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MATHEMATICAL PROGRAMMING AND POLYHEDRAL OPTIMIZATION OF SECOND ORDER DISCRETE AND DIFFERENTIAL INCLUSIONS

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Abstract: The present paper is devoted to a difficult and interesting field-second order polyhedral optimization described by ordinary discrete and differential inclusions. The posed problems and the corresponding optimality conditions are new. The stated second order discrete problem is reduced to the polyhedral minimization problem with polyhedral geometric constraints and in terms of the polyhedral Euler-Lagrange inclusions, necessary and sufficient conditions for optimality are derived. Derivation of the sufficient conditions for the second order polyhedral differential inclusions is based on the discrete-approximation method. The transversality condition is formulated separately, a fact peculiar to problems involving higher order derivatives.

Key words: *polyhedral, discrete-approximation, Euler-Lagrange, multivalued, transversality*

Mathematics Subject Classification: *49K20, 49K24, 49J52, 49M25, 90C31*

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1 Introduction

In the first part of the paper, we deal with certain large classes of second order discrete optimizations, which often arise in applications:

$$\text{minimize} \sum_{t=2}^{T-1} g(x_t, t) \tag{1.1}$$

(P_D) subject to

$$x_{t+2} \in F(x_t, x_{t+1}), \quad t = 0, 1, \dots, T-2$$
$$x_0 = \tilde{\alpha}_0, x_1 = \tilde{\alpha}_1 \tag{1.2}$$

where

$$F(x, v_1) = \{v_2 : P_0 x + P_1 v_1 - Q v_2 \leq d\} \tag{1.3}$$

is a polyhedral multivalued mapping $F : \mathbb{R}^n \times \mathbb{R}^n \to P(\mathbb{R}^n)$ (see [24, 27, 28], where $P(\mathbb{R}^n)$ is a set of polyhedral subsets of $\mathbb{R}^n$), P_0, P_1 and Q are $m \times n$ dimensional matrices with rows P_0^i, P_1^i, Q_i $i = 1, \dots, m$ respectively , d is a m-dimensional column-vector with components d_i, $i = 1, \dots, m$ and $\tilde{\alpha}_0, \tilde{\alpha}_1$ are fixed vectors. Moreover, $g(\cdot, t) : \mathbb{R}^n \to \mathbb{R}$ is a polyhedral function that is its epigraph epi $g(\cdot, t)$ is polyhedral set in R^{n+1}. It is required to find a sequence of points $\{\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_T\} \equiv \{\tilde{x}_t\}_{t=0}^T$ of problem (1.1) – (1.3) that minimizes the sum of functions $\sum_{t=2}^{T-1} g(x_t, t)$. We label this problem as (P_D).

In Section 3 we deal with the optimization problem for second order polyhedral differential inclusions:

$$\text{minimize} J(x(\cdot)) = \int_0^1 g(x(t), t)dt + \varphi_0(x(1)), \tag{1.4}$$

(P_C) subject to

$$x''(t) \in F(x(t), x'(t)), \quad \text{a.e.} \quad t \in [0, 1], \tag{1.5}$$

$$x(0) = \alpha_0, \ x'(0) = \alpha_1 \tag{1.6}$$

where the functions and multivalued mapping encountered in Problem (1.4), (1.5) are polyhedral. The problem is to find an arc $\tilde{x}(t)$ of the Cauchy problem $(1.4) - (1.6)$ for the second order differential inclusions satisfying (1.5) almost everywhere (a.e.) on $[0, 1]$ and the initial conditions (1.6) on $[0, 1]$ that minimizes the Bolza functional $J(x(\cdot))$. We label this problem as (P_C). To this end we first derive necessary optimality conditions in the sequences of the discrete-approximation problem (P_{DA}) and then establish, by passing to the limit(formally) as $h \to 0$ (h is the discrete step), sufficient optimality conditions for the original optimal control problem (P_C) described by the second order polyhedral differential inclusions (1.5). Here, a feasible trajectory $x(\cdot)$ is understood to be an absolutely continuous function on a time interval $[0, 1]$ together with the first order derivatives for which $x''(\cdot) \in L_1^n$. Here $L_1^n = L_1^n([0, 1])$ is a Banach space of functions integrable on the time interval $[0, 1]$ in the Lebesgue sense.

Notice that such a class of functions $W_{1,2}^n([0, 1])$ is Banach space, endowed with the different equivalent norms. For instance, one of the norms can be defined as follows $||x(\cdot)|| = |x(0)| + |x'(0)| + ||x''(\cdot)||_1$ or $||x(\cdot)|| = \sum_{k=0}^{2} ||x^{(k)}(\cdot)||_1$, where $||x^{(k)}(\cdot)||_1 = \int_0^1 |x^{(k)}(t)|dt$ and $|x|$ is an Euclidean norm in $\mathbb{R}^n$.

Note that the problems associated with the higher order differential and discrete inclusions are more complicated due to the higher order derivatives and their discrete analogues. A convenient procedure for eliminating this complication in optimal control theory involving higher order derivatives is a formal reduction of these problems by substitution to the system of first order differential inclusions or equations. However in practice returning to the original higher order problem and expressing the resulting optimality conditions in terms of the original problem data is in general very difficult.

To our best knowledge, control problems for higher order differential inclusions have not been studied in the literature and on the whole only the qualitative problems with second order differential inclusions have been investigated. The first viability results for second order differential inclusions were given by Haddad and Yarou [15]. The nonconvex case for second order differential inclusions has been studied by Lupulescu [19] and Cernea [7]. In [19] existence of viable solutions was proved for an autonomous second-order functional differential inclusion in the case when the multifunction that define the inclusion is upper semicontinuous compact valued and contained in the subdifferential of a proper lower semicontinuous convex function. In the paper [6] the existence of solutions for initial and boundary value problems for second order impulsive functional differential inclusions in Banach spaces are investigated. Here a fixed point theorem for contraction multivalued maps due to Covitz and Nadler [9] is used. The theory of impulsive differential equations has seen considerable development; see the monograph of Lakshmikantham, et al. [17]. The paper [5] gives necessary and sufficient conditions ensuring the existence of solutions to the second order differential inclusions with state constraint. In [29] the existence of Lyapunov functions for second-order differential inclusions is analyzed by using the methodology of the viability theory.

The problems (P_D) and (P_C) have a wider class of applications. For example, they can be applied in the investigations of the so-called von Neumann economic dynamics model [30] the graph of which is a polyhedral cone $K = \{(x, v_1, v_2) : x \geq B_0\lambda,\ v_1 \geq B_1\lambda,\ v_2 = A\lambda,\ \lambda \geq 0, \lambda \in \mathbb{R}^m\}$, where A, B_0, B_1 are $n \times m$ matrices with nonnegative elements and λ is a vector with components $\lambda_j, j = 1, \ldots, m$. Moreover, the problems (P_D) and (P_C) can be applied in the linear discrete ($x_{t+2} = A_0 x_t + A_1 x_{t+1} + Bu_t, u_t \in U \subset \mathbb{R}^r,\ t = 0, \ldots, T-1, A_0, A_1 - n\times n$ and $B - n \times r$ matrices,respectively) or linear differential optimal control problem ($x'' = A_0 x + A_1 x' + Bu,\ u = u(t) \in U, t \in [0, 1]$) where a control domain U is a polyhedral set. The key problem for the investigation of the problem (P_C) is the discrete-approximation problem. Note that in [24, 27, 28] some properties of nondegenerate polyhedral mappings, where $F(x, v_1) \equiv F(x)$ (the number of constraints defining every vertex of polytope $F(x)$ is n) are studied. Although the given problems $(P_D), (P_C)$ are governed by multivalued mappings we do not use the LAM notion (see [20], [22]- [27]) in formulation of any optimality conditions. In general, in the last decade discrete and continuous time processes with lumped and distributed parameters have found wide application in the field of mathematical economics and in problems of control dynamic system optimization and differential games (see, for example [1, 2, 10–14, 17, 18, 29, 30, 34, 37, 40, 41]. Note that for different problems described by set-valued mappings the reader can consult Aubin and Cellina [3], Aubin and Frankowska [4],Clarke [8], Mordukhovich [31–33], Mahmudov [27], Rockafellar [36], Vinter and Zheng [39] and the bibliography therein.

The paper is organized as follows. In Section 2, by converting the problem (P_D) into a problem with geometric constraints and applying Farkas theorem [27, p.22] we formulate necessary and sufficient conditions for a convex minimization problem with linear inequality constraints. Then with the problem data, we are able to obtain the conditions of optimality for polyhedral second order discrete inclusions, where a Slater condition of convex analysis about existence of an interior point is not needed in this case.

In Section 3 the necessary and sufficient conditions of optimality for discrete-approximation problem (P_{DA}) are formulated using the approximation method for the continuous polyhedral problem (P_C). Note that an important role of the discrete-approximation method for different type of ordinary and partial differential inclusions is demonstrated in [20], [22]- [27], [31, 33].

In Section 4 in order to formulate necessary and sufficient conditions for second order differential inclusions, Theorem 3.1 plays a significant role. Thus by passing to formally limit the sufficient conditions of optimality for second order polyhedral optimization is obtained.

2 Optimization of Second Order Polyhedral Discrete Inclusions

We reduce the problem (P_D) to a convex mathematical programming problem with constraints consisting of the linear inequalities. Let us denote A and D by

$$A = \begin{pmatrix} P_0 & P_1 & -Q & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & P_0 & P_1 & -Q & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & 0 & P_0 & P_1 & -Q \end{pmatrix}, \quad D = \begin{pmatrix} d \\ \vdots \\ \vdots \\ d \end{pmatrix}$$

where A is the partitioned into submatrices $P_0, P_1, -Q$ and $m \times n$ zero matrices 0, D is a $m(T-1)$ dimensional column-vector and A is a matrix with size $m(T-1) \times n(T+1)$; the number of rows and the number of columns are equal to T-1 and T+1, respectively. If we introduce a vector $w = (x_0, x_1, \dots, x_T) \in \mathbb{R}^{n(T+1)}$, then the problem can be reduced to the problem with geometric constraints and with the objective function $f(w) = \sum_{t=2}^{T-1} g(x_t, t)$; the problem (P_D) can be replaced by the following equivalent problem in Euclidean space $\mathbb{R}^{n(T+1)}$:

$$\begin{aligned} &\text{minimize} \quad f(w), \\ &\text{subject to} \quad w \in M \cap N_0 \cap N_1, \; w \in \mathbb{R}^{n(T+1)}, \end{aligned} \tag{2.1}$$

where $M = \{w = (x_0, \dots, x_T) : Aw \le D\}$, $N_0 = \{w = (x_0, \dots, x_T) : x_0 = \tilde{\alpha}_0\}$, $N_1 = \{w = (x_0, \dots, x_T) : x_1 = \tilde{\alpha}_1\}$.

Thus, if $\{\tilde{x}_t\}_{t=0}^{T}$ is a solution of problem (1.1)-(1.3), then $\tilde{w} = (\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_T)$ is a solution of the problem (2.1) and vice versa. Obviously, $M = \bigcap_{t=0}^{T-2} M_t$, where $M_t = \{w = (x_0, \dots, x_T) : P_0 x_t + P_1 x_{t+1} - Q x_{t+2} \le d\}$, $t = 0, \dots, T-2$. Let $K_M(\tilde{w}), \tilde{w} \in M; K_{N_0}(\tilde{w}), \tilde{w} \in N_0$ and $K_{N_1}(\tilde{w}), \tilde{w} \in N_1$ be the cone of tangent directions [20]- [27], [35]. Then by Theorem 3.4. [27, p.99] there exist a vector $w_f^* \in \partial_w f(\tilde{w})$ and vectors $w_M^* \in K_M^*(\tilde{w}), w_0^* \in K_{N_0}^*(\tilde{w}), w_1^* \in K_{N_1}^*(\tilde{w})$ such that $w_f^* = w_0^* + w_M^* + w_1^*$. Here, $K_S^*(\tilde{w})$ is the dual cone to the cone of tangent directions $K_S(\tilde{w})$ =cone$(S - \tilde{w})$ at a point $\tilde{w} \in S$, i.e., $K_S^*(\tilde{w}) = \{w^* = (x_0^*, \dots, x_T^*) : \langle \overline{w}, w^* \rangle \ge 0, \; \forall\, \overline{w} \in K_S(\tilde{w})\}$, where $\langle \cdot, \cdot \rangle$ is a scalar product. On the other hand, the cone of tangent directions $K_{M_t}(\tilde{w})$, $t = 0, \dots, T-2$ are polyhedral cones and so by Lemma 1.22 [27, p.23] $K_M^*(\tilde{w}) = \sum_{t=0}^{T-2} K_{M_t}^*(\tilde{w})$. Thus, we have a formula

$$w_f^* = w_0^* + \sum_{t=0}^{T-2} w_{M_t}^* + w_1^*, \quad w_{M_t}^* \in K_{M_t}^*(\tilde{w}). \tag{2.2}$$

Note that for $\tilde{w}$ to be a point minimizing f over $M \cap N_0 \cap N_1$ in problem (2.1), it is necessary and sufficient that the condition (2.2) is fulfilled. Clearly $w_f^* \in \partial_w f(\tilde{w})$ implies that $w_f^* = (x_{f0}^*, x_{f1}^*, \dots, x_{fT}^*)$, $x_{ft}^* \in \partial_x g(\tilde{x}_t, t)$, $t = 2, \dots, T$.

Now, we shall compute the dual cones $K_M^*(\tilde{w}), K_{N_0}^*(\tilde{w})$ and $K_{N_1}^*(\tilde{w})$. First, by using Farkas Theorem [27, p.22] , we prove the following result.

Lemma 2.1. For a polyhedral set M_t at a point $\tilde{w} \in M_t$ one has

$$K_{M_t}^*(\tilde{w}) = \{w^*(t) : \; x_t^*(t) = -P_0^* \lambda_t, \; x_{t+1}^*(t) = -P_1^* \lambda_t, x_{t+2}^*(t) = Q^* \lambda_t, \; x_k^* = 0,$$

$$k \ne t, t+1, t+2, \; \lambda_t \ge 0, \; \lambda_t \in \mathbb{R}^m, \langle P_0 \tilde{x}_t + P_1 \tilde{x}_{t+1} - Q \tilde{x}_{t+2} - d, \; \lambda_t \rangle \; = 0, \; t = 0, \dots, T-2\}.$$

Proof. By the definition of the cone of tangent directions, we infer

$$K_{M_t}(\tilde{w}) = \{\overline{w} : P_0(\tilde{x}_t + \lambda \overline{x}_t) + P_1(\tilde{x}_{t+1} + \lambda \overline{x}_{t+1}) - Q(\tilde{x}_{t+2} + \lambda \overline{x}_{t+2}) \le d \tag{2.3}$$

for a small $\lambda > 0\}$, $t = 0, \dots, T-2$.

Let $I(\tilde{w})$ denote the set of active indices, i.e.

$$I(\tilde{w}) = \{i : P_0^i \tilde{x}_t + P_1^i \tilde{x}_{t+1} - Q_i \tilde{x}_{t+2} = d_i \;, \; i = 1, \dots, m\}.$$

It follows from (2.3) that for indices $i \in I(\tilde{w})$, the inequality

$$P_0^i(\tilde{x}_t + \lambda\overline{x}_t) + P_1^i(\tilde{x}_{t+1} + \lambda\overline{x}_{t+1}) - Q_i(\tilde{x}_{t+2} + \lambda\overline{x}_{t+2})$$
$$= d_i + \lambda(P_0^i\overline{x}_t + P_1^i\overline{x}_{t+1} - Q_i\overline{x}_{t+2}) \le d_i \ , \ t = 0, \dots, T-2$$

holds, if

$$P_0^i\overline{x}_t + P_1^i\overline{x}_{t+1} - Q_i\overline{x}_{t+2} \le 0 \ , \ i \in I(\tilde{w}). \qquad (2.4)$$

Clearly, if $i \notin I(\tilde{w})$, the inequality

$$P_i(\tilde{x}_t + \lambda\overline{x}_t) - Q_i(\tilde{x}_{t+1} + \lambda\overline{x}_{t+1}) = P_i\tilde{x}_t - Q_i\tilde{x}_{t+1} + \lambda(P_i\overline{x}_t - Q_i\overline{x}_{t+1}) < d_i$$

holds for small λ, regardless choosing $(\overline{x}_t, \overline{x}_{t+1}, \overline{x}_{t+2})$. Thus, a cone $K^*_{M_t}(\tilde{w})$ is defined completely by the system of inequalities (2.4). Now we rewrite (2.4) in the form

$$\langle \overline{x}_t, -P_0^i\rangle + \langle \overline{x}_{t+1}, -P_1^i\rangle + \langle \overline{x}_{t+2}, Q_i\rangle \ \ge 0, \quad i \in I(\tilde{w}).$$

Then applying Farkas Theorem [27, p.22] and taking into account that $\overline{x}_k$, $k \ne t, t+1, t+2$ are arbitrary, it follows from the latter inequality that $w^* = (x_0^*, x_1^*, \dots, x_T^*) \in K^*_{M_t}(\tilde{w})$ if and only if

$$x_t^* = -\sum_{i\in I(\tilde{w})} P_0^{i*}\lambda_i^t, \ x_{t+1}^* = -\sum_{i\in I(\tilde{w})} P_1^{i*}\lambda_i^t, \ x_{t+2}^*(t) = \sum_{i\in I(\tilde{w})} Q_i^*\lambda_i^t, \ \lambda_i^t \ge 0, \qquad (2.5)$$

where $P_0^{i*}, P_1^{i*}, Q_i^*$ are transposed vectors of P_0^i, P_1^i, Q_i respectively. Thus, taking $\lambda_i^t = 0$ for $i \notin I(\tilde{w})$ and denoting λ_t a vector with the components λ_i^t, formula (2.5) can be rewritten in the equivalent form

$$K^*_{M_t}(\tilde{w}) = \{w^*(t) = (0, \dots, 0, x_t^*(t), x_{t+1}^*(t), x_{t+2}^*(t), 0, \dots, 0) : x_t^*(t) = -P_0^*\lambda_t,$$
$$x_{t+1}^*(t) = -P_1^*\lambda_t, x_{t+2}^*(t) = Q^*\lambda_t, \ \lambda_t \ge 0, \ \lambda_t \in \mathbb{R}^m, \ t = 0, \dots, T-2,$$
$$\langle P_0\tilde{x}_t + P_1\tilde{x}_{t+1} - Q\tilde{x}_{t+2} - d, \ \lambda_t\rangle \ = 0\}. \qquad (2.6)$$

The proof of the lemma is completed. □

The necessary and sufficient condition of optimality for the problem given by second order polyhedral discrete inclusions can be resumed as following theorem.

Theorem 2.2. Let F be a polyhedral mapping defined by (1.3) and $g(\cdot, t)$ be a polyhedral function. Then, for the $\{\tilde{x}_t\}_{t=0}^T$ to be an optimal trajectory of the second order discrete polyhedral optimization problem (P_D), it is necessary that there exist vectors $x_t^*, t = 0, \dots, T-1$ simultaneously not all equal to zero satisfying the discrete Euler-Lagrange and transversality inclusions:

$$x_t^* = P_0^*\lambda_t + P_1^*\lambda_{t-1} + u_t^*, \lambda_t \ge 0, \ u_t^* \in \partial g(\tilde{x}_t, t), \ \partial g(\tilde{x}_0, 0) = \partial g(\tilde{x}_1, 1) = 0,$$
$$x_{t+2}^* = Q^*\lambda_t \ , \quad \lambda_{-1} = 0 \ , \ t = 0, \dots, T-2.$$
$$\langle P_0\tilde{x}_t + P_1\tilde{x}_{t+1} - Q\tilde{x}_{t+2} - d, \lambda_t\rangle = 0,$$
$$-P_1^*\lambda_{T-2} + x_{T-1}^* \in \partial g(\tilde{x}_{T-1}, T-1) \ , \quad x_T^* = 0.$$

Proof. The cones $K^*_{N_0}(\tilde{w})$, $K^*_{N_1}(\tilde{w})$ should be computed to give the component-wise representation of the relationship (2.2). In fact on the definition of cone of tangent directions $\overline{w} \in K_{N_0}(w), w \in N_0$ if and only if $w + \lambda\overline{w} \in N_0$ for a sufficiently small $\lambda > 0$. It follows that $x_0 + \lambda\overline{x}_0 = \tilde{\alpha_0}$. On the other hand $w \in N_0$ and so $x_0 = \tilde{\alpha_0}$. Thus $\overline{x}_0 = 0$. As a result we have $K_{N_0}(w) = \{\overline{w} = (\overline{x}_0, \dots, \overline{x}_T) : \overline{x}_0 = 0\}$, whence

$$\begin{aligned} K^*_{N_0}(w) &= \{w^* = (x^*_0, \dots, x^*_T) : \ \langle \overline{w}, w^* \rangle \geq 0, \overline{w} \in K_{N_0}(w)\} \\ &= \Big\{ \sum_{t=0}^{T} \langle \overline{x}_t, x^*_t \rangle \geq 0, \overline{w} \in K_{N_0}(w) \Big) \} \\ &= \{w^* = (x^*_0, \dots, x^*_T) : \ x^*_t = 0, t = 1, \dots, T\}. \end{aligned}$$

In analogy to calculation of $K^*_{N_0}(w)$, we derive the following formula

$$K^*_{N_1}(w) = \{w^* = (x^*_0, \dots, x^*_T) : \ x^*_t = 0, t \neq 1\}.$$

Then, it easily be seen that in accordance with Lemma 2.1 and formulas (1.3), we can write

$$\begin{aligned} w^*_0 &= (x^*_{00}, 0, \dots, 0), \ w^*_1 = (0, x^*_{11}, 0 \dots, 0), \\ w^*(t) &= (0, \dots, 0, x^*_t(t), x^*_{t+1}(t), x^*_{t+2}(t), 0, \dots, 0), \ t = 0, \dots, T-2. \end{aligned} \tag{2.7}$$

In turn, by using the structure of the vectors (2.7) the component-wise representation of (2.2) imply

$$\begin{aligned} 0 &= x^*_{00} + x^*_0(0), \\ 0 &= x^*_{11} + x^*_1(1) + x^*_1(0), \end{aligned} \tag{2.8}$$

$$x^*_{ft} = x^*_t(t) + x^*_t(t-1) + x^*_t(t-2) \ , \quad t = 2, \dots, T-2. \tag{2.9}$$

Further, by Lemma 2.1

$$x^*_t(t) = -P^*_0\lambda_t, \ x^*_{t+1}(t) = -P^*_1\lambda_t, \ x^*_{t+2}(t) = Q^*\lambda_t \ , \ \lambda_t \geq 0.$$

Then by denoting $x^*_{t+2}(t) \equiv x^*_{t+2}, \ t = 1, \dots, T-2, \ x^*_{ft} \equiv u^*_t, x^*_{00} \equiv x^*_0$ in view of (2.9) we obtain

$$\begin{aligned} x^*_t &= P^*_0\lambda_t + P^*_1\lambda_{t-1} + u^*_t, \quad u^*_t \in \partial g(\tilde{x}_t, t), \\ x^*_{t+2} &= Q^*\lambda_t \quad , \lambda_t \geq 0 \ , \quad t = 0, \dots, T-2. \end{aligned} \tag{2.10}$$

where

$$\langle P_0\tilde{x}_t + P_1\tilde{x}_{t+1} - Q\tilde{x}_{t+2} - d, \lambda_t \rangle = 0$$

On the other hand,it is easy to see that by setting $g(\tilde{x}_0, 0) = g(\tilde{x}_1, 1) = 0, \ x^*_{00} = x^*_0 \ , \lambda_{-1} \equiv 0, \ x^*_{11} \equiv x^*_1$ and using the relations (2.8), the formula (2.9) can be generalized to the case $t = 0, 1$. Finally for $t = T-1$ we have $x^*_{(T-1)0} = x^*_{T-1}(T-2) + x^*_{T-1}(T-3)$ or on the accepted notations

$$x^*_{(T-1)0} = x^*_{T-1} - P^*_1\lambda_{T-2}.$$

It is obvious that since $g(\tilde{x}_T, T) \equiv 0$, it follows that $x^*_T = 0$. Thus, we have

$$\begin{aligned} &\partial g(\tilde{x}_0, 0) = \partial g(\tilde{x}_1, 1) = 0, \\ &x^*_{T-1} - P^*_1\lambda_{T-2} \in \partial g(\tilde{x}_{T-1}, T-1) \quad x^*_T = 0. \end{aligned}$$

Now, these relations and formulas (2.10) justify the validity of the theorem. □

Remark 2.3. Suppose we have the problem (P_D) with the objective function $\sum_{t=2}^{T} g(x_t, t)$ that is $g(x_T, T) \neq 0$ for second order polyhedral discrete inclusions. Then it is easy to see that in this case $x_T^* \neq 0$ and $x_T^* \in \partial g(x_T, T)$. This means that transversality condition for such problems consists of the following inclusions $x_{T-1}^* - P_1^* \lambda_{T-2} \in \partial g(\tilde{x}_{T-1}, T-1)$ and $x_T^* \in \partial g(\tilde{x}_T, T)$. Also in the problem (P_D), we can replace $\sum_{t=2}^{T} g(x_t, t)$ by $\sum_{t=2}^{T-2} g(x_t, t) + \overline{g}(x_{T-1}, x_T)$, where $\overline{g}(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a polyhedral function. Clearly for the problem (P_D) with the objective function $\sum_{t=2}^{T-2} g(x_t, t) + \overline{g}(x_{T-1}, x_T)$ the transversality condition of Theorem 2.2 has a form

$$(x_{T-1}^* - P_1^* \lambda_{T-2}, x_T^*) \in \partial \overline{g}(\tilde{x}_{T-1}, \tilde{x}_T).$$

3 Optimization of Second Order Polyhedral Discrete-approximation Problem

Let h be a step on the t-axis and $x_h(t)$ is a grid function on a uniform grid on $[0, 1]$. We introduce the following first and second order difference operators

$$\begin{aligned} \Delta_h x_h(t) &= \frac{1}{h}\big(x_h(t+h) - x_h(t)\big) \\ \Delta_h^2 x_h(t) &= \frac{1}{h^2}\big(x_h(t+2h) - 2x_h(t+h) + x_h(t)\big) \end{aligned}$$

We associate with the continuous problem (P_C) the following discrete- approximation problem

$$\begin{aligned} &\text{minimize} \sum_{t=0}^{1-2h} hg(x_h(t), t) + \varphi_0(x_h(1-h)), \\ (P_{DA}) \qquad &\text{subject to} \\ &P_0 x_h(t) + P_1 \Delta_h x_h(t) - Q \Delta_h^2 x_h(t) \leq d, \\ &x_h(0) = \alpha_0, \\ &\Delta_h x_h(0) = \alpha_1, \quad t = 0, h, \ldots, 1-2h \end{aligned} \tag{3.1}$$

We label this problem as (P_{DA}) and we formulate the optimality conditions for it.

Theorem 3.1. Let F be a polyhedral multivalued mapping and g be a polyhedral function with respect to x. In order that $\{\tilde{x}_h(t)\}_{t=0}^{1}$ be a solution of the second order polyhedral discrete-approximation optimization problem (P_{DA}), it is necessary and sufficient that there exists an adjoint trajectory of vectors $\{x_h^*(t)\}_{t=0}^{1}$ simultaneously not all equal to zero satisfying the approximate Euler-Lagrange inclusions and the transversality condition:

(a) $\Delta_h^2 x_h^*(t) \in P_0^* \lambda_h(t) - P_1^* \Delta_h \lambda_h(t-h) + \partial g(\tilde{x}_h(t), t),\ \lambda_h(t) \geq 0,$

(b) $-\Delta_h x_h^*(1) - P_1^* \lambda_h(1-2h) \in \partial \varphi_0(\tilde{x}_h(1-h)),\ x_h^*(1) = 0,$

(c) $\langle P_0 \tilde{x}_h(t) + P_1 \Delta_h \tilde{x}_h(t) - Q \Delta_h^2 \tilde{x}_h(t) - d, \lambda_h(t) \rangle = 0,\ t = 2h, \ldots, 1-2h.$

Proof. We apply the Theorem 2.2 to formulate the necessary and sufficient conditions for problem (3.1). First we reformulate this problem in the form of the (P_D) problem:

$$\text{minimize} \sum_{t=0}^{1-2h} hg(x_h(t),t) + \varphi_0(x_h(1-h)),$$

subject to

$$\begin{gathered}\big[h^2P_0 - hP_1 - Q\big]x_h(t) + \big[hP_1 + 2Q\big]x_h(t+h) - Qx_h(t+2h) \le h^2 d,\\ x_h(0) = \alpha_0,\ x_h(h) = \alpha_0 + h\alpha_1,\\ t = 0, h, \dots, 1-2h.\end{gathered} \tag{3.2}$$

Let $\{\tilde{x}_h(t)\}, t = 0, h, \dots, 1$ be an optimal solution of the problem (3.2). Obviously, an adjoint discrete inclusions Theorem 2.2 for second order polyhedral problem with discrete approximation inclusions (3.2) has a form

$$\begin{gathered}x_h^*(t) \in \big[h^2P_0 - hP_1 - Q\big]^*\lambda_h(t) + \big[hP_1 + 2Q\big]^*\lambda_h(t-h) + h\partial g(\tilde{x}_h(t),t),\\ x_h^*(t+2h) = Q^*\lambda_h(t) \quad,\ \lambda_h(t) \ge 0,\ t = 0, h, \dots, 1-h,\\ \big\langle \big(h^2P_0 - hP_1 - Q\big)\tilde{x}_h(t) + \big(hP_1 + 2Q\big)\tilde{x}_h(t+h) - Q\tilde{x}_h(t+2h) - h^2 d,\ \lambda_h(t)\big\rangle = 0.\end{gathered} \tag{3.3}$$

We rewrite the first inclusions in (3.3) in more convenient form, that is to express it in terms of second order difference operators. Clearly

$$\begin{gathered}x_h^*(t) \in h^2P_0^*\lambda_h(t) - hP_1^*\lambda_h(t) - Q^*\lambda_h(t) + hP_1^*\lambda_h(t-h)\\ +2Q^*\lambda_h(t-h) + h\partial g(\tilde{x}_h(t),t)\end{gathered}$$

which yields

$$\begin{gathered}Q^*\lambda_h(t) - 2Q^*\lambda_h(t-h) + x_h^*(t) \in\ h^2P_0^*\lambda_h(t) + hP_1^*\lambda_h(t-h)\\ -hP_1^*\lambda_h(t) + h\partial g(\tilde{x}_h(t),t),\ t = 0, h, \dots, 1-2h.\end{gathered}$$

On the other hand, since $x_h^*(t+2h) = Q^*\lambda_h(t)$ it follows from the last inclusion that

$$\begin{gathered}x_h^*(t+2h) - 2x_h^*(t+h) + x_h^*(t) \in\ h^2P_0^*\lambda_h(t) - hP_1^*\lambda_h(t)\\ +hP_1^*\lambda_h(t-h) + h\partial g(\tilde{x}_h(t),t),\ t = 0, h, \dots, 1-2h.\end{gathered}$$

Dividing both sides of the latter relation by h^2, we have

$$\Delta_h^2 x_h^*(t) \in P_0^*\lambda_h(t) - P_1^*\Delta_h\lambda_h(t-h) + \partial g(\tilde{x}_h(t),t),\ t = 2h, \dots, 1-2h. \tag{3.4}$$

Here it is taken into account that $h\lambda_h(t)$ and $hx_h^*(t)$ are denoted again by $\lambda_h(t)$ and $x_h^*(t)$, respectively.

Similarly, the second formula in (3.3) can be rewritten as follows

$$\langle P_0\tilde{x}_h(t) + P_1\Delta_h\tilde{x}_h(t) - Q\Delta_h^2\tilde{x}_h(t) - d, \lambda_h(t)\rangle = 0,\ t = 2h, \dots, 1-2h.$$

On the other hand, transversality condition of Theorem (2.2) imply:

$$-P_1^*\lambda_h(1-2h) + x_h^*(1-h) \in h\partial\varphi_0(\tilde{x}_h(1-h)),\ x_h^*(1) = 0.$$

Since $x_h^*(1) = 0$ the second assertion (2) of theorem follows from the last relation which can be rewritten as

$$-\Delta_h x_h^*(1) - P_1^*\lambda_h(1-2h) \in \partial\varphi_0(\tilde{x}_h(1-h)),\ x_h^*(1) = 0.$$

Therefore, we have obtained the desired result. □

Remark 3.2. By the analogy in Remark 2.3 the result of Theorem 3.1 can be easily developed to the problem (P_{DA}) with the objective function $\sum_{t=0}^{1-2h} hg(x_h(t), t) + \overline{\varphi}_0(x_h(1-h), x_h(1))$, where $\overline{\varphi}_0(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a polyhedral function having the form $\overline{\varphi}_0(x_h(1-h), x_h(1)) = \varphi_0(x_h(1-h), \Delta x_h(1))$. Thus according to the Remark 2.3 we have a new form of the transversality condition for the problem (P_{DA}) :

$$(x_h^*(1-h) - P_1^*\lambda_h(1-2h), x_h^*(1)) \in \partial\varphi_0(\tilde{x}_h(1-h), \Delta_h\tilde{x}_h(1)).$$

Theorem 3.3. Let $\overline{\varphi}_0(\cdot,\cdot)$ be a polyhedral function defined by relation $\overline{\varphi}_0(x, v_1) \equiv \varphi_0\Big(x, \dfrac{x - v_1}{h}\Big)$ and $z_0 = (x^0, v_1^0) \in dom\overline{\varphi}_0$. Then the following inclusions are equivalent:

(i) $(\overline{x}^*, \overline{v}_1{}^*) \in \partial\overline{\varphi}_0(z_0)$,

(ii) $(\overline{x}^* + \overline{v}_1{}^*, -h\overline{v}_1{}^*) \in \partial\varphi_0\Big(x^0, \dfrac{x^0 - v_1^0}{h}\Big)$.

Proof. Note that $\partial_z\overline{\varphi}(z_0)$ is a convex closed set and for $z_0 \in ri(dom\overline{\varphi}_0)$ is bounded [27,33,35]. By the definition of subdifferential sets

$$\begin{aligned}\partial_z\overline{\varphi}_0(z_0) = \Big\{(\overline{x}^*, \overline{v}_1{}^*) : \overline{\varphi}_0(z) - \overline{\varphi}_0(z_0) \geq \ \langle\overline{x}^*, x - x^0\rangle + \langle\overline{v}_1{}^*, v_1 - v_1^0\rangle, \\ \forall z = (x, v_1) \in \mathbb{R}^{2n}, z_0 = (x^0, v_1^0)\Big\}\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\partial\varphi_0\Big(x^0, \frac{x^0 - v_1^0}{h}\Big) &= \Big\{(x^*, v_1{}^*) : \varphi_0\Big(x, \frac{x - v_1}{h}\Big) - \varphi_0\Big(x, \frac{x^0 - v_1^0}{h}\Big) \\ &\geq \Big\langle x^*, x - x^0\Big\rangle + \Big\langle v_1{}^*, \frac{x - v_1}{h} - \frac{x^0 - v_1^0}{h}\Big\rangle \ , \ \forall z \in \mathbb{R}^{2n}\Big\}\end{aligned}$$

which yield

$$\begin{aligned}\partial\varphi_0\Big(x^0, \frac{x^0 - v_1^0}{h}\Big) &= \Big\{(x^*, v_1{}^*) : \varphi_0\Big(x, \frac{x - v_1}{h}\Big) - \varphi_0\Big(x^0, \frac{x^0 - v_1^0}{h}\Big) \\ &\geq \Big\langle x^* + \frac{v_1{}^*}{h}, x - x^0\Big\rangle + \Big\langle -\frac{v_1^*}{h}, v_1 - v_1^0\Big\rangle \ , \ \forall (x, v_1) \in \mathbb{R}^{2n}\Big\}.\end{aligned} \tag{3.6}$$

Comparing (3.5) and (3.6) we derive that

$$\overline{x}^* = x^* + \frac{v_1{}^*}{h}, \quad \overline{v}_1{}^* = -\frac{v_1^*}{h},$$

whence

$$x^* = \overline{x}^* + \overline{v}_1{}^*, \quad v_1{}^* = -h\overline{v}_1{}^*$$

This means that $(\overline{x}^*, \overline{v}_1{}^*) \in \partial_z\overline{\varphi}_0(z_0)$ if and only if $(\overline{x}^* + \overline{v}_1{}^*, -h\overline{v}_1{}^*) \in \partial\varphi_0\Big(x^0, \dfrac{x^0 - v_1^0}{h}\Big)$. The proof of the theorem is ended. □

4 Optimization of Polyhedral Differential Inclusions

By passing to the formally limit in conditions of Theorem 3.1 as $h \to 0$, we have the adjoint polyhedral Euler-Lagrange inclusion with the transversality condition. It occurs that these conditions are sufficient for the optimality of $\tilde{x}(t),\ t \in [0,1]$ in the problem for polyhedral differential inclusions (P_C). We remind that our notation and terminology are generally consistent with those in Mordukhovich [33], Mahmudov [27] and Pshenichnyi [35] for first order differential inclusions. In what follows, we assume that $x^*(t),\ t \in [0,1]$ is absolutely a continuous function together with the first order derivatives for which $x^{*''}(\cdot) \in L_1^n$.

Theorem 4.1. Let F be a polyhedral multivalued mapping defined by (1.3) and $g(\cdot,t)$ and $\varphi_0(\cdot)$ be a polyhedral functions. Then, in order for the trajectory $\tilde{x}(t)$, $t \in [0,1]$,lying interior to $domF$ to be an optimal in Bolza problem with the second order polyhedral differential inclusion (P_C), it is sufficient that there exists an absolutely continuous function $x^*(t)$ satisfying the following polyhedral Euler-Lagrange inclusion and the transversality condition almost everywhere

(a) $\dfrac{d^2x^*(t)}{dt^2} \in P_0^*\lambda(t) - P_1^*\frac{d\lambda(t)}{dt} + \partial g(\tilde{x}(t),t)$ a.e. $t \in [0,1], x^*(t) = Q^*\lambda(t)\ \lambda(t) \geq 0,$

(b) $-\frac{dx^*(1)}{dt} - P_1^*\lambda(1) \in \partial\varphi_0(\tilde{x}(1)),\ x^*(1) = 0,$

(c) $\Big\langle P_0\tilde{x}(t) + P_1\frac{d\tilde{x}(t)}{dt} - Q\frac{d^2\tilde{x}(t)}{dt^2} - d, \lambda(t)\Big\rangle = 0$, a.e. $t \in [0,1]$.

Proof. By definition of subdifferential for all feasible solutions, we rewrite the Euler-Lagrange polyhedral inclusion (a) in the form

$$g(x(t),t) - g(\tilde{x}(t),t) \geq \Big\langle \frac{d^2x^*(t)}{dt^2} - P_0^*\lambda(t) + P_1^*\frac{d\lambda(t)}{dt}, x(t) - \tilde{x}(t)\Big\rangle. \tag{4.1}$$

In turn, it follows from the condition (b), that

$$\varphi_0(x(1)) - \varphi_0(\tilde{x}(1)) \geq -\Big\langle \frac{dx^*(1)}{dt} + P_1^*\lambda(1), x(1) - \tilde{x}(1)\Big\rangle. \tag{4.2}$$

Now we transform, the right hand side of the inequality (4.1) :

$$\Big\langle \frac{d^2x^*(t)}{dt^2} - P_0^*\lambda(t) + P_1^*\frac{d\lambda(t)}{dt}, x(t) - \tilde{x}(t)\Big\rangle = \Big\langle \frac{d^2x^*(t)}{dt^2}, x(t) - \tilde{x}(t)\Big\rangle$$
$$-\langle \lambda(t), P_0x(t) - P_0\tilde{x}(t)\rangle + \Big\langle \frac{d\lambda(t)}{dt}, P_1x(t) - P_1\tilde{x}(t)\Big\rangle. \tag{4.3}$$

Moreover, for all feasible solutions $x(\cdot)$, the inequality

$$P_0x(t) + P_1\frac{dx(t)}{dt} - Q\frac{d^2x(t)}{dt^2} \leq d$$

holds a.e. Then for $\lambda(t) \geq 0, t \in [0,1]$ we have

$$\Big\langle P_0x(t) + P_1\frac{dx(t)}{dt}, \lambda(t)\Big\rangle \ \leq\ \Big\langle Q\frac{d^2x(t)}{dt^2} + d, \lambda(t)\Big\rangle.$$

Using the third condition (c) of theorem, we can write

$$\Big\langle P_0\tilde{x}(t) + P_1\frac{d\tilde{x}(t)}{dt}, \lambda(t)\Big\rangle \ =\ \Big\langle Q\frac{d^2\tilde{x}(t)}{dt^2} + d, \lambda(t)\Big\rangle$$

and then by subtracting the last two relations, we derive that

$$\Big\langle P_0 x(t) - P_0\tilde{x}(t) + P_1\frac{dx(t)}{dt} - P_1\frac{d\tilde{x}(t)}{dt}, \lambda(t)\Big\rangle \le \Big\langle Q\frac{d^2x(t)}{dt^2} - Q\frac{d^2\tilde{x}(t)}{dt^2}, \lambda(t)\Big\rangle.$$

Recalling that $x^*(t) = Q^*\lambda(t)$, the last inequality can be rewritten as follows:

$$\Big\langle P_0 x(t) - P_0\tilde{x}(t) + P_1\frac{dx(t)}{dt} - P_1\frac{d\tilde{x}(t)}{dt}, \lambda(t)\Big\rangle \le \Big\langle \frac{d^2x(t)}{dt^2} - \frac{d^2\tilde{x}(t)}{dt^2}, x^*(t)\Big\rangle. \tag{4.4}$$

Then from (4.3) and (4.4), we conclude that

$$\Big\langle \frac{d^2x^*(t)}{dt^2} - P_0^*\lambda(t) + P_1^*\frac{d\lambda(t)}{dt}, x(t) - \tilde{x}(t)\Big\rangle \ge \Big\langle \frac{d^2x^*(t)}{dt^2}, x(t) - \tilde{x}(t)\Big\rangle$$
$$+\Big\langle P_1\frac{dx(t)}{dt} - P_1\frac{d\tilde{x}(t)}{dt}, \lambda(t)\Big\rangle - \Big\langle \frac{d^2x(t)}{dt^2} - \frac{d^2\tilde{x}(t)}{dt^2}, x^*(t)\Big\rangle + \Big\langle \frac{d\lambda(t)}{dt}, P_1x(t) - P_1\tilde{x}(t)\Big\rangle$$

or by rewriting it, we have

$$\Big\langle \frac{d^2x^*(t)}{dt^2} - P_0^*\lambda(t) + P_1^*\frac{d\lambda(t)}{dt}, x(t) - \tilde{x}(t)\Big\rangle \ge \Big\langle \frac{d^2x^*(t)}{dt^2}, x(t) - \tilde{x}(t)\Big\rangle$$
$$-\Big\langle \frac{d^2x(t)}{dt^2} - \frac{d^2\tilde{x}(t)}{dt^2}, x^*(t)\Big\rangle + \frac{d}{dt}\langle P_1x(t) - P_1\tilde{x}(t), \lambda(t)\rangle.$$

Then (4.1) can be rewritten as follows

$$g(x(t),t) - g(\tilde{x}(t),t) \ge \Big\langle \frac{d^2x^*(t)}{dt^2}, x(t) - \tilde{x}(t)\Big\rangle - \Big\langle \frac{d^2x(t)}{dt^2} - \frac{d^2\tilde{x}(t)}{dt^2}, x^*(t)\Big\rangle$$
$$+\frac{d}{dt}\langle P_1x(t) - P_1\tilde{x}(t), \lambda(t)\rangle. \tag{4.5}$$

Thus, integrating the inequality (4.5) over the interval $[0,1]$ and taking into account that $x(\cdot), \tilde{x}(\cdot)$ are feasible ($x(0) = \tilde{x}(0) = \alpha_0$), we obtain

$$\int_0^1 \big[g(x(t),t) - g(\tilde{x}(t),t)\big]dt \ge \int_0^1 \Big\langle \frac{d^2x^*(t)}{dt^2}, x(t) - \tilde{x}(t)\Big\rangle dt$$
$$-\int_0^1 \Big\langle \frac{d^2x(t)}{dt^2} - \frac{d^2\tilde{x}(t)}{dt^2}, x^*(t)\Big\rangle dt + \langle P_1x(1) - P_1\tilde{x}(1), \lambda(1)\rangle. \tag{4.6}$$

Furthermore, by summing the inequalities (4.2) and (4.6), we deduce that

$$\int_0^1 \big[g(x(t),t) - g(\tilde{x}(t),t)\big]dt + \varphi_0(x(1)) - \varphi_0(\tilde{x}(1)) \ge -\Big\langle \frac{dx^*(1)}{dt}, x(1) - \tilde{x}(1)\Big\rangle$$
$$+\int_0^1 \Big[\Big\langle \frac{d^2x^*(t)}{dt^2}, x(t) - \tilde{x}(t)\Big\rangle - \Big\langle \frac{d^2x(t)}{dt^2} - \frac{d^2\tilde{x}(t)}{dt^2}, x^*(t)\Big\rangle\Big]dt. \tag{4.7}$$

Now, since $x^*(t) = Q^*\lambda(t)$ and $x(t), t \in [0,1]$ is feasible $(x(0) = \tilde{x}(0) = \alpha_0), (x'(0) = \tilde{x}'(0) =$

α_1), the integral on the right hand side of (4.7) can be evaluated as follows

$$\begin{aligned}
&\int_0^1 \Big[\Big\langle \frac{d^2x^*(t)}{dt^2}, x(t)-\tilde{x}(t)\Big\rangle - \Big\langle \frac{d^2(x(t)-\tilde{x}(t))}{dt^2}, x^*(t)\Big\rangle\Big]dt \\
= &\int_0^1 \Big[\Big\langle \frac{d^2\lambda(t)}{dt^2}, Q(x(t)-\tilde{x}(t))\Big\rangle - \Big\langle Q\frac{d^2(x(t)-\tilde{x}(t))}{dt^2}, \lambda(t)\Big\rangle\Big]dt \\
= &\int_0^1 \Big[\frac{d}{dt}\Big\langle \frac{d\lambda(t)}{dt}, Q(x(t)-\tilde{x}(t))\Big\rangle - \frac{d}{dt}\Big\langle Q\frac{d(x(t)-\tilde{x}(t))}{dt}, \lambda(t)\Big\rangle\Big]dt \\
= &\Big\langle \frac{d\lambda(1)}{dt}, Q(x(1)-\tilde{x}(1))\Big\rangle - \Big\langle \frac{d\lambda(0)}{dt}, Q(x(0)-\tilde{x}(0))\Big\rangle \\
&+\Big\langle Q\frac{d(x(0)-\tilde{x}(0))}{dt}, \lambda(0)\Big\rangle - \Big\langle Q\frac{d(x(1)-\tilde{x}(1))}{dt}, \lambda(1)\Big\rangle \\
= &\Big\langle Q^*\frac{d\lambda(1)}{dt}, x(1)-\tilde{x}(1)\Big\rangle - \Big\langle \frac{d(x(1)-\tilde{x}(1))}{dt}, Q^*\lambda(1)\Big\rangle. \qquad (4.8)
\end{aligned}$$

Consequently, recalling that $\dfrac{dx^*(1)}{dt} = Q^*\dfrac{d\lambda(1)}{dt}$ and by condition (b) of theorem $x^*(1) = Q^*\lambda(1) = 0$, the relation (4.8) implies

$$\int_0^1 \Big[\Big\langle \frac{d^2x^*(t)}{dt^2}, x(t)-\tilde{x}(t)\Big\rangle - \Big\langle \frac{d^2(x(t)-\tilde{x}(t))}{dt^2}, x^*(t)\Big\rangle\Big]dt = \Big\langle \frac{dx^*(1)}{dt}, x(1)-\tilde{x}(1)\Big\rangle. \qquad (4.9)$$

Therefore, taking into account (4.9), from the inequality (4.7) we have

$$\int_0^1 g(x(t),t)dt + \varphi_0(x(1)) \geq \int_0^1 g(\tilde{x}(t),t)dt + \varphi_0(\tilde{x}(1)),$$

i.e. $J(x(\cdot)) - J(\tilde{x}(\cdot)) \geq 0$ for all feasible solutions $x(t)$ and so $\tilde{x}(t)$ is optimal. □

Remark 4.2. The result of Theorem 4.1 can be easily generalized to the polyhedral optimization problem (P_C) with the function $\overline{J}(x(\cdot)) = \int_0^1 g(x(t),t)dt + \varphi_0(x(1),x'(1))$, where $\varphi_0(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a polyhedral function. Indeed according to the equivalence conditions of Theorem 3.3, the transversality condition for problem (P_{DA}) , $(x_h^*(1-h) - P_1^*\lambda_h(1-2h), x_h^*(1)) \in \partial\overline{\varphi}_0(\tilde{x}_h(1-h), \tilde{x}_h(1))$ (see Remark 3.2) has a form:

$$\big(-\Delta_h x_h^*(1) - P_1^*\lambda_h(1-2h), x_h^*\big) \in \partial\varphi_0(\tilde{x}_h(1-h), \Delta_h\tilde{x}_h(1)\big).$$

Here as a result of formally limit as $h \to 0$, we have the transversality condition for the second order polyhedral optimization problem (P_C) with the objective function $\overline{J}(x(\cdot))$:

$$\Big(-\frac{dx^*(1)}{dt} - P_1^*\lambda(1), x^*(1)\Big) \in \partial\varphi_0(\tilde{x}(1), \tilde{x}'(1))$$

or equivalently

$$\begin{aligned}
\varphi_0\big(x(1),x'(1)\big) - \varphi_0\big(\tilde{x}(1),\tilde{x}'(1)\big) \geq &-\Big\langle \frac{dx^*(1)}{dt}, x(1)-\tilde{x}(1)\Big\rangle - \Big\langle P_1^*\lambda(1), x(1)-\tilde{x}(1)\Big\rangle \\
&+\Big\langle x^*(1), x'(1)-\tilde{x}'(1)\Big\rangle. \qquad (4.10)
\end{aligned}$$

To justify this, recall that by (4.8)

$$\int_0^1 \Big[\Big\langle \frac{d^2x^*(t)}{dt^2}, x(t)-\tilde{x}(t)\Big\rangle - \Big\langle \frac{d^2(x(t)-\tilde{x}(t))}{dt^2}, x^*(t)\Big\rangle\Big] dt$$
$$= \Big\langle \frac{dx^*(1)}{dt}, x(1)-\tilde{x}(1)\Big\rangle - \Big\langle x^*(1), x'(1)-\tilde{x}'(1)\Big\rangle, \tag{4.11}$$

where $x^*(1) = Q^*\lambda(1) \neq 0$. Then taking into account this relation, summing the inequalities (4.6) and (4.10), it can easily be seen that

$$\int_0^1 g(x(t),t)dt + \varphi_0(x(1),x'(1)) \geq \int_0^1 g(\tilde{x}(t),t)dt + \varphi_0(\tilde{x}(1),\tilde{x}'(1))$$

that is, $\overline{J}(x(t)) \geq \overline{J}(\tilde{x}(t)),\ t \in [0,1]$ for all feasible solutions $x(\cdot)$.

References

[1] A. Agrachev, Geometry of optimal control problems and Hamiltonian systems, in *Nonlinear and Optimal Control Theory*, Lect. Notes in Math. CIME, 1932, Springer Verlag, 2008, pp.1–59.

[2] N.U. Ahmed and X. Xiang, Necessary conditions of optimality for differential inclusionsin Banach spaces, *Nonlinear Anal.* 30 (1997) 5437–5445.

[3] J.P. Aubin and A. Cellina, *Differential Inclusions*, Springer-Verlag, Berlin, 1984

[4] J. P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, 1990.

[5] A. Auslender, J. Mechler, Second order viability problems for differential inclusions, *J. Math. Anal. Appl.* 181 (1994) 205–218.

[6] M. Benchohra and A. Ouahab, Initial boundary value problems for second order impulsive functional differential inclusions, *E. J. Qualitative Theory of Diff. Equat.* 3 (2003) 1–10.

[7] A. Cernea, On the existence of viable solutions for a class of second order differential inclusions, *Discuss. Math., Differ. Incl.* 22 (2002) 67–78.

[8] F.H. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley and Sons Inc. New York 1983.

[9] H. Covitz and S.B. Nadler Jr., Multivalued contraction mappings in generalized metric spaces, *Israel J. Math.* 8 (1970) 5–11.

[10] M.S. De Queiroz , et. al., *Optimal Control, Stabilization and Nomsmooth Analysis*, Springer Berlin / Heidelberg, 2004.

[11] M.D.R. De Pinco, M.M.A. Ferreira and F. A.C.C. Fontes, An Euler-Lagrange inclusion for optimal control problems with state constraints, *J. Dynam. Control Systems* 8 (2002) 23–45.

[12] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, North-Holland Pub. 1999.

[13] M. Fukushima and L. Qi (eds), *Reformulation: Nonsmooth, Piecewise Smooth, Semismooth and Smoothing Methods (Applied Optimization)*, Springer, New York, 2010.

[14] G. Gabor and A. Grudzka, Structure of the solution set to impulsive functional differential inclusions on the half-line, *Nonlinear diff. Equat. Appl. NoDEA* 19 (2012) 609–627.

[15] T. Haddad and M. Yarou, Existence of solutions for nonconvex second-order differential inclusions in the infinite dimensional space, *Electronic J. Diff. Equat.* 2006 (2006) 1–8.

[16] P.B. Kurzhanski and T.F. Filippova, *Differential Inclusions with State Constraints* The Singular Perturbation Method, Trudy Matem. Inst. Ross. Akad. Nauk, 1995, (in Russian).

[17] V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.

[18] I. Lasiecka and R. Triggiani, *Control theory for PDEs: Vol.2*, Cambridge University Press, Cambridge, 2000.

[19] V. Lupulescu, A viability result for nonconvex second order differential inclusions, *Electronic J. Diff. Equat.* 76 (2002) 1–12.

[20] E.N. Mahmudov, Approximation and optimization of higher order discrete and differential inclusions, *Nonlinear Diff. Equat. and Appl.* NoDEA, DOI:10.1007/s00030-013-0234-1

[21] E.N. Mahmudov, *Single Variable Differential and Integral Calculus*, Springer, 2013.

[22] E. N. Mahmudov, Duality in the problems of optimal control for systems described by convex differential inclusions with delay, *Probl. Control Inf. Theory* 16 (1987) 411–422.

[23] E.N. Mahmudov, On duality in problems of optimal control described by convex differential inclusions of Goursat-Darboux type, *J. Math. Anal. Appl.* 307 (2005) 628–640.

[24] E.N. Mahmudov and B.N. Pshenichnyi , Necessary condition of extremum and evasion problem, Preprint, Institute Cybernetcis of Ukraine SSR, Kiev, 1978, pp. 3–22.

[25] E.N. Mahmudov, The optimality principle for discrete and first order differential inclusions, *J. Math. Anal. Appl.* 308 (2005) 605-619.

[26] E.N. Mahmudov, Necessary and sufficient conditions for discrete and differential inclusions of elliptic type, *J. Math. Anal. Appl.* 323 (2006) 768–789.

[27] E.N. Mahmudov, *Approximation and Optimization of Discrete and Differential Inclusions*, Elsevier, Amsterdam, 2011.

[28] E.N. Mahmudov and B.N. Pshenichnyi, Polyhedral mappings, *Izv. Academy of Sci. of Azerbaijan* No.2, (1979) 10–14.

[29] L. Marco and J.A. Murillo, Lyapunov functions for second-order differential inclusions: A viability approach, *J. Math. Anal. Appl.* 262 (2001) 339–354.

[30] V.L. Makarov and A.M. Rubinov, *The Mathematical Theory of Economic Dynamics and Equilibrium*, Nauka, Moscow, 1977, English transl., Springer-Verlag, Berlin.

[31] B.S. Mordukhovich, Discrete approximations and refined Euler-Lagrange conditions for nonconvex differential inclusions, *SIAM J. Control and Optim.* 33 (1995) 882–915.

[32] B.S. Mordukhovich, D. Wang and L. Wang, Optimization of delaydifferential inclusions in infinite dimensions, *Pacific Journ. of Optim.* 6 (2010) 353–374.

[33] B.S. Mordukhovich, *Variational Analysis and Generalized Differentiation, I: Basic Theory; II: Applications*, Grundlehren Series (Fundamental Principles of Mathematical Sciences), Vol. 330 and 331, Springer, Berlin, 2006.

[34] P.M. Pardalos, T.M. Rassias and A.A. Khan, *Nonlinear Analysis and Variational Problems*, Springer, Berlin, 2010.

[35] B.N. Pshenichnyi, *Convex Analysis and Extremal Problems*, Nauka, Moscow, 1980.

[36] R.T. Rockafellar, Variational analysis and its applications, *Set-Valued Analysis*12, (2004) 1–4.

[37] J.D.L. Rowland and R.B. Vinter, Dynamic optimization problems with free time and active state constraints, *SIAM J. Control Optimi.* 31 (1993), 677–691.

[38] H.D. Tuan, Contingent and intermediate tangent cones in hyperbolic differential inclusions and necessary optimality conditions, *J. Math. Anal. Appl.* 185 (1994) 86–106.

[39] R. Vinter and H.H. Zheng, Necessary conditions for free end-time measurably time dependent optimal control problems with state constraints, Set-Valued Anal. 8 (2000) 11–29.

[40] L. Wang, Discrete approximations to optimization of neutral functional differential inclusions, *J. Math. Anal. Appl.* 309 (2005) 474–488.

[41] H.H. Zheng, Second-order necessary conditions for differential inclusion problems, *Applied Mathem. and Optimi.* 30 (1994) 1-14.

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