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SUPERLINEAR SEPARATION AND DUAL CHARACTERIZATIONS OF RADIANT FUNCTIONS

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Dedicated to A.M. Rubinov on his 65th birthday.

Abstract: Superlinear functionals are used to separate points from a radiant set according to both a strict and a weak version. Strict separation characterizes closed radiant sets; weak separation is used to define evenly radiant sets, which are characterized by means of a property of the tangent cone to the set at points of the boundary. The separation properties can be described via a polarity relation between a normed space X and the set L of continuous superlinear functionals defined on X. Radiant functions are the ones which are increasing along rays, i.e. the ones whose lower level sets are radiant and so they extend the class of quasiconvex functions with minimum at the origin. We study two particular subclasses: the one of l.s.c. radiant functions, whose lower level sets are closed and radiant and the one of evenly radiant functions, whose lower levels are evenly radiant. We introduce a conjugate function (defined on L), in two different versions, and prove the coincidence between a function and its second conjugate when the function belongs to one of the classes mentioned above. The conjugate function is then used to give global optimality conditions for problems described by radiant objective and constraints.

Key words: *radiant sets, radiant functions, abstract convexity, nonlinear separation, polarity, generalized conjugacy, duality*

Mathematics Subject Classification: *26B25, 49N15, 90C46*

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1 Introduction

Radiant function are the ones whose lower level sets are radiant, i.e. those which are increasing along rays. This is a very large class of functions, which seemingly enjoyies very few regularity properties. For their dual description the space of linear functional, which is used for convex and quasiconvex functions, is largely insufficient. Nevertheless we show that a dual description in terms of a conjugate function is indeed possible if we define the latter on the set of superlinear continuous functions.

In recent years a number of papers have been devoted to the study of conjugation schemes for quasiconvex functions and in particular those with minimum point at the origin, which are a subclass of radiant functions (see e.g. [4, 8, 9, 16, 21] and references therein) with the aim of finding global optimality conditions for optimization problems described by quasiconvex functions.

These schemes are strictly related to the separation properties of convex sets by means of linear functionals and therefore the conjugate function is usually defined on the dual space X' or in $X' \times I\!R$.

Analogous separation results (see [26]), using superlinear functions, hold for radiant sets: namely if A is a closed radiant set of a normed vector space X then every point x not belonging to A can be separated from A by means of a superlinear continuous function p such that $p(x) > 1$ and $p(a) \leq 1$ for every $a \in A$.

This result extends the well-known characterization of closed convex sets containing the origin and can be described in terms of a polarity relation. The polar set of A is defined in the convex cone L of continuous superlinear functions defined on X and A coincide with its bipolar if and only if it is closed and radiant.

This result is the basis of a conjugation scheme, in which the space X is paired with the set L and an extended-real valued function f defined on X coincides with its second conjugate if and only if it is lower semicontinuous and radiant, that is its lower level sets are closed and radiant, and satisfies $f(0) = -\infty$.

Since the conjugate function is defined in such a way that its lower level sets are polar to the level sets of f and such polar sets are always convex and closed in L, then the conjugate function is l.s.c. and quasiconvex.

In close analogy to similar construction for convex sets, we also introduce the class of evenly radiant sets as those radiant sets A such that, for every $x \notin A$ there exists a continuous superlinear function p with $p(x) \geq 1$ and $p(a) < 1$ for all $a \in A$. Thus an evenly radiant set A is the intersection of open level sets $[p < 1]$, with $p \in L$ and clearly a closed radiant set is evenly radiant. Evenly radiant sets can be characterized in primal terms by a property of the tangent cone at points not belonging to the set.

Since evenly radiant sets can be described in terms of an appropriate polarity relation, we can introduce a second type of conjugate function, which can be used to characterize the functions (we call them evenly radiant) whose lower level sets are evenly radiant. Since the conjugates we introduce are derived from a polarity, they fit in the general scheme described in [20] and thus a number of properties follow from the theory developed therein. Besides them we can prove a version of the Toland-Singer formula which relates the infimum value of the difference of two function to the infimum of the difference of their conjugates. This formula can be used, as in [8], to develop global optimality conditions for a number of set constrained problems described by radiant functions. We do not follow this line of research. but rather apply the previous concepts to obtain necessary and sufficient optimality conditions for a constrained maximization problem in which both the objective and the constraint functions are radiant. Such conditions are given both in terms of the conjugate function and by means of appropriate subgradients.

The outline of the paper is the following: in Section 2 we describe the separation properties of radiant sets and introduce two polarity relations between X and L. Section 3 is devoted to some particular classes of radiant functions: those which are lower semicontinuous (thus having closed level sets) and the ones, that we call evenly radiant, whose lower level sets are evenly radiant. We also study a subclass of the latter formed by the functions whose *strict* lower level sets $[f < k]$ are evenly radiant. These functions can be characterized as consistently increasing along rays, a property which is equivalent to the requirement that the lower Hadamard directional derivative $f_H^-(x, x)$ is nonnegative for every $x \in X$. In Section 4 we introduce conjugate functions. These are closely related to polarity, and thus we obtain two different definitions of conjugate functions and study their properties. We will see that the functions studied in Section 3 are precisely those which coincide with the second conjugate, under the further assumption that their value at the origin is $-\infty$. Section 5 introduces the notion of subdifferential (related to the conjugate function in the usual way) and discusses an application to global optimization.

$\boxed{2}$ Separation and Polarity for Radiant Sets

Consider a real normed vector space X with topological dual space X', endowed with the norm $\|x^*\| = \sup\{|x^*(x)|, \|x\| \leq 1\}$. We will denote by B the closed unit ball in X and by $B_\delta = \delta B$ the closed ball of radius $\delta > 0$. We will denote by $\overline{I\!R}$ the set $I\!R \cup \{+\infty\} \cup \{-\infty\}$ of extended real numbers. For any extended-real valued function g defined on X and $k \in I\!R$, we will denote by $[g \leq k] = \{x \in X : g(x) \leq k\}$ and $[g < k] = \{x \in X : g(x) < k\}$ the lower and, respectively, strict lower level set of g. Given a set $A \subseteq X$ the cone generated by A is the set $\operatorname{cone} A = \{y \in X : y = \lambda x, x \in A, \lambda > 0\}$ and the shadow of A is the set shw A=$\{y \in X : y = tx, x \in A, t \geq 1\}$. A set $A \subseteq X$ is said to be *radiant* if $x \in A$ and $\alpha \in [0, 1]$ imply $\alpha x \in A$. A number of properties of radiant sets are studied in [18, 12, 13]. In this section we are mainly concerned with their separation properties, which offer great analogies with those which hold for convex sets and naturally lead us to single out two important subclasses: the one of closed radiant sets and the one of evenly radiant sets.

$\boxed{2.1}$ Closed Radiant Sets

As every closed convex set can be separated by points not belonging to it by means of an open halfspace or, more precisely, by a continuous linear functional, analogously every closed radiant set can be separated by points not belonging to it by means of an open convex cone or, equivalently, by (positive) level sets of a continuous superlinear function. These results where proved in [26]. The proof of the geometric version of this result is given for completeness. The analytic version follows from standard separation arguments.

Proposition 2.1. *[26] For any closed and radiant set $A \subseteq X$ and any $x \notin A$, there exists an open convex cone K with $x \in K$ and some $\beta \in (0, 1)$ such that $A \cap (\beta x + K) = \emptyset$.*

Proof: The claim is trivially true if A is empty, by taking $K = X$. So let $A \neq \emptyset$. Since A is closed there exists some open ball U around x, with $A \cap U = \emptyset$. Moreover, since A is radiant, then $A \cap \operatorname{shw} U = \emptyset$. Let $K = \operatorname{cone} U$. Then K is an open convex cone with $x \in K$. Moreover there exists $\beta \in (0, 1)$ such that $\beta x \in U$ and $\beta x + K \subseteq \operatorname{shw} U$. □

Using standard separation arguments the set $\beta x + K$ mentioned in the proof of Proposition 2.1 can be seen as a (positive) upper level set of a superlinear function, i.e. (see [26]) if $K \neq X$ is an open convex cone and $z \in K$, then there exists a superlinear continuous funtion $p : X \to I\!R$ such that $z + K = \{x \in X : p(x) > 1\}$.

The analytical version of the separation result given in Proposition 2.1 reads now as follows.

Corollary 2.2. *For any nonempty closed radiant set $A \subseteq X$ and any point $x \notin A$, there exists a superlinear continuous function $p : X \to I\!R$ such that $p(a) \leq 1$ for every $a \in A$ and $p(x) > 1$.*

Corollary 2.2 extends to infinite dimensional spaces an analogous result by Shveidel [19], which makes use of particular superlinear functions. See also [13] for an equivalent formulation of the latter. These authors consider the class of those superlinear functions (defined on $I\!R^n$) which can be written as $p(x) = \min_{i=1,\dots,k} \langle x, \ell_i \rangle$, $k \leq n$, that is the minimum of at most n linear functions. This special feature of p offers a number of advantages for practical calculation purposes, but the resulting class $H_n \subseteq L$ is not a convex set. Our more general approach allows instead to consider a convex set L of superlinear functions (indeed a convex cone) and this plays an important role in the sequel. Indeed the separation result given in

Corollary 2.2 can be reinterpreted in terms of a polarity relation between the subsets of the space X and the subsets of the space L of superlinear functionals. In general a polarity (see e.g. [7, 8, 20] and the references therein) between two sets Z and W is a correspondance P which associates a subset of W to a subset of Z and, for every family of sets $A_i \subseteq Z$, $i \in I$, satisfies the equality

$$P\left(\bigcup_{i\in I} A_i\right) = \bigcap_{i\in I} P(A_i).$$

Consider the convex cone L of continuous superlinear functions defined on the normed space X endowed with the topology of pointwise convergence, so that $b_n \to b$ if and only if $b_n(x) \to b(x)$ for every $x \in X$. The set L is a convex cone in the set H of continuous positively homogeneous functions from X to $\mathbb{R}$, and its lineality space $L \cap -L$ coincides with X', the (normed) space of continuos linear functionals on X. We can endow H with the componentwise ordering relation, i.e. $h_1 \geq h_2$ if $h_1(x) \geq h_2(x)$ for all $x \in X$; this relation is induced by the closed convex cone $K \subset H$ of functions with nonnegative values. With respect to this order, X' is the set of maximal elements in L, i.e. if $p_0 \in X'$, there is no $p \in L$ such that $p \neq p_0$ and $p \geq p_0$. Moreover it holds $L \cap K = \{0\}$.

We will be interested in a different order relation on L which refines the componentwise order in that it considers only positive values. Given $p, q \in L$ we will write $p \geq_1 q$ if $[p > 1] \supseteq [q > 1]$. It is easy to show that $\geq_1$ is a reflexive and transitive binary relation on L. Moreover $p \geq_1 q$ means that $p \geq q$ in the set where they are both positive and therefore $p \geq q$ implies $p \geq_1 q$. To see that the converse implication is not true in general one can consider the functions $p(x) = 3x - |x|$ and $q(x) = x$. It is easy to show that $p \geq_1 q$ holds, though $p \geq q$ is not verified.

Definition 2.3. *Given some set $A \subset X$ we define the polar set of A as*

$$A^{\vee} = \{p \in L : p(a) \leq 1, \forall a \in A\}.$$

The following properties are easily verified: for any set A, we have that $A^{\vee} \subset L$ is closed (for the topology of pointwise convergence), radiant and convex, in that if $p_1, p_2 \in A^{\vee}$, then $[tp_1 + (1-t)p_2](a) \leq 1$ for all $t \in [0,1]$ and all $a \in A$. Moreover every polar set $A^{\vee}$ is downward in L with respect to the componentwise ordering on L in the sense that if $p_2 \in A^{\vee}$ and $p_1 \leq p_2$, then also $p_1 \in A^{\vee}$. Moreover it is downward with respect to the order $\geq_1$. An open question, whose answer has interesting consequences, is whether or not such properties are sufficient to characterize those sets in L which are the polar of some set in X.

For any set $B \subset L$ we can define the dual polarity in a completely analogous way (and this justifies the use of the same symbol) as

$$B^{\vee} = \{x \in X : p(x) \leq 1, \forall p \in B\}.$$

We note that $B^{\vee}$ is closed and radiant in X for all sets $B \subset L$.

For any set $A \subseteq X$ we can introduce the bipolar $(A^{\vee})^{\vee} = A^{\vee\vee}$. It can easily be seen that it holds $A \subseteq A^{\vee\vee}$ and that, given $A \neq \emptyset$, A is closed and radiant if and only if $A^{\vee\vee} = A$ (as an application of Corollary 2.2). Since the intersection of radiant sets is itself radiant, we can consider the radiant hull, $\operatorname{rad} A$, of any set $A \subseteq X$. The use of the operations of polarity on a set $A \subset X$ gives the closed radiant hull of A: the set $A^{\vee\vee} = \operatorname{cl}\operatorname{rad} A$ is the smallest closed and radiant set containing A.

The polarity relation just defined will be exploited in Section 3 to introduce a conjugation scheme which is well suited to analyse some classes of functions with radiant level sets.

We can compare the $\vee$-polarity with the usual polarity notion defined between subsets of X and X'. Given some set $A \subset X$, let the set $A^\circ = \{\ell \in X' : \ell(a) \leq 1\}$ be the polar set of A according to the classical definition of Convex Analysis. It is readily seen that $A^\circ = A^\vee \cap X'$. One can characterize convex sets among those which are closed and radiant by looking at their polar $A^\vee$.

Proposition 2.4. *Let $A \subseteq X$ be closed and radiant. Then A is convex if and only if for every $p \in A^\vee$ there exists a linear functional $\ell \in X'$ such that $\ell \in A^\vee$ and $\ell \geq p$.*

Proof: Let A be convex and take $p \in A^\vee$. Since $p(a) \leq 1$ for all $a \in A$, then the sets A and $P = [p > 1]$ are disjoint. If $[p > 1] = \emptyset$ then $p \leq 0$ and the thesis holds with $\ell = 0_{X'}$. If $[p > 1]$ is nonempty, then it is an open convex set, which can be separated from A. More precisely there exists $\nu \in X'$ and $\alpha \in \mathbb{R}$ such that $\nu(a) \leq \alpha$ for all $a \in A$ and $\nu(x) > \alpha$ for all $x \in P$. Since $0 \in A$ it holds $\alpha \geq 0$. We need to prove that α can be taken positive. Thus let

$$I = \{r \in \mathbb{R} : \nu(a) \leq r < \nu(x), \forall a \in A, \forall x \in P\},$$

which is nonempty, and assume that $I = \{0\}$. Then $\inf\{\nu(x), x \in P\} = 0$, i.e. there exists a sequence $\{x_n\} \subseteq P$ such that $\nu(x_n) \to 0$.

On the other hand, it is easy to see that $\operatorname{cone} P = [p > 0]$ and that $\nu(k) > 0$ for all $k \in \operatorname{cone} P$.

Since p is continuous, for a fixed $-1 < \eta < 0$, we find $\varepsilon > 0$ such that $p(z) > \eta$ for all $z \in \varepsilon B$. By applying p to the set $P + \varepsilon B$, we obtain

$$p(x + z) \geq p(x) + p(z) > 1 + \eta > 0, \qquad \forall x \in P, \quad \forall z \in \varepsilon B$$

so that $P + \varepsilon B \subseteq [p > 0]$ and ν must be positive on $P + \varepsilon B$. We obtain a contradiction when we evaluate ν on the sets $x_n + \varepsilon B$. Since $\nu(x_n) \to 0$ then eventually it must hold $\nu(x_n + z) < 0$ for some $z \in \varepsilon B$.

Thus we find $0 \neq r \in I$ and $s = \sup I > 0$. Setting $\ell = \nu/s$, we obtain $\ell(a) \leq 1$ for all $a \in A$, $\ell(x) > 1$ for all $x \in P$ and $\inf\{\ell(x), x \in P\} = 1$. We deduce from this that $p(x) = 1$ implies $\ell(x) \geq 1$ and then $[p \geq 1] \subseteq [\ell \geq 1]$. Moreover for every $\varepsilon > 0$ there exists $\bar{x} \in X$ such that $p(\bar{x}) = 1$ and

$$1 = p(\bar{x}) \leq \ell(\bar{x}) < 1 + \varepsilon. \tag{1}$$

Making use of positive homogeneity and continuity of both ℓ and p, we obtain also that $\ell(x) \geq p(x)$ for all $x \in [\ell \geq 0]$ and hence $p(y) \leq 0$ when $y \in H = \{x \in X : \ell(x) = 0\}$.

Reasoning by contradiction, suppose now that there exists $\bar{z} \in X$ such that $\ell(\bar{z}) < p(\bar{z})$. It must be $\ell(\bar{z}) < 0$.

Choose $\varepsilon > 0$ such that

$$0 < -\varepsilon\ell(\bar{z}) < p(\bar{z}) - \ell(\bar{z})$$

and fix $\bar{x}$ such that (1) holds. Since $\bar{x} \notin H$ there exist $y \in H$ and $0 \neq \beta \in \mathbb{R}$ such that $\bar{z} = y + \beta\bar{x}$. Since $\ell(\bar{z}) < 0$ and $\ell(\bar{x}) > 0$, it follows $\beta < 0$.

Since p is superadditive we have

$$p(y) = p(y + \beta\bar{x} - \beta\bar{x}) \geq p(y + \beta\bar{x}) + p(-\beta\bar{x})$$

whence

$$p(\bar{z}) = p(y + \beta\bar{x}) \leq p(y) - p(-\beta\bar{x}) = \beta p(\bar{x}) = \beta. \tag{2}$$

Since $\ell(\bar{z}) = \beta\ell(\bar{x})$, we deduce from (1), that $\beta(1+\varepsilon) < \ell(\bar{z}) \leq \beta$ and hence

$$\ell(\bar{z}) < p(\bar{z}) + \varepsilon\ell(\bar{z}) \leq \beta + \varepsilon\beta = \beta(1+\varepsilon) < \ell(\bar{z}),$$

which is a contradiction.

Thus $\ell \geq p$ and necessity is proved.

Conversely, to prove convexity it will be enough to show that any point not belonging to A can be separated from A by means of a linear continuous functional. Take $x \notin A$. From Corollary 2.2 there exists $p \in L$ such that $p(x) > 1$ and $p(a) \leq 1$ for all $a \in A$. Thus $p \in A^\vee$ and, from the assumptions, there exists some $\ell \in X'$ such that $\ell(a) \leq 1$ for all $a \in A$ and $\ell(x) \geq p(x) > 1$. □

The previous result can be given a more geometric interpretation: let $N \subset L$ be the convex cone of nonpositive superlinear functions, $N = L \cap -K$. Then Proposition 2.4 can be stated as follows: a closed radiant set A is convex if and only if $A^\vee = A^\circ + N$.

$\boxed{2.2}$ Evenly Radiant Sets

In analogy to the case of convex sets and with the aim of application to conjugation theory, we introduce the family of evenly radiant sets.

In convex analysis a set is called evenly convex if it can be seen as the intersection of open halfspaces. Thus every open convex set and every closed convex set is evenly convex. A characterization of even convexity which does not entail separation properties is given in [1] in terms of the tangent cone to the set at points of its boundary. We will follow the same scheme: introduce evenly radiant sets in terms of separation and characterize them by means of the tangent cone at points of the boundary.

Definition 2.5. *A subset $A \subset X$ is called evenly radiant if for each $x \notin A$ there exists a continuous superlinear functional $p \in L$ such that $p(x) \geq 1$ and $p(a) < 1$ for all $a \in A$. By convention we will consider both the sets X and $\emptyset$ as evenly radiant.*

It is immediate from the definitions that an evenly radiant set is radiant, and from Corollary 2.2 that every closed radiant set is evenly radiant. The following example shows that the same is not true for open radiant sets.

Example 2.6. Let $D \subset \mathbb{R}^2$ be the set $D = \{(x,x),\ x \geq 1\}$ and U be the open ball around the origin of radius 2. Then $A = U \backslash D$ is open and radiant. However the points (x,x) with $1 \leq x < \sqrt{2}$, though not belonging to A, cannot be separated from A by means of a continuous superlinear functional p such that $p(x,x) \geq 1$ and A is contained in the strict lower level set $[p < 1]$. To see this, it is enough to note that the interior of set $[p \geq 1]$ is nonempty.

Since the intersection of any family of evenly radiant sets is itself evenly radiant, Definition 2.5 allows to introduce another hull operation: we say that the evenly radiant hull of a set $A \subset X$, $\operatorname{erad} A$, is the intersection of all strict lower level sets $[p < 1]$ containing A, with $p \in L$. Obviously we have that, provided $A \neq \emptyset$, A is evenly radiant if and only if $A = \operatorname{erad} A$.

By analogy with even convexity, we defined evenly radiant sets by means of their separation property. In [1] an evenly convex set A is characterized through a property of the tangent cone of A at points of its boundary. An analogous description can be given for evenly radiant sets. We recall that the (Bouligand) tangent cone to the set A at the point x is the set

$$T(A,x) = \{v \in X : \forall r > 0, \forall \delta > 0, \exists s \in (0,r), \exists u \in B_\delta : x + s(v+u) \in A\}.$$

Theorem 2.7. *Let $A \subset X$ be a radiant set. Then A is evenly radiant if and only if $x \in cl A \backslash A$ implies $x \notin T(A, x)$.*

An important step in order to prove Theorem 2.7 is to show that, for a radiant set A, the local information given by the tangent cone, turns into global information. We single out this step which will also be useful in the next section.

Lemma 2.8. *If the set $A \subseteq X$ is radiant and $x \notin T(A, x)$, then there exists a closed, convex cone C with $x \in int C$ such that*

$$A \cap (x + C) \subseteq \{x\}.$$

Proof: Let $G = x + [0, r] \cdot (x + B_\delta(0))$. If $x \notin T(A, x)$, then $A \cap G \subseteq \{x\}$ for some $r > 0$ and $\delta > 0$ and, since A is radiant, $A \cap \operatorname{shw} G \subseteq \{x\}$. Since G is closed and $0 \notin G$, then the cone $C = \{z \in X : z = \lambda g, \lambda \geq 0, g \in G\}$ is closed. Moreover $x \in \operatorname{int} C$. Therefore it only remains to prove that $x + C \subseteq \operatorname{shw} G$. To prove the latter relation take $y \in x + C$. Then

$$y = x + \lambda(x + s(x + \delta b)) = (1 + \lambda + \lambda s)x + \lambda s \delta b,$$

with $\lambda \geq 0$, $s \in [0, r]$, $b \in B$. By taking $\alpha = 1 + \lambda \geq 0$ and $s' = \lambda s/(1 + \lambda) \in [0, r]$, we obtain

$$y = \alpha(1 + s')x + \alpha s' \delta b = \alpha(x + s'(x + \delta b)) \in \operatorname{shw} G$$

and the proof is complete. □

Proof of Theorem 2.7: We prove sufficiency first.

If A is closed then there is no point in $\operatorname{cl} A \backslash A$ and the implication is vacuously satisfied; on the other hand a closed radiant set is evenly radiant.

Take a point $x \in \operatorname{cl} A \backslash A$; from the assumptions it holds $x \neq 0$ and $x \notin T(A, x)$. Lemma 2.8 implies the existence of some closed convex cone C, with $x \in \operatorname{int} C$, such that

$$A \cap (x + C) = \{x\}. \tag{3}$$

To prove that A is evenly radiant it is enough to see (as in [26]) that there exists a superlinear function $p \in L$ such that $[p \geq 1] = x + C$ and then $p(x) = 1$ and $p(a) < 1$ for all $a \in A$.

We turn now to prove necessity:

Given some $x \in \operatorname{cl} A \setminus A$, suppose that there exists some $p \in L$ such that $p(x) = 1$ and $p(a) < 1$ for all $a \in A$. Suppose moreover that $x \in T(A, x)$. Then there exist sequences $d_n \to x$ and $\lambda_n \to 0^+$ such that $x + \lambda_n d_n \in A$. This yields $p(x + \lambda_n d_n) < 1$ and

$$1 > p(x + \lambda_n d_n) \geq p(x) + \lambda_n p(d_n) = 1 + \lambda_n p(d_n)$$

which implies $p(d_n) < 0$ and $p(x) = \lim p(d_n) \leq 0$ which is a contradiction. □

The problem to characterize those radiant sets which are evenly radiant is strictly connected to the problem of separating some set A from a point x belonging to its boundary by means of a convex cone. This is extensively treated in [13]. The condition $x \notin T(A, x)$ is used in [19] to characterize such separation.

Definition 2.9. *A radiant set $A \subseteq X$ has the the cone support property at the point $x \notin int A$, $x \neq 0$, if there exists a closed, convex cone C, with $x \in int C$, such that*

$$A \cap (C + x) \subseteq \{x\}.$$

It is immediate to note, just comparing the definitions, that a set A with the cone support property is always evenly radiant and that, if A is evenly radiant and $A \subseteq B \subseteq \operatorname{cl} A$, then B has the cone support property (and is therefore evenly radiant).

Example 2.10. By means of Theorem 2.7 we can illustrate another example of a set which is radiant and open but is not evenly radiant. Let $A \subset \mathbb{R}^2$ be

$$A = \{(x_1, x_2) : x_2 < 1 + \sqrt{|x_1|}\}$$

and consider the point $x = (x_1, x_2) = (0, 1)$. We will show that, though $x \notin A$, it holds $x \in T(A, x)$. To see this it is enough to consider the sequence $\{x_n\} = \{(4/n^2, 1 + 1/n)\} \subset A$, with limit $(0, 1)$, and the sequence $\{t_n\} = \{n\}$ and find

$$\lim t_n(x_n - x) = \lim n \left(\frac{4}{n^2}, \frac{1}{n}\right) = (0, 1) = x \in T(A, x).$$

Evenly radiant sets can be described by means of a polarity defined by a strict inequality.

Definition 2.11. *Given some set $A \subset X$, its strict polar is the set $A^\wedge \subseteq L$ given by*

$$A^\wedge = \{p \in L : p(a) < 1, \forall a \in A\}.$$

The (strict) $\wedge$-polar of a set $A \subseteq X$ has similar properties to the ones seen for the $\vee$-polar: it is convex, radiant and downward with respect to the order $\geq_1$. We can define the bipolar $A^{\wedge\wedge} = (A^\wedge)^\wedge$ and check that $A \subseteq A^{\wedge\wedge}$ for every set $A \subset X$ and that the equality $A = A^{\wedge\wedge}$ holds if and only if A is evenly radiant, provided $A \neq \emptyset$.

It is important to stress that a convex set which is evenly radiant is not necessarily evenly convex, as shown by the simple example of a set given by the union of some open halfspace containing the origin and just one point in its boundary.

3 Radiant Functions

We are interested here in the following class of functions.

Definition 3.1. *A function $f : X \to \overline{\mathbb{R}}$ is called* **radiant** *if its lower level sets $[f \leq k]$ are radiant for every $k \in \mathbb{R}$.*

As it is easily checked, the following characterizations hold.

Proposition 3.2. *[26] For a function $f : X \to \overline{\mathbb{R}}$ the following are equivalent:*

1. *the lower level sets $[f \leq k]$ are radiant for every $k \in \mathbb{R}$;*
2. *the strict lower level sets $[f < k]$ are radiant for every $k \in \mathbb{R}$;*
3. *for every $x \in X$, the function $f_x : \mathbb{R}^+ \to \mathbb{R}$ given by:*

$$f_x(\alpha) \equiv f(\alpha x), \qquad \alpha \geq 0$$

is non decreasing.

Following Proposition 3.2 we will say that a function f is increasing along rays to mean that it is radiant.

Obviously, if the function f is radiant, then either $f(0) = -\infty$ or 0 is a global minimum point for f or $f(x) = +\infty$ everywhere.

Among radiant functions we are primarily interested in two subclasses: those which are lower semicontinuous, in that their lower level sets are closed and radiant, and the following ones.

Definition 3.3. *A function $f : X \to \overline{I\!R}$ is called evenly radiant if its lower level sets $[f \leq k]$ are evenly radiant for every $k \in I\!R$.*

It is obvious that every l.s.c. radiant function is evenly radiant. The indicator function of some set $C \subseteq X$, which is evenly radiant but not closed, shows that the converse is not true.

We can study evenly radiant functions by the tools of Abstract Convexity. For this purpose we need first to recall some related definitions.

Given a set H of functions defined on the space X, a function $f : X \to \overline{I\!R}$ is called abstract convex with respect to H (or H-convex for short) if it holds

$$f(x) = \sup\{h(x) |\, h \leq f,\, h \in H\}. \tag{4}$$

If, for every x such that $f(x) < +\infty$, the equality in (4) holds with max instead of sup, i.e. if there exists some $h \in H$ such that $h \leq f$ and $h(x) = f(x)$, then we will say that f is exactly H-convex. For any function f, we call H-support of f, denoted by $H(f)$, the set of elementary functions $h \in H$ which minorize f on X.

These concepts find their origin in the well-known characterization of a lower semicontinuous convex (sublinear) functions as the supremum of its affine (linear) minorants and have been proved to be a very useful way to extend many global properties of convex functions to various classes of nonconvex functions.

We mainly refer to the monographs [7, 13, 21] for a complete treatment of the theory and examples. Note that convex functions which are finite and continuous on X are exactly convex with respect to the family H of affine functions, since the (convex) subdifferential is nonempty at every point $x \in X$ in this case.

To study radiant functions in the framework of Abstract Convexity we need to describe the family of elementary functions which generate them by means of sup envelopes. To characterize in this framework lower semicontinuous radiant functions, one should consider a family of l.s.c. functions. As in the general scheme proposed in [14] we will consider the following family $\mathcal{P}$ of 'two steps' elementary functions: consider a continuous superlinear function $p : X \to I\!R$ and $c \geq c' \in I\!R \cup \{-\infty\}$ and the function

$$s_{p,c,c'}(x) = \begin{cases} c & \text{if } p(x) > 1 \\ c' & \text{otherwise} \end{cases} \tag{5}$$

It has been shown in [26] that $\mathcal{P}$ is indeed a supremal generator for the set of l.s.c. radiant functions defined on X, i.e. a function $f : X \to \overline{I\!R}$ is l.s.c. and radiant if and only if it is $\mathcal{P}$-convex. To characterize evenly radiant functions, we introduce the family $\mathcal{P}'$ whose elements are the functions

$$s'_{p,c,c'}(x) = \begin{cases} c & \text{if } p(x) \geq 1 \\ c' & \text{otherwise} \end{cases} \tag{6}$$

where p is some superlinear continuous function which maps X to $I\!R$ and $c \geq c' \in I\!R \cup \{-\infty\}$.

Proposition 3.4. *A function $f : X \to \overline{I\!R}$ is evenly radiant if and only if it is $\mathcal{P}'$-convex.*

Proof: Since every function in $\mathcal{P}'$ is evenly radiant, and for any $\mathcal{P}'$-convex function f it holds

$$[f \leq k] = \bigcap_{s' \in \mathcal{P}'(f)} [s' \leq k],$$

where $\mathcal{P}'(f)$ is the $\mathcal{P}'$-support of f, then every $\mathcal{P}'$-convex function is evenly radiant.

For the converse we need to show that, if f is evenly radiant, $f(x) \in I\!R \cup \{+\infty\}$ and $k < f(x)$, then there exists some $s' \in \mathcal{P}'$ such that $s' \leq f$ and $s'(x) \geq k$. Since $x \notin [f \leq k]$ then there exists $p \in L$ such that $p(x) \geq 1$ and $p(z) < 1$ for all $z \in [f \leq k]$, which implies $f(z) > k$ for all $z \in X$ with $p(z) \geq 1$. Hence we can form s' as in (6) by taking $c = k$ and $c' \leq f(0)$. If $f(x) = -\infty$ it is enough to consider $s' \equiv -\infty$. $\square$

We noticed already that all l.s.c. radiant functions are evenly radiant. On the other hand it is not easy to use the characterization given by Theorem 2.7 to see what type of restriction is imposed to a radiant function which is not l.s.c. by the requirement that it is evenly radiant. Just by rewording the condition $x \notin T(A, x)$, where $A = [f \leq k]$, it is possible to say that a function f is evenly radiant if and only if the inequality $f(x) > k$ implies $f(x + t_n x_n) > k$ for all sequences $\{t_n\}$ converging to 0^+ and all sequences $\{x_n\}$ converging to x.

To have a better intuition of how broad is the class of evenly radiant functions, one can notice that it contains all l.s.c. radiant functions and moreover it contains the following functions, which are defined by means of a particular monotonicity property.

Definition 3.5. *A radiant function $f : X \to \overline{I\!R}$ is called consistently radiant if, for every $0 \neq x \in X$, there exists a neighbourhood $U(x)$ such that*

$$f(x + tz) \geq f(x), \qquad \forall t > 0,\ \forall z \in U(x). \tag{7}$$

Consistently radiant functions can be characterized by means of their strict level sets and in terms of Abstract Convexity.

Theorem 3.6. *For a function $f : X \to \overline{I\!R}$ the following are equivalent:*

a) f is consistently radiant;

b) the strict lower level sets $[f < k]$ are evenly radiant for every $k \in I\!R$;

c) f is exactly $\mathcal{P}'$-convex.

Proof:

(a) $\Leftrightarrow$ (b) Take $k \in I\!R$ and $f(x) \geq k$ (if $f(x) < k$ for all $x \in X$, then $[f < k] = X$ is evenly radiant). Then (7) can be rewritten as

$$[x + (0, +\infty)(x + U(0))] \cap [f < k] = \emptyset$$

and this implies $x \notin T(A, x)$, with $A = [f < k]$. Hence the set $[f < k]$ is evenly radiant.

To prove the converse, fix $x \in X$ and $k = f(x)$. Then $x \notin [f < k]$ and, from Lemma 2.8, we find a closed convex cone C, with $x \in \operatorname{int} C$, such that $[f < k] \cap x + C = \emptyset$ and (7) follows.

(b) $\Leftrightarrow$ (c) For any $k \in I\!R$, consider the set $[f < k]$ and a point x with $f(x) \geq k$, that is $x \notin [f < k]$ (if $f(x) < k$ for all $x \in X$, then $[f < k] = X$ is evenly radiant). If $f(x) \in I\!R$ and f is exactly $\mathcal{P}'$-convex, then there exist $p \in L$ and $c \geq c' \in I\!R \cup \{-\infty\}$ such that the elementary function $s'_{p,c,c'} \in \mathcal{P}'$ minorizes f and satisfies $s'_{p,c,c'}(x) = f(x)$. If $f(x) = c'$, then x is a minimum point of f. Since $f(x) \geq k$, then the set $[f < k]$ is empty and, by convention, it is evenly radiant.

If $f(x) = c$, then $p(x) \geq 1$. Since $s'_{p,c,c'} \leq f$, then $f(z) \geq f(x)$ for all $z \in [p \geq 1]$, or equivalently,

$$f(z) < f(x) \quad \Rightarrow \quad p(z) < 1, \tag{8}$$

Since $f(x) \geq k$, then $[f < k] \subseteq [f < f(x)]$ and (8) can be rephrased as: there exists $p \in L$ such that $p(x) \geq 1$ and $p(z) < 1$ for all $z \in [f < k]$, that is the level set $[f < k]$ is evenly radiant.

If $f(x) = +\infty$ then, for all $M > 0$ there exists some $s' = s'_{p,c,c'}$ in the $\mathcal{P}'$-support of f such that $s' \leq f$ and $s'(x) \geq M$. Unless f is identically $+\infty$, then it holds $s'(x) = c \geq M$ and $p(x) \geq 1$ and the previous argument still holds.

Conversely suppose that for any $k \in I\!R$, the set $[f < k]$ is evenly radiant and take any $x \in X$. If $f(x) = f(0) \in \overline{I\!R}$ then the function $s'(z) = f(0)$ minorizes f and coincides with it at the point x. So let $f(x) > f(0)$; in this case $f(x) > -\infty$. If $f(x) = +\infty$, then $x \notin [f < M]$, whatever is $M > 0$ and hence there exists $p \in L$ such that the function $s'_{p,c,c'} \in \mathcal{P}'$, with $c = M$ and $c' = -\infty$ satisfies $s' \leq f$ so that $f(x) = \sup\{s' \in \mathcal{P}' : s' \leq f\}$. If $f(x) \in I\!R$ consider the level set $[f < k]$ with $f(x) = k$. Then there exists $p \in L$ such that $p(x) \geq 1$ and $p(z) < 1$ for all $z \in [f < k]$; this means that the function $s'_{p,c,c'} \in \mathcal{P}'$ with $c = f(x)$ and $c' = -\infty$ minorizes f on X and satisfies $s'_{p,c,c'}(x) = f(x)$ and thus f is exactly $\mathcal{P}'$-convex. $\square$

We can use the equivalence between (a) and (b) in Theorem 3.6 to show that all consistently radiant functions are evenly radiant. Indeed the class of evenly radiant sets is closed under intersection and the equality

$$[f \leq k] = \bigcap_{s > k} [f < s],$$

which holds for all functions, shows that all (weak) lower level sets $[f \leq k]$ are evenly radiant if the strict lower level sets $[f < s]$ have this property.

For an example which shows that the converse relation is not generally true, we may refer to the following (which was given in [1], concerning evenly quasiconvex functions).

Example 3.7. Let $f : I\!R^2 \to I\!R$ be defined as:

$$f(x,y) = \begin{cases} 1 & y > x \\ y/x & 0 < y \leq x \\ 0 & y \leq 0 \text{ and } x \geq y \end{cases}$$

The function f is nonnegative, positively homogeneous of degree zero (hence constant on every rays and then radiant) and lower semicontinuous. Thus its lower level sets $[f \leq k]$ are closed and radiant for every $k \in I\!R$. Since every closed radiant set is evenly radiant, the function f is evenly radiant. To show that f is not consistently radiant, consider any

$k \in (0,1]$ and the strict level set $[f < k]$. If we consider a point $P = (x, kx)$, with $x > 0$, the function f takes the value k for all points which stay on the ray defined by P and condition (7) is not satisfied since $f(P) = k$ and for every δ we can find $t \in (0, \delta)$ and $z \in B(P, \delta)$ such that $f(P + tz) < k$.

Radiant functions may conveniently be described in terms of directional derivatives. For a function $f : X \to I\!R$ the lower Dini directional derivative and the lower Hadamard directional derivative in the direction $d \in X$ are given by

$$f_D^-(x,d) = \liminf_{t \to 0^+} \frac{f(x+td) - f(x)}{t},$$

and

$$f_H^-(x,d) = \liminf_{\substack{t \to 0^+ \\ v \to d}} \frac{f(x+tv) - f(x)}{t}. \tag{9}$$

Theorem 3.8. *Let $f : X \to I\!R$ be continuous on each ray. Then it holds*

a) f is radiant if and only if $f_D^-(x,x) \geq 0$ for all $x \in X$;

b) f is consistently radiant if and only if $f_H^-(x,x) \geq 0$ for all $x \in X$.

The proof of Theorem 3.8 is based on the following lemma, whose proof can be found in [2].

Lemma 3.9. *Let $\phi : I\!R \to I\!R$ be defined and continuous on the closed interval $[a,b]$. If*

$$\phi'_-(\alpha) \equiv \liminf_{\beta \searrow \alpha} \frac{\phi(\beta) - \phi(\alpha)}{\beta - \alpha} \geq 0 \qquad \forall \alpha \in (a,b),$$

then $\phi(b) \geq \phi(a)$.

Proof of Theorem 3.8:

a) Necessity is obvious. For sufficiency we need to prove that, for all $\beta > 1$, it holds

$$f(\beta x) \geq f(x).$$

Let $\phi(\alpha) = f(x + \alpha x)$ for $\alpha \in [0, \beta - 1]$. Since $f_D^-(x + \alpha x, x) = \phi'_-(\alpha)$ and, by positive homogeneity of the function $f_D^-(x, \cdot)$,

$$(1+\alpha) f_D^-(x + \alpha x, x) = f_D^-(x + \alpha x, x + \alpha x) \geq 0,$$

it holds $\phi'_-(\alpha) \geq 0$ for all $\alpha \in (0, \beta - 1)$. Hence, by Lemma 3.9, it follows

$$f(\beta x) = \phi(\beta - 1) \geq \phi(0) = f(x).$$

b) Again necessity is obvious. To prove sufficiency note first that $f_D^-(x,x) \geq f_H^-(x,x) \geq 0$ implies that f is radiant. Moreover the inequality $f_H^-(x,x) \geq 0$ means that there exist $\delta > 0$ and $\eta > 0$ such that $f(z) \geq f(x)$ for all $z \in G \equiv x + [0,\delta] \cdot (x + B_\eta(0))$. Hence $x \notin T(A,x)$ with $A = [f < f(x)]$ and Lemma 2.8 yields that f is consistently radiant. $\square$

If f is locally Lipschitz around x then $f_H^-(x,d) = f_D^-(x,d)$ for all $d \in X$, while if f is Fréchet differentiable at x with gradient $\nabla f(x)$, then we have

$$f_D^-(x,d) = f_H^-(x,d) = \nabla f(x) \cdot d.$$

Thus a real valued differentiable function f is radiant if and only if it holds $\nabla f(x) \cdot x \geq 0$ for all $x \in X$ and moreover both locally Lipshitz radiant functions and differentiable radiant functions are always consistently radiant. We will see in the next example that this is not always the case for a continuous radiant function.

Example 3.10. We can use Thorem 3.8 to show that the Minkowski gauge

$$\mu_A(x) = \inf\{\lambda > 0 : \, x \in \lambda A\}$$

of the set $A = \{(x_1, x_2) : \, x_2 < 1 + \sqrt{|x_1|}\}$ seen in Example 2.10, is radiant and continuous but is not consistently radiant. To see that μ_A is radiant and continuous it is enough to note that $\mu_{\mathrm{cl}\, A} = \mu_A$ and that the set $\mathrm{cl}\, A$ is radiative (see [12]) in that it contains the origin in its interior and every ray starting from the origin meets the boundary of A at most in one point. To see that μ_A is not consistently radiant we will show that the lower Hadamard derivative $f_H^-(x,x)$ is negative at the point $x = (0,1)$. To show this we will see that the differential quotient (9) is negative if we take $x = (0,1)$, $t_n = 1/n$ and $d_n = (4/n, 1) \to (0,1) = x$. Indeed it holds $\mu_A(x) = 1$ since x is on the boundary of A and

$$\mu_A(x + t_n d_n) = \mu_A(4/n^2, 1 + 1/n) = \frac{n^2 + n + 2 - 2\sqrt{n^2 + n + 1}}{n^2}$$

which yields

$$\lim \frac{\mu_A(x + t_n d_n) - \mu_A(x)}{t_n} = -1.$$

Concerning the comparison between l.s.c. radiant functions and consistently radiant functions, which are different restrictions of radiant functions, it is easy to find examples which show that neither of these two classes contains the other. For instance the indicator function of an open convex set containing the origin is a consistently radiant function which is not lower semicontinuous, while the function μ_A of Example 3.10 is radiant and continuous but not consistently radiant.

4 Conjugate Functions

Many authors have dealt with the problem of introducing a dual for a quasiconvex optimization problem and with the related question of defining a conjugate function which is appropriate to quasiconvex functions (see [4, 5, 8, 9, 11, 10, 16, 22, 23, 24, 25, ...] and references therein). For quasiconvex functions with minimal point at the origin (a subset of radiant functions) a simplified definition can be given, which is strictly related to the (convex) polarity relation (see e.g. [20, 16, 8]). Following this approach, we introduce a conjugate function based on the polarity relations between subsets of X and L studied in Section 2.

Given $f : X \to \overline{I\!R}$, let $f^\vee : L \to \overline{I\!R}$ be the function

$$f^\vee(p) = \sup\{-f(x) : \, p(x) > 1\}$$

and $f^\wedge : L \to \overline{I\!R}$ be

$$f^\wedge(p) = \sup\{-f(x) : \, p(x) \geq 1\}.$$

These functions can be interpreted within the Fenchel scheme if we introduce the coupling functionals $c^{\vee} : X \times L \to I\!R \cup \{-\infty\}$, given by

$$c^{\vee}(x,p) = \begin{cases} 0 & p(x) > 1 \\ -\infty & p(x) \le 1 \end{cases}$$

and $c^{\wedge} : X \times L \to I\!R \cup \{-\infty\}$, given by

$$c^{\wedge}(x,p) = \begin{cases} 0 & p(x) \ge 1 \\ -\infty & p(x) < 1. \end{cases}$$

Indeed we obtain that

$$f^{\times}(p) = -\inf_{X}\{f(x) - c^{\times}(x,p)\},$$

where $\times$ stands for $\vee$ or $\wedge$ and we adopt the rule $+\infty - \infty = +\infty$ for the addition among extended real numbers.

The coupling functions $c^{\vee}$ and $c^{\wedge}$ are closely related to the elementary functions $s \in \mathcal{P}$ and $s' \in \mathcal{P}'$. Indeed we have, posing $c' = -\infty$ in (5) and in (6), that $c^{\vee}(x,p) + c = s_{p,c,c'}$ and $c^{\wedge}(x,p)+c = s'_{p,c,c'}$ so that we can use the family of functions $c^{\vee}(\,\cdot\,,p)+c$, for $p \in L$ and $c \in \overline{I\!R}$ as the supremal generator for the class of l.s.c. radiant functions with $f(0) = -\infty$ and the family $c^{\wedge}(\,\cdot\,,p)+c$ to generate the class of evenly radiant functions with $f(0) = -\infty$.

The conjugate functions studied in [8, 9, 15, 22, 23, 24] are closely related to our definitions. More precisely they coincide (at least for $p \neq 0$) with the restriction of $f^{\vee}$ or $f^{\wedge}$ to the set $X' \subseteq L$.

The following properties of conjugate functions (in which $\times \in \{\vee, \wedge\}$) follow immediately from the definitions.

1. If $f_i : X \to \overline{I\!R}$, $i \in I$, is an arbitrary family of functions, then

$$\left(\inf_{i\in I} f_i\right)^{\times}(p) = \sup_{i\in I} f_i^{\times}(p) \qquad \text{for all } p \in L. \tag{10}$$

2. If $f : X \to \overline{I\!R}$ and $c \in \overline{I\!R}$, then

$$(f+c)^{\times}(p) = -\left(c - f^{\times}(p)\right) \qquad \text{for all } p \in L \tag{11}$$

which becomes

$$(f+c)^{\times}(p) = f^{\times}(p) - c \qquad \text{for all } p \in L$$

when $c \in I\!R$.

3. If $f_i : X \to \overline{I\!R}$, for $i = 1, 2$, then

$$f_1 \le f_2 \Rightarrow f_1^{\times} \ge f_2^{\times}.$$

4. Young inequality. If $f : X \to \overline{I\!R}$ then, for all $x \in X$ and all $p \in L$ it holds:

$$f(x) + f^{\times}(p) \ge c^{\times}(x,p). \tag{12}$$

5. If $f : X \to \overline{I\!R}$, then

$$f^{\vee}(p) \le f^{\wedge}(p) \qquad \text{for all } p \in L. \tag{13}$$

6. If $f : X \to \overline{I\!R}$, then $f^{\times}$ is nondecreasing with respect to the order $\geq_1$.

From (10) and (11) we obtain that both the $\vee$- and the $\wedge$-conjugate are a conjugation in the sense of Singer [20]. We can give a simple condition (see [8]), which guarentees the equality in (13). For a radiant function it simply means that f is upper semicontinuous along every ray.

Proposition 4.1. *If the function $f : X \to \overline{I\!R}$ satisfies the requirement that for every $x \in X\backslash\{0\}$ and every $s > f(x)$ there exists $t > 1$ such that $f(tx) < s$, then it holds*

$$f^{\vee}(p) = f^{\wedge}(p) \qquad \forall p \in L. \tag{14}$$

Proof: Let $p \in L$, with $p \neq 0$ and $s < f^{\wedge}(p)$. Then we can find $x \in X$ such that $p(x) \geq 1$ and $-f(x) > s$. Then we can find $t > 1$ such that $-f(tx) > s$. Since $p(tx) > 1$ we obtain $f^{\vee}(p) > s$ and hence (14) holds for all $p \neq 0$. Moreover $f^{\vee}(0) = f^{\wedge}(0) = -\infty$ and the result is proved. $\square$

In analogy to the conjugation scheme for quasiconvex function proposed in [8, 9], the main property of the conjugate functions $f^{\vee}$ and $f^{\wedge}$ is that their lower level sets are the $\vee$-polar and $\wedge$-polar, respectively, to the level sets of f.

Theorem 4.2. *Given a function $f : X \to \overline{I\!R}$ and its conjugate $f^{\times}$ as defined above, for $\times \in \{\vee, \wedge\}$, it holds*

$$[f^{\times} \le -k] = [f < k]^{\times}$$

and

$$[f^{\times} < -k] = \bigcup_{s>k} [f \le s]^{\times}.$$

Proof: We prove the result for the $\vee$-conjugate. The other case is completely analogous.

The first equality is proved by the following coimplications:

$$\begin{aligned}
p \in [f^{\vee} \le -k] &\iff f^{\vee}(p) \le -k \\
&\iff (p(x) > 1 \Rightarrow -f(x) \le -k) \\
&\iff (p(x) > 1 \Rightarrow f(x) \ge k) \\
&\iff (f(x) < k \Rightarrow p(x) \le 1) \\
&\iff p(x) \le 1, \forall x \in [f < k] \\
&\iff p \in [f < k]^{\vee}
\end{aligned}$$

For the second statement:

$$\begin{aligned}
p \in [f^{\vee} < -k] &\iff f^{\vee}(p) < -k \\
&\iff \exists s > k : f^{\vee}(p) < -s \\
&\iff \exists s > k : (p(x) > 1 \Rightarrow -f(x) < -s) \\
&\iff \exists s > k : (p(x) > 1 \Rightarrow f(x) > s) \\
&\iff \exists s > k : (f(x) \le s \Rightarrow p(x) \le 1) \\
&\iff \exists s > k : p \in [f \le s]^{\vee} \\
&\iff p \in \bigcup_{s>k} [f \le s]^{\vee}.
\end{aligned}$$

$\square$

As any polar set $A^\vee$ and $A^\wedge$ is convex and radiant in L, an important consequence of Theorem 4.2 is that both conjugate functions $f^\vee$ and $f^\wedge$ are radiant and quasiconvex on L; moreover $f^\vee$ is lower semicontinuous.

Observe moreover that, as both the level sets $[p \geq 1]$ and $[p > 1]$ are empty for $p \equiv 0_L$, then $f^\vee(0) = f^\wedge(0) = \sup \emptyset = -\infty$ by an usual convention.

By the same scheme we can introduce the second conjugate of a function $f : X \to \overline{I\!R}$ as the conjugate of its conjugate, i.e. $f^{\times\times}(x) = (f^\times)^\times(x)$. From the following result we will derive the main relations between a function and its second conjugate.

Theorem 4.3. *Given a function $f : X \to \overline{I\!R}$, its biconjugate $f^{\times\times}$, for $\times = \vee$ or $\times = \wedge$, satisfies:*

$$[f^{\times\times} \leq k] = \bigcap_{s>k} [f < s]^{\times\times}.$$

Moreover it holds $f^{\vee\vee} = f$ if and only if f is radiant and l.s.c. with $f(0) = -\infty$ and $f^{\wedge\wedge} = f$ if and only if f is evenly radiant with $f(0) = -\infty$.

Proof: It holds

$$\begin{aligned} [f^{\times\times} \leq k] &= [f^\times < -k]^\times \\ &= \left(\bigcup_{s>k} [f^\times \leq -s]\right)^\times \\ &= \bigcap_{s>k} [f^\times \leq -s]^\times \\ &= \bigcap_{s>k} [f < s]^{\times\times}. \end{aligned}$$

This relation shows that $f^{\vee\vee}$ is radiant l.s.c. in that its level sets can be expressed as the intersection of closed radiant sets and shows that $f^{\wedge\wedge}$ is evenly radiant since the level sets $[f^{\wedge\wedge} \leq k]$ are intersections of evenly radiant sets.

For both conjugates it holds $f^{\vee\vee}(0) = f^{\wedge\wedge}(0) = -\infty$.

Moreover, since $[f \leq k] \subseteq [f < s]$ for all $s > k$ and taking the bipolar is a monotone operator on sets, it holds

$$[f \leq k] \subseteq [f \leq k]^{\times\times} \subseteq \bigcap_{s>k} [f < s]^{\times\times} = [f^{\times\times} \leq k].$$

Hence for any function f it holds $f^{\times\times} \leq f$.

Now we will prove that the equality $f = f^{\vee\vee}$ holds. The proof for the $\wedge$-biconjugate is analogous and hence omitted.

Let f be radiant and l.s.c. with $f(0) = -\infty$ and suppose that $f(x) > f^{\vee\vee}(x)$. Take $k \in I\!R$ such that $f(x) > k > f^{\vee\vee}(x)$. Since $x \notin [f \leq k]$, then there exists $\bar{p} \in L$ such that $\bar{p}(x) > 1$ and $\bar{p}(z) \leq 1$ for all $z \in [f \leq k]$, which entails $f(z) > k$ for all z with $\bar{p}(z) > 1$.

Hence we have

$$f^\vee(\bar{p}) = \sup\{-f(x) : \bar{p}(x) > 1\} \leq -k.$$

Since $\bar{p}(x) > 1$, then

$$f^{\vee\vee}(x) = \sup\{-f^\vee(p) : p(x) > 1\} \geq -f^\vee(\bar{p}).$$

Putting together the above inequalities, we obtain

$$k > f^{\vee\vee}(x) \geq -f^{\vee}(\bar{p}) \geq k$$

which is a contradiction. $\square$

As stated in Theorem 4.3 not every l.s.c. radiant function f coincides with its second $\vee$-conjugate but only the ones with $f(0) = -\infty$. To obtain this coincidence we should modify f to the function $\tilde{f}$ such that $\tilde{f}(x) = f(x)$ for $x \neq 0$ and $\tilde{f}(0) = -\infty$. This shows that, for every l.s.c. radiant function f it holds $f(x) = f^{\vee\vee}(x)$ for all $x \neq 0$. In [22] and in [15] the value at 0 of the conjugate function (defined there on the space X') is modified in order to obtain the coincidence between some function f and its second conjugate in a larger class of functions for which the value $f(0)$ may be greater than $-\infty$.

5 An Application to Global Optimality

Among the properties of the conjugate functions defined above, which could often be derived as consequences of a more general theory of conjugation, we would like to stress the importance of the following, an extension of the Toland-Singer formula, which establishes the equality among the infimum of the difference of two functions and the difference of their conjugates. This is the basis of a number of results giving global optimality conditions for various set constrained optimization problems described by radiant functions. Some instances of such conditions, applied to quasiconvex conjugation but readily extendable to this setting are given in [8, 9].

Consider the problem

$$(\mathcal{P}) \qquad \text{minimize } g(x) - h(x) : \; x \in X,$$

where g, h are extended-real valued functions defined on X and we adopt the convention that $\infty - \infty = \infty$. In the following result, we need the definition of a subdifferential, which is the particular case related to our setting of the standard abstract definition as given for instance in [25, 20, 13].

Definition 5.1. *Given a function $f : X \to \overline{I\!R}$ and a point x_0 where f is finite we call $\times$-subgradient of f at x_0 an element $p \in L$ such that $c^{\times}(x_0, p)$ is finite and*

$$f(x) \geq f(x_0) + c^{\times}(x, p) - c^{\times}(x_0, p), \qquad \forall x \in X, \tag{15}$$

where, as above, the symbol $\times$ stands for $\vee$ or $\wedge$. We call $\times$-subdifferential and denote it by $\partial^{\times} f(x_0)$ the set of such $\times$-subgradients.

It is easy to see that we obtain the following equalities:

$$\partial^{\vee} f(x_0) = \{p \in L : p(x_0) > 1, \, f(x) \geq f(x_0), \, \forall x \in [p > 1]\}$$

and

$$\partial^{\wedge} f(x_0) = \{p \in L : p(x_0) \geq 1, \, f(x) \geq f(x_0), \, \forall x \in [p \geq 1]\}$$

so that $p \in \partial^{\vee} f(x_0)$ (resp. $p \in \partial^{\wedge} f(x_0)$) if and only if x_0 is the minimal point of f in $[p > 1]$ (resp. in $[p \geq 1]$). We can also equivalently say that $p \in \partial^{\times} f(x_0)$ if and only if $c^{\times}(x_0, p)$ is finite and the Young inequality (12) holds as an equality:

$$f(x_0) + f^{\times}(p) = 0. \tag{16}$$

In the sequel we will mainly refer to the $\wedge$-subdifferential. This is due to the fact that the $\vee$-subdifferential is empty under very weak assumptions on f, or equivalently is nonempty only for very special functions.

Indeed $c^{\vee}(x_0, p)$ is finite if and only if $p(x_0) > 1$. If this holds, there exists some $t < 1$ such that $p(tx_0) > 1$ and, in case f is strictly increasing on the ray $\{\alpha x_0, \alpha > 0\}$, then $f(tx_0) < f(x_0)$. This yields $\partial^{\vee} f(x_0) = \emptyset$.

For the same reason, if f is stictly increasing on the ray $\{\alpha x_0, \alpha > 0\}$ and $p \in \partial^{\wedge} f(x_0)$ then $p(x_0) = 1$.

Moreover it is easy to see that the strict subdifferential $\partial^{\wedge}$ is nonempty for every consistently radiant function f at every point x where f is finite.

Proposition 5.2. *Let $f : X \to \overline{I\!R}$ be consistently radiant and finite at the point x_0. Then $\partial f(x) \neq \emptyset$.*

Proof: By Theorem 3.6 we know that f is exactly $\mathcal{P}'$-convex and this means that for every x there exists $p \in L$ such that the function $s'_{p,c,c'}$ with $c = f(x_0)$ and $c' = -\infty$ minorizes f and coincides with it at the point x_0. Hence $p(x_0) \geq 1$ and $f(x) \geq f(x_0)$ for all $x \in [p \geq 1]$ and therefore $p \in \partial^{\wedge} f(x_0)$. □

Theorem 5.3. *Let g, h be extended-real valued functions defined on X, with $h^{\times\times} = h$, where $\times$ stands for $\vee$ or $\wedge$. Then*

$$\inf_{x\in X} (g(x) - h(x)) = \inf_{p\in L} \left(h^{\times}(p) - g^{\times}(p)\right). \tag{17}$$

Moreover if $p \in \partial^{\times} g(x_0) \cap \partial^{\times} h(x_0)$, then x_0 is a solution of $\mathcal{P}$ if and only if p is a minimizer of $h^{\times} - g^{\times}$.

Proof: As $h^{\times\times} = h$, we have

$$\begin{aligned}
\inf_{x\in X} (g(x) - h(x)) &= \inf_{x\in X} \left(g(x) + \inf_{p\in L} \left(h^{\times}(p) - c^{\times}(x,p)\right)\right) \\
&= \inf_{x\in X} \inf_{p\in L} \left(g(x) + h^{\times}(p) - c^{\times}(x,p)\right) \\
&= \inf_{p\in L} \left(h^{\times}(p) + \inf_{x\in X} \left(g(x) - c^{\times}(x,p)\right)\right) \\
&= \inf_{p\in L} \left(h^{\times}(p) - g^{\times}(p)\right).
\end{aligned}$$

The last assertion follows from the fact that, when $p \in \partial^{\times} g(x_0) \cap \partial^{\times} h(x_0)$, the functions g and h (resp. $g^{\times}$ and $h^{\times}$) are finite at x_0 (resp. p) and

$$\begin{aligned}
g(x_0) + g^{\times}(p) &= 0, \\
h(x_0) + h^{\times}(p) &= 0,
\end{aligned}$$

and the result follows by subtraction. □

The proof of Theorem 5.3 is an adaptation to the one in [8] for the quasiconjugate function defined on X'. The Toland-Singer formula can be used, as in [8], to give global optimality conditions for a number of problems which can be reformulated as the minimization of a difference of extended-real valued functions. For instance the minimizations of a function f over the complement of a radiant set or the maximization of a radiant function under a set constraint. We rather apply the results about conjugate functions and subdifferentials of radiant functions to give global optimality conditions for the following constrained maximization problem:

$$(\mathcal{P}_b) \qquad f(x) \to \sup \qquad \text{sub} \quad g(x) < b,$$

where $f, g : X \to \overline{I\!R}$ are radiant functions. A point $x_0 \in X$ is considered to be a solution for $(\mathcal{P}_b)$ if $g(x_0) = b$ and

$$f(x_0) = \sup\{f(x) : g(x) < b\}.$$

The use of strict inequality in $(\mathcal{P}_b)$ has the consequence that the solution of the problem is not an admissible point for the constraint. On the other hand if f is continuous and $[g \leq b] \subseteq \operatorname{cl}[g < b]$ then

$$\sup_{g(x)<b} f(x) = \sup_{g(x)\leq b} f(x)$$

and x_0 is admissible for the new problem.

In the sequel we will call regular a function f such that $k \in I\!R$ and $[f < k] \neq \emptyset$ imply $[f \leq k] \subseteq \operatorname{cl}[f < k]$. It is easy to see that f is regular if it is strictly increasing along rays.

Problem $(\mathcal{P}_b)$ extends convex and quasiconvex maximization on a convex set, which has received great attention in recent years, mainly in D.C. optimization and global optimization. In order to obtain global optimality conditions for problem $(\mathcal{P}_b)$, we follow the approach developed in [17], where quasiconvex radiant functions f and g were considered. See also [3] were a reverse convex optimization problem is studied in which the admissible region is given by a strict inequality.

Let $c \in I\!R$ be the value of $(\mathcal{P}_b)$, that is $\sup_{g(x)<b} f(x) = c$ and consider the dual problem:

$$(\mathcal{D}_{-c}) \qquad g^\wedge \to \sup \qquad \text{sub} \quad f^\wedge(p) < -c.$$

Let moreover $\phi : I\!R \to \overline{I\!R}$ be the value function of the problem $(\mathcal{P}_b)$, i.e.

$$\phi(b) = \sup_{g(x)<b} f(x).$$

Lemma 5.4. *If $\phi(b) = c$ and $b' > b$ implies $\phi(b') > \phi(b)$, then*

$$\sup_{f^\wedge(p)<-c} g^\wedge(p) = -b.$$

Proof: Let $c' > c$. Then $\sup\{f(x) : g(x) < b\} < c'$ and $[g < b] \subseteq [f < c']$, which yields

$$[f^\wedge \leq -c'] = [f < c']^\wedge \subseteq [g < b]^\wedge = [g^\wedge \leq -b].$$

If $p \in [f^\wedge < -c]$, there exists $c' > c$ such that $p \in [f^\wedge \leq -c']$ and therefore $g^\wedge(p) \leq -b$, which implies that

$$\sup\{g^\wedge(p) : f^\wedge(p) < -c\} \leq -b. \tag{18}$$

If the strict inequality would hold in (18), then there would exists $b' > b$ such that $\sup\{g^\wedge(p) : f^\wedge(p) < -c\} < -b' < -b$ and hence

$$[f^\wedge < -c] \subseteq [g^\wedge < -b']$$

which yields

$$[g \leq b'] \subseteq [f \leq c]$$

and $\sup\{f(x) : g(x) \leq b'\} \leq c$. But this implies that $\phi(b') \leq c = \phi(b)$ against strict monotonicity. $\square$

Theorem 5.5. *Let $f : X \to \overline{\mathbb{R}}$ and $g : X \to \overline{\mathbb{R}}$ be consistently radiant, strictly increasing along rays and continuous and let $c \in \mathbb{R}$ be the value of $(\mathcal{P}_b)$, for some $b \in \mathbb{R}$. Then $x_0 \neq 0$ is a solution of $(\mathcal{P}_b)$ if and only if there exists a common $\wedge$-subgradient p for f and g at x_0 and p is a solution of $(\mathcal{D}_{-c})$.*

Proof: Since f is strictly increasing along rays then it is regular and $[f \leq k] \subseteq \mathrm{cl}\,[f < k]$ for all $k \in \mathbb{R}$. If x_0 is a solution to $(\mathcal{P}_b)$ then $f(x_0) = c$ and $g(x_0) = b$. Since $\sup\{f(x) : g(x) < b\} = c$, then

$$[g < b] \subseteq [f \leq c] \subseteq \mathrm{cl}\,[f < c]$$

so that x_0 is on the boundary of $[f < c]$ but does not belong to it. Since f is consistently radiant, there exists $p \in L$ such that $p(x_0) = 1$ and

$$[f < c] \subseteq [p < 1], \tag{19}$$

which implies

$$[g < b] \subseteq [f \leq c] \subseteq \mathrm{cl}\,[f < c] \subseteq \mathrm{cl}\,[p < 1] = [p \leq 1]. \tag{20}$$

We deduce from (19) that $p(x) \geq 1$ implies $f(x) \geq f(x_0)$. Since $p(x_0) = 1$, then $p \in \partial^\wedge f(x_0)$.

From (20) we have $[p > 1] \subseteq [g \geq b]$. Since g is continuous, the set $[g \geq b]$ is closed and

$$[p \geq 1] = \mathrm{cl}\,[p > 1] \subseteq [g \geq b],$$

which yields $p \in \partial^\wedge g(x_0)$ so that p is a common $\wedge$-subgradient for f and g at the point x_0.

From the definition of subgradient and (16) we obtain $f^\wedge(p) + f(x_0) = g^\wedge(p) + g(x_0) = 0$ and then

$$f^\wedge(p) = -f(x_0) = -c$$

and

$$g^\wedge(p) = -g(x_0) = -b.$$

Let $b' > b$; since g is continuous, there exists a neighbourhood $U(x_0)$ such that $g(z) < b'$ for all $z \in U(x_0)$. Moreover there exists $\beta > 1$ such that $\beta x_0 \in U$ and $f(\beta x_0) > f(x_0)$, which yields

$$\phi(b') \geq f(\beta x_0) > f(x_0) = \phi(b).$$

Then, by Lemma 5.4, the value of the dual problem $(\mathcal{D}_{-c})$ is exactly $-b$. Hence p is a solution of the dual.

Conversely let x_0 be such that $p \in \partial^\wedge f(x_0) \cap \partial^\wedge g(x_0)$ and p is a solution of $(\mathcal{D}_{-c})$. Then the following equalities are verified:

$$f^\wedge(p) + f(x_0) = g^\wedge(p) + g(x_0) = 0; \qquad g^\wedge(p) = -b; \qquad f^\wedge(p) = -c.$$

Then $f(x_0) = c$ and $g(x_0) = b$. To prove that x_0 is a solution of $(\mathcal{P}_b)$ we need to show that $f(x) \leq c$ for all $x \in [g < b]$ and that $\sup\{f(x), g(x) < b\} = f(x_0)$. By the assumptions we have that $[f^\wedge < -c] \subseteq [g^\wedge \leq -b]$ and hence

$$\bigcup_{s>c} [f \leq s]^\wedge = [f^\wedge < -c] \subseteq [g^\wedge \leq -b] = [g < b]^\wedge$$

which, taking polars on both sides, yields

$$\begin{aligned} [g < b] &= [g < b]^{\wedge\wedge} \subseteq \left[\bigcup_{s>c} [f \leq s]^\wedge\right]^\wedge = & (21) \\ &= \bigcap_{s>c} [f \leq s]^{\wedge\wedge} = \bigcap_{s>c} [f \leq s] = [f \leq c]. & (22) \end{aligned}$$

In (22) we used the fact that the weak level sets $[f \leq s]$ of f are evenly radiant.

To finish consider a sequence $\{t_n\}$ converging to 1 from below. Hence $g(t_n x_0) < b$ for all n and moreover $f(t_n x_0) \to f(x_0) = c$ so that $\sup\{f(x), g(x) < b\} = f(x_0)$. □

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