reals and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{-\infty\}$ is its two point extension with the infinite elements. We put also $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{R}_- = (-\infty, 0]$. For the norm and the dual pairing in the normed spaces X, Y and Z we write $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$. We denote by B and S the open unit ball and the unit sphere in the considered spaces. From the context it should be clear to exactly which spaces these notations are applied. Recall that the dual pairing in a normed space Y is a mapping $(\xi, y) \in Y^* \times Y \to \langle \xi, y \rangle$ where Y^* is the dual space of Y. When Y is a Euclidean space, Y^* can be identified with Y and $\langle \cdot, \cdot \rangle$ with the scalar product in Y.

We recall that for a given closed convex cone $C \subseteq Y$, its positive polar cone is defined by $C' = \{\xi \in Y^* \mid \langle \xi, y \rangle \geq 0 \text{ for all } y \in C\}$.

For a given $w \in G(x^0) \cap (-K)$, we consider the closed convex cones $K'(w) = \{\xi \in K' | \langle \xi, w \rangle = 0\}$ and $K(w) = (K'(w))'$. The cone $K(w)$ plays a crucial role when we deal with svp (1) and we observe that $K(w)$ is not necessarily pointed.
We focus on the following concepts of solutions for problem (1), where we assume that the considered point x^0 is feasible for svp (1). The pair (x^0, y^0), $y^0 \in F(x^0)$, is said to be w-minimizer (respectively e-minimizer) if there exists a neighbourhood U of x^0 such that for every feasible $x \in U$, $F(x) \cap \big(y^0 - \operatorname{int} C\big) = \emptyset$ (respectively $F(x) \cap \big(y^0 - (C \setminus \{0\}\big) = \emptyset$). In vector optimization w-minimizers are called weakly efficient points and e-minimizers efficient points. Obviously, if $C \neq Y$, each e-minimizer is a w-minimizer.

The above definitions can be given with arbitrary, not necessarily closed cones. Then the e-minimizers are independent of the norm in Y. The w-minimizers depend on the norm in Y through $\operatorname{int} C$. Since equivalent norms define the same topology, the w-minimizers are invariant with respect to equivalent norms.

Define now the weakly efficient frontier (w-frontier), w-$\mathrm{Min}_C A$, and the efficient frontier (e-frontier) e-$\mathrm{Min}_C A$, of a set $A \subseteq Y$ with respect to the cone C by $w\text{-}\mathrm{Min}_C A = \{y \in A \mid A \cap (y - \operatorname{int} C) = \emptyset\}$ and $e\text{-}\mathrm{Min}_C A = \{y \in A \mid A \cap (y - (C \setminus \{0\})) = \emptyset\}$. If $C \neq Y$, then

$\operatorname{int} C \subseteq C \setminus \{0\}$, whence $w\text{-Min}_C A \supseteq e\text{-Min}_C A$. For vector optimization theory based on the notions of efficient frontiers see Luc [19].

It is clear that if (x^0, y^0) is a w-minimizer (respectively e-minimizer) for svp (1) then y^0 belongs to the w-frontier (respectively e-frontier) of the set $F(x^0)$. Thus, if the couple (x^0, y^0), $y^0 \in F(x^0)$, is a minimizer of some type for svp (1), then frontier-type limitations for the point y^0 do occur.

For a set $A \subseteq Y$ the distance from $y \in Y$ to A is given by $d(y, A) = \inf\{\|a - y\| \mid a \in A\}$. It is convenient to allow also value $+\infty$ of the distance function, putting $d(y, \emptyset) = +\infty$. The oriented distance from y to A is defined by $D(y, A) = d(y, A) - d(y, Y \setminus A)$. It takes values in $\overline{\mathbb{R}}$ and in particular $D(y, \emptyset) = +\infty$ and $D(y, Y) = -\infty$. The function D is introduced in Hiriart-Urruty [11], [12], and it has been often used in vector optimization. Ginchev, Hoffmann [9] apply the oriented distance to study the approximation of set-valued functions by single-valued ones and, in the case of a convex cone C, show the representation $D(y, -C) = \sup\{\langle \xi, y \rangle \mid \|\xi\| = 1, \xi \in C'\}$.

We define next the oriented distance, $D(M, A)$, from a set $M \subseteq Y$ to the set $A \subseteq Y$ by $D(M, A) = \inf\{D(y, A) \mid y \in M\}$.

A characterization of the w-minimizers can be obtained in terms of the oriented distance.

Proposition 2.1 *[4] Let x^0 be feasible for svp (1) and consider the scalar function*

$$\varphi : X \to \mathbb{R}, \quad \varphi(x) = D(F(x) - y^0, -C) \,. \tag{4}$$

The pair (x^0, y^0), $y^0 \in F(x^0)$, is a w-minimizer of svp (1) with $C \neq Y$ if and only if $\varphi(x^0) = 0$ and $\varphi(x) - \varphi(x^0) \geq 0$, for every $x \in X$ with $G(x) \cap (-K) \neq \emptyset$ (i.e. x^0 is a minimizer for the function φ).

Next we recall the notion of a properly efficient point (p-minimizer) for vvp (3), when a possibly non pointed cone C is considered (see e.g. [6] and [8]).

Let $C \subseteq Y$ be a cone and let a be a real number. Define the set

$$C(a) = \{y \in Y \mid D(y, C) \leq a \,\|y\|\} \,. \tag{5}$$

The set $C(a)$ is a closed but not necessarily convex cone, which is a consequence of the positive homogeneity of the oriented distance $D(\cdot, C)$ and the norm $\|\cdot\|$.

It can be shown (see e.g. [4], [8]) that in the case of a pointed closed convex cone C any p-minimizer is a properly efficient point in a commonly accepted sense [10], [19]. The advantages of the p-minimizers from the above definition are in the generality and the simple analytic description of the cones (5), but the price we pay is to eventually deal with non-convex cones $C(a)$.

Definition 2.1 *We say that the point (x^0, y^0), $y^0 \in F(x^0)$, is a p-minimizer for svp (1) if there exists a, $0 < a < 1$, and a neighbourhood U of x^0, such that for every feasible $x \in U$ and $y \in F(x)$, $y - y^0 \notin -\operatorname{int} C(a)$.*

It can be shown [4] that the notion of a p-minimizer is invariant with respect to equivalent norms in Y. Given a set $A \subseteq Y$ we define the properly efficient frontier (p-frontier) of A with respect to C by $p\text{-Min}_C A = \{y \in A \mid A \cap (y - C(a)) = \{y\} \text{ for some } a,\ 0 < a < 1\}$. Obviously $e\text{-Min}_C A \supseteq p\text{-Min}_C A$.

For $x = x^0$ the definition of a p-minimizer gives that if (x^0, y^0), $y^0 \in F(x^0)$, is a p-minimizer for svp (1), then $y^0 \in p\text{-Min}_C F(x^0)$.

Another concept of optimality is that of an isolated minimizer (i-minimizer), which can be generalized from vector to set-valued optimization as follows.

Definition 2.2 *We say that* (x^0, y^0), $y^0 \in F(x^0)$, *is an* i*-minimizer for svp (1) if there is a neighbourhood* U *of* x^0 *and a constant* $A > 0$ *such that* $D(F(x) - y^0, -C) \geq A\,\|x - x^0\|$ *and* $y^0 \in p\text{-Min}_C F(x^0)$ *for every feasible* $x \in U$.

The i-minimizers are invariant with respect to equivalent norms in Y. Generally, if $\psi : X \to \mathbb{R}$ is any scalar function, the point $x^0 \in X$ is said to be an isolated minimizer of order $\kappa > 0$ for ψ, if there exists a neighbourhood U of x^0 and a constant $A > 0$, such that $\psi(x) - \psi(x^0) \geq A\,\|x - x^0\|^\kappa$ for $x \in U$. In this paper we deal only with isolated minimizers of order 1. The notion of isolated minimizer has been popularized by Auslender [2]. For vector functions it has been extended by Ginchev [5], Ginchev, Guerraggio, Rocca [6], [7], [8] and under the name of strict efficiency by Jiménez [16], [17], and Jiménez, Novo [18]. We prefer to use the original name of *isolated minimizer* given by Auslender. Besides, the concept of a strict minimizer has been used in vector optimization in the context of another meaning, see e.g. [3].

In the definition of an i-minimizer for svp (1) there appears explicitly the requirement $y^0 \in p\text{-Min}_C F(x^0)$. This can be explained as follows. For vvp (3) with locally Lipschitz function f, each i-minimizer is also a p-minimizer, see [6]. In order that similar relation occurs for svp (1), see Theorem 2.1 below, we need to explicitly insert this assumption, which is necessarily satisfied for a p-minimizer and does not follow from inequality $D(F(x) - y^0, -C) \geq A\,\|x - x^0\|$.

We recall [1] that the svf $F : X \rightsquigarrow Y$ is locally Lipschitz at $x^0 \in X$ if there exists a neighborhood U of x^0 and a constant $L > 0$, such that for x^1, $x^2 \in U$, $F(x^2) \subseteq F(x^1) + L\,\|x^2 - x^1\|\,\mathrm{cl}\,B$. The svf $F : X \rightsquigarrow Y$ is locally Lipschitz if it is locally Lipschitz at each $x^0 \in X$.

Further, Theorem 2.1 gives a relation between i-minimizers and p-minimizers in the case when the image space Y is finite dimensional. In advance we need some lemmas. Lemma 2.1 is proved in [4].

Lemma 2.1 *Let* C *be a closed cone in a finite dimensional Euclidean space* Y *and* a_1, $a_2 \geq 0$ *be two nonnegative numbers. Then* $C(a_1)(a_2) \subseteq C(a_1 + a_2)$.

The next Lemma 2.2 is applied in the proof of Theorem 2.1, but it plays also a crucial role in the remaining part of the paper.

Lemma 2.2 *Let* Y *be a finite dimensional Euclidean space. Let the svf* F *be Lipschitz with constant* L *in a neighborhood* U *of* x^0 *and* $y^0 \in F(x^0)$. *Assume that for some* σ *with* $0 < \sigma < 1/2$, $F(x^0) \cap (y^0 - C(2\sigma)) = \{y^0\}$. *Then, for each feasible* $x \in U$ *and each* $y \in F(x) \cap (y^0 - C(\sigma))$,

$$\|y - y^0\| \leq \frac{L(1+\sigma)}{\sigma}\|x - x^0\|. \tag{6}$$

Proof Let $x \in U$ be feasible, $y \in F(x) \cap (y^0 - C(\sigma))$ and denote by y' a projection of y on $\mathrm{cl}\,(y^0 - C(2\sigma))^c$, which means $D(y, \mathrm{cl}\,(y^0 - C(2\sigma))^c) = \|y - y'\|$ (the existence of y' follows from the closedness of $\mathrm{cl}\,(y^0 - C(2\sigma))^c$). Then,

$$\|y - y'\| \leq D(y, F(x^0)) \leq L\|x - x^0\|\,.$$

The second inequality comes from F being Lipschitz on U. Lemma 2.1 gives $-C(\sigma)(\sigma) \subseteq -C(2\sigma)$. The point $y' - y^0$ is not contained in the interior of $-C(2\sigma)$, whence $D(y' - y^0, -C(\sigma) \geq \sigma\|y' - y^0\|$. Since also $y - y^0 \in -C(\sigma)$, we obtain

$$\|y - y'\| \geq D(y' - y^0, -C(\sigma)) \geq \sigma\|y' - y^0\| \geq \sigma\left(\|y - y^0\| - \|y - y'\|\right)\,.$$

From here and from the preceding inequality we get

$$\frac{\sigma}{1+\sigma}\,\|y-y^0\| \le \|y-y'\| \le L\,\|x-x^0\|\,,$$

which gives (6) straightforwardly. □

The next Theorem gives a relation between i-minimizers and p-minimizers and illustrates an application of Lemma 2.2.

Theorem 2.1 *Consider svp (1) with finite dimensional image space Y. Let the svf $F : X \rightsquigarrow Y$ be locally Lipschitz and (x^0, y^0), $y^0 \in F(x^0)$, be an i-minimizer for (1). Then (x^0, y^0) is also a p-minimizer of (1).*

Proof We may assume without loss of generality that Y is endowed by a Euclidean norm, since the i-minimizers and p-minimizers are invariant with respect to equivalent norms in the image space and all the norms in a finite dimensional space are equivalent.

The assumption that (x^0, y^0) is i-minimizer implies that there exists a neighbourhood U of x^0 and constants $A > 0$ and $\sigma > 0$, such that $D(F(x) - y^0, -C) \ge A\,\|x - x^0\|$ for $x \in U$, and $F(x^0) \cap (y^0 - C(2\sigma)) = \{y^0\}$. We suppose also that F is Lipschitz in the neighborhood U. Assume that (x^0, y^0) is not a p-minimizer of (1). Therefore there exist sequences $x^k \to x^0$, $y^k \in F(x^k)$ and $\varepsilon_k \to 0^+$ such that $x^k \in U$, $\varepsilon_k < \sigma$, $y^k \in y^0 - \operatorname{int} C(\varepsilon_k)$. The latter inclusion gives in particular $y^k \ne y^0$. Now with regard to $y^k \in y^0 - \operatorname{int} C(\varepsilon_k) \subseteq y^0 - C(\sigma)$ and applying Lemma 2.2 we get

$$\begin{aligned} D(F(x^k) - y^0, -C) &\le D(y^k - y^0, -C) \le \varepsilon_k\,\|y^k - y^0\| \\ &\le \varepsilon_k\,\frac{L\,(1+\sigma)}{\sigma}\,\|x^k - x^0\|\,. \end{aligned}$$

From this chain of inequalities we get $x^k \ne x^0$, since otherwise we would have the contradictory inequalities $0 < \|y^k - y^0\| \le 0$. However, if $x^k \ne x^0$ from the inequalities

$$A\,\|x^k - x^0\| \le D(F(x^k) - y^0, -C) \le \varepsilon_k\,\frac{L\,(1+\sigma)}{\sigma}\,\|x^k - x^0\|$$

we get $0 < A \le \varepsilon_k\,L\,(1 + 1/\sigma)$. Taking the limit as $k \to \infty$ we get the contradiction $0 < A \le 0$. □

3 First-order Optimality Conditions

In this section we give optimality conditions for svp (1). We define the (upper) Dini derivative of a svf $\Phi : X \rightsquigarrow Y$ at (x^0, y^0), $y^0 \in \Phi(x^0)$, in the direction $u \in X$ as the upper limit:

$$\Phi'(x^0, y^0; u) = \operatorname*{Limsup}_{t \to 0^+} \frac{1}{t}\left(\Phi(x^0 + tu) - y^0\right)\,.$$

Basic properties of the Dini derivative have been given in [4]. There, this notion has also been compared with the concept of a contingent epiderative that is widespread in most of the investigations on set-valued optimization (see e.g. [13], [14], [15]). In conjunction with svp (1) we consider the set-valued function $H : X \rightsquigarrow Y \times Z$ defined by $H(x) = F(x) \times G(x)$.

Hence, for given $x^0 \in X$, $y^0 \in F(x^0)$, $w^0 \in G(x^0)$ and $u \in X$, the first-order Dini derivative of H is

$$H'\Big(x^0, (y^0, w^0); u\Big) = \operatorname*{Limsup}_{t \to 0^+} \frac{H(x^0 + tu) - (y^0, w^0)}{t}$$

We observe that $H'(x^0, (y^0, w^0); u) \subseteq F'(x^0, y^0; u) \times G'(x^0, w^0; u)$.

Theorem 3.1 (Necessary conditions for w-minimizers) *Let Z be a finite dimensional space, let $x^0 \in X$ be feasible for problem (1) and (x^0, y^0), $y^0 \in F(x^0)$ be a w-minimizer. Then $\forall w^0 \in G(x^0) \cap (-K)$ and $\forall u \in X$,*

$$H'\Big(x^0, (y^0, w^0), u\Big) \cap \Big(-(\operatorname{int} C \times \operatorname{int} K(w^0))\Big) = \emptyset\,. \tag{7}$$

Proof Assume that for some $u \in X$ and $w^0 \in G(x^0) \cap (-K)$ there exists some $(z^0, v^0) \in -\big(\operatorname{int} C \times \operatorname{int} K(x^0)\big) \cap H'\Big(x^0, (y^0, w^0); u\Big)$. Therefore one can write, for some sequences $y^n \in F(x^0 + t_n u)$ and $w^n \in G(x^0 + t_n u)$:

$$z^0 = \lim_{n \to +\infty} \frac{y^n - y^0}{t_n} \quad \text{and} \quad v^0 = \lim_{n \to +\infty} \frac{w^n - w^0}{t_n}\ .$$

We claim now that there exists some n_0 such that $G(x^0 + t_n u) \cap -(\operatorname{int} K) \neq \emptyset$ for all $n > n_0$, that is $x^0 + t_n u$ is feasible for $n > n_0$. Set $\Gamma_{K'} := \big\{\xi \in K' \mid \|\xi\| = 1\big\}$ and let $\bar{\xi} \in \Gamma_{K'}$. We show that there exist a positive integer $n(\bar{\xi})$ and a neighbourhood $V(\bar{\xi})$, such that $\langle \xi, w^n \rangle < 0$, for $n > n(\bar{\xi})$ and $\xi \in V(\bar{\xi})$. Recalling $K'\left(w^0\right) \subseteq K'$ we split the proof in two parts.

1. Let first assume $\bar{\xi} \in \Gamma_{K'(w^0)}$. We have $\langle \bar{\xi}, v^0 \rangle < -\delta < 0$, for some $\delta = \delta(\bar{\xi}) > 0$, and so

$$\lim_{n \to +\infty} \frac{1}{t_n} \langle \bar{\xi}, w^n - w^0 \rangle = \lim_{n \to +\infty} \frac{1}{t_n} \langle \bar{\xi}, w^n \rangle = \langle \bar{\xi}, v^0 \rangle < 0.$$

Hence, there exists $n(\bar{\xi})$ such that $\forall n > n(\bar{\xi})$, $\langle \bar{\xi}, w^n \rangle < 0$.
Now let $\langle \bar{\xi}, w^n \rangle < -\varepsilon < 0$ for some $\varepsilon > 0$ and $n > n\left(\bar{\xi}\right)$. Then

$$\begin{aligned} \langle \xi, w^n \rangle &= \langle \bar{\xi}, w^n \rangle + \langle \xi - \bar{\xi}, w^n \rangle < -\varepsilon + \left\|\xi - \bar{\xi}\right\| \left\|w^n - w^0 + w^0\right\| \\ &\leq -\varepsilon + \left\|\xi - \bar{\xi}\right\| \left(\left\|w^n - w^0\right\| + \left\|w^0\right\|\right). \end{aligned}$$

Since clearly $w^n \to w^0$, we have that, for every $\beta > 0$, there exists $n(\beta) > 0$ such that $\left\|w^n - w^0\right\| < \beta$. Now we consider $\bar{n} = \max\{n(\beta), n(\bar{\xi})\}$ and we get

$$\langle \xi, w^n \rangle < -\varepsilon + \left\|\xi - \bar{\xi}\right\| \left(\beta + \left\|w^0\right\|\right) < -\frac{1}{2}\varepsilon,$$

as long as $\left\|\xi - \bar{\xi}\right\| < \dfrac{\varepsilon}{2\big(\beta + \|w^0\|\big)}$, which defines $V(\bar{\xi})$.

2. Let now assume $\bar{\xi} \in \Gamma_{K'} \setminus \Gamma_{K'(w^0)}$. We have now $\langle \bar{\xi}, w^0 \rangle < -\varepsilon < 0$, for some $\varepsilon = \varepsilon(\bar{\xi}) > 0$. Then:

$$\begin{aligned} \langle \xi, w^n \rangle &= \langle \bar{\xi}, w^0 \rangle + \langle \xi, w^n - w^0 \rangle + \langle \xi - \bar{\xi}, w^0 \rangle < -\varepsilon + \left\|w^n - w^0\right\| + \left\|\xi - \bar{\xi}\right\| \left\|w^0\right\| \\ &< -\varepsilon + o(n) + \left\|\xi - \bar{\xi}\right\| \left\|w^0\right\| < -\varepsilon + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < 0, \end{aligned}$$

for n large enough, i.e. $n > n(\bar{\xi})$ and $\left\|\xi - \bar{\xi}\right\| < -\dfrac{\varepsilon}{3\left\|w^0\right\|}$, which defines $V\left(\bar{\xi}\right)$.

The assumption that Z is finite dimensional implies that $\Gamma_{K'}$ is a compact set and hence we can find a finite number of elements $\xi_1, \dots, \xi_s \in \Gamma_{K'}$ such that $\Gamma_{K'} \subseteq \bigcup_{i=1}^{s} V(\xi_i)$. Let $n_0 = \max\{n(\xi_i), i = 1, \dots, s\}$. For $n > n_0$, $\langle \xi, w^n \rangle < 0$, $\forall \xi \in \Gamma_{K'}$ and hence, $\forall \xi \in K'$. This shows that $w^n \in -\operatorname{int} K \subseteq -K$ and so points $x^0 + t_n u$ are feasible for $n > n_0$.

From the assumptions, we have $z^0 \in -\operatorname{int} C$, which implies the contradiction $y^n - y^0 \in -\operatorname{int} C$, for n large enough. □

Remark 3.1 The following dual form of (7) can be easily derived. For all $(z^0, v^0) \in H'\Big(x^0, (y^0, w^0); u\Big)$, $\exists \lambda \in C'$, $\eta \in K'(w^0)$, $(\xi, \eta) \neq (0, 0)$ such that

$$\langle \xi, z^0 \rangle + \langle \eta, v^0 \rangle \geq 0 \tag{8}$$

We now present sufficient conditions in terms of $H'\Big(x^0, (y^0, w^0); u\Big)$ for (x^0, y^0) to be an i-minimizer for the constrained problem (1). We need to first deal with the unconstrained problem (2).

Theorem 3.2 (Sufficient conditions for i-minimizers, unconstrained case) *Consider svp (2) with Y finite dimensional and $F : X \rightsquigarrow Y$ locally Lipschitz. Suppose that (x^0, y^0), $y^0 \in F(x^0)$, is such that $y^0 \in p\text{-}\mathrm{Min}_C F(x^0)$ and*

$$\forall\, u \in X \setminus \{0\} : F'(x^0, y^0; u) \cap (-C) = \emptyset\,. \tag{9}$$

Then (x^0, y^0) is an i-minimizer for (2).

Proof Like in the proof of Theorem 2.1, we may assume without loss of generality that Y is an Euclidean space (note that not only the notion of an i-minimizer is invariant with respect to equivalent norms, but also the set $p\text{-}\mathrm{Min}_C F(x^0)$).

We may assume that F is Lipschitz with constant $L > 0$ on $x^0 + r \operatorname{cl} B$. Suppose that x^0 is not an i-minimizer. Choose a monotone decreasing sequence $\varepsilon_k \to 0^+$. Then there exist sequences $t_k \to 0^+$ $(t_k < r)$, and $u^k \in X \cap S$, such that

$$D\left(F\left(x^0 + t_k u^k\right) - y^0, -C\right) < \varepsilon_k t_k\,.$$

Since u^k are unit vectors in Y, passing to a subsequence we may assume also that $u^k \to u^0$. By the Lipschitz property $D(F(x^0 + t_k u^0) - y^0, -C) < \varepsilon_k t_k + L\,\|u^k - u^0\|\, t_k$, and from the positive homogeneity of $D(\,\cdot\,, -C)$, we obtain

$$D\left(\frac{1}{t_k}\left(F(x^0 + t_k u^0) - y^0\right), -C\right) < \varepsilon_k + L\,\|u^k - u^0\|\,.$$

Let $y^k \in F(x^0 + t_k u^0)$ be such that $D(\bar{y}^k, -C) < \varepsilon_k + L\|u^k - u^0\|$, where $\bar{y}^k = (1/t_k)(y^k - y^0)$. The sequence $\{\bar{y}^k\}$ is bounded, which follows from the following reasoning. Since $y^0 \in p\text{-}\mathrm{Min}_C F(x^0)$, there exists $\sigma > 0$, such that $F(x^0) \cap (y^0 - C(2\sigma)) = \{y^0\}$. Eventually diminishing σ, we may assume that $0 < \sigma < 1/2$. Let k be such that $\varepsilon_k + L\,\|u^k - u^0\| < L$, whence $D(y^k - y^0, -C) < L t_k$. Then $\|\bar{y}^k\| \leq L\,(1 + 1/\sigma)$. Indeed, assume on the contrary that $\|\bar{y}^k\| > L\,(1 + 1/\sigma)$, or equivalently $\|y^k - y^0\| > L\,(1 + 1/\sigma)\, t_k$. We have

$$D(y^k - y^0, -C) < L t_k\, \frac{\sigma}{L t_k (1 + \sigma)}\, \|y^k - y^0\| < \sigma\, \|y^k - y^0\|\,.$$

This inequality shows that $y^k - y^0 \in -C(\sigma)$, whence, from Lemma 2.2 we get

$$\|y^k - y^0\| \le \frac{L\,(1+\sigma)}{\sigma}\,\|(x^0 + t_k u^0) - x^0\| = L\,\left(1 + \frac{1}{\sigma}\right)\,t_k\,,$$

a contradiction.

We proved that the sequence $\{\bar{y}^k\}$ is bounded and $\|\bar{y}^k\| \le L\,(1 + 1/\sigma)$ for all sufficiently large k. Passing to a subsequence, we may assume that $\bar{y}^k \to \bar{y}^0$, whence $\|\bar{y}^0\| \le L\,(1+1/\sigma)$ and $\bar{y}^0 \in F'(x^0, y^0; u^0)$. In other words $\bar{y}^0 \in F'(x^0, y^0; u^0) \cap L\,(1 + 1/\sigma)\,\mathrm{cl}\,B$. This set is compact (recall that $F'(x^0, y^0; u)$ is closed as a consequence of the general properties of the upper limit, see the representation in [1, page 41]). From the compactness and the property $F'(x^0, y^0; u) \cap (-C) = \emptyset$, we have

$$D(\bar{y}^0, -C) \ge D\left(F'(x^0, y^0; u) \cap L\Big(1 + \frac{1}{\sigma}\Big)\,\mathrm{cl}\,B, -C\right) > 0\,.$$

On the other hand, taking a limit in the inequality $D(\bar{y}^k, -C) \le \varepsilon_k + L\,\|u^k - u^0\|$, we get $D(\bar{y}^0, -C) \le 0$, a contradiction. □

In order to extend the previous result to the constrained svp (1), we need the following lemma.

Lemma 3.1 *Let x^0 be feasible for problem (1). Assume there exist $y^0 \in F(x^0)$ and $w^0 \in G\left(x^0\right) \cap (-K)$, such that for some positive numbers A and α it holds*

$$D\left(H\left(x\right) - h^0, -\left(C \times K\left(w^0\right)\right)\right) \ge A\left\|x - x^0\right\|^{\alpha} \quad \forall x \in U\left(x^0\right) \setminus \left\{x^0\right\},$$

where $h^0 = \left(y^0, w^0\right)$. Then there exists a positive number A' such that

$$D\left(F\left(x\right) - y^0, -C\right) \ge A'\left\|x - x^0\right\|^{\alpha} \qquad \forall x \in U\left(x^0\right) \setminus \left\{x^0\right\}.$$

Proof Assume there exists A such that

$$D\left(\left(F\left(x\right) \times G\left(x\right)\right) - \left(y^0, w^0\right), -\left(C \times K\left(w^0\right)\right)\right) \ge A\left\|x - x^0\right\|^{\alpha} \forall x \in U\left(x^0\right) \setminus \left\{x^0\right\}. \quad (10)$$

Set $\theta = (y, w) \in F\left(x\right) \times G\left(x\right)$ and $\xi = (\xi_1, \xi_2) \in \left(C' \times K'(w^0)\right) \cap S$ (S denotes here the unit sphere in $Y \times Z$). From the definition of the oriented distance, we get

$$\max\{\langle \xi_1, y - y^0\rangle + \langle \xi_2, w - w^0\rangle \mid (\xi_1, \xi_2) \in (C' \times K'(w^0)) \cap S\} \ge A\left\|x - x^0\right\|^{\alpha}$$

$\forall\,(y, w) \in F\left(x\right) \times G\left(x\right)$. Let x now be any feasible point, asume that is there exists, eventually dependent on x, some $w\left(x\right) \in G\left(x\right) \cap (-K)$. We can now evaluate the previous inequality along any couple $(y, w\left(x\right))$, $y \in F\left(x\right)$. Clearly $\langle \xi_2, w(x) - w^0\rangle = \langle \xi_2, w(x)\rangle \le 0$. For every feasible $x \in U$, $x \ne 0$ and every couple $(y, w(x)) \in H(x)$, there exists $(\hat{\xi}_1, \hat{\xi}_2) \in \left(C' \times K'(w^0)\right) \cap S$ (eventually dependent on x, y, w), such that the maximum in inequality (10) is attained, i.e.

$$\langle \hat{\xi}_1, y - y^0\rangle + \langle \hat{\xi}_2, w(x) - w^0\rangle \ge A\left\|x - x^0\right\|^{\alpha}.$$

Therefore

$$\langle \hat{\xi}_1, y - y^0\rangle \ge A\left\|x - x^0\right\|^{\alpha} \quad \forall x,\, \forall y \in F\left(x\right) \quad (11)$$

and $\hat{\xi}_1 \ne 0$. In fact, if $\hat{\xi}_1 = 0$, then we would get the contradiction $0 \ge \langle \hat{\xi}_2, w\left(x\right) - w^0\rangle \ge A\left\|x - x^0\right\| > 0$.

Now, since $(\hat{\xi}_1, \hat{\xi}_2) \in S$, we have

$$0 < \sup\left\{\left\|\hat{\xi}_1\right\| \mid x \in U,\ x \text{ feasible}, y \in F(x)\right\} < \tau < +\infty .$$

Hence, $\forall y \in F(x)$, from equation (11) one finally gets

$$\frac{1}{\left\|\hat{\xi}_1\right\|}\langle\hat{\xi}_1, y - y^0\rangle \geq \frac{A}{\left\|\hat{\xi}_1\right\|}\left\|x - x^0\right\|^{\alpha} \geq \frac{A}{\tau}\left\|x - x^0\right\|^{\alpha} .$$

Putting $A' = \dfrac{A}{\tau}$, we can write for every feasible $x \in U$ and every $y \in F(x)$,

$$\max\{\langle\xi, y - y^0\rangle | \xi \in C', \|\xi\| = 1\} \geq A'\|x - x^0\|^{\alpha} ,$$

which is equivalent to $D(F(x), -C) \geq A'\|x - x^0\|^{\alpha}$ and the proof is complete.

□

Theorem 3.3 (Sufficient conditions for i-minimizers, constrained case) *Consider svp (1), with Y and Z finite dimensional spaces and let $F : X \rightsquigarrow Y$ and $G : X \rightsquigarrow Z$ be locally Lipschitz. Assume x^0 is feasible for svp (1) and $y^0 \in p\text{-Min}_C F\left(x^0\right)$. If for some $w^0 \in G\left(x^0\right) \cap (-K)$,*

$$H'\left(x^0, \left(y^0, w^0\right); u\right) \cap \left(-\left(C \times K(w^0)\right)\right) = \emptyset, \quad \forall u \in X \setminus \{0\}, \tag{12}$$

then $\left(x^0, y^0\right)$ is an i-minimizer.

Proof The assumptions guarantee that $\left(x^0, \left(y^0, w^0\right)\right) \in p\text{-Min}_{C\times K(w^0)} H\left(x^0\right)$. Then Theorem 3.2 ensures that $\left(x^0, \left(y^0, w^0\right)\right)$ is an i-minimizer for the unconstrained problem $\min_{C\times K(w^0)} H(x)$, $x \in X$. Applying Lemma 3.1, the proof is complete. □

Remark 3.2 Condition (12) can be expressed also in dual form requiring that for all $\left(z^0, v^0\right) \in H'\left(x^0, \left(y^0, w^0\right); u\right)$ and for all $u \in X$, there exists a couple $\left(\xi^0, \eta^0\right) \in \left(C' \times K'\left(w^0\right)\right)$, $\left(\xi^0, \eta^0\right) \neq (0, 0)$ such that

$$\langle\xi^0, z^0\rangle + \langle\eta^0, v^0\rangle > 0$$

Dealing with isolated minimizers of svp (1) we can also prove a reversal of the previous sufficient conditions. However we need to assume the following constraint qualification holds.

Definition 3.1 *Let x^0 be feasible for problem (1) and $w^0 \in G(x^{)} \cap (-K)$. If $v^0 \in -K(w^0)$, $v^0 = \lim \frac{w^k - w^0}{t_k}$, $t_k \to 0^+$, $w^k \in G(x^0 + t_k u)$, implies there exists a sequence $u^k \in X$, $u^k \to u$, with $G(x^0 + t_k u^k) \cap (-K) \neq \emptyset$, then we say that the constraint qualification $\mathcal{Q}$ holds at (x^0, w^0).*

Constraint qualification $\mathcal{Q}$ can be regarded as an extension of the classical Kuhn-Tucker constraint qualification (see e. g. [20]).
The proof of the necessary condition for i minimizers is based on the following lemma.

Lemma 3.2 *Let E^k be a sequence of sets in Y such that $D\left(E^k,-C\right)\geq A$, for all k, and let $u^k\in X$ be a sequence converging to some $u\in X$. Then, for any positive number L, there exists a positive number A' such that*

$$D\left(E^k+L\left\|u^k-u\right\|B,-C\right)\geq A'$$

for k sufficiently large.

Proof Assume ab absurdo that there exists a sequence $\varepsilon_k\to 0^+$ such that

$$D\left(E^k+L\left\|u^k-u\right\|B,-C\right)\leq\varepsilon_k$$

and recall that, by definition,

$$D\left(E^k+L\left\|u^k-u\right\|B,-C\right)=\inf\left\{D\left(y,-C\right)\mid y\in E^k+L\left\|u^k-u\right\|B\right\}.$$

Therefore, for every k, there exists $y^k\in E^k+L\left\|u^k-u\right\|B$ such that

$$D\left(y^k,-C\right)\leq D\left(E^k+L\left\|u^k-u\right\|B,-C\right)+\frac{1}{k},$$

that is,

$$\max\{\langle\xi,y^k\rangle\mid\xi\in C'\cap S\}\leq D\left(E^k+L\left\|u^k-u\right\|B,-C\right)+\frac{1}{k}\leq\varepsilon_k+\frac{1}{k}.$$

If we put $y^k=e^k+L\left\|u^k-u^0\right\|b^k$, $e^k\in E^k$, $b^k\in B$, we obtain

$$\begin{aligned}
D(E^k,-C) &\leq \max\{\langle\xi,e^k\rangle\mid\xi\in C'\cap S\}\\
&= \max\{\langle\xi,e^k+L\|u^k-u\|b^k-L\|u^k-u\|b^k\rangle\mid\xi\in C'\cap S\}\\
&\leq \max\{\langle\xi,e^k+L\|u^k-u\|b^k\rangle\mid\xi\in C'\cap S\}\\
&+ \max\{\langle\xi,-L\|u^k-u\|b^k\rangle\mid\xi\in C'\cap S\}\\
&\leq \varepsilon_k+\frac{1}{k}+\max\{\langle\xi,-L\|u^k-u\|b^k\rangle\mid\xi\in C'\cap S\}\to 0\,,
\end{aligned}$$

which contradicts $D\left(E^k,-C\right)\geq A>0$. □

Theorem 3.4 *Let x^0 be feasible for svp (1). Assume constraint qualification $\mathcal{Q}$ holds for svp (1) at $\left(x^0,w^0\right)$, $w^0\in G\left(x^0\right)\cap(-K)$ and suppose the couple $\left(x^0,y^0\right)$, $y^0\in F\left(x^0\right)$, is an i-minimizer for problem (1) and F is locally Lipschitz. Then $y^0\in p\text{-}\mathrm{Min}_C F\left(x^0\right)$ and condition (12) holds.*

Proof There exists a neighborhood U of x^0 such that, for every feasible $x\in U$,

$$D\left(F\left(x\right)-y^0,-C\right)\geq A\left\|x-x^0\right\|.$$

Assume, by contradiction, that condition (12) does not hold. Then there exists a vector $u\in X\backslash\{0\}$ and a couple $\left(z^0,v^0\right)\in H'\left(x^0,\left(y^0,w^0\right);u\right)$ such that $\left(z^0,v^0\right)\in-\left(C\times K\left(w^0\right)\right)$. Hence $v^0\in-K\left(w^0\right)$ and

$$v^0=\lim_{k\to+\infty}\frac{w^k-w^0}{t_k}$$

for some $w^k \in G\left(x^0+t_k u\right)$. Since the constraint qualification $\mathcal{Q}$ holds, there exists some sequence $u^k \to u$, such that $G\left(x^0+t_k u^k\right) \cap (-K) \neq \emptyset$. It follows that

$$D\left(F\left(x^0+t_k u^k\right)-y^0,-C\right) \geq A t_k \left\|u^k\right\|$$

and hence

$$D\left(\frac{1}{t_k}\left(F\left(x^0+t_k u^k\right)-y^0,-C\right)\right) \geq A\left\|u^k\right\|.$$

Since F is assumed to be locally Lipschitz, we have

$$\frac{1}{t_k}\left(F\left(x^0+t_k u\right)-y^0\right) \subseteq \frac{1}{t_k}\left(F\left(x^0+t_k u^k\right)-y^0\right)+L\left\|u^k-u\right\| B,$$

and then, for some positive number A',

$$D\left(\frac{F\left(x^0+t_k u\right)-y^0}{t_k\|u^k\|},-C\right) \geq D\left(\frac{F\left(x^0+t_k u^k\right)-y^0}{t_k\|u^k\|}+L\left\|\frac{u^k}{\|u^k\|}-\frac{u}{\|u^k\|}\right\| B,-C\right) \geq A'$$

(the last inequality follows from Lemma 3.2). Hence, we also have:

$$D\left(\frac{y^k-y^0}{t_k\|u^k\|},-C\right) \geq D\left(\frac{F\left(x^0+t_k u\right)-y^0}{t_k\|u^k\|},-C\right) \geq A',$$

where $y^k \in F\left(x^0+t_k u^0\right)$, is such that

$$z^0=\lim \frac{y^k-y^0}{t_k}.$$

Hence,

$$D\left(z^0,-C\right) \geq A'\|u\|>0,$$

whence $z^0 \notin -C$, which completes the proof. □

The next example shows the importance of constraint qualification $\mathcal{Q}$ in Theorem 3.4 even for single-valued functions.

Example 3.1 *Consider vvp (3), with $f, g: \mathbb{R}^2 \to \mathbb{R}^2$, $C=K=\mathbb{R}_+^2$. Assume $f=(f_1, f_2)$ with $f_1(x_1, x_2)=x_1^2$, if $x_1 \leq 0$, $f_1(x_1, x_2)=x_1$, if $x_1>0$ and $f_2(x_1, x_2)=-x_1^2-x_2^2$. Further let $g=(g_1, g_2)$, with $g_1(x_1, x_2)=-x_1^3+x_2$ and $g_2(x_1, x_2)=-x_2$. Let $x^0=(0,0)$; constraint qualification $\mathcal{Q}$ does not hold at $(x^0, g(x^0))$ and the point $(x^0, f(x^0))$ is an i-minimizer. However, condition (12) does not hold, since for $u=(-1,0)$, we have*

$$\langle \xi^0, f'\left(x^0\right) u\rangle+\langle \eta^0, g'\left(x^0\right) u\rangle=0$$

whatever $\left(\xi^0, \eta^0\right) \in \left(C' \times K'\left(g(x^0)\right)\right)$ (we have used the fact that in case of a single valued directionally differentiable function, the defined Dini derivative coincides with the classical directional derivative).

4 Optimality Under Convexity Type Conditions

Theorem 3.2 reverts the necessary conditions for i-minimizers from Theorem 3.4 under the stronger assumptions that Y is finite dimensional. It is natural to ask whether, similarly, the necessary conditions for w-minimizers from Theorem 3.1 can be reverted. We show that this is possible if in addition a convexity type condition for the svf F is assumed and C and K are closed convex pointed cones. In convex analysis, convexity type conditions are usually associated with global minimizers. By analogy, in Theorem 4.1, we propose a result, which concerns global w-minimizers.

The pair (x^0, y^0), $y^0 \in F(x^0)$, is said to be a global w-minimizer for svp (1) if, for every feasible $x \in X$, $F(x) \cap (y^0 - \operatorname{int} C) = \emptyset$. Similarly, one can define global versions of all the optimality concepts introduced in Section 2.

We say that the svf $F : X \rightsquigarrow Y$ is C-convex-along-rays at (x^0, y^0) if $(1-t)y^0 + tF(x) \subseteq F((1-t)x^0 + tx) + C$ for all $x \in U$ and $0 < t < 1$. The concept of a convex-along-rays scalar-valued function is introduced in Rubinov [21] and is used there for the purposes of abstract convexity and global optimization. The next lemma has been proved in [4].

Lemma 4.1 *Let C be a pointed closed convex cone in the finite dimensional space Y. Then, for any a^1, $a^2 \in Y$ the set $(a^1 - C) \cap (a^2 + C)$ is bounded.*

Theorem 4.1 *Let Y and Z be finite dimensional normed spaces and let $C \subseteq Y$ and $K \subseteq Z$ be closed convex and pointed cones. Let x^0 be feasible for svp (1), $y^0 \in F\left(x^0\right)$ and $w^0 \in G\left(x^0\right) \cap (-K)$. Assume F is C-convex along-rays at $\left(x^0, y^0\right)$ and G is K-convex along-rays at $\left(x^0, w^0\right)$. Suppose also that $\forall u \in X \setminus \{0\}$ there exist $f_u \in Y$ and $g_u \in Z$ such that*

$$\begin{aligned} F\left(x^0 + tu\right) &\subseteq y^0 + tf_u + C\,, \\ G\left(x^0 + tu\right) &\subseteq w^0 + tg_u + K\,. \end{aligned} \tag{13}$$

Then, if condition (7) is satisfied, $\left(x^0, y^0\right)$ is a w-minimizer for svp (1)

Proof By convexity along-rays of F and G, it follows that the map $H\left(x\right) = \left(F\left(x\right) \times G\left(x\right)\right)$ is $C \times K$-convex along-rays starting at $\left(x^0, h^0\right)$, where $h^0 = (y^0, w^0)$. We have

$$tH\left(x\right) - th^0 \subseteq H\left(x^0 + t\left(x - x^0\right)\right) - h^0 + \left(C \times K\right),$$

$\forall x \in X$, $\forall t \in (0,1)$ and hence, for $u = x - x^0$,

$$H\left(x\right) - h^0 \subseteq \frac{1}{t}\left\{H\left(x^0 + tu\right) - h^0\right\} + \left(C \times K\right).$$

It follows that $\forall h \in H(x)$ and $\forall t \in (0,1)$, there exist $h^t = (y^t, w^t) \in H\left(x^0 + tu\right)$ and $(\delta^t, \gamma^t) \in (C \times K)$ such that

$$\frac{1}{t}\left\{h^t - h^0\right\} = h - h^0 - \left(\delta^t, \gamma^t\right)\,. \tag{14}$$

By (13), $\exists h_u = (f_u, g_u)$ such that

$$\frac{1}{t}\left\{H\left(x^0 + tu\right) - h^0\right\} \subseteq th_u + \left(C \times K\right)$$

and therefore

$$\frac{1}{t}\left(h^t - h^0\right) \in \left[h - h^0 - \left(C \times K\right)\right] \cap \left[h_u + \left(C \times K\right)\right].$$

The latter is, by Lemma 4.1, a bounded set. Hence, there exists some sequence $t_k \to 0^+$ such that, setting $h^k = h^{t_k}$,

$$\frac{1}{t_k}\left(h^k - h^0\right) \to \left(z^0, v^0\right) \in H'\left(x^0, h^0; u\right) .$$

By (14), putting $\delta^k = \delta^{t_k}$, $\gamma^k = \gamma^{t_k}$, we obtain for every $(\xi, \eta) \in C' \times K'$ and for every $h = (y, w) \in H(w)$

$$\begin{aligned} \langle \xi, y - y^0 \rangle + \langle \eta, w - w^0 \rangle &= \frac{1}{t_k}\left(\langle \xi, y^k - y^0 \rangle + \langle \eta, w^k - w^0 \rangle\right) + \langle \xi, \delta^k \rangle + \langle \eta, \gamma^k \rangle \\ &\geq \frac{1}{t_k}\left(\langle \xi, y^k - y^0 \rangle + \langle \eta, w^k - w^0 \rangle\right) . \end{aligned}$$

Taking the limit as $t_k \to 0^+$, by (7), $\forall h \in H(x)$, $\xi \in C'$, $\eta \in K'\left(w^0\right) \subseteq K'$, we get

$$\langle \xi, y - y^0 \rangle + \langle \eta, w - w^0 \rangle \geq 0 ,$$

and moreover $\langle \eta, w^0 \rangle = 0$. Assume x is feasible and let $w \in G(x) \cap (-K)$. Then, $\forall y \in F(x)$

$$\langle \xi, y - y^0 \rangle \geq \langle \eta, -w \rangle \geq 0.$$

That is $y - y^0 \notin -\operatorname{int} C$, i.e., finally,

$$F(x) \cap \left(y^0 - \operatorname{int} C\right) = \emptyset$$

for every feasible x. □

The next example shows that condition (13) is important for the validity of Theorem 4.1.

Example 4.1 *Consider vvp (3) with $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $Z = \mathbb{R}$, $K = \mathbb{R}_+$, $g(x) = x$ and $f(x) = (x, -\sqrt{|x|})$. Then the assumptions of Theorem 4.1 are satisfied at the point $(x^0, (y^0, w^0))$, where $x^0 = 0$, $y^0 = (0,0)$ and $w^0 = 0$. Condition (7) is satisfied, since $H'(x^0, (y^0, w^0); u) = \emptyset$, for $u \neq 0$. At the same time (x^0, y^0) is not a w-minimizer.*

The next example shows an application of Theorem 4.1.

Example 4.2 *Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $Z = \mathbb{R}$, $C = \mathbb{R}^2_+$, $K = \mathbb{R}_+$, $F : X_0 \rightsquigarrow \mathbb{R}^2$ be given by*

$$F(x) = \begin{cases} [0,\, 1] \times [0,\, 1] & , \quad x \neq 0, \\ ([-1,\, 0] \times \{0\}) \cup (\{0\} \times [-1,\, 0]) & , \quad x = 0. \end{cases}$$

and $G(x) = |x| - 1$. Put $x^0 = 0$ and $y^0 = (0,\, 0)$, $w^0 = -1$. It can be easily checked that G fulfils the assumptions of Theorem 4.1. To show the C-convexity-along-rays of F at $\left(x^0, y^0\right)$ we must check that $tF(x) \subseteq F(tx) + \mathbb{R}^2_+$ for $0 < t < 1$. For $x \neq x^0$ this is the true inclusion $[0,\, 1] \times [0,\, 1] \subseteq ([0,\, 1] \times [0,\, 1]) + \mathbb{R}^2_+$. For $x = x^0$ the validity follows from the true inclusion $[-t,\, 0] \subseteq [-1,\, 0]$. Easy calculations yield

$$F'\left(x^0, y^0; u\right) = \begin{cases} \mathbb{R}^2_+ & , \quad u \neq 0, \\ (\mathbb{R}_- \times \{0\}) \cup (\{0\} \times \mathbb{R}_-) & , \quad u = 0, \end{cases}$$

and $G'\left(x^0, w^0; u\right) = |u|$, whence it is obvious that condition (7) is satisfied. Further for $u \neq 0$ the vectors $f_u = (0,\, 0)$ and $g_u = 0$ satisfy the conditions (13). Then $\left(x^0, y^0\right)$ is a global w-minimizer, which follows from Theorem 4.1.

5 Conclusions

First order optimality conditions in set-valued optimization are mostly developed in the framework of epiderivatives. This dual approach seems to be less general than a primal one. The present paper presents an attempt to develop a concept of directional derivative (of Dini type) and to apply it to optimization.

Besides, we think the results proved strengthen the idea set-valued optimization is a generalization of vector-optimization. Therefore, the most developed studies in multiobjective optimization can be a guideline for further researches in this field. The common approach we used in this paper allows obtaining some multiobjective results as a special case of those we proved here. Finally, within the scheme we present, it is highlighted the role of isolated minimizer. This notion seems to be fairly new for set-valued optimization, yet it is well established in vector optimization.

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Giovanni P. Crespi
Université de la Vallée d'Aoste, Facoltà di Scienze Economiche, Strada Cappuccini 2A, 11100 Aosta, Italy
E-mail address: g.crespi@univda.it

Ivan Ginchev
University of Insubria, Department of Economics, Via Monte Generoso, 71, 21100 Varese, Italy
E-mail address: iginchev@eco.uninsubria.it

Matteo Rocca
University of Insubria, Department of Economics, Via Monte Generoso, 71, 21100 Varese, Italy
E-mail address: mrocca@eco.uninsubria.it