Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 11, No. 3, pp. 495–510, July 2015

# OPTIMALITY CONDITIONS AND DUALITY FOR APPROXIMATE SOLUTIONS OF VECTOR OPTIMIZATION PROBLEMS*

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**Abstract:** In this paper, we study approximate solutions of vector optimization problems. We introduce the concept of cone convex functions with respect to (in short w.r.t.) a mapping. Under this kind of cone convexity assumption, we obtain the Karush-Kuhn-Tucker type necessary and sufficient optimality conditions for quasiminimal solutions w.r.t. a mapping of vector optimization problems. We formulate approximate Mond-Weir type dual problem and establish the duality results. We also consider vector optimization problems with perturbed cone constraint. The necessary and sufficient optimality conditions and duality results for quasi-solutions w.r.t. a mapping of vector optimization problems with perturbed cone constraint are established.

**Key words:** *vector optimization, approximate solutions, optimality conditions, approximate duality, quasi-minimal solutions*

**Mathematics Subject Classification:** *90C29, 90C30, 90C46*

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## 1 Introduction

In recent years, more and more attention has been focused on the study of approximate solutions for vector optimization problems. There are two main reasons. One is computational challenge. Many iterative algorithms or heuristic algorithms will provide approximate solutions rather than exact solutions. Another is the efficient solutions set may be empty, but under very weak requirements, approximate efficient solutions set can be nonempty. Moreover, sometimes we do not need to find an exact optimal solution even if it does exist due to the fact that it is often very expensive to find an exact solution. The first and most fashionable concept of approximation solution was introduced by Kutateladze[12], where a fixed tolerance error has been taken, independent of the choice of the decision variable. This seems rather a stringent condition. Hence, Loridan[15] introduced the concept of approximate quasi efficient solutions, by making the error dependent on the decision variable, one can provide more flexibility to the idea of approximate efficiency, which extend the concept of $\varepsilon$-quasiminimal of scalar optimization problems introduced by himself[16].

In the very recent years, there have been an increasing interest in studying approximate quasi efficiency solutions of vector optimization problems. Dutta and Vetrivel[2] introduced

*This work is supported by the National Natural Science Foundation of China (11201379, 11271391) and the Fundamental Research Funds for the Central Universities (JBK140926,JBK120504,JBK130401).

the concept of weak quasi efficient solution in finite dimensional space, and gave the optimality conditions for this type of solutions under convexity assumptions. In infinite dimensional space, Liu and Lee[13] established Lagrange multiplier rules for (weak) approximate quasi efficient solution of nonsmooth vector optimization. Gutiérrez, Loépez and Novo[6] introduced the concept of (weak) generalized $\varepsilon$-quasi-solution of vector optimization problems, that extends the approximate efficiency notions of the literature[15, 2, 7], and developed optimality conditions for a particular case of these kinds of weak generalized $\varepsilon$-quasi-solution in nonsmooth convex problems. A new kind of generalized approximate quasi efficient solution for vector optimization was introduced by Gutiérrez, Jiménez and Novo[8], named minimal quasi-solution w.r.t. a mapping, they characterized these solutions through scalarization and Lagrange multiplier rules. It is well known that convexity and generalized convexity play a crucial role in establishing optimality conditions for optimization problems. In order to deal with approximation quasi solutions of scalar optimization problems or vector optimization by the scalarization methods, $\varepsilon$-convexity[11], approximate convexity[17], approximate quasiconvexity[5] and approximate pseudoconvexity[5] were introduced one after another. In this paper, in order to deal with quasi-solutions w.r.t. a mapping, we introduce a new class of generalized convexity for vector-valued functions, named $C$-convexity w.r.t. a mapping. Under the convexity assumption, we establish Karush-Kuhn-Tucker (in short KKT) type necessary and sufficient optimality conditions for minimal quasi-solution w.r.t. a mapping.

Duality theory provides a rich framework for the development of solution methods for optimization problems. In scalar optimization problems, Scovel, Hush and Steinwart[18] provided the Lagrangian duality theory for approximate solutions. Son, Strodiot and Nguyen[19] formulated a Wolfe-type dual problem, presented $\varepsilon$-duality theorems. As for the vector case, there are a few articles to deal with approximate quasi efficient solutions of vector optimization problems by considering dual problems, for example, please see[4, 14, 20] and references therein. In this paper, based on the obtained KKT type necessary and sufficient optimality conditions for quasiminimal solutions w.r.t. a mapping of vector optimization, we formulate Mond-Weir type dual problems and study the duality theory for this kind of quasiminimal solutions.

The paper is structured as follows. In Section 2 some well-known definitions and results used in the sequel are recalled. In Section 3 the concept of generalized cone convex functions w.r.t. a mapping is defined and several properties are given. In Section 4 we consider the vector optimization problems with cone constraint and perturbation cone constraint, respectively. For them, the KKT type necessary and sufficient optimality conditions for quasiminimal solutions w.r.t. a mapping are obtained. In Section 5 we formulate two approximate Mond-Weir type dual problems and establish duality results for vector optimization problems with cone convexity w.r.t. a mapping.

## 2 Preliminaries

Throughout this paper, let $X, Y, Z$ be real Banach spaces. Let $X^*$,$Y^*$,$Z^*$ be the dual spaces of $X, Y, Z$, respectively. The apex of all cones considered in this paper will be at the origin. For a cone $C \subset Y$, we set

$$\begin{aligned} C^* &= \{y^* \in Y^* : \langle y^*, y\rangle \geq 0, \forall y \in C\}, \\ C^{*i} &= \{y^* \in Y^* : \langle y^*, y\rangle > 0, \forall y \in C\backslash\{0\}\}. \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual product between $Y$ and $Y^*$. Let $\Gamma$ be a nonempty subset of $X$ and $f : \Gamma \to Y$ a vector-valued mapping. Let $C$ be a nontrivial pointed convex cone in $Y$, which induces a partial order relation $\leqslant_C$ on $Y$ as follows: $\forall y_1, y_2 \in Y$,

$$y_1 \leqslant_C y_2 \Longleftrightarrow y_2 - y_1 \in C,$$

$$y_1 \leqslant_{C\backslash\{0\}} y_2 \Longleftrightarrow y_2 - y_1 \in C\backslash\{0\},$$

$$y_1 \leqslant_{intC} y_2 \Longleftrightarrow y_2 - y_1 \in intC.$$

**Definition 2.1.** Let $\bar{x} \in X$, $f : X \to Y$ is said to be G$\hat{a}teaux$ differentiable at $\bar{x}$ if for any $v \in X$,

$$\lim_{t \to 0^+} \frac{f(\bar{x} + tv) - f(\bar{x})}{t}$$

exists and the limit is denoted by $f^{'}(\bar{x}; v)$.

**Definition 2.2.** A subset $\Gamma$ of $X$ is said to be convex if, for any $x, y \in \Gamma$ and $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y \in \Gamma$.

**Definition 2.3** ([1])**.** The function $f$ is said to be $C$-convex on convex set $\Gamma$, if for any $x, y \in \Gamma$ and $t \in [0, 1]$,

$$tf(x) + (1 - t)f(y) - f(tx + (1 - t)y) \in C.$$

**Definition 2.4** ([9, 10])**.** The function $f : \Gamma \to Y$ is said to be $C$-convexlike on $\Gamma$, if, for any $x, y \in \Gamma$ and $t \in [0, 1]$, there exists $z \in \Gamma$ such that

$$tf(x) + (1 - t)f(y) - f(z) \in C.$$

Obviously, every $C$-convex function is $C$-convexlike. The following alternative theorem is a special case of the convexlike results given by Jeyakumar [10].

**Lemma 2.5.** Let $\Gamma$ be a convex subset of $X$ and let $f : \Gamma \to Y$ be a $C$-convex function. Then, exactly one of the following statements holds:
(i) $\exists x_0 \in \Gamma$, such that $f(x_0) \in -intC$;
(ii) $\exists y^* \in C^*\backslash\{0\}$, such that $\langle y^*, f(x)\rangle \geq 0, \forall x \in \Gamma$.

By the definition of $C$-convexity, it is easily to obtain the following lemma.

**Lemma 2.6.** Let $C$ be closed, the function $f : \Gamma \to Y$ $C$-convex on convex set $\Gamma$. Suppose that $f$ is G$\hat{a}teaux$ differentiable at $\bar{x} \in \Gamma$. Then, for any $x \in \Gamma$,

$$f(x) - f(\bar{x}) - f'(\bar{x}; x - \bar{x}) \in C.$$

## 3 Generalized Cone Convex Functions

In this section, we will introduce a new class of vector-valued cone convex functions, named $C$-convex functions with respect to (for short w.r.t.) a mapping $\psi : \Gamma \times \Gamma \to R_+$, as a generalization of cone convex functions.

In the following, we always assume $e \in C\backslash\{0\}$, $\psi_z := \psi(\cdot, z)$, $\psi_z e := \psi(\cdot, z)e$, where $z \in \Gamma$.

**Definition 3.1.** Let $\Gamma \subset X$ be a convex set and $z \in \Gamma$. The function $f : \Gamma \to Y$ is said to be $C$-convex w.r.t. a mapping $\psi_z e$ if the function $f(\cdot) + \psi(\cdot, z)e$ is $C$-convex.

**Remark 3.2.** 1) If $\psi_z$ is convex, then $\psi_z e$ is C-convex. Let $h(x) = f(x) + \psi(x,z)e$. Then $f(x) = h(x) - \psi(x,z)e$. If $f$ is $C$-convex w.r.t. $\psi_z e$ and $\psi_z$ is convex, then $f$ is a vector-valued $DC$ function[3].
2) If $f$ is $C$-convex and $\psi_z$ is convex, then $f$ is $C$-convex w.r.t. $\psi_z e$. Specially, if taking $\psi_z = \|x-z\|$, then $\psi_z$ is convex. Let $f$ be $C$-convex, then $x \mapsto f(x) + \psi(x,z)e = f(x) + \|x-z\|e$ is $C$-convex.
3)When $f$ is $C$-convex w.r.t. $\psi_z e$, $f$ is not necessarily $C$-convex. Please see the following examples.

**Example 3.3.** Let $f : R \to R^2$ and $\psi : R \times R \to R_+$ be defined by, respectively,

$$f(x) = \begin{cases} (1,1), & x = 0, \\ (0,0), & x \neq 0. \end{cases}$$

$$\psi(x,z) = \begin{cases} 0, & x = 0, z \in R, \\ 1, & x \neq 0, z \in R. \end{cases}$$

Let $C = R^2_+$, $e = (1,1)$. Then, $f(x) + \psi(x,z)e \equiv (1,1)$. Hence, $f$ is $R^2_+$-convex w.r.t. $\varphi_z e$. But $f$ is not $C$-convex, because there exist $x_1 = 1$, $x_2 = -1$, $\lambda = \frac{1}{2}$ such that

$$f(\lambda x_1 + (1-\lambda)x_2) = f(0) = (1,1) \nleq_{R^2_+} (0,0) = \lambda f(x_1) + (1-\lambda)f(x_2).$$

In Example 3.3, $\psi_z$ is convex. Next, we give another example with nonconvex $\psi_z$.

**Example 3.4.** Let $f : [-1,1] \to R^2$ and $\psi : [-1,1] \times [-1,1] \to R_+$ be defined by, respectively,

$$f(x) = (x^3, x^2),$$
$$\psi(x,z) = -x^3 + x^2 + z.$$

Let $C = R^2_+$, $e = (1,0)$. Then, $f(x) + \psi(x,z)e \equiv (x^2+z, x^2)$. Hence, $f$ is $R^2_+$-convex w.r.t. $\psi_z e$. But $f$ is not $C$-convex, because there exist $x_1 = -1$, $x_2 = \frac{1}{2}$, $\lambda = \frac{1}{2}$ such that

$$f(\lambda x_1 + (1-\lambda)x_2) = f(-\frac{1}{4}) = (-\frac{1}{64}, \frac{1}{16}) \nleq_{R^2_+} (-\frac{7}{16}, \frac{5}{8}) = \lambda f(x_1) + (1-\lambda)f(x_2).$$

In the following, we always suppose that $\Gamma \subset X$ is a convex set, $C \subset Y$ is a pointed closed convex cone with nonempty interior. Now we give some properties of $C$-convex functions w.r.t. a mapping $\psi_z e$.

**Theorem 3.5.** Let $\bar{x} \in \Gamma$. If $f : \Gamma \to Y$ is $C$-convex w.r.t. $\psi_{\bar{x}} e$ on $\Gamma$ and G$\hat{a}teaux$ differentiable at $\bar{x}$, $\psi$ is G$\hat{a}teaux$ differentiable at $(\bar{x},\bar{x})$, then

$$f(x) - f(\bar{x}) + (\psi(x,\bar{x}) - \psi(\bar{x},\bar{x}))e - f'(\bar{x}; x-\bar{x}) - \psi'((\bar{x},\bar{x}); (x-\bar{x},0))e \in C.$$

*Proof.* Since the function $f$ is $C$-convex w.r.t. $\psi_{\bar{x}} e$ on $\Gamma$, for any $x \in \Gamma$ and for any $t \in (0,1)$, we have

$$t(f(x) + \psi(x,\bar{x})e) + (1-t)(f(\bar{x}) + \psi(\bar{x},\bar{x})e) - f(tx + (1-t)\bar{x}) - \psi(tx + (1-t)\bar{x}, \bar{x})e \in C.$$

Hence

$$f(x) - f(\bar{x}) - \frac{f(\bar{x} + t(x-\bar{x})) - f(\bar{x})}{t} + (\psi(x,\bar{x}) - \psi(\bar{x},\bar{x}))e - \frac{\psi(\bar{x} + t(x-\bar{x}), \bar{x}) - \psi(\bar{x},\bar{x})}{t} e \in C.$$

Because $C$ is closed, letting $t \to 0^+$, we get

$$f(x) - f(\bar{x}) + (\psi(x,\bar{x}) - \psi(\bar{x},\bar{x}))e - f'(\bar{x}; x-\bar{x}) - \psi'((\bar{x},\bar{x}); (x-\bar{x},0))e \in C.$$

□

**Corollary 3.6.** Let $\bar{x} \in \Gamma$. And let $f : \Gamma \to Y$ be $C$-convex w.r.t. $\psi_{\bar{x}}e$ on $\Gamma$ and G*âteaux* differentiable at $\bar{x}$. In addition, let $\psi(x, y) = 0$ if and only if $x = y$. Then

$$f(x) - f(\bar{x}) + \psi(x, \bar{x})e - f'(\bar{x}; x - \bar{x}) \in C.$$

Considering the following vector optimization problem:

$$(VOP) \qquad \begin{array}{ll} \min & f(x) \\ \text{s. t.} & x \in \Gamma. \end{array}$$

where $\Gamma \subset X$, $f : \Gamma \to Y$, $X, Y$ are two real Banach spaces, $C \subset Y$ is a pointed closed convex cones with nonempty interior and induces the ordering relationship in $Y$.

Firstly, we recall some concepts of solution for (VOP).

**Definition 3.7.** A point $\bar{x} \in \Gamma$ is said to be an efficient solution of (VOP) if

$$(f(\Gamma) - f(\bar{x})) \cap (-C \backslash \{0\}) = \emptyset,$$

and a weak efficient solution of (VOP) if

$$(f(\Gamma) - f(\bar{x})) \cap (-intC) = \emptyset.$$

**Definition 3.8.** ([13]) Let $\varepsilon \in R$, $\varepsilon \geq 0$ and $e \in C\backslash\{0\}$. $\bar{x} \in S$ is said to be an $\varepsilon e$-quasi efficient solution(resp. weak $\varepsilon e$-quasi efficient solution) of (VOP) if

$$f(x) - f(\bar{x}) + \varepsilon\|x - \bar{x}\|e \notin -C\backslash\{0\}, \ \forall x \in \Gamma$$

$$(resp. \quad f(x) - f(\bar{x}) + \varepsilon\|x - \bar{x}\|e \notin -intC, \ \forall x \in \Gamma).$$

When $\varepsilon = 0$, $\varepsilon e$-quasi efficient solution(resp. weak $\varepsilon e$-quasi efficient solution) of (VOP) will coincide with efficient solution (resp. weak efficient solution) of (VOP).

Recently, Gutiérrez, Jiménez and Novo[8] introduced a concept of minimal quasi-solution for (VOP) with set-valued perturbations and characterized this solutions through scalarization and Lagrange multiplier rules.

**Definition 3.9.** ([8]) Let $\varphi : \Gamma \times \Gamma \to R_+$ be such that $\varphi(x, y) = 0$ if and only if $x = y$. $\bar{x} \in \Gamma$ is said to be a minimal quasi-solution of (VOP) w.r.t. $\varphi$ if

$$f(x) - f(\bar{x}) - \varphi(x, \bar{x})q \notin -C\backslash\{0\}, \ \forall x \in \Gamma, \quad \forall q \in C\backslash\{0\},$$

and a weak minimal quasi-solution of (VOP) w.r.t. $\varphi$ if

$$f(x) - f(\bar{x}) - \varphi(x, \bar{x})q \notin -intC, \ \forall x \in \Gamma, \quad \forall q \in C\backslash\{0\}.$$

As a particular case of Definition 3.9, in this paper we discuss the following quasiminimal solutions of (VOP) w.r.t. a mapping $\varphi e$, where $\varphi e := \varphi(\cdot, \cdot)e$, and establish optimality conditions and duality results for it.

**Definition 3.10.** Let $e \in C\backslash\{0\}$ and $\varphi : \Gamma \times \Gamma \to R_+$ be such that $\varphi(x, y) = 0$ if and only if $x = y$. $\bar{x} \in \Gamma$ is said to be a quasiminimal solution of (VOP) w.r.t. $\varphi e$ if

$$f(x) - f(\bar{x}) + \varphi(x, \bar{x})e \notin -C\backslash\{0\}, \ \forall x \in \Gamma,$$

and a weak quasiminimal solution of (VOP) w.r.t. $\varphi e$ if

$$f(x) - f(\bar{x}) + \varphi(x, \bar{x})e \notin -intC, \ \forall x \in \Gamma.$$

**Remark 3.11.** If we take $\varphi(x,y)=\varepsilon\|x-y\|$, $\varepsilon\geq 0$, then (weak) quasiminimal solution of (VOP) w.r.t. $\varphi e$ coincide with (weak) $\varepsilon e$-quasi efficient solution.

**Definition 3.12.** Let $e\in C\backslash\{0\}$ and $\varphi:\Gamma\times\Gamma\to R_+$ be such that $\varphi(x,y)=0$ if and only if $x=y$. $\bar{x}\in\Gamma$ is said to be a locally quasiminimal solution of (VOP) w.r.t. $\varphi e$ if there is a neighborhood $U$ of $\bar{x}$ such that

$$f(x)-f(\bar{x})+\varphi(x,\bar{x})e\notin -C\backslash\{0\},\ \forall x\in\Gamma\cap U,$$

and a locally weak quasiminimal solution of (VOP) w.r.t. $\varphi e$ if

$$f(x)-f(\bar{x})+\varphi(x,\bar{x})e\notin -intC,\ \forall x\in\Gamma\cap U.$$

**Definition 3.13.** Let $e\in C\backslash\{0\}$ and $\varphi:\Gamma\times\Gamma\to R_+$ be such that $\varphi(x,y)=0$ if and only if $x=y$. $\bar{x}\in\Gamma$ is said to be a quasimaximal solution of $\{maxf(x):x\in\Gamma\}$ w.r.t. $\varphi e$ if

$$f(x)-f(\bar{x})-\varphi(\bar{x},x)e\notin C\backslash\{0\},\ \forall x\in\Gamma,$$

and a weak quasimaximal solution of $\{maxf(x):x\in\Gamma\}$ w.r.t. $\varphi e$ if

$$f(x)-f(\bar{x})-\varphi(\bar{x},x)e\notin intC,\ \forall x\in\Gamma.$$

**Theorem 3.14.** Let $\Gamma$ be a convex set, $\bar{x}\in\Gamma$ a locally quasiminimal solution of (VOP) w.r.t. $\varphi e$, and let $f:\Gamma\to Y$ be $C$-convex w.r.t. $\varphi_{\bar{x}}e$ on $\Gamma$. Then $\bar{x}$ is a globally quasiminimal solution of (VOP) w.r.t. $\varphi e$.

*Proof.* Let $\bar{x}\in\Gamma$ be a locally quasiminimal solution of (VOP) w.r.t. $\varphi e$. Then, there is a neighborhood $U$ of $\bar{x}$ such that

$$f(x)-f(\bar{x})+\varphi(x,\bar{x})e\notin -C\backslash\{0\},\quad \forall x\in\Gamma\cap U. \tag{3.1}$$

Suppose that $\bar{x}$ is not a globally quasiminimal solution of (VOP) w.r.t. $\varphi e$. Then, there is a $y\in\Gamma$ such that

$$f(y)-f(\bar{x})+\varphi(y,\bar{x})e\in -C\backslash\{0\}. \tag{3.2}$$

Because the function $f$ is $C$-convex w.r.t. $\varphi_{\bar{x}}e$ and $\varphi(\bar{x},\bar{x})=0$, by Definition 3.1, we get

$$f(\bar{x}+t(y-\bar{x}))-f(\bar{x})+\varphi(\bar{x}+t(y-\bar{x}),\bar{x})e-t(f(y)-f(\bar{x})+\varphi(y,\bar{x})e)\in -C,\ \forall t\in[0,1]. \tag{3.3}$$

(3.2) and (3.3) imply that, for any $t\in(0,1)$,

$$f(\bar{x}+t(y-\bar{x}))-f(\bar{x})+\varphi(\bar{x}+t(y-\bar{x}),\bar{x})e\in -C-C\backslash\{0\}\subset -C\backslash\{0\}. \tag{3.4}$$

For small enough $t$, $\bar{x}+t(y-\bar{x})\in U$, then (3.4) contradicts (3.1). □

Similar to the proof of Theorem 3.14, we get the following theorem about weak quasiminimal solution of (VOP) w.r.t. $\varphi e$.

**Theorem 3.15.** Let $\Gamma$ be a convex set, $\bar{x}\in\Gamma$ a locally weak quasiminimal solution of (VOP) w.r.t. $\varphi e$, and let $f:\Gamma\to Y$ be $C$-convex w.r.t. $\varphi_{\bar{x}}e$ on $\Gamma$. Then $\bar{x}$ is a globally weak quasiminimal solution of (VOP) w.r.t. $\varphi e$.

## 4 Optimality Conditions

Consider the following vector optimization problem with cone constraint:

$$\text{(VP)} \qquad \begin{array}{ll} \min & f(x) \\ \text{s. t.} & g(x) \in -D, \\ & x \in \Gamma, \end{array}$$

where $\Gamma \subset X$, $f : \Gamma \to Y$, $g : \Gamma \to Z$. $X, Y, Z$ are three real Banach spaces, $C \subset Y$, $D \subset Z$ are pointed closed convex cones with nonempty interior and induce the ordering relationships in $Y$ and $Z$, respectively. We denote $S := \{x \in \Gamma : g(x) \in -D\}$, i.e. $S$ is the feasible set of (VP).

Now we give the necessary and sufficient optimality conditions for quasiminimal solution of (VP) w.r.t. $\varphi e$. In the following, we assume that $\Gamma$ is a convex subset of $X$.

**Theorem 4.1.** Let $\bar{x} \in \Gamma$ be a weak quasiminimal solution of (VP) w.r.t. $\varphi e$. Suppose that $f : \Gamma \to Y$ is $C$-convex w.r.t. $\varphi_{\bar{x}} e$, $g : \Gamma \to Z$ is $D$-convex, and $f, g$ are G$\hat{a}teaux$ differentiable at $\bar{x}$, $\varphi$ is G$\hat{a}teaux$ differentiable at $(\bar{x}, \bar{x})$. Then, there exists $(\lambda, \mu) \in C^* \times D^*$ with $(\lambda, \mu) \neq (0, 0)$ such that

$$\langle \lambda, f'(\bar{x}; x - \bar{x}) \rangle + \langle \mu, g'(\bar{x}; x - \bar{x}) \rangle + \varphi'((\bar{x}, \bar{x}); (x - \bar{x}, 0)) \langle \lambda, e \rangle \geq 0, \quad \forall\, x \in \Gamma, \tag{4.1}$$

$$\langle \mu, g(\bar{x}) \rangle = 0. \tag{4.2}$$

*Proof.* Since $\bar{x} \in \Gamma$ is a weak quasiminimal solution of (VP) w.r.t. $\varphi e$, we have

$$f(x) - f(\bar{x}) + \varphi(x, \bar{x}) e \notin -intC, \quad \forall x \in S := \{x \in \Gamma : g(x) \in -D\}.$$

Hence, there exists no $x \in \Gamma$ such that

$$(f(x) - f(\bar{x}) + \varphi(x, \bar{x}) e, \ g(x)) \in -int(C \times D).$$

By Lemma 2.5, there exists $(\lambda, \mu) \in C^* \times D^*$ with $(\lambda, \mu) \neq (0, 0)$ such that

$$\langle \lambda, f(x) - f(\bar{x}) + \varphi(x, \bar{x}) e \rangle + \langle \mu, g(x) \rangle \geqslant 0, \ \forall x \in \Gamma. \tag{4.3}$$

Taking $x = \bar{x}$ in (4.3), we get

$$\langle \mu, g(\bar{x}) \rangle \geqslant 0. \tag{4.4}$$

On the other hand, $g(\bar{x}) \in -D$, $\mu \in D^*$, then

$$\langle \mu, g(\bar{x}) \rangle \leqslant 0. \tag{4.5}$$

The inequalities (4.4) and (4.5) imply $\langle \mu, g(\bar{x}) \rangle = 0$, i.e. (4.2) holds. Since $\Gamma$ is a convex set, for any $x \in \Gamma$, for any $t \in (0, 1)$, $\bar{x} + t(x - \bar{x}) \in \Gamma$. By (4.3) and $\langle \mu, g(\bar{x}) \rangle = 0$, we obtain

$$\langle \lambda, f(\bar{x} + t(x - \bar{x})) - f(\bar{x}) \rangle + \langle \mu, g(\bar{x} + t(x - \bar{x})) - g(\bar{x}) \rangle + \varphi(\bar{x} + t(x - \bar{x}), \bar{x}) \langle \lambda, e \rangle \geqslant 0.$$

Consequently,

$$\left\langle \lambda, \frac{f(\bar{x} + t(x - \bar{x})) - f(\bar{x})}{t} \right\rangle + \left\langle \mu, \frac{g(\bar{x} + t(x - \bar{x})) - g(\bar{x})}{t} \right\rangle + \frac{\varphi(\bar{x} + t(x - \bar{x}), \bar{x}))}{t} \langle \lambda, e \rangle \geqslant 0.$$

Letting $t \to 0^+$, we get

$$\langle \lambda, f'(\bar{x}; x - \bar{x}) \rangle + \langle \mu, g'(\bar{x}; x - \bar{x}) \rangle + \varphi'((\bar{x}, \bar{x}); (x - \bar{x}, 0)) \langle \lambda, e \rangle \geq 0, \quad \forall\, x \in \Gamma,$$

i.e. (4.1) holds. ∎

**Corollary 4.2.** If the conditions of Theorem 4.1 are satisfied, in addition, $\varphi_{\bar{x}}$ is convex, then there exists $(\lambda, \mu) \in C^* \times D^*$ with $(\lambda, \mu) \neq (0, 0)$ such that

$$\langle \lambda, f'(\bar{x}; x - \bar{x}) \rangle + \langle \mu, g'(\bar{x}; x - \bar{x}) \rangle + \varphi(x, \bar{x}) \langle \lambda, e \rangle \geq 0, \quad \forall\, x \in \Gamma,$$

$$\langle \mu, g(\bar{x}) \rangle = 0.$$

**Theorem 4.3.** Let $f$, $g$ and $\varphi$ be as in Theorem 4.1. Suppose that $\bar{x} \in \Gamma$ be a weak quasiminimal solution of (VP) w.r.t. $\varphi e$ and there is $\hat{x} \in \Gamma$ such that $g(\hat{x}) \in -intD$. Then there exist $\lambda \in C^* \backslash \{0\}$, $\mu \in D^*$ such that

$$\langle \lambda, f'(\bar{x}; x - \bar{x}) \rangle + \langle \mu, g'(\bar{x}; x - \bar{x}) \rangle + \varphi'((\bar{x}, \bar{x}); (x - \bar{x}, 0)) \langle \lambda, e \rangle \geq 0, \quad \forall\, x \in \Gamma, \tag{4.6}$$

$$\langle \mu, g(\bar{x}) \rangle = 0. \tag{4.7}$$

*Proof.* Let $\bar{x}$ be a weak quasiminimal solution of (VP) w.r.t. $\varphi e$. It follows from Theorem 4.1 that there exists $(\lambda, \mu) \in C^* \times D^*$ with $(\lambda, \mu) \neq (0, 0)$ such that (4.6) and (4.7) hold. Next we prove $\lambda \neq 0$. Assume, on the contrary, $\lambda = 0$, then $\mu \neq 0$. Thus, letting $x = \hat{x}$ in (4.6), we get

$$\langle \mu, g'(\bar{x}; \hat{x} - \bar{x}) \rangle \geqslant 0. \tag{4.8}$$

Since $g$ is $D$-convex, by Lemma 2.6, we have

$$g(\hat{x}) - g(\bar{x}) - g'(\bar{x}; \hat{x} - \bar{x}) \in D.$$

Hence,

$$\langle \mu, g(\hat{x}) - g(\bar{x}) \rangle \geqslant \langle \mu, g'(\bar{x}; \hat{x} - \bar{x}) \rangle. \tag{4.9}$$

By (4.7), (4.8) and (4.9), we obtain

$$\langle \mu, g(\hat{x}) \rangle \geqslant 0. \tag{4.10}$$

On the other hand, $g(\hat{x}) \in -intD$ and $0 \neq \mu \in D^*$ imply $\langle \mu, g(\hat{x}) \rangle < 0$, which contradicts (4.10). So, $\lambda \neq 0$. □

**Theorem 4.4.** Let $\bar{x} \in \Gamma$, and let $f : \Gamma \to Y$ be $C$-convex w.r.t. $\varphi_{\bar{x}} e$, $g : \Gamma \to Y$ $D$-convex on convex set $\Gamma$, $f$ and $g$ G$\hat{a}teaux$ differentiable at $\bar{x}$, $\varphi$ is G$\hat{a}teaux$ differentiable at $(\bar{x}, \bar{x})$. If there exist $\lambda \in C^* \backslash \{0\}$, $\mu \in D^*$, $\bar{x} \in S$ such that (4.6) and (4.7) hold, then $\bar{x}$ is a weak quasiminimal solution of (VP) w.r.t. $\varphi e$.

*Proof.* Since $f$ is $C$-convex w.r.t. $\varphi_{\bar{x}} e$ and $g : \Gamma \to Y$ is $D$-convex, by Theorem 3.5, we get

$$f(x) - f(\bar{x}) + \varphi(x, \bar{x}) e - f'(\bar{x}; x - \bar{x}) - \varphi'((\bar{x}, \bar{x}); (x - \bar{x}, 0)) e \in C,$$

$$g(x) - g(\bar{x}) - g'(\bar{x}; x - \bar{x}) \in D.$$

Hence, for $\lambda \in C^*$, $\mu \in D^*$,

$$\langle \lambda, f(x) - f(\bar{x}) + \varphi(x, \bar{x}) e \rangle - \langle \lambda, f'(\bar{x}; x - \bar{x}) \rangle - \varphi'((\bar{x}, \bar{x}); (x - \bar{x}, 0)) \langle \lambda, e \rangle \geq 0. \tag{4.11}$$

$$\langle \mu, g(x) - g(\bar{x}) \rangle - \langle \mu, g'(\bar{x}; x - \bar{x}) \rangle \geq 0. \tag{4.12}$$

Adding (4.11) and (4.12), by (4.6), (4.7), we obtain

$$\langle \lambda, f(x) - f(\bar{x}) + \varphi(x, \bar{x}) e \rangle + \langle \mu, g(x) \rangle \geq 0.$$

For any feasible solution $x \in S$ of (VP), $\langle \mu, g(x) \rangle \leq 0$. Hence,

$$\langle \lambda, f(x) - f(\bar{x}) + \varphi(x, \bar{x})e \rangle \geq 0.$$

Since $0 \neq \lambda \in C^*$, we have

$$f(x) - f(\bar{x}) + \varphi(x, \bar{x})e \notin -intC, \quad \forall x \in S,$$

which shows that $\bar{x}$ is a weak quasiminimal solution of (VP) w.r.t. $\varphi e$. □

**Theorem 4.5.** Let $f, g$ and $\varphi$ be as in Theorem 4.4. If there exist $\lambda \in C^{*i}$, $\mu \in D^*$, $\bar{x} \in S$ such that (4.6) and (4.7) hold, then $\bar{x}$ is a quasiminimal solution of (VP) w.r.t. $\varphi e$.

The proof of Theorem 4.5 is similar to Theorem 4.4, here it is omitted.

Next, we consider the optimization problem with perturbation cone constraint:

$$(\mathrm{VP})_{\bar{x}} \qquad \begin{array}{ll} \min & f(x) \\ \text{s. t.} & g(x) + \omega(x, \bar{x})p \in -D, \\ & x \in \Gamma, \end{array}$$

where $\Gamma$ is a convex subset of $X$, $\bar{x} \in \Gamma$, $f : \Gamma \to Y$, $g : \Gamma \to Z$, $\omega : \Gamma \times \Gamma \to R_+$ and $\omega(x, y) = 0$ if and only if $x = y$, $p \in D \backslash \{0\}$. We denote $S_{\bar{x}} := \{x \in \Gamma : g(x) + \omega(x, \bar{x})p \in -D\}$, i.e. $S_{\bar{x}}$ is the feasible set of $(\mathrm{VP})_{\bar{x}}$. Let $\omega_{\bar{x}}p := \omega(\cdot, \bar{x})p$.

**Theorem 4.6.** Let $\bar{x} \in \Gamma$ be a weak quasiminimal solution of $(VP)_{\bar{x}}$ w.r.t. $\varphi e$. Suppose that $f : \Gamma \to Y$ is $C$-convex w.r.t. $\varphi_{\bar{x}}e$, $g : \Gamma \to Z$ is $D$-convex w.r.t. $\omega_{\bar{x}}p$, $f, g$ are G*âteaux* differentiable at $\bar{x}$ and $\varphi, \omega$ are G*âteaux* differentiable at $(\bar{x}, \bar{x})$. Then, there exists $(\lambda, \mu) \in C^* \times D^*$ with $(\lambda, \mu) \neq (0, 0)$ such that

$$\langle \lambda, f'(\bar{x}; x - \bar{x}) \rangle + \langle \mu, g'(\bar{x}; x - \bar{x}) \rangle + \varphi'((\bar{x}, \bar{x}); (x - \bar{x}, 0)) \langle \lambda, e \rangle + \omega'((\bar{x}, \bar{x}); (x - \bar{x}, 0)) \langle \mu, p \rangle \geq 0, \quad \forall\, x \in \Gamma, \tag{4.13}$$

$$\langle \mu, g(\bar{x}) \rangle = 0. \tag{4.14}$$

*Proof.* Suppose that $\bar{x} \in \Gamma$ is a weak quasiminimal solution of $(VP)_{\bar{x}}$ w.r.t. $\varphi e$, we have

$$f(x) - f(\bar{x}) + \varphi(x, \bar{x})e \notin -intC, \quad \forall x \in S_{\bar{x}} := \{x \in \Gamma : g(x) + \omega(x, \bar{x})p \in -D\}.$$

Hence, there exists no $x \in \Gamma$ such that

$$(f(x) - f(\bar{x}) + \varphi(x, \bar{x})e,\ g(x) + \omega(x, \bar{x})p) \in -int(C \times D).$$

By the cone convexity of $f$ and $g$ w.r.t. a mapping, and Lemma 2.5, there exists $(\lambda, \mu) \in C^* \times D^*$ with $(\lambda, \mu) \neq (0, 0)$ such that

$$\langle \lambda, f(x) - f(\bar{x}) + \varphi(x, \bar{x})e \rangle + \langle \mu, g(x) + \omega(x, \bar{x})p \rangle \geqslant 0,\ \forall x \in \Gamma. \tag{4.15}$$

Taking $x = \bar{x}$ in (4.15), we get

$$\langle \mu, g(\bar{x}) \rangle \geqslant 0. \tag{4.16}$$

On the other hand, since $g(\bar{x}) + \omega(x, \bar{x})p \in -D$, $\mu \in D^*$, $p \in D \backslash \{0\}$, we have

$$\langle \mu, g(\bar{x}) \rangle \leqslant 0. \tag{4.17}$$

The inequalities (4.16) and (4.17) imply $\langle \mu, g(\bar{x}) \rangle = 0$, i.e. (4.14) holds. Since $\Gamma$ is a convex set, then, for any $x \in \Gamma$, for any $t \in (0,1)$, $\bar{x} + t(x - \bar{x}) \in \Gamma$. By (4.15) and $\langle \mu, g(\bar{x}) \rangle = 0$, we obtain

$$\begin{aligned}&\langle \lambda, f(\bar{x} + t(x - \bar{x})) - f(\bar{x}) \rangle + \langle \mu, g(\bar{x} + t(x - \bar{x})) - g(\bar{x}) \rangle \\ &+\varphi(\bar{x} + t(x - \bar{x}), \bar{x}) \langle \lambda, e \rangle + \omega(\bar{x} + t(x - \bar{x}), \bar{x}) \langle \mu, p \rangle \geqslant 0.\end{aligned}$$

Consequently,

$$\begin{aligned}&\Big\langle \lambda, \frac{f(\bar{x} + t(x - \bar{x})) - f(\bar{x})}{t} \Big\rangle + \Big\langle \mu, \frac{g(\bar{x} + t(x - \bar{x})) - g(\bar{x})}{t} \Big\rangle \\ &+ \frac{\varphi(\bar{x} + t(x - \bar{x}), \bar{x})}{t} \langle \lambda, e \rangle + \frac{\omega(\bar{x} + t(x - \bar{x}), \bar{x})}{t} \langle \mu, p \rangle \geqslant 0.\end{aligned}$$

Letting $t \to 0^+$, we get

$$\langle \lambda, f'(\bar{x}; x - \bar{x}) \rangle + \langle \mu, g'(\bar{x}; x - \bar{x}) \rangle + \varphi'((\bar{x}, \bar{x}); (x - \bar{x}, 0)) \langle \lambda, e \rangle + \omega'((\bar{x}, \bar{x}); (x - \bar{x}, 0)) \langle \mu, p \rangle \geq 0, \quad \forall\, x \in \Gamma,$$

i.e. (4.13) holds. □

**Corollary 4.7.** If the conditions of Theorem 4.6 are satisfied, in addition, $\varphi_{\bar{x}}$ and $\omega_{\bar{x}}$ are convex, then there exists $(\lambda, \mu) \in C^* \times D^*$ with $(\lambda, \mu) \neq (0, 0)$ such that

$$\langle \lambda, f'(\bar{x}; x - \bar{x}) \rangle + \langle \mu, g'(\bar{x}; x - \bar{x}) \rangle + \varphi(x, \bar{x}) \langle \lambda, e \rangle + \omega(x, \bar{x}) \langle \mu, p \rangle \geq 0, \quad \forall\, x \in \Gamma,$$

$$\langle \mu, g(\bar{x}) \rangle = 0.$$

**Theorem 4.8.** Let $f, g$ and $\varphi, \omega$ be as in Theorem 4.6. Suppose that $\bar{x} \in \Gamma$ is a weak quasiminimal solution of $(VP)_{\bar{x}}$ w.r.t. $\varphi e$ and there is $\hat{x} \in \Gamma$ such that $g(\hat{x}) + \omega(\hat{x}, \bar{x}) p \in -intD$. Then there exist $\lambda \in C^* \backslash \{0\}$, $\mu \in D^*$ such that

$$\langle \lambda, f'(\bar{x}; x - \bar{x}) \rangle + \langle \mu, g'(\bar{x}; x - \bar{x}) \rangle + \varphi'((\bar{x}, \bar{x}); (x - \bar{x}, 0)) \langle \lambda, e \rangle + \omega'((\bar{x}, \bar{x}); (x - \bar{x}, 0)) \langle \mu, p \rangle \geq 0, \quad \forall\, x \in \Gamma, \tag{4.18}$$

$$\langle \mu, g(\bar{x}) \rangle = 0. \tag{4.19}$$

*Proof.* Let $\bar{x}$ be a weak quasiminimal solution of $(\mathrm{VP})_{\bar{x}}$ w.r.t. $\varphi e$. It follows from Theorem 4.6 that there exists $(\lambda, \mu) \in C^* \times D^*$ with $(\lambda, \mu) \neq (0, 0)$ such that (4.18) and (4.19) hold. Next we prove $\lambda \neq 0$. Assume, on the contrary, $\lambda = 0$, then $\mu \neq 0$. Thus, letting $x = \hat{x}$ in (4.18), we get

$$\langle \mu, g'(\bar{x}; \hat{x} - \bar{x}) \rangle + \omega'((\bar{x}, \bar{x}); (\hat{x} - \bar{x}, 0)) \langle \mu, p \rangle \geqslant 0. \tag{4.20}$$

Since $g$ is a $D$-convex w.r.t. $\omega_{\bar{x}} p$, by Theorem 3.5, we get

$$g(\hat{x}) - g(\bar{x}) + \omega(\hat{x}, \bar{x}) p - g'(\bar{x}; \hat{x} - \bar{x}) - \omega'((\bar{x}, \bar{x}); (\hat{x} - \bar{x}, 0)) p \in D.$$

Hence,

$$\langle \mu, g(\hat{x}) - g(\bar{x}) + \omega(\hat{x}, \bar{x}) p \rangle \geqslant \langle \mu, g'(\bar{x}; \hat{x} - \bar{x}) \rangle + \omega'((\bar{x}, \bar{x}); (\hat{x} - \bar{x}, 0)) \langle \mu, p \rangle. \tag{4.21}$$

By (4.19), (4.20) and (4.21), we obtain

$$\langle \mu, g(\hat{x}) + \omega(\hat{x}, \bar{x}) p \rangle \geqslant 0. \tag{4.22}$$

On the other hand, $g(\hat{x}) + \omega(\hat{x}, \bar{x}) p \in -intD$ and $0 \neq \mu \in D^*$ imply $\langle \mu, g(\hat{x}) + \omega(\hat{x}, \bar{x}) p \rangle < 0$, which contradicts (4.22). So, $\lambda \neq 0$. □

**Theorem 4.9.** Let $\bar{x} \in \Gamma$, $f$ and $g$ be *Gâteaux* differentiable at $\bar{x}$, $\varphi, \omega$ are *Gâteaux* differentiable at $(\bar{x}, \bar{x})$, and let $f : \Gamma \to Y$ be $C$-convex w.r.t. $\varphi_{\bar{x}} e$, $g : \Gamma \to Y$ $D$-convex w.r.t. $\omega_{\bar{x}} p$ on $\Gamma$. If there exist $\lambda \in C^* \backslash \{0\}$, $\mu \in D^*$ such that (4.18) and (4.19) hold, then $\bar{x}$ is a weak quasiminimal solution of $(VP)_{\bar{x}}$ w.r.t. $\varphi e$.

*Proof.* Since $f$ is $C$-convex w.r.t. $\varphi_{\bar{x}} e$ and $g : \Gamma \to Y$ is $D$-convex w.r.t. $\omega_{\bar{x}} p$, by Theorem 3.5, we get

$$f(x) - f(\bar{x}) + \varphi(x, \bar{x})e - f'(\bar{x}; x - \bar{x}) - \varphi'((\bar{x}, \bar{x}); (x - \bar{x}, 0))e \in C.$$

$$g(x) - g(\bar{x}) + \omega(x, \bar{x})p - g'(\bar{x}; x - \bar{x}) - \omega'((\bar{x}, \bar{x}); (x - \bar{x}, 0))p \in D.$$

Hence, for $\lambda \in C^*$, $\mu \in D^*$,

$$\langle \lambda, f(x) - f(\bar{x}) + \varphi(x, \bar{x})e \rangle - \langle \lambda, f'(\bar{x}; x - \bar{x}) \rangle - \varphi'((\bar{x}, \bar{x}); (x - \bar{x}, 0))\langle \lambda, e \rangle \geq 0. \quad (4.23)$$

$$\langle \mu, g(x) - g(\bar{x}) + \omega(x, \bar{x})p \rangle - \langle \mu, g'(\bar{x}; x - \bar{x}) \rangle - \omega'((\bar{x}, \bar{x}); (x - \bar{x}, 0))\langle \lambda, p \rangle \geq 0. \quad (4.24)$$

Adding (4.23) and (4.24), by (4.18) and (4.19), we obtain

$$\langle \lambda, f(x) - f(\bar{x}) + \varphi(x, \bar{x})e \rangle + \langle \mu, g(x) + \omega(x, \bar{x})p \rangle \geq 0.$$

For any feasible solution $x \in S_{\bar{x}}$, $\langle \mu, g(x) + \omega(x, \bar{x})p \rangle \leq 0$. Hence

$$\langle \lambda, f(x) - f(\bar{x}) + \varphi(x, \bar{x})e \rangle \geq 0.$$

Since $0 \neq \lambda \in C^*$, then

$$f(x) - f(\bar{x}) + \varphi(x, \bar{x})e \notin -intC, \quad \forall x \in S_{\bar{x}},$$

which shows that $\bar{x}$ is a weak quasiminimal solution of $(VP)_{\bar{x}}$ w.r.t. $\varphi e$. □

**Theorem 4.10.** Let $f, g, \varphi, \omega$ be as in Theorem 4.9. If there exist $\lambda \in C^{*i}$, $\mu \in D^*$, $\bar{x} \in S$ such that (4.18) and (4.19) hold, then $\bar{x}$ is a quasiminimal solution of $(VP)_{\bar{x}}$ w.r.t. $\varphi e$.

The proof of Theorem 4.10 is similar to Theorem 4.9, here it is omitted.
The following example illustrate Theorem 4.3 and 4.4.

**Example 4.11.** Consider a (VP) as follows:

$$\begin{array}{ll} \min & f(x) = (x, x^2) \\ \text{s. t.} & g(x) = x^2 - \frac{1}{2} \leq 0, \\ & x \in [-1, 1], \end{array}$$

We take $C = R^2_+$, $D = R_+$, $\varphi(x, \bar{x}) = |x - \bar{x}|$, $e = (1, 0)$.

We can verify that $\bar{x} = 0 \in [-1, 1]$ is a weak quasiminimal solution of (VP) w.r.t. $\varphi e$. $f$ is $R^2_+$-convex w.r.t. $\varphi_{\bar{x}} e$, $g$ is convex, $f$ and $g$ are *Gâteaux* differentiable at $\bar{x} = 0$, $\nabla f(0) = (1, 0)^T$, $g'(0) = 0$, $\varphi$ is *Gâteaux* differentiable at $(0, 0)$, for any $x \in [-1, 1]$, $\varphi'((0, 0); (x, 0)) = |x|$, there exists $\hat{x} = \frac{1}{4}$ such that $g(\hat{x}) + \omega(\hat{x}, \bar{x})p = (\frac{1}{4})^2 - \frac{1}{2} + |x| = -\frac{3}{16} < 0$. Now, there exist $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_i > 0$, $i = 1, 2$, $\mu = 0$ such that for any $x \in [-1, 1]$,

$$\langle \lambda, f'(\bar{x}; x - \bar{x}) \rangle + \langle \mu, g'(\bar{x}; x - \bar{x}) \rangle + \varphi'((\bar{x}, \bar{x}); (x - \bar{x}, 0))\langle \lambda, e \rangle = \lambda_1 x + \lambda_1 |x| \geq 0.$$

$$\langle \mu, g(\bar{x}) \rangle = 0.$$

The following example illustrate Theorem 4.8 and 4.9.

**Example 4.12.** Consider a $(\mathrm{VP})_{\bar{x}}$ as follows:

$$\begin{array}{ll} \min & f(x) \\ \text{s. t.} & g(x)+\omega(x,\bar{x})p \leq 0, \\ & x \in [-1,1], \end{array}$$

where $f(x)=(x,x^2)$, $g(x)=x^2-\frac{1}{2}$, $\omega(x,\bar{x})=|x-\bar{x}|$, $p=1$. We take $C=R^2_+$, $D=R_+$, $\varphi(x,\bar{x})=|x-\bar{x}|$, $e=(1,0)$.

We can verify that $\bar{x}=0\in[-1,1]$ is a weak quasiminimal solution of $(VP)_{\bar{x}}$ w.r.t. $\varphi e$. $f$ is $R^2_+$-convex w.r.t. $\varphi_{\bar{x}}e$, $g$ is convex w.r.t. $\omega_{\bar{x}}p$, $f$ and $g$ are Gâteaux differentiable at $\bar{x}=0$, $\nabla f(0)=(1,0)^T$, $g'(0)=0$, $\varphi,\omega$ are Gâteaux differentiable at $(0,0)$, for any $x\in[-1,1]$, $\varphi'((0,0);(x,0))=|x|$, $\omega'((0,0);(x,0))=|x|$, there exists $\hat{x}=\frac{1}{4}$ such that $g(\hat{x})+\omega(\hat{x},\bar{x})p=(\frac{1}{4})^2-\frac{1}{2}+|x|=-\frac{3}{16}<0$. Now, there exist $\lambda=(\lambda_1,\lambda_2)$ with $\lambda_i>0,\ i=1,2,\ \mu=0$ such that for any $x\in[-1,1]$,

$$\langle\lambda,f'(\bar{x};x-\bar{x})\rangle+\langle\mu,g'(\bar{x};x-\bar{x})\rangle+\varphi'((\bar{x},\bar{x});(x-\bar{x},0))\langle\lambda,e\rangle+\omega'((\bar{x},\bar{x});(x-\bar{x},0))\langle\mu,p\rangle$$
$$=\lambda_1 x+\lambda_1|x|\geq 0.$$
$$\langle\mu,g(\bar{x})\rangle=0.$$

## 5 Approximate Duality

Consider the following approximate Mond-Weir type dual problem:

$$(\mathrm{AVD})\quad \begin{array}{ll} \max & f(y) \\ \text{s. t.} & \langle\lambda,f'(y;x-y)\rangle+\langle\mu,g'(y;x-y)\rangle+\varphi'((y,y);(x-y,0))\langle\lambda,e\rangle\geq 0,\quad \forall\, x\in\Gamma, \\ & \langle\mu,g(y)\rangle\geq 0, \\ & y\in\Gamma,\ 0\neq\lambda\in C^*,\ \mu\in D^*, \end{array}$$

where $e\in C\backslash\{0\}$, $\Gamma\subset X$, $f:\Gamma\to Y$, $g:\Gamma\to Z$, $\varphi:X\times X\to R_+$ and $\varphi(x,y)=0$ if and only if $x=y$.

**Theorem 5.1** (Approximate Weak Duality)**.** Let $x\in\Gamma$ be a feasible solution of (VP) and $(y,\lambda,\mu)$ be a feasible solution of (AVD). Let $f:\Gamma\to Y$ be $C$-convex w.r.t. $\varphi_y e$ and $g:\Gamma\to Z$ $D$-convex on convex set $\Gamma$, $f,g$ *Gâteaux* differentiable at $\bar{x}$ and $\varphi$ *Gâteaux* differentiable at $(y,y)$. Then

$$f(x)-f(y)+\varphi(x,y)e\notin -intC.$$

*Proof.* Since $f$ is $C$-convex w.r.t. $\varphi_y e$ and $g$ is $D$-convex, by Theorem 3.5 and Lemma **??**, for the dual feasible solutions $x\in\Gamma$ and $(y,\lambda,\mu)$, we get

$$f(x)-f(y)+\varphi(x,y)e-f'(y;x-y)-\varphi'((y,y);(x-y,0))e\in C,$$

$$g(x)-g(y)-g'(y;x-y)\in D.$$

As $\lambda\in C^*$, $\mu\in D^*$,

$$\langle\lambda,f(x)-f(y)+\varphi(x,y)e\rangle-\langle\lambda,f'(y;x-y)\rangle-\varphi'((y,y);(x-y,0))\langle\lambda,e\rangle\geq 0,\tag{5.1}$$

$$\langle\mu,g(x)-g(y)\rangle-\langle\mu,g'(y;x-y)\rangle\geq 0.\tag{5.2}$$

Adding (5.1) and (5.2), by feasibility of $(y,\lambda,\mu)$ for (AVD), we obtain

$$\langle\lambda,f(x)-f(y)+\varphi(x,y)e\rangle+\langle\mu,g(x)\rangle\geq 0.$$

According to feasibility of $x$, we get $\langle \mu, g(x) \rangle \leq 0$. Hence,

$$\langle \lambda, f(x) - f(y) + \varphi(x,y)e \rangle \geq 0.$$

Because $\lambda \in C^*$ and $\lambda \neq 0$, we get $f(x) - f(y) + \varphi(x,y)e \notin -intC$. The proof is completed. □

**Theorem 5.2** (Approximate Strong Duality). Let $\bar{x} \in \Gamma$ be a weak quasiminimal solution of (VP) w.r.t. $\varphi e$. Suppose that $f : \Gamma \to Y$ is $C$-convex w.r.t. $\varphi_{\bar{x}} e$, $g : \Gamma \to Z$ is $D$-convex on $\Gamma$, $f, g$ are Gâteaux differentiable at $\bar{x}$ and $\varphi$ is Gâteaux differentiable at $(\bar{x}, \bar{x})$. If there is some $\hat{x} \in \Gamma$, such that $g(\hat{x}) \in -intD$, then there exist $0 \neq \bar{\lambda} \in C^*$ and $\bar{\mu} \in D^*$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a weak quasimaximal solution of (AVD) w.r.t. $\varphi e$.

*Proof.* Since the conditions of Theorem 4.3 are satisfied, there exist $0 \neq \bar{\lambda} \in C^*$ and $\bar{\mu} \in D^*$ such that (4.6) and (4.7) hold. Thus $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is dual feasible. If $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is not a weak quasimaximal solution of (AVD) w.r.t. $\varphi e$, then there is a feasible solution $(y, \lambda, \mu)$ of (AVD), such that

$$f(y) - f(\bar{x}) - \varphi(\bar{x}, y)e \in intC,$$

which contradicts the Approximate Weak Duality. □

**Theorem 5.3** (Approximate Converse Duality). Let $(\bar{y}, \bar{\lambda}, \bar{\mu})$ is a weak quasimaximal solution of (AVD) w.r.t. $\varphi e$. Suppose that $f : \Gamma \to Y$ is $C$-convex w.r.t. $\varphi_{\bar{y}} e$, $g : \Gamma \to Z$ is $D$-convex on $\Gamma$, $f, g$ are Gâteaux differentiable at $\bar{x}$ and $\varphi$ is Gâteaux differentiable at $(\bar{y}, \bar{y})$. If there is some $\hat{x} \in \Gamma$, such that $g(\hat{x}) \in -intD$, then $\bar{y} \in \Gamma$ is a weak quasiminimal solution of (VP) w.r.t. $\varphi e$.

*Proof.* On the contrary, assuming that $\bar{y} \in \Gamma$ is not a weak quasiminimal solution of (VP) w.r.t. $\varphi e$, then there is a feasible solution $x \in S$ such that

$$f(x) - f(\bar{y}) + \varphi(x, \bar{y})e \in -intC.$$

Because $0 \neq \bar{\lambda} \in C^*$, we get

$$\langle \bar{\lambda}, f(x) - f(\bar{y}) + \varphi(x, \bar{y})e \rangle < 0. \tag{5.3}$$

According to the convexity of $g$ and $f$ w.r.t. $\varphi_{\bar{y}} e$ and the feasibility of $(\bar{y}, \bar{\lambda}, \bar{\mu})$ for (AVD), similar to the proof of Theorem 5.1, we obtain

$$\langle \bar{\lambda}, f(x) - f(\bar{y}) + \varphi(x, \bar{y})e \rangle \geq 0.$$

which contradicts (5.3). □

Consider optimization problems $(\text{VP})_y$ with perturbation cone constraint, we formulate the following approximate Mond-Weir type dual problem:

$$(\text{AVD})_c \quad \begin{array}{ll} \max & f(y) \\ \text{s. t.} & \langle \lambda, f'(y; x-y) \rangle + \langle \mu, g'(y; x-y) \rangle \\ & +\varphi'((y,y); (x-y, 0))\langle \lambda, e \rangle + \omega'((y,y); (x-y,0))\langle \mu, p \rangle \geq 0, \quad \forall\, x \in \Gamma, \\ & \langle \mu, g(y) \rangle \geq 0, \\ & y \in \Gamma,\ 0 \neq \lambda \in C^*,\ \mu \in D^*, \end{array}$$

where $e \in C \backslash \{0\}$, $p \in D \backslash \{0\}$, $\Gamma \subset X$, $f : \Gamma \to Y$, $g : \Gamma \to Z$, $\varphi : \Gamma \times \Gamma \to R_+$ and $\varphi(x, y) = 0 \Leftrightarrow x = y$, $\omega : \Gamma \times \Gamma \to R_+$ and $\omega(x, y) = 0 \Leftrightarrow x = y$.

**Theorem 5.4** (Approximate Weak Duality)**.** Let $x \in \Gamma$ be a feasible solution of $(VP)_y$ and $(y, \lambda, \mu)$ be a feasible solution of $(AVD)_c$. Let $f : \Gamma \to Y$ be $C$-convex w.r.t. $\varphi_y e$, $g : \Gamma \to Z$ $D$-convex w.r.t. $\omega_y p$ on convex set $\Gamma$, $f, g$ G$\hat{a}teaux$ differentiable at $y$ and $\varphi, \omega$ G$\hat{a}teaux$ differentiable at $(y, y)$. Then

$$f(x) - f(y) + \varphi(x, y)e \notin -intC.$$

*Proof.* Since $f$ is $C$-convex w.r.t. $\varphi_y e$ and $g$ is $D$-convex w.r.t. $\omega_y p$, by Theorem 3.5, for the dual feasible solutions $x \in \Gamma$ and $(y, \lambda, \mu)$, we get

$$f(x) - f(y) + \varphi(x, y)e - f'(y; x - y) - \varphi'((y, y); (x - y, 0))e \in C,$$

$$g(x) - g(y) + \omega(x, y)p - g'(y; x - y) - \omega'((y, y); (x - y, 0))p \in D.$$

As $\lambda \in C^*$, $\mu \in D^*$,

$$\langle \lambda, f(x) - f(y) + \varphi(x, y)e\rangle - \langle \lambda, f'(y; x - y)\rangle - \varphi'((y, y); (x - y, 0))\langle \lambda, e\rangle \geq 0, \tag{5.4}$$

$$\langle \mu, g(x) - g(y) + \omega(x, y)p\rangle - \langle \mu, g'(y; x - y)\rangle - \omega'((y, y); (x - y, 0))\langle \mu, p\rangle \geq 0. \tag{5.5}$$

Adding (5.4) and (5.5), by feasibility of $(y, \lambda, \mu)$ for $(\text{AVD})_c$, we obtain

$$\langle \lambda, f(x) - f(y) + \varphi(x, y)e\rangle + \langle \mu, g(x) + \omega(x, y)p\rangle \geq 0.$$

According to feasibility of $x$ for $(\text{VP})_y$, we get $\langle \mu, g(x) + \omega(x, y)p\rangle \leq 0$. Hence,

$$\langle \lambda, f(x) - f(y) + \varphi(x, y)e\rangle \geq 0.$$

Because $\lambda \in C^*$ and $\lambda \neq 0$, we get $f(x) - f(y) + \varphi(x, y)e \notin -intC$. The proof is complete. □

**Theorem 5.5** (Approximate Strong Duality)**.** Let $\bar{x} \in \Gamma$ be a weak quasiminimal solution of $(VP)_{\bar{x}}$ w.r.t. $\varphi e$. Suppose that $f : \Gamma \to Y$ is $C$-convex w.r.t. $\varphi_{\bar{x}} e$, $g : \Gamma \to Z$ is $D$-convex w.r.t. $\omega_{\bar{x}} p$ on $\Gamma$, $f, g$ are G$\hat{a}teaux$ differentiable at $\bar{x}$ and $\varphi, \omega$ are G$\hat{a}teaux$ differentiable at $(\bar{x}, \bar{x})$. If there is some $\hat{x} \in \Gamma$, such that $g(\hat{x}) + \omega(\hat{x}, \bar{x}) \in -intD$, then there exist $0 \neq \bar{\lambda} \in C^*$ and $\bar{\mu} \in D^*$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a weak quasimaximal solution of $(AVD)_c$ w.r.t. $\varphi e$.

*Proof.* Since the conditions of Theorem 4.8 are satisfied, there exist $0 \neq \bar{\lambda} \in C^*$ and $\bar{\mu} \in D^*$ such that (4.18) and (4.19) hold. Thus $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is dual feasible. If $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is not a weak quasimaximal solution of $(AVD)_c$ w.r.t. $\varphi e$, then there is a feasible solution $(y, \lambda, \mu)$ of $(AVD)_c$, such that

$$f(y) - f(\bar{x}) - \varphi(\bar{x}, y)e \in intC,$$

which contradicts the Approximate Weak Duality (Theorem 5.4). □

**Theorem 5.6** (Approximate Converse Duality)**.** Let $(\bar{y}, \bar{\lambda}, \bar{\mu})$ is a weak quasimaximal solution of $(AVD)_c$ w.r.t. $\varphi e$. Suppose that $f : \Gamma \to Y$ is $C$-convex w.r.t. $\varphi_{\bar{y}} e$, $g : \Gamma \to Z$ is $D$-convex w.r.t. $\omega_{\bar{y}} p$ on $\Gamma$, $f, g$ are G$\hat{a}teaux$ differentiable at $\bar{y}$ and $\varphi, \omega$ are G$\hat{a}teaux$ differentiable at $(\bar{y}, \bar{y})$. If there is some $\hat{x} \in \Gamma$, such that $g(\hat{x}) + \omega(\hat{x}, \bar{y}) \in -intD$, then $\bar{y} \in \Gamma$ be a weak quasiminimal solution of $(VP)_{\bar{y}}$ w.r.t. $\varphi e$.

*Proof.* On the contrary, assuming that $\bar{y} \in \Gamma$ is not a weak quasiminimal solution of $(\text{VP})_{\bar{y}}$ w.r.t. $\varphi e$, then there is a feasible solution $x \in S_{\bar{y}}$ such that

$$f(x) - f(\bar{y}) + \varphi(x, \bar{y})e \in -intC.$$

Because $0 \neq \bar{\lambda} \in C^*$, $\bar{\mu} \in D^*$ and $x \in S_{\bar{y}}$, we get

$$\langle \bar{\lambda}, f(x) - f(\bar{y}) + \varphi(x, \bar{y})e \rangle + \langle \bar{\mu}, g(x) + \omega(x, \bar{y})p \rangle < 0. \tag{5.6}$$

According to the convexity of $f$ w.r.t. $\varphi_{\bar{y}}e$ and $g$ w.r.t. $\omega_{\bar{y}}p$ and the feasibility of $(\bar{y}, \bar{\lambda}, \bar{\mu})$ for $(\text{AVD})_c$, similar to the proof of Theorem 5.4, we obtain

$$\langle \bar{\lambda}, f(x) - f(\bar{y}) + \varphi(x, \bar{y})e \rangle + \langle \bar{\mu}, g(x) + \omega(x, \bar{y})p \rangle \geq 0.$$

which contradicts (5.6). □

## Acknowledgments

We thank the anonymous referees for their helpful comments and suggestions.

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*Manuscript received 20 February 2014*
*revised 10 June 2014*
*accepted for publication 14 July 2014*



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