Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 11, No. 4, pp. 737–749, October 2015

DIFFERENTIAL PROPERTIES OF THE METRIC PROJECTORS OVER THE EPIGRAPH OF THE WEIGHTED ℓ_1 AND ℓ_∞ NORMS

Yong-Jin Liu*, Ning Han, Shiyun Wang† and Caihua Chen‡

Abstract: The optimization problems on the epigraphs of the weighted ℓ_1 and ℓ_∞ norms have recently found many applications in diversifying areas, such as compressive sensing and image processing. In this paper, we investigate some key differential properties of the metric projector over the epigraph of the weighted ℓ_1/ℓ_∞ norm, including its directional derivative, B-subdifferential and Clarke's generalized Jacobian. Our study not only plays an important role in designing numerical methods for the related problems, but also lays a theoretical foundation for further study on the stability and sensitivity analysis of optimization problems involving the epigraphs of the weighted ℓ_1 and ℓ_∞ norms.

Key words: *metric projector, subdifferential, epigraph, weighted ℓ_1/ℓ_∞ norm*

Mathematics Subject Classification: *90C25, 90C30, 65K05*

1 Introduction

Let $\mathcal{K}$ be a closed convex cone in a finite dimensional Hilbert space. Properties, especially differential properties, of the metric operator onto $\mathcal{K}$ play an important role not only in developing numerical methods for the related optimization problems, see, e.g., [12, 17, 21, 25], but also in carrying out the stability and sensitivity analysis of the related problems including conic programming problems, variational inequalities and complementarity problems, and equilibrium problems, see, e.g., [6, 16, 18, 22].

There have been fruitful results achieved on the properties of the metric projectors over symmetric cones. It was proved in [3, 4] that the metric projector over the second-order cone is strongly semismooth. In [16], Pang, Sun and Sun derived the directional derivative and B-subdifferential of this projection. Furthermore, Hayashi, Yamashita and Fukushima [9] gave an explicit expression of its Clarke's generalized Jacobian. For the cone of symmetric positive semidefinite matrices, the projection was shown to be strongly semismooth in [19]. Moreover, Pang, Sun and Sun [16] characterized the directional derivative and B-subdifferential of the

*The author is supported by the National Natural Science Foundation of China under grants No. 11001180 and No. 11371255, the Program for Liaoning Excellent Talents in University, and the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry.

†The author is supported by Scientific Research Project of the Education Department of Liaoning Province under grant L2013080

‡Corresponding author. This author is supported in part by the Natural Science Foundation of Jiangsu Province under grant No. BK20130550.

projection, and Malick and Sendov [13] derived its Clarke's generalized Jacobian. For the general case of symmetric cones, Sun and Sun [20] proved the strong semismoothness of the projection. Kong, Tunçel and Xiu [11] obtained the exact representation for the Clarke's generalized Jacobian of this projection. Very recently, Ding, Sun and Toh [7] introduced several matrix norm cones which are not in the class of symmetric cones, and discussed the strong semismoothness and directional derivative of the projections. In his thesis [6], Ding studied the B-subdiffential and Clarke's generalized Jacobian of the projections over several matrix norm cones.

Motivated by the aforementioned works and the wide applications of the weighted ℓ_1/ℓ_∞ optimization problems in diversifying areas such as compressive sensing and image processing [1, 2], we consider the metric operators over the epigraph of the weighted ℓ_1 norm $\mathcal{K}_1^w$ defined by

$$\mathcal{K}_1^w := \{(x;t) \in \Re^n \times \Re : \|Wx\|_1 \le t\}$$

and its dual $\mathcal{K}_\infty^w$ (see Section 2 for its definition), where $W = \text{diag}(w_1, w_2, \dots, w_n)$ is a diagonal matrix with its i-th diagonal entry $w_i > 0$ for $i = 1, 2, \dots, n$, and $\|\cdot\|_1$ denotes the ℓ_1 norm, i.e., for any $u \in \Re^n$, $\|u\|_1 = \sum_{i=1}^n |u_i|$. More specifically, we focus on the study of differential properties including the directional derivative, B-subdifferential and Clarke's generalized Jacobian of the metric operators over $\mathcal{K}_1^w$ and $\mathcal{K}_\infty^w$.

The major contributions of our paper are threefold. Firstly, we provide a simple approach to establish the closed-form solution of the metric projector over the epigraph of the weighted ℓ_1/ℓ_∞ norm. Note that the closed-form solution firstly appeared in the thesis [23]. Secondly, we generalize the existing result on the directional derivative of the metric projector over the epigraph of ℓ_1/ℓ_∞ norm in [7] to the weighted case. The directional derivative of the metric projector over $\mathcal{K}_1^w/\mathcal{K}_\infty^w$ plays crucial roles in the sensitivity analysis of the weighted ℓ_1/ℓ_∞ norm involved optimization problems. Thirdly, we also derive the explicit expressions of the B-subdifferential and Clarke's generalized Jacobian of the metric operators over $\mathcal{K}_1^w$ and $\mathcal{K}_\infty^w$. The subdifferentials can be further used in the design of the semismooth Newton method for the weighted ℓ_1/ℓ_∞ norm involved optimization problems. To the best of our knowledge, this is the first work in the literature to characterize exactly the directional derivative, B-subdifferetial and Clark's generalized Jacobian of the metric projectors over $\mathcal{K}_1^w$ and $\mathcal{K}_\infty^w$.

The rest of this paper is organized as follows. Section 2 presents some preliminary results on the metric projection and convex analysis needed in the subsequent discussion. In Section 3, we derive the closed-form solutions of the metric projectors over $\mathcal{K}_1^w$ and $\mathcal{K}_\infty^w$. Section 4 is devoted to studying the differential properties of the metric projectors over the epigraphs of the weighted ℓ_1 and ℓ_∞ norms. We make final conclusions and discussions in Section 5.

Notation. For any $z \in \Re^n$, we use $|z|$ to denote the vector in $\Re^n$ whose i-th component is $|z_i|$, $i = 1, \dots, n$. We denote the sign vector of z by $\text{sgn}(z)$, i.e., $(\text{sgn}(z))_i = 1$ if $z_i > 0$, $(\text{sgn}(z))_i = 0$ if $z_i = 0$, and $(\text{sgn}(z))_i = -1$ otherwise. For any index set $\mathcal{I} \subseteq \{1, \dots, n\}$, we use $|\mathcal{I}|$ to represent the cardinality of $\mathcal{I}$, i.e., the number of elements contained in $\mathcal{I}$. We also use $z_\mathcal{I} \in \Re^{|\mathcal{I}|}$ to denote the sub-vector of z obtained by removing all the components of z not in $\mathcal{I}$. By "$\circ$" we denote the Hadamard product, i.e., for any $x, y \in \Re^n$, the i-th component of $z := x \circ y \in \Re^n$ is $z_i = x_i y_i$.

2 Preliminaries

Let $\mathcal{H}$ be a finite dimensional real Euclidean space with an inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\|\cdot\|$. Let C be a closed convex set in $\mathcal{H}$. Then, the dual of C is defined as:

$$C^* := \{y \in \mathcal{H} : \langle x, y \rangle \geq 0, \forall x \in C\}$$

and the polar is $C^o := -C^*$. It is well known that the dual of $\mathcal{K}_1^w$ is the epigraph of the weighted ℓ_∞ norm defined by

$$\mathcal{K}_\infty^w := \{(x,t) \in \Re^n \times \Re : \|W^{-1}x\|_\infty \leq t\}$$

and the polar of $\mathcal{K}_\infty^w$ is given by

$$(\mathcal{K}_\infty^w)^\circ := -\mathcal{K}_1^w = \{(x,t) \in \Re^n \times \Re : \|Wx\|_1 \leq -t\},$$

where $W^{-1} = \mathrm{diag}(w_1^{-1}, w_2^{-1}, \ldots, w_n^{-1})$ and $\|\cdot\|_\infty$ denotes the ℓ_∞ norm.

For any $x \in \mathcal{H}$, the metric projector of x onto C, denoted as $\Pi_C(x)$, is the unique optimal solution to the following convex optimization problem:

$$\min_{y \in C} \quad \frac{1}{2}\|y - x\|^2$$

The metric projector $\Pi_C(\cdot)$ is globally Lipschitz continuous with modulus 1 (c.f. [24]). When C is a nonempty closed convex cone, due to Moreau [14], we have the following Moreau decomposition:

$$x = \Pi_C(x) + \Pi_{C^o}(x).$$

Consequently, for any $(x,t) \in \Re^n \times \Re$, it is easily seen that

$$\Pi_{\mathcal{K}_1^w}(x,t) = (x,t) + \Pi_{\mathcal{K}_\infty^w}(-x,-t). \tag{2.1}$$

Thus, we only need to study the metric operator over $\mathcal{K}_\infty^w$ since corresponding results on the metric operator over $\mathcal{K}_1^w$ can be directly obtained by (2.1).

Let $\widetilde{\mathcal{H}}$ be another finite dimensional real Euclidean space and $\mathcal{O}$ be an open set in $\mathcal{H}$. Suppose that $g : \mathcal{O} \to \widetilde{\mathcal{H}}$ is a locally Lipschitz continuous function in $\mathcal{O}$. Then, it follows from Rademacher's theorem that g is almost everywhere F(réchet)-differentiable in $\mathcal{O}$. Let $\mathcal{F}_g$ denote the set of points in $\mathcal{O}$ where g is F-differentiable. Let $g'(x)$ be the derivative of g at $x \in \mathcal{F}_g$. Then, the B(ouligand)-subdifferential of g at $x \in \mathcal{O}$ is defined by:

$$\partial_B g(x) := \{\lim_{\mathcal{F}_g \ni x^k \to x} g'(x^k)\}.$$

The Clarke's generalized Jacobian of g at $x \in \mathcal{O}$ is the convex hull of $\partial_B g(x)$, i.e.,

$$\partial g(x) := \mathrm{conv}\{\partial_B g(x)\},$$

where "conv" stands for the convex hull.

The following result characterizes the tangent cone of the epigraph of any given positive homogeneous convex function, see e.g., [5, Theorem 2.4.9] for its proof.

Proposition 2.1. Let $f : \Re^n \to \Re$ be a positively homogeneous convex function. Denote $C := \{(x,t) \in \Re^n \times \Re : f(x) \leq t\}$. Then, C is a closed convex cone. Moreover, for any $(\bar{x}, \bar{t}) \in bd(C)$, the tangent cone of C at $(\bar{x}, \bar{t})$ can be characterized by:

$$T_C(\bar{x}, \bar{t}) = \{(d, \xi) \in \Re^n \times \Re : f'(\bar{x}; d) \leq \xi\}.$$

The following proposition due to [8, Theorem 2] and [15, Lemma 5] is crucial for the study of the directional derivative of the metric projectors over $\mathcal{K}_1^w$ and $\mathcal{K}_\infty^w$.

Proposition 2.2. Let $C \subseteq \Re^n \times \Re$ be a polyhedral set. For any $(x,t) \in \Re^n \times \Re$, let $(\bar{x},\bar{t}) := \Pi_C(x,t)$. Then, for any $(h,\eta) \in \Re^n \times \Re$, the directional derivative of $\Pi_C(\cdot,\cdot)$ at (x,t) along the direction (h,η) is given by

$$\Pi'_C((x,t);(h,\eta)) = \Pi_{\widehat{C}}(h,\eta),$$

where $\widehat{C} := T_C(\bar{x},\bar{t}) \cap \big((x,t)-(\bar{x},\bar{t})\big)^\perp$ is the critical cone of C at (x,t).

3 Projections Over $\mathcal{K}_1^w$ and $\mathcal{K}_\infty^w$

In this section, we derive the metric projectors over two closed convex cones. These results will be used in the study of directional derivative of the metric projector over $\mathcal{K}_\infty^w$.

For any real vector $u \in \Re^n$, denote the closed convex cone $\mathcal{C}_n^u$ by

$$\mathcal{C}_n^u := \{(x,t) \in \Re^n \times \Re : u_i^{-1} x_i \le t, i = 1,2,\dots,n\},$$

where $0 \ne u_i \in [-\infty, +\infty]$ for any i. For notational simplicity, $(\pm\infty)^{-1}$ is defined to be 0 and $(\pm\infty)\cdot 0 = 0$.

The next proposition extends the result [7, Propsition 3.2] on the calculation of a special case $\Pi_{\mathcal{C}_n^u}(\cdot,\cdot)$ with all entries of u being one to the general case.

Proposition 3.1. Assume that $(x,t) \in \Re^n \times \Re$ and $u \in \Re^n$ are given. Let π be a permutation of $\{1,2,\dots,n\}$ such that $u_{\pi(i)}^{-1} x_{\pi(i)} \ge u_{\pi(i+1)}^{-1} x_{\pi(i+1)}, i = 1,2,\dots,n-1$ and π^{-1} be the inverse of π. Denote $u_{\pi(0)}^{-1} x_{\pi(0)} = +\infty$ and $u_{\pi(n+1)}^{-1} x_{\pi(n+1)} = -\infty$. Let $\bar{k}$ be the smallest integer $k \in \{0,1,\dots,n\}$ such that

$$u_{\pi(k+1)}^{-1} x_{\pi(k+1)} \le \frac{t + \sum_{j=1}^k u_{\pi(j)} x_{\pi(j)}}{1 + \sum_{j=1}^k u_{\pi(j)}^2} < u_{\pi(k)}^{-1} x_{\pi(k)}.$$

Define

$$\bar{\tau} := \frac{t + \sum_{j=1}^{\bar{k}} u_{\pi(j)} x_{\pi(j)}}{1 + \sum_{j=1}^{\bar{k}} u_{\pi(j)}^2} \quad \text{and} \quad \bar{y}_i := \begin{cases} u_i \bar{\tau}, & u_i^{-1} x_i > \bar{\tau}, \\ x_i, & \textit{otherwise}, \end{cases} \quad i = 1,2,\dots,n.$$

Then, $\Pi_{\mathcal{C}_n^u}(x,t)$ is computed by $\Pi_{\mathcal{C}_n^u}(x,t) = (\bar{y},\bar{\tau})$.

Proof. The proof can be done by a similar way in [7, Propsition 3.2]. We omit it here. □

Using the result of Proposition 3.1, we can readily obtain the expression of $\Pi_{\mathcal{K}_\infty^w}(\cdot,\cdot)$, which can be also found in [23].

Proposition 3.2. Assume that $(x,t) \in \Re^n \times \Re$ is given. For any positive vector $w \in \Re^n$, let π be a permutation of $\{1,2,\dots,n\}$ such that $w_{\pi(i)}^{-1}|x_{\pi(i)}| \ge w_{\pi(i+1)}^{-1}|x_{\pi(i+1)}|, i = 1,2,\dots,n-1$. Then, the metric projectors $\Pi_{\mathcal{K}_\infty^w}(x,t)$ can be computed as

$$\Pi_{\mathcal{K}_\infty^w}(x,t) = (\bar{x},\bar{t}),$$

and hence $\Pi_{\mathcal{K}_1^w}(-x,-t)$ is given by

$$\Pi_{\mathcal{K}_1^w}(-x,-t) = (\bar{x}-x, \bar{t}-t),$$

where $\bar{t} \in \Re_+$ and $\bar{x} \in \Re^n$ are defined by

$$\bar{t} := \max\{\vartheta(x,t), 0\}, \quad \bar{x}_i := \begin{cases} \operatorname{sgn}(x_i) w_i \bar{t}, & |x_i| > w_i \bar{t}, \\ x_i, & \text{otherwise}, \end{cases} \quad i = 1,2,\ldots,n,$$

in which $\vartheta(\cdot,\cdot)$ is defined by

$$\vartheta(x,t) := \frac{t + \sum_{j=1}^{\bar{k}} w_{\pi(j)} |x_{\pi(j)}|}{1 + \sum_{j=1}^{\bar{k}} w_{\pi(j)}^2} \tag{3.1}$$

and $\bar{k}$ is the smallest integer $k \in \{0,1,\ldots,n\}$ such that

$$w_{\pi(k+1)}^{-1} |x_{\pi(k+1)}| \le \frac{t + \sum_{j=1}^{k} w_{\pi(j)} |x_{\pi(j)}|}{1 + \sum_{j=1}^{k} w_{\pi(j)}^2} < w_{\pi(k)}^{-1} |x_{\pi(k)}|. \tag{3.2}$$

4 Differential Properties of the Projector Over $\mathcal{K}_\infty^w$

In this section, we shall study some key differential properties, including the directional derivative, B-subdifferential and Clarke's generalized Jacobian, of the metric projector over $\mathcal{K}_\infty^w$.

For any $(x,t) \in \Re^n \times \Re$ and $w \in \Re^n_{++}$, we define three index sets α, β and γ by

$$\alpha := \{i : w_i^{-1}|x_i| > \vartheta(x,t)\}, \beta := \{i : w_i^{-1}|x_i| = \vartheta(x,t)\}, \gamma := \{1,2,\ldots,n\} \setminus (\alpha \cup \beta),$$

where the function $\vartheta(\cdot,\cdot)$ is given in (3.1).

4.1 The directional derivative

In this subsection, we discuss the directional derivative of the metric projector over $\mathcal{K}_\infty^w$. It should be pointed out that Ding, Sun and Toh [7] achieved the results on the directional derivative of $\Pi_{\mathcal{K}_\infty^w}(\cdot,\cdot)$ in the case where $w_i = 1, i = 1,2,\ldots,n$. By using Proposition 2.2, we extend their results to the general cases in the following theorem. The proof is analogous to that of [7, Proposition 3.2], which is included here for completeness.

Theorem 4.1. Assume that $w \in \Re^n_{++}$ and $(x,t) \in \Re^n \times \Re$ are given. For any $(h,\eta) \in \Re^n \times \Re$, denote $\varrho := 1/\sqrt{1 + \sum_{i\in\alpha} w_i^2}$ and $\hat{h} := \operatorname{sgn}(x) \circ h$. Let

$$\hat{\eta} := \begin{cases} \varrho(\eta + \sum_{i\in\alpha} w_i \hat{h}_i), & \text{if } t \ge -\|Wx\|_1, \\ 0, & \text{otherwise}. \end{cases}$$

Then, the directional derivative of $\Pi_{\mathcal{K}_\infty^w}(\cdot,\cdot)$ at (x,t) along the direction $(h,\eta) \in \Re^n \times \Re$ is given by

$$\Pi'_{\mathcal{K}_\infty^w}((x,t);(h,\eta)) = (\bar{h}, \bar{\eta}), \tag{4.1}$$

where $(\bar{h},\bar{\eta}) \in \Re^n \times \Re$ is computed as

$$\bar{h}_i = \text{sgn}(x_i) w_i \bar{\eta}, \; i \in \alpha \;\; \text{and} \;\; \bar{h}_i = h_i, \; i \in \gamma \tag{4.2}$$

and

$$((\text{sgn}(x) \circ \bar{h})_\beta, \varrho^{-1}\bar{\eta}) = \begin{cases} \Pi_{\mathcal{C}^{u_\beta}_{|\beta|}}(\hat{h}_\beta, \hat{\eta}), & \text{if } \; t > -\|Wx\|_1, \\ \Pi_{\mathcal{D}^{u_\beta}_{|\beta|}}(\hat{h}_\beta, \hat{\eta}), & \text{otherwise}, \end{cases} \tag{4.3}$$

in which $u = \varrho w$ and for the case of $\beta = \emptyset$, $\mathcal{C}^{u_\beta}_{|\beta|} = \Re$ and $\mathcal{D}^{u_\beta}_{|\beta|} = \Re_+$.

Proof. Let $(\bar{x},\bar{t}) := \Pi_{\mathcal{K}^w_\infty}(x,t)$. Since $\mathcal{K}^w_\infty$ is a polyhedral set, we know from Proposition 2.2 that

$$\Pi'_{\mathcal{K}^w_\infty}\big((x,t);(h,\eta)\big) = \Pi_{\widehat{\mathcal{K}^w_\infty}}(h,\eta), \tag{4.4}$$

where $\widehat{\mathcal{K}^w_\infty} := T_{\mathcal{K}^w_\infty}(\bar{x},\bar{t}) \cap ((x,t) - (\bar{x},\bar{t}))^\perp$.

Let $f(z) := \|W^{-1}z\|_\infty, z \in \Re^n$. For any $z \in \Re^n$, denote

$$I(z) := \{i : |w_i^{-1} z_i| = \|W^{-1}z\|_\infty, i = 1,\ldots,n\}. \tag{4.5}$$

Then, for any $d \in \Re^n$, it is easy to deduce that

$$f'(z;d) = \begin{cases} \max\{\text{sgn}(z_i) w_i^{-1} d_i, i \in I(z)\}, & \text{if } z \neq 0, \\ \|W^{-1}d\|_\infty, & \text{if } z = 0. \end{cases} \tag{4.6}$$

From Proposition 2.1, we know that the tangent cone of $\mathcal{K}^w_\infty$ at $(z, f(z))$ can be characterized as

$$T_{\mathcal{K}^w_\infty}(z, f(z)) = \{(d,\xi) \in \Re^n \times \Re : f'(z;d) \le \xi\}. \tag{4.7}$$

We next present the directional derivative of $\Pi_{\mathcal{K}^w_\infty}(\cdot,\cdot)$ by considering the following five cases:

(i). $t > \|W^{-1}x\|_\infty$. In this case, $(\bar{x},\bar{t}) = (x,t)$ and hence $\widehat{\mathcal{K}^w_\infty} = T_{\mathcal{K}^w_\infty}(\bar{x},\bar{t}) = \Re^n \times \Re$. Thus, (4.4) implies that

$$\Pi'_{\mathcal{K}^w_\infty}\big((x,t);(h,\eta)\big) = (h,\eta).$$

In this case, $\bar{k} = 0$ and $\vartheta(x,t) = t$, which imply that

$$\alpha = \emptyset, \quad \beta = \emptyset \;\; \text{and} \;\; \gamma = \{1,2,\ldots,n\}.$$

Thus, $\varrho = 1$ and $\hat{\eta} = \eta$. Since $\mathcal{C}^{u_\beta}_{|\beta|} = \Re$ in this case, it is easy to see that $(\bar{h},\bar{\eta}) = (h,\eta)$ and hence (4.1) holds.

(ii). $t = \|W^{-1}x\|_\infty$. In this case, $(\bar{x},\bar{t}) = (x,t)$ and hence $\widehat{\mathcal{K}^w_\infty} = T_{\mathcal{K}^w_\infty}(\bar{x},\bar{t})$. From (4.6) and (4.7), we have that

$$\widehat{\mathcal{K}^w_\infty} = T_{\mathcal{K}^w_\infty}(\bar{x},\bar{t}) = \begin{cases} \{(d,\xi) \in \Re^n \times \Re : \text{sgn}(x_i) w_i^{-1} d_i \le \xi, i \in I(x)\}, & x \neq 0, \\ \mathcal{K}^w_\infty, & x = 0. \end{cases}$$

Furthermore, in this case, $\bar{k} = 0$ and $\vartheta(x,t) = \|W^{-1}x\|_\infty$ and hence

$$\alpha = \emptyset, \quad \beta = I(x) \;\; \text{and} \;\; \gamma = \{1,2,\ldots,n\} \setminus I(x).$$

Thus, $\varrho = 1$ and $\hat{\eta} = \eta$. We can easily verify that $(\bar{h},\bar{\eta})$ satisfies (4.2) and (4.3).

(iii). $-\|Wx\|_1 < t < \|W^{-1}x\|_\infty$. In this case, $(\bar{x},\bar{t}) \neq (0,0)$ and $\mathrm{sgn}(x_i) = \mathrm{sgn}(\bar{x}_i), i = 1,2,\ldots,n$. Let $J_1 = \{\pi^{-1}(i) : i = 1,\ldots,\bar{k}\}$. Then, the definitions of $\bar{k}$ and $I(\bar{x})$ imply that

$$J_1 \subseteq I(\bar{x})$$

and

$$\begin{aligned}\big((x,t)-(\bar{x},\bar{t})\big)^\perp &= \{(d,\xi) \in \Re^n \times \Re : \sum_{i\in J_1}(x_i-\bar{x}_i)d_i + (t-\bar{t})\xi = 0\}\\ &= \{(d,\xi) \in \Re^n \times \Re : \sum_{i\in J_1}(x_i-\bar{x}_i)d_i + \sum_{i\in J_1}(w_i^2\bar{t} - w_i|x_i|)\xi = 0\}\\ &= \{(d,\xi) \in \Re^n \times \Re : \sum_{i\in J_1}\mathrm{sgn}(x_i)(|x_i|-|\bar{x}_i|)d_i + \sum_{i\in J_1}w_i(|\bar{x}_i|-|x_i|)\xi = 0\}\\ &= \{(d,\xi) \in \Re^n \times \Re : \sum_{i\in J_1}w_i(|x_i|-|\bar{x}_i|)(-\xi + \mathrm{sgn}(x_i)w_i^{-1}d_i) = 0\},\end{aligned}$$

where the second equality holds since in this case $\bar{t} = \vartheta(x,t)$ and the third equality is valid due to $|\bar{x}_i| = w_i\bar{t}$ for any $i \in J_1$. This, together with (4.5), (4.7), and the facts that $\bar{t} = \|W^{-1}\bar{x}\|_\infty$ and $|x_i| > |\bar{x}_i|$ for each $i \in J_1$, implies that

$$\widehat{\mathcal{K}_\infty^w} = \{(d,\xi) \in \Re^n \times \Re : \mathrm{sgn}(x_i)w_i^{-1}d_i = \xi, i \in J_1; \mathrm{sgn}(x_i)w_i^{-1}d_i \leq \xi, i \in I(\bar{x}) \setminus J_1\}.$$

In this case, we further know that

$$\alpha = J_1, \quad \beta = I(\bar{x}) \setminus J_1 \ \text{ and } \ \gamma = \{1,2,\ldots,n\} \setminus (\alpha \cup \beta).$$

By Proposition 3.1, we know from a simple calculation that $(\bar{h},\bar{\eta})$ can be computed as (4.1) such that (4.2) and (4.3).

(iv). $t = -\|Wx\|_1 \neq 0$. In this case, $(\bar{x},\bar{t}) = (0,0)$ and hence $T_{\mathcal{K}_\infty^w}(\bar{x},\bar{t}) = \mathcal{K}_\infty^w$. Let $J_1 = \{i : x_i \neq 0, i = 1,\ldots,n\}$ and $J_2 = \{1,2,\ldots,n\} \setminus J_1$. Since

$$\big((x,t)-(\bar{x},\bar{t})\big)^\perp = \{(d,\xi) \in \Re^n \times \Re : \langle x,d\rangle - \|Wx\|_1\xi = 0\},$$

after simple calculation, we can easily derive that

$$\widehat{\mathcal{K}_\infty^w} = \{(d,\xi) \in \Re^n \times \Re : \mathrm{sgn}(x_i)w_i^{-1}d_i = \xi, i \in J_1; \|(W^{-1}d)_{J_2}\|_\infty \leq \xi\}.$$

In this case, $\bar{k} = |J_1|$ and $\vartheta(x,t) = 0$. We further know that

$$\alpha = J_1, \quad \beta = \{1,2,\ldots,n\} \setminus J_1 \ \text{ and } \ \gamma = \emptyset.$$

After simple transformation, by Proposition 3.1, we can easily derive that $(\bar{h},\bar{\eta})$ can be computed as (4.1) such that (4.2) and (4.3).

(v). $t < -\|Wx\|_1$. In this case, $(\bar{x},\bar{t}) = (0,0)$ and hence $T_{\mathcal{K}_\infty^w}(\bar{x},\bar{t}) = \mathcal{K}_\infty^w$, which implies that $\widehat{\mathcal{K}_\infty^w} = \{(0,0)\}$. Hence, (4.4) implies that

$$\Pi'_{\widehat{\mathcal{K}_\infty^w}}\big((x,t);(h,\eta)\big) = (0,0).$$

In this case, we further know that $\alpha = \{1,2,\ldots,n\}$, $\beta = \emptyset$ and $\gamma = \emptyset$. Since $\hat{\eta} = 0$ and $\mathcal{D}_{|\beta|}^{u_\beta} = \Re_+$, we have that $\bar{\eta} = 0$ and hence $\bar{h} = 0$, which implies that (4.1) holds.

This completes the proof of this theorem. ∎

4.2 The subdifferential

In this subsection, we focus on the characterization of the B-subdifferential and Clarke's generalized Jacobian of $\Pi_{\mathcal{K}_\infty^w}(\cdot,\cdot)$. We begin with studying the F-differentiability of $\Pi_{\mathcal{K}_\infty^w}(\cdot,\cdot)$. For this purpose, we first give the following definitions.

For two index sets β_1, β_2 that partition β, let $P_{\beta_1,\beta_2} \in \Re^{(n+1)\times(n+1)}$ be a permutation matrix (which is orthogonal) such that

$$P_{\beta_1,\beta_2}(x,t) = (x_\gamma, x_{\beta_1}, x_{\beta_2}, x_\alpha, t),$$

where $x_\alpha \in \Re^{|\alpha|}, x_{\beta_1} \in \Re^{|\beta_1|}, x_{\beta_2} \in \Re^{|\beta_2|}, x_\gamma \in \Re^{|\gamma|}$. We will simply use P for the case $\beta = \emptyset$.

By using the results of Theorem 4.1 directly, we easily get the following results on the conditions under which the projector $\Pi_{\mathcal{K}_\infty^w}(\cdot,\cdot)$ is F-differentiable.

Theorem 4.2. Assume that $w \in \Re^n_{++}$ is given. For any given $(x,t) \in \Re^n \times \Re$, the metric projector $\Pi_{\mathcal{K}_\infty^w}(\cdot,\cdot)$ is continuously differentiable at (x,t) if and only if (x,t) satisfies one of the following three conditions: (i) $t > \|W^{-1}x\|_\infty$; (ii) $t < -\|Wx\|_1$; or (iii) $-\|Wx\|_1 < t < \|W^{-1}x\|_\infty$ and $\beta = \emptyset$. Moreover,

(i) if $t > \|W^{-1}x\|_\infty$, then $\Pi'_{\mathcal{K}_\infty^w}(x,t) = I_{n+1}$;

(ii) if $t < -\|Wx\|_1$, then $\Pi'_{\mathcal{K}_\infty^w}(x,t) = 0_{(n+1)\times(n+1)}$;

(iii) if $-\|Wx\|_1 < t < \|W^{-1}x\|_\infty$ and $\beta = \emptyset$, then $\Pi'_{\mathcal{K}_\infty^w}(x,t) = P^TVP$, where V is given by:

$$V := \begin{bmatrix} I_{|\gamma|} & 0_{|\gamma|\times|\alpha|} & 0_{|\gamma|\times 1} \\ 0_{|\alpha|\times|\gamma|} & \multicolumn{2}{c}{\multirow{2}{*}{$s_{\alpha\cup\{n+1\}}s^T_{\alpha\cup\{n+1\}}$}} \\ 0_{1\times|\gamma|} & & \end{bmatrix},$$

in which $s \in \Re^{n+1}$ is defined by

$$s := \varrho(\alpha)(\mathrm{sgn}(x)\circ w, 1)$$

with $\varrho(\alpha) = 1/\sqrt{1+\sum_{i\in\alpha} w_i^2}$.

Since $\Pi_{\mathcal{K}_\infty^w}(\cdot,\cdot)$ is continuously differentiable everywhere in $\mathcal{F}_{\Pi_{\mathcal{K}_\infty^w}}$, we know that $\partial_B\Pi_{\mathcal{K}_\infty^w}(\cdot,\cdot)$ at $(x,t) \in \mathcal{F}_{\Pi_{\mathcal{K}_\infty^w}}$ is a singleton consisting of $\{\Pi'_{\mathcal{K}_\infty^w}(x,t)\}$. Thus, we only need to consider $\partial_B\Pi_{\mathcal{K}_\infty^w}(\cdot,\cdot)$ at the point $(x,t) \notin \mathcal{F}_{\Pi_{\mathcal{K}_\infty^w}}$, which will be exactly characterized in the following theorem.

Theorem 4.3. Assume that $(x,t) \notin \mathcal{F}_{\Pi_{\mathcal{K}_\infty^w}}$ is given. Then, $V \in \partial_B\Pi_{\mathcal{K}_\infty^w}(x,t)$ if and only if there exist two index sets β_1 and β_2 that partition β such that

$$V = P^T_{\beta_1,\beta_2}\begin{bmatrix} I_{|\bar\gamma|} & 0_{|\bar\gamma|\times|\beta_2|} & 0_{|\bar\gamma|\times|\alpha|} & 0_{|\bar\gamma|\times 1} \\ 0_{|\beta_2|\times|\bar\gamma|} & & & \\ 0_{|\alpha|\times|\bar\gamma|} & & \bar s_{\bar\alpha\cup\{n+1\}}\bar s^T_{\bar\alpha\cup\{n+1\}} & \\ 0_{1\times|\bar\gamma|} & & & \end{bmatrix}P_{\beta_1,\beta_2}, \tag{4.8}$$

where $\bar\alpha := \beta_2\cup\alpha$, $\bar\gamma := \gamma\cup\beta_1$ and $\bar s \in \Re^{n+1}$ is defined by

$$\bar s := \varrho(\bar\alpha)(\overline{\mathrm{sgn}}(x)\circ w, 1)$$

with

$$\varrho(\bar{\alpha}) := \begin{cases} 1, & \bar{\alpha} = \emptyset, \\ 0, & |\bar{\alpha}| = n \text{ and } \vartheta(x,t) < 0, \\ 0 \text{ or } 1/\sqrt{1+\sum_{i=1}^n w_i^2}, & |\bar{\alpha}| = n \text{ and } \vartheta(x,t) = 0, \\ 1/\sqrt{1+\sum_{i\in\bar{\alpha}} w_i^2}, & \textit{otherwise}, \end{cases}$$

and for $x \in \Re^n$, $(\overline{\text{sgn}}(x))_i = 1$ if $x_i > 0$, $(\overline{\text{sgn}}(x))_i = -1$ if $x_i < 0$, $(\overline{\text{sgn}}(x))_i = \pm 1$ otherwise.

Proof. Let V be an element of $\partial_B \Pi_{\mathcal{K}_\infty^w}(x,t)$. Then, there exists a sequence $\{(x^\nu, t^\nu)\}$ such that $\mathcal{F}_{\Pi_{\mathcal{K}_\infty^w}} \ni (x^\nu, t^\nu) \to (x,t)$ and

$$V = \lim_{\nu\to\infty} V^\nu := \Pi'_{\mathcal{K}_\infty^w}(x^\nu, t^\nu).$$

Let $\bar{k}^\nu$ be the smallest integer $k \in \{0,1,\ldots,n\}$ such that (3.2) in which (x,t) is replaced by (x^ν, t^ν). Define $\alpha^\nu := \{i : w_i^{-1}|x_i^\nu| > \vartheta(x^\nu, t^\nu)\}$, $\beta^\nu := \{i : w_i^{-1}|x_i^\nu| = \vartheta(x^\nu, t^\nu)\}$, and $\gamma^\nu := \{1,2,\ldots,n\} \setminus (\alpha^\nu \cup \beta^\nu)$, where the function $\vartheta(\cdot,\cdot)$ is given in (3.1).

We proceed to characterize $\partial_B \Pi_{\mathcal{K}_\infty^w}(x,t)$ by considering three cases.

Case 1: $-\|Wx\|_1 < t < \|W^{-1}x\|_\infty$ and $\beta \neq \emptyset$. In this case, $(\bar{x}, \bar{t}) \neq (0,0)$ and $\vartheta(x,t) > 0$. It can be easily seen that $-\|Wx^\nu\|_1 < t^\nu < \|W^{-1}x^\nu\|_\infty$ for sufficiently large ν. By passing to a subsequence if necessary, from item (iii) of Theorem 4.2 we know that $\beta^\nu = \emptyset$ and there exist two index sets β_1 and β_2 that partition β such that $\alpha^\nu = \bar{\alpha}(:= \beta_2 \cup \alpha)$ and $\gamma^\nu = \bar{\gamma}(:= \gamma \cup \beta_1)$ for each ν. Then, by Theorem 4.2 (iii), we know that

$$\Pi'_{\mathcal{K}_\infty^w}(x^\nu, t^\nu) = P_{\beta_1,\beta_2}^T \begin{bmatrix} I_{|\bar{\gamma}|} & 0_{|\bar{\gamma}|\times|\bar{\alpha}|} & 0_{|\bar{\gamma}|\times 1} \\ \begin{matrix} 0_{|\bar{\alpha}|\times|\bar{\gamma}|} \\ 0_{1\times|\bar{\gamma}|} \end{matrix} & \multicolumn{2}{c}{s^\nu_{\bar{\alpha}\cup\{n+1\}}(s^\nu_{\bar{\alpha}\cup\{n+1\}})^T} \end{bmatrix} P_{\beta_1,\beta_2}, \tag{4.9}$$

where s^ν is defined by

$$s^\nu := \varrho(\alpha^\nu)(\text{sgn}(x^\nu) \circ w, 1)$$

with $\varrho(\alpha^\nu) := 1/\sqrt{1+\sum_{i\in\bar{\alpha}} w_i^2}$. The fact that $x_i \neq 0$ for all $i \in \alpha \cup \beta$ implies that

$$\lim_{\nu\to\infty} \text{sgn}(x_i^\nu) = \text{sgn}(x_i) \quad \forall i \in \bar{\alpha}.$$

Therefore, by taking limits on both sides of (4.9), we get that (4.8) holds.

Case 2: $t = \|W^{-1}x\|_\infty$. In this case, $\alpha = \emptyset, \beta = \{i : w_i^{-1}|x_i| = \|W^{-1}x\|_\infty\}, \gamma = \{1,2,\ldots,n\} \setminus \beta$ and $\vartheta(x,t) = \|W^{-1}x\|_\infty$.

Case 2.1: $t \neq 0$. In this case, $\vartheta(x,t) > 0$. By choosing a subsequence if necessary, we know that (x^ν, t^ν) satisfies (i): $\|W^{-1}x^\nu\|_\infty < t^\nu$ or (ii): $-\|Wx^\nu\|_1 < t^\nu < \|W^{-1}x^\nu\|_\infty$ and $\beta^\nu = \emptyset$ for each ν. For case (i), V^ν is the identity matrix I_{n+1} for each ν. Let $\beta_1 = \beta, \beta_2 = \emptyset$. Then, β_1, β_2 partition β and V given by (4.8) is reduced to be I_{n+1}, as desired. For case (ii), similar arguments to Case 1 show that (4.8) holds.

Case 2.2: $t = 0$. In this case, $\vartheta(x,t) = 0$. Without loss of generality, we assume that the sequence $\{(x^\nu, t^\nu)\}$ satisfies one of the following three cases: (i) $\|W^{-1}x^\nu\|_\infty < t^\nu$; (ii) $-\|Wx^\nu\|_1 < t^\nu < \|W^{-1}x^\nu\|_\infty$ and $\beta^\nu = \emptyset$; and (iii) $t^\nu < -\|Wx^\nu\|_1$. The proof of case (i) can be obtained by the same way in Case 2.1. For case (ii), again by Theorem 4.2 (iii), we know that there exist two index sets β_1 and β_2 that partition β such that (4.9) in which $\bar{\alpha} = \beta_2$ and $\bar{\gamma} = \beta_1$ holds. Since $x_i = 0$ for all $i \in \beta$, one has

$$\lim_{\nu\to\infty} \operatorname{sgn}(x_i^\nu) = 1 \text{ or } -1 \quad \forall i \in \bar{\alpha}.$$

In particular, if $|\bar{\alpha}| = n$, then $\vartheta(x^\nu, t^\nu) > 0$ for all ν and hence $\varrho(\bar{\alpha}) = 1/\sqrt{1 + \sum_{l=1}^n w_l^2}$. For case (iii), we know that $\vartheta(x^\nu, t^\nu) < 0$ and $V^\nu = 0_{(n+1)\times(n+1)}$ for each ν. Let $\beta_1 = \emptyset$ and $\beta_2 = \{1, 2, \ldots, n\}$. Then, β_1, β_2 partition β and V given by (4.8) is reduced to be $0_{(n+1)\times(n+1)}$, as desired.

Case 3: $0 \neq t = -\|Wx\|_1$. In this case, $\alpha = \{i : x_i \neq 0\} \neq \emptyset, \beta = \{1, 2, \ldots, n\} \setminus \alpha$ and $\gamma = \emptyset$. By passing a sequence to a sequence if necessary, we know that (x^ν, t^ν) satisfies either (i) $t^\nu < -\|Wx^\nu\|_1$; or (ii) $-\|Wx^\nu\|_1 < t^\nu < \|W^{-1}x^\nu\|_\infty$ and $\beta^\nu = \emptyset$ for each ν. The proof can be similarly obtained as in Case 2.2. We omit it here.

This completes the proof. □

By Proposition 2.2 and (4.4), for any given $(x, t) \in \Re^n \times \Re$, we easily obtain that the following results on $\partial_B \Pi_{\mathcal{K}_\infty^w}(x, t)$ are analogous to [16, Lemma 11] for the cone of symmetric positive semidefinite matrices and [16, Lemma 12] for the second order cone.

Proposition 4.4. Let $(x, t) \in \Re^n \times \Re$ be given. Then, it holds that

$$\partial_B \Pi_{\mathcal{K}_\infty^w}(x, t) = \partial_B \psi(0, 0),$$

where for any $(h, \eta) \in \Re^n \times \Re$, $\psi(h, \eta) := \Pi'_{\mathcal{K}_\infty^w}\big((x, t); (h, \eta)\big)$.

From Proposition 4.4, we easily know that

$$\partial \Pi_{\mathcal{K}_\infty^w}(x, t) = \partial \psi(0, 0). \tag{4.10}$$

We are now in the position to characterize the Clarke's generalized Jacobian of $\Pi_{\mathcal{K}_\infty^w}(\cdot, \cdot)$ at any $(x, t) \in \Re^n \times \Re$. For this purpose, for given $(x, t) \in \Re^n \times \Re$, we let $P_{\gamma,\alpha,\beta}$ be a permutation matrix such that

$$P_{\gamma,\alpha,\beta}(x, t) = (x_\gamma, x_\alpha, x_\beta, t). \tag{4.11}$$

Theorem 4.5. Assume that $(x, t) \in \Re^n \times \Re$ is given. Let $P_{\gamma,\alpha,\beta}$ be a permutation matrix such that (4.11). Then, $V \in \partial \Pi_{\mathcal{K}_\infty^w}(x, t)$ if and only if there exists $W \in \partial \Pi_{\mathcal{K}}(0, 0)$, for any $(d, \xi) \in \Re^n \times \Re$, one has

$$V(d, \xi) = P_{\gamma,\alpha,\beta}^T \begin{bmatrix} I_{|\gamma|} & 0_{|\gamma|\times(n+1-|\gamma|)} \\ 0_{|\gamma|\times(n+1-|\gamma|)}^T & A^T W A \end{bmatrix} P_{\gamma,\alpha,\beta}(d, \xi),$$

where $\mathcal{K} := \mathcal{C}_{|\beta|}^{u_\beta}$ if $t > -\|Wx\|_1$ and $\mathcal{K} := \mathcal{D}_{|\beta|}^{u_\beta}$ otherwise, the matrix $A \in \Re^{(|\beta|+1)\times(n+1-|\gamma|)}$ is given by

$$A := \begin{bmatrix} 0_{|\beta|\times|\alpha|} & \operatorname{diag}(\operatorname{sgn}(x_\beta)) & 0_{|\beta|\times 1} \\ \varrho(\operatorname{sgn}(x)\circ w)_\alpha^T & 0_{1\times|\beta|} & \varrho \end{bmatrix}, \tag{4.12}$$

here, ϱ is given as in Theorem 4.1.

Proof. From the definition of ψ, for any $(h,\eta)\in\Re^n\times\Re$, together with (4.1)–(4.3), we readily derive that

$$\psi(h,\eta)=P^T_{\gamma,\alpha,\beta}\begin{bmatrix} h_\gamma \\ A^T\Pi_{\mathcal{K}}(A(h_\alpha,h_\beta,\eta))\end{bmatrix},$$

where A is given by (4.12). We define the function $g:\Re^n\times\Re\to\Re$ by

$$g(h,\eta):=\frac{1}{2}\|h_\gamma\|^2+\frac{1}{2}\|\Pi_{\mathcal{K}}(A(h_\alpha,h_\beta,\eta))\|^2.$$

It is easy to see that $g(\cdot,\cdot)$ is continuously differentiable with its gradient given by

$$\nabla g(h,\eta)=\psi(h,\eta). \tag{4.13}$$

Thus, we have that $\partial\psi(h,\eta)=\partial^2 g(h,\eta)$. Furthermore, it follows from [10, Example 2.5] that for any $(d,\xi)\in\Re^n\times\Re$, one must have $\partial^2 g(h,\eta)(d,\xi)=\hat{\partial}^2 g(h,\eta)(d,\xi)$, where

$$\hat{\partial}^2 g(h,\eta)(d,\xi):=\left\{P^T_{\gamma,\alpha,\beta}\begin{bmatrix} I_{|\gamma|} & 0\\ 0^T & A^TWA\end{bmatrix}P_{\gamma,\alpha,\beta}(d,\xi): W\in\Pi_{\mathcal{K}}(A(h_\alpha,h_\beta,\eta))\right\}.$$

Consequently, combining (4.10) with (4.13), we get the desired results. The proof is complete. □

5 Conclusions

This paper discussed the following differential properties of the metric projections over the epigraph of the weighted ℓ_1 and ℓ_∞ norms: the directional derivative, the B-subdifferential, and the Clarke's generalized Jacobian. The results obtained in this paper can be used to carry out the stability and sensitivity analysis of optimization problems over the epigraph of the (weighted) ℓ_1/ℓ_∞ norm.

References

[1] E.J. Candès, M.B. Wakin and S. Boyd, Enhancing sparsity by reweighted l_1 minimization, *Journal of Fourier Analysis and Applications* 14 (2008) 877–905.

[2] R. Chartrand and W. Yin, Iteratively reweighted algorithms for compressive sensing, *Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing*, 2008, pp. 3869–3872.

[3] J. Chen, X. Chen and P. Tseng, Analysis of nonsmooth vector-valued functions associated with second-order cones, *Mathematical Programming, Series B* 101 (2004) 95–117.

[4] X.D Chen, D. Sun and J. Sun, Complementarity functions and numerical experiments on some smoothing Newton methods for second-order cone complementarity problems, *Journal of Computational Optimization and Applications* 25 (2003) 39–56.

[5] F.H. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley & Sons, New York, 1983.

[6] C. Ding, *An introduction to a Class of matrix optimization problems*, PhD thesis, National University of Singapore, 2012.

[7] C. Ding, D. Sun and K.-C. Toh, An introduction to a class of matrix cone programming, *Mathematical Programming* 144 (2014), 141–179.

[8] P.T. Haraux, How to differentiate the projection on a convex set in Hilbert space, *Journal of the Mathematical Society of Japan* 29 (1977) 615–631.

[9] S. Hayashi, N. Yamashita and M.A. Fukushima, A combined smoothing and regularization method for monotone second-order cone complementarity problems, *SIAM Journal on Optimization* 15 (2005) 593–615.

[10] J.-B. Hiriart-Urruty, J.-J. Strodiot and V.H. Nguyen, Generalized Hessian matrix and second-order optimality conditions for problems with $C^{1,1}$ data, *Applied Mathematics and Optimization* 11 (1984) 43–56.

[11] L.C. Kong, L. Tunçel and N.H. Xiu, Clarke generalized Jacobian of the projection onto symmetric cones, *Set-Valued Analysis* 17 (2009), 135–151.

[12] Y.J. Liu and L.W. Zhang, Convergence of the augmented Lagrangian method for nonlinear optimization problems over second-order cones, *Journal of Optimization Theory and Applications* 139 (2008) 557–575.

[13] J. Malick and H.S. Sendov, Clarke generalized Jacobian of the projection onto the cone of positive semidefinite matrices, *Set-valued Analysis* 14 (2006) 273–293.

[14] J.J. Moreau, *Décomposition orthogonale d'un espace hilbertien selon deux cones mutuellement polaires*, *Comptes Rendus de l'Académie des Sciences* 255 (1962) 238–240.

[15] J.-S. Pang, *Newton's method for B-differentiable equations*, *Mathematics of Operations Research* 15 (1990), 149–160.

[16] J.-S. Pang, D.F. Sun and J. Sun, Semismooth homeomorphisms and strong stability of semidefinite and Lorentz complementarity problems, *Mathematics of Operations Research* 28 (2003) 39–63.

[17] J. Povh, F. Rendl and A. Wiegele, A boundary point method to solve semidefinite programs, *Computing* 78 (2006) 277–286.

[18] DF. Sun, The strong second-order sufficient condition and constraint nondegeneracy in nonlinear semidefinite programming and their implications, *Mathematics of Operations Research* 31 (2006) 761–776.

[19] D.F. Sun and J. Sun, Semismooth matrix-valued functions, *Mathematics of Operations Research* 27 (2002) 150–169.

[20] D.F Sun and J. Sun, Löwner's operator and spectral functions on Euclidean Jordan algebras, *Mathematics of Operations Research* 33 (2008), 421–445.

[21] D.F. Sun, J. Sun and L.W. Zhang, The rate of convergence of the augmented Lagrangian method for nonlinear semidefinite programming, *Mathematical Programming* 114 (2008) 349–391.

[22] Y. Wang and L.W. Zhang, Properties of equation reformulation of the Karush-Kuhn-Tucker condition for nonlinear second order cone optimization problems, *Mathematical Methods of Operations Research* 70 (2009) 195–218.

[23] Y.N. Wang, *Analysis on several nonsymmetric matrix cones*, PhD thesis, Beijing Jiaotong University, 2011 (in Chinese).

[24] E.H. Zarantonello, Projections on convex sets in Hilbert space and spectral theory I and II, in *Contributions to Nonlinear Functional Analysis*, (E.H. Zarantonello, ed.), Academic Press, New York, 1971, pp. 237–424.

[25] X.Y. Zhao, D.F. Sun and K.C. Toh, , A Newton-CG augmented Lagrangian method for semidefinite programming, *SIAM Journal on Optimization*, 20 (2010) 1737–1765.

Manuscript received 28 May 2014
revised 23 November 2014
accepted for publication 23 November 2014

Yong-Jin Liu
Department of Mathematics, School of Science
Shenyang Aerospace
University Shenyang, 110136, China
E-mail address: yjliu@sau.edu.cn

Ning Han
Department of Mathematics, School of Science
Shenyang Aerospace University
Shenyang, 110136, China
E-mail address: hanning0806@126.com

Shiyun Wang
Department of Mathematics, School of Science
Shenyang Aerospace University
Shenyang, 110136, China
E-mail address: wsy0902@163.com

Caihua Chen
Department of Management Science and Engineering
School of Management and Engineering
Nanjing University, Nanjing, 210093, China
E-mail address: chchen@nju.edu.cn