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STRONG KUHN-TUCKER OPTIMALITY IN NONSMOOTH MULTIOBJECTIVE OPTIMIZATION PROBLEMS*

Ke Quan Zhao

Abstract: In this paper, a nonsmooth multiobjective optimization problem with inequality constraints is considered. Two types of regularity conditions are proposed in terms of Clarke derivative and relations with some other regularity conditions are investigated. A strong Kuhn-Tucker necessary optimality condition is derived under one regularity condition and some examples are given to illustrate the main results.

Key words: *nonsmooth multiobjective optimization, regularity conditions, strong Kuhn-Tucker conditions, Geoffrion properly efficient solution*

Mathematics Subject Classification: *90C29, 90C46*

1 Introduction

Multiobjective optimization theory has been playing an important role in many fields such as economics and engineering design (see [10, 13, 18]). We say that strong Kuhn-Tucker conditions hold when all the multipliers corresponding to the objective functions are positive. In general, it needs some conditions to ensure strong Kuhn-Tucker necessary optimality conditions. These conditions are called regularity conditions (RC) when they have to be fulfilled by both the objective and the constraints. Regularity conditions are essential in deriving the Kuhn-Tucker necessary optimality conditions of multiobjective optimization problems.

Maeda [14] was the first to introduce a generalized Guignard regularity condition and established strong Kuhn-Tucker necessary optimality conditions for a differentiable multiobjective optimization problem with inequality constraints. Afterwards, researchers proposed various regularity conditions to derive Kuhn-Tucker necessary optimality conditions of multiobjective optimization problems. Especially, by using some nonsmooth analysis tools such as Clarke generalized gradients, Mordukhovich subdifferentials, etc, these regularity conditions were generalized to the nonsmooth case (see [9, 11, 12, 16]). How to propose weaker regularity condition and obtain the strong Kuhn-Tucker optimality conditions becomes one

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of the interesting topics. Recently, Burachik and Rizvi [2] proposed two types of new regularity conditions and established strong Kuhn-Tucker necessary optimality conditions for a smooth multiobjective optimization problem with inequality constraints.

Motivated by the works of [2, 8, 9, 12], this paper aims at a nonsmooth multiobjective optimization problem with inequality constraints. We propose two types of regularity conditions in terms of Clarke derivative and discuss relations with other regularity conditions. Furthermore, we apply one regularity condition to establish a strong Kuhn-Tucker necessary optimality condition.

2 Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, $\mathbb{R}^n_+$ be the points in $\mathbb{R}^n$ with all coordinates positive or null. For $x, y \in \mathbb{R}^n$, we use the following notations:

$$x \geqq y \Leftrightarrow x_i \geqq y_i,\ \forall\ i = 1, 2, ..., n,$$

$$x \geq y \Leftrightarrow x \geqq y \text{ and } x \neq y,$$

$$x > y \Leftrightarrow x_i > y_i,\ \forall\ i = 1, 2, ..., n.$$

Let $A \subseteq \mathbb{R}^p$. conv$A$ and cl$A$ represent the convex hull and the closure of A, respectively. The generated cone of A is defined by cone$A = \{\lambda a |\ \lambda \geqq 0, a \in A\}$. In this paper, we consider the following nonsmooth multiobjective optimization problem (P):

$$\text{(P)}\ \ \min f(x) = (f_1(x), f_2(x), \cdots, f_l(x))$$

$$\text{s.t. } g_j(x) \leqq 0,\ j = 1, 2, \cdots, m,$$

where $f_i : \mathbb{R}^n \to \mathbb{R},\ i \in I = \{1, 2, \cdots, l\}$ and $g_j : \mathbb{R}^n \to \mathbb{R},\ j \in J = \{1, 2, \cdots, m\}$ are locally Lipschitz. Let $S = \{x \in \mathbb{R}^n | g_j(x) \leqq 0, j \in J\}$ and $J(\bar{x}) = \{j \in J | g_j(\bar{x}) = 0\}$.

Definition 2.1 ([5])**.** Let $C \subseteq \mathbb{R}^n$ and $\mathbb{R}_{++} = \mathbb{R}_+ \setminus \{0\}$. The contingent cone to C at $\bar{x} \in \text{cl}C$ is defined by

$$T(C, \bar{x}) = \{d \in \mathbb{R}^n | \exists \{(x_n, t_n)\} \subseteq (C \times \mathbb{R}_{++}) \text{ such that } x_n \to \bar{x} \text{ and } t_n(x_n - \bar{x}) \to d\}.$$

Definition 2.2 ([5])**.** $\bar{x} \in S$ is said to be efficient for (P) if there is no $x \in S$ such that $f(x) \leq f(\bar{x})$.

Definition 2.3 ([7])**.** $\bar{x} \in S$ is said to be Geoffrion properly efficient for (P) if it is efficient and there exists $M > 0$ such that, for each i,

$$\frac{f_i(x) - f_i(\bar{x})}{f_j(\bar{x}) - f_j(x)} \leqq M,$$

for some j such that $f_j(\bar{x}) < f_j(x)$ whenever $x \in S$ and $f_i(\bar{x}) > f_i(x)$.

Definition 2.4 ([4])**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz and $\bar{x}, \nu \in \mathbb{R}^n$. The Clarke derivative of f at $\bar{x}$ in the direction ν is

$$f^\circ(\bar{x}, \nu) = \limsup_{y \to \bar{x}, t \to 0^+} \frac{f(y + t\nu) - f(y)}{t}.$$

The Clarke subdifferential of f at $\bar{x}$ is $\partial_c f(\bar{x}) = \{\xi \in \mathbb{R}^n | \langle \xi, \nu \rangle \leq f^\circ(\bar{x}, \nu), \forall \nu \in \mathbb{R}^n\}$.

Lemma 2.5 ([4])**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz at $x \in \mathbb{R}^n$. Then,*

(i) *The function $\nu \to f^\circ(x,\nu)$ is finite, positively homogeneous and subadditive;*

(ii) $f^\circ(x,\nu)$ is upper semicontinuous as a function of (x,ν)*;*

(iii) *For every $\nu \in \mathbb{R}^n$, one has $f^\circ(x,\nu) = \max\{\langle \xi,\nu\rangle | \xi \in \partial_c f(x)\}$.*

Lemma 2.6 ([4])**.** *Let $x, y \in \mathbb{R}^n$, and suppose that f is Lipschitz on an open set containing the line segment $[x,y]$. Then there exists a point u in the line segment (x,y) such that*

$$f(y) - f(x) \in \langle \partial_c f(u), y - x\rangle .$$

Lemma 2.7. *Let $h : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz. Assume that*

(i) $z_n \to \bar{z}$,

(ii) $h(z_n) \geqq h(\bar{z})$,

(iii) $\nu = \lim_{n\to\infty} s_n(z_n - \bar{z})$ *with $s_n > 0$, for all n.*

Then

$$h^\circ(\bar{z},\nu) \geqq 0.$$

Proof. From (ii) and Lemma 2.6, it follows that there exist u_n in the line segment $(\bar{z}, z_n)$ and $\xi_n \in \partial_c h(u_n)$ such that

$$\langle \xi_n, z_n - \bar{z}\rangle \geqq 0, \tag{2.1}$$

where $u_n = \bar{z} + \bar{\lambda}_n(z_n - \bar{z})$ for some $\bar{\lambda}_n \in (0,1)$. From (2.1) and Lemma 2.5 (iii), we have $h^\circ(u_n, z_n - \bar{z}) \geqq 0$. By Lemma 2.5 (i), for $s_n > 0$,

$$h^\circ(u_n, s_n(z_n - \bar{z})) = s_n h^\circ(u_n, z_n - \bar{z}) \geqq 0. \tag{2.2}$$

From (i) and (iii), we can obtain $u_n = \bar{z} + \bar{\lambda}_n(z_n - \bar{z}) \to \bar{z}$ and $s_n(z_n - \bar{z}) \to \nu$. Hence, by Lemma 2.5 (ii) and from (2.2), it follows that $0 \leqq \overline{\lim_{n\to\infty}} h^\circ(u_n, s_n(z_n - \bar{z})) \leqq h^\circ(\bar{z},\nu)$. □

Remark 2.8. Clearly, Lemma 2.7 generalizes Lemma 3.1 in [2] to the nonsmooth case.

3 Regularity Conditions

In this section, we generalize the regularity conditions in [2] to the nonsmooth case in terms of Clarke derivative and address some relations with other regularity conditions.

Maeda [14] and Burachik and Rizvi [2] introduced the following sets. For $\bar{x} \in S$, for $i \in I$,

$$\begin{aligned}
Q^i(\bar{x}) &= \{x \in \mathbb{R}^n | g(x) \leqq 0, f_k(x) \leqq f_k(\bar{x}), k \in I \text{ and } k \neq i\},\\
Q(\bar{x}) &= \{x \in \mathbb{R}^n | g(x) \leqq 0, f(x) \leqq f(\bar{x})\},\\
M^i(\bar{x}) &= \{x \in \mathbb{R}^n | g(x) \leqq 0, f_i(x) \leqq f_i(\bar{x})\},\\
M(\bar{x}) &= \bigcap_{i\in I} M^i(\bar{x}).
\end{aligned}$$

Definition 3.1. The linearizing cone to $M^i(\bar{x})$ at $\bar{x}$ is the set defined by

$$L(M^i(\bar{x}),\bar{x}) = \{d \in \mathbb{R}^n | f_i^\circ(\bar{x},d) \leqq 0, g_j^\circ(\bar{x},d) \leqq 0, j \in J(\bar{x})\}, i \in I$$

and $L(M(\bar{x}),\bar{x}) = \{d \in \mathbb{R}^n | f_i^\circ(\bar{x},d) \leqq 0, i \in I, g_j^\circ(\bar{x},d) \leqq 0, j \in J(\bar{x})\}$.

We propose the following two types of regularity conditions for (P).

$$L(M^i(\bar{x}),\bar{x}) \subseteq \text{clconv}T(M^i(\bar{x}),\bar{x}), \text{for at least one } i, \tag{3.1}$$

$$L(M(\bar{x}),\bar{x}) \subseteq \bigcap_{i\in I} T(M^i(\bar{x}),\bar{x}). \tag{3.2}$$

We call (3.1) a generalized Guignard regularity condition(GGRC) and (3.2) an extended generalized Abadie regularity condition(EGARC). Next, we discuss relations among (GGRC), (EGARC) and some other regularity conditions.

(GARC) Generalized Abadie Regularity Condition: $L(Q(\bar{x}),\bar{x}) \subseteq \bigcap_{i\in I} T(Q^i(\bar{x}),\bar{x})$.

(GCRC) Generalized Cottle Regularity Condition: for each $i \in I$, the system

$$\begin{cases} f_k^\circ(\bar{x},d) < 0, k \in I\setminus\{i\}, \\ g_j^\circ(\bar{x},d) < 0, j \in J(\bar{x}), \end{cases}$$

has a solution $d^i \in \mathbb{R}^n$.

Remark 3.2. It is clear that (GARC) and (GCRC) generalize (ACQ) and (CCQ) in [14] to the nonsmooth case, respectively.

Theorem 3.3. *(GARC) implies (EGARC).*

Proof. Since $M(\bar{x}) = Q(\bar{x})$, we have $L(Q(\bar{x}),\bar{x}) = L(M(\bar{x}),\bar{x})$. For any fixed $i \in I$, it follows from $Q^i(\bar{x}) = \bigcap_{\substack{j\in I\\ j\neq i}} M^j(\bar{x})$ that $Q^i(\bar{x}) \subseteq M^j(\bar{x}), \forall j \in I\backslash\{i\}$. Hence,

$$T(Q^i(\bar{x}),\bar{x}) \subseteq T(M^j(\bar{x}),\bar{x}), \forall j \in I\backslash\{i\}.$$

Therefore,

$$T(Q^i(\bar{x}),\bar{x}) \subseteq \bigcap_{\substack{j\in I\\ j\neq i}} T(M^j(\bar{x}),\bar{x}).$$

Thus,

$$\bigcap_{i\in I} T(Q^i(\bar{x}),\bar{x}) \subseteq \bigcap_{i\in I}\bigcap_{\substack{j\in I\\ j\neq i}} T(M^j(\bar{x}),\bar{x}) = \bigcap_{i\in I} T(M^i(\bar{x}),\bar{x}).$$

□

Theorem 3.4. *(GCRC) implies (GGRC).*

Proof. Since (GCRC) holds, then for any fixed $i \in I$, there exists $d^i \in \mathbb{R}^n$ such that

$$\begin{cases} f_k^\circ(\bar{x},d_i) < 0,\ k \in I\setminus\{i\}, \\ g_j^\circ(\bar{x},d_i) < 0,\ j \in J(\bar{x}). \end{cases}$$

If (GGRC) does not hold, then for given $k \in I\backslash\{i\}$, there exists $\nu^k \in L(M^k(\bar{x}),\bar{x})$ such that

$$\nu^k \notin \text{clconv}T(M^k(\bar{x}),\bar{x}). \tag{3.3}$$

From $\nu^k \in L(M^k(\bar{x}), \bar{x})$, we have $f_k^\circ(\bar{x}, \nu^k) \leqq 0$ and $g_j^\circ(\bar{x}, \nu^k) \leqq 0 (j \in J(\bar{x}))$. For any $t_n > 0$, $t_n \to 0 (n \to \infty)$, define $d_n^k = \nu^k + t_n d^i$. From Lemma 2.5 (i), for small t_n,

$$f_k^\circ(\bar{x}, d_n^k) \leqq f_k^\circ(\bar{x}, \nu^k) + t_n f_k^\circ(\bar{x}, d^i) < 0, \tag{3.4}$$
$$g_j^\circ(\bar{x}, d_n^k) \leqq g_j^\circ(\bar{x}, \nu^k) + t_n g_j^\circ(\bar{x}, d^i) < 0, j \in J(\bar{x}). \tag{3.5}$$

For each d_n^k and any positive sequence $\{\mu_{nm}^k\}_{m=1}^\infty$ converging to 0, define $x_{nm}^k = \bar{x} + \mu_{nm}^k d_n^k$. Hence from (3.4)-(3.5) and the continuity of $g_j (j \notin J(\bar{x}))$ it follows that there exists $N_0 \in N$ such that for $m > N_0$,

$$f_k(x_{nm}^k) < f_k(\bar{x}), g_j(x_{nm}^k) < 0 (j \in J).$$

Hence, $x_{nm}^k \in M^k(\bar{x})$ for $m > N_0$ and

$$\lim_{m \to \infty} x_{nm}^k = \bar{x} \in \mathrm{cl} M^k(\bar{x}).$$

Moreover, $\lim_{m \to \infty} \frac{(x_{nm}^k - \bar{x})}{\mu_{nm}^k} = d_n^k$ implies $d_n^k \in T(M^k(\bar{x}), \bar{x})$. Hence,

$$\lim_{n \to \infty} d_n^k = \lim_{n \to \infty} \left(\nu^k + t_n d^i\right) = \nu^k \in T(M^k(\bar{x}), \bar{x}).$$

Hence for any $i \in I$, $\nu^i \in T(M^i(\bar{x}), \bar{x}) \subseteq \mathrm{clconv} T(M^i(\bar{x}), \bar{x})$ which contradicts to (3.3). □

Theorem 3.5. *(GCRC) implies (GARC).*

Proof. The proof is similar with the proof of Theorem 3.4 and is omitted. □

We summarize the relations of the regularity conditions as follows:

$$\text{EGARC} \Longleftarrow \text{GARC} \Longleftarrow \text{GCRC} \Longrightarrow \text{GGRC}$$

Remark 3.6. (GGRC) is different from (GARC) and the following example can illustrate this point.

Example 3.7. Consider the following problem:

$$\min f(x) = (f_1(x), f_2(x), f_3(x))$$
$$\text{s.t.} \quad g_1(x) \leqq 0, g_2(x) \leqq 0, x \in \mathbb{R}^2,$$

where $f_1(x) = (x_1 - 1)^2$, $f_2(x) = \mid x_2 \mid$, $f_3(x) = -x_1$, $g_1(x) = -x_1^3$, $g_2(x) = -x_2^3$. Here, $\bar{x} = (0, 0)$ is a feasible solution. Furthermore,

$$\begin{aligned} M^1(\bar{x}) &= \{x \in \mathbb{R}_+^2 | 0 \leqq x_1 \leqq 2\}, M^2(\bar{x}) = \{x \in \mathbb{R}_+^2 | x_2 = 0\}, M^3(\bar{x}) = \mathbb{R}_+^2; \\ Q^1(\bar{x}) &= \{x \in \mathbb{R}_+^2 | x_2 = 0\}, Q^2(\bar{x}) = \{x \in \mathbb{R}_+^2 | 0 \leqq x_1 \leqq 2\}, \\ Q^3(\bar{x}) &= \{x \in \mathbb{R}_+^2 | 0 \leqq x_1 \leqq 2, x_2 = 0\}; \\ T(M^1(\bar{x}), \bar{x}) &= T(M^3(\bar{x}), \bar{x}) = \mathbb{R}_+^2, T(M^2(\bar{x}), \bar{x}) = \{d \in \mathbb{R}_+^2 | d_2 = 0\}; \\ T(Q^1(\bar{x}), \bar{x}) &= T(Q^3(\bar{x}), \bar{x}) = \{d \in \mathbb{R}_+^2 | d_2 = 0\}, T(Q^2(\bar{x}), \bar{x}) = \mathbb{R}_+^2; \\ L(M^1(\bar{x}), \bar{x}) &= L(M^3(\bar{x}), \bar{x}) = \{d \in \mathbb{R}^2 | d_1 \geqq 0\}, L(M^2(\bar{x}), \bar{x}) = \{d \in \mathbb{R}^2 | d_2 = 0\}. \end{aligned}$$

We can verify that

$$L(M(\bar{x}), \bar{x}) = L(Q(\bar{x}), \bar{x}) = \bigcap_{i=1}^3 T(Q^i(\bar{x}), \bar{x}) = \{d \in \mathbb{R}_+^2 | d_2 = 0\},$$

i.e., (GARC) holds at $\bar{x}$. However,

$$\begin{aligned}
&L(M^1(\bar{x}),\bar{x}) = \{d \in \mathbb{R}^2 | d_1 \geqq 0\} \nsubseteq \text{clconv}T(M^1(\bar{x}),\bar{x}) = \mathbb{R}^2_+,\\
&L(M^2(\bar{x}),\bar{x}) = \{d \in \mathbb{R}^2 | d_2 = 0\} \nsubseteq \text{clconv}T(M^2(\bar{x}),\bar{x}) = \{d \in \mathbb{R}^2_+ | d_2 = 0\},\\
&L(M^3(\bar{x}),\bar{x}) = \{d \in \mathbb{R}^2 | d_1 \geqq 0\} \nsubseteq \text{clconv}T(M^3(\bar{x}),\bar{x}) = \mathbb{R}^2_+,
\end{aligned}$$

i.e., (GGRC) does not hold.

Example 3.8. Consider the problem in Example 4.1 in [2]. Clearly, $\bar{x} = (2,0)$ is a feasible solution. Moreover,

$$\begin{aligned}
M^1(\bar{x}) &= \{x \in \mathbb{R}^2 | x_2 = 0\},\ M^2(\bar{x}) = M^3(\bar{x}) = \{(2,0)\};\\
Q^1(\bar{x}) &= Q^2(\bar{x}) = Q^3(\bar{x}) = \{(2,0)\},\ M(\bar{x}) = Q(\bar{x}) = \{(2,0)\};\\
T(M^1(\bar{x}),\bar{x}) &= \{d \in \mathbb{R}^2 | d_2 = 0\},\ T(M^2(\bar{x}),\bar{x}) = T(M^3(\bar{x}),\bar{x}) = \{(0,0)\};\\
T(Q^1(\bar{x}),\bar{x}) &= T(Q^2(\bar{x}),\bar{x}) = T(Q^3(\bar{x}),\bar{x}) = \{(0,0)\};\\
L(M^1(\bar{x}),\bar{x}) &= L(M^2(\bar{x}),\bar{x}) = L(M^3(\bar{x}),\bar{x}) = \{d \in \mathbb{R}^2 | d_2 = 0\}.
\end{aligned}$$

We can verify that

$$L(M^1(\bar{x}),\bar{x}) \subseteq \text{clconv}T(M^1(\bar{x}),\bar{x}) = \{d \in \mathbb{R}^2 | d_2 = 0\},$$

i.e., (GGRC) holds at $\bar{x}$. However,

$$L(Q(\bar{x}),\bar{x}) = \{d \in \mathbb{R}^2 | d_2 = 0\} \nsubseteq \bigcap_{i=1}^{3} T(Q^i(\bar{x}),\bar{x}) = \{(0,0)\},$$

i.e., (GARC) does not hold.

Remark 3.9. It is well known that (ACQ) always implies that (GACQ) for a nonlinear optimization problem, see Proposition 2.8 in [6]. Furthermore, Maeda first proposed some constraint qualifications including as (ACQ) and (GACQ) based on the sets Q and Q^i for a multiobjective optimization problem with inequality constraints and obtained some relations among them, see Figure 1 in [14]. It is clear that constraint qualifications of multiobjective case inherits those relations of a single objective optimization problem. However, Burachik and Rizvi proposed Guignard regularity condition (GRC) based on the sets M and M^i in [2] (see [14] and [3] for a similar type of condition, where M^i is replaced by Q^i). It is because of the different definitions and therefore some relations may not be true. In consequence, some relations also may not be true for the nonsmooth case. In fact, Remark 3.6 has shown it.

4 Strong Kuhn-Tucker Conditions

By means of some scalarization methods, some scholars have studied some characterizations of Geoffrion properly efficient solution of multiobjective optimization problems. The following result is one of the classic results, see [1].

If f_i, g_j are convex, then $\bar{x}$ is a Geoffrion properly efficient solution of (P) if and only if there exist $\lambda_i > 0, i \in I$ such that $\bar{x}$ is an optimal solution of the problem

$$\min\left\{\sum_{i=1}^{l} \lambda_i f_i(x) \middle| g_j(x) \leqq 0, \forall j \in J\right\}.$$

Moreover, by making use of the nonlinear version of the weighted sum method, Zarepisheh et al. also obtained some characterizations of Geoffrion properly efficient solution under some suitable generalized convexity in [17].

In the following, without any convexity, we establish a strong Kuhn-Tucker necessary optimality condition of Geoffrion properly efficient solution for (P) under (EGARC).

Theorem 4.1. *Let $\bar{x} \in S$. Assume that (EGARC) holds at $\bar{x}$ and $\bar{x}$ is a Geoffrion properly efficient solution of (P), then for each $i \in I$, the following system has no solution $d \in \mathbb{R}^n$.*

$$\begin{cases} f_i^\circ(\bar{x}, d) < 0, \\ f_k^\circ(\bar{x}, d) \leqq 0, \forall k \in I \backslash \{i\}, \\ g_j^\circ(\bar{x}, d) \leqq 0, \forall j \in J(\bar{x}). \end{cases} \tag{4.1}$$

Proof. On the contrary, assume that there exists $i \in I$ such that the system (4.1) has a solution $d \in \mathbb{R}^n$. Without loss of generality one may assume

$$\begin{cases} f_1^\circ(\bar{x}, d) < 0, \\ f_k^\circ(\bar{x}, d) \leqq 0, k \in I \backslash \{1\}, \\ g_j^\circ(\bar{x}, d) \leqq 0, \forall j \in J(\bar{x}). \end{cases} \tag{4.2}$$

From (4.2), it is clear that $d \in L(M(\bar{x}), \bar{x})$. Since (EGARC) holds, it shows that $d \in L(M^i(\bar{x}), \bar{x}), \forall i \in I$. For any $k \in I \backslash \{1\}$, we have

$$d \in T(M^k(\bar{x}), \bar{x}).$$

This means that there exist a sequence $\{x_n\}_{n \in N} \subseteq M^k(\bar{x})$ with $\lim\limits_{n \to \infty} x_n = \bar{x}$ and $t_n > 0$ for all n, such that

$$\lim_{n \to \infty} t_n(x_n - \bar{x}) = d.$$

Fix $k_0 \in \{2, 3, \cdots, l\}$ and for this k_0, take the corresponding sequence $\{x_n\}_{n \in N} \subseteq M^{k_0}(\bar{x})$. It is clear that for all n, $f_{k_0}(x_n) - f_{k_0}(\bar{x}) \leqq 0$. For any $n \in N$, consider the set

$$\Gamma_n = \{k \geqq 2 | f_k(x_n) > f_k(\bar{x})\}.$$

Clearly, $\Gamma_n \neq \varnothing$ for all $n \in N$ and we can assume Γ_n is constant for all n. By Lemma 2.7, we can obtain that

$$f_k^\circ(\bar{x}, d) \geqq 0, \forall k \in \Gamma_n.$$

From (4.2) it must be that $f_k^\circ(\bar{x}, d) = 0, \forall k \in \Gamma_n$.

On the other hand, by Lemma 2.6, there exist ν_n^1 in line segment $(\bar{x}, x_n)$ and $\xi_n^1 \in \partial c f_1(\nu_n^1)$ such that

$$f_1(x_n) - f_1(\bar{x}) = \langle \xi_n^1, x_n - \bar{x} \rangle, \tag{4.3}$$

where $\nu_n^1 = \bar{x} + \lambda_n^1(x_n - \bar{x})$ for some $\lambda_n^1 \in (0, 1)$. Clearly, $f_1(x_n) < f_1(\bar{x})$. Otherwise, it contradicts to $f_1^\circ(\bar{x}, d) < 0$. Hence from (4.3), we obtain that

$$-\langle \xi_n^1, x_n - \bar{x} \rangle > 0. \tag{4.4}$$

Again by Lemma 2.6, there exist ν_n^k in line segment $(\bar{x}, x_n)$ and $\xi_n^k \in \partial c f_k(\nu_n^k)$ such that

$$f_k(x_n) - f_k(\bar{x}) = \langle \xi_n^k, x_n - \bar{x} \rangle, \forall k \in \Gamma_n, \tag{4.5}$$

where $\nu_n^k = \bar{x} + \lambda_n^k(x_n - \bar{x})$ for some $\lambda_n^k \in (0,1)$. Hence, from Lemma 2.5 (iii) and combine with (4.3) and (4.5), we have

$$f_1^\circ(x_n - \bar{x}, \nu_n^1) \geqq \langle \xi_n^1, x_n - \bar{x} \rangle$$

and

$$f_k^\circ(x_n - \bar{x}, \nu_n^k) \geqq \langle \xi_n^k, x_n - \bar{x} \rangle > 0, \forall k \in \Gamma_n. \tag{4.6}$$

Therefore, from (4.4) and (4.6), for any $k \in \Gamma_n$,

$$\begin{aligned}\frac{f_1(x_n) - f_1(\bar{x})}{f_k(\bar{x}) - f_k(x_n)} &= \frac{-(f_1(x_n) - f_1(\bar{x}))}{f_k(x_n) - f_k(\bar{x})} = \frac{-\langle \xi_n^1, x_n - \bar{x} \rangle}{\langle \xi_n^k, x_n - \bar{x} \rangle} \geqq \frac{-\langle \xi_n^1, x_n - \bar{x} \rangle}{f_k^\circ(\nu_n^k, x_n - \bar{x})} \\ &\geqq \frac{-f_1^\circ(x_n - \bar{x}, \nu_n^1)}{f_k^\circ(x_n - \bar{x}, \nu_n^k)} = \frac{-f_1^\circ(t_n(x_n - \bar{x}), \nu_n^1)}{f_k^\circ(t_n(x_n - \bar{x}), \nu_n^k)}.\end{aligned} \tag{4.7}$$

Since, $\nu_n^1 \to \bar{x}$, $\nu_n^k \to \bar{x}$, and $t_n(x_n - \bar{x}) \to d$ when n tends to ∞, from Lemma 2.5 (ii) and (4.6), we can obtain

$$\overline{\lim_{n\to\infty}} f_1^\circ(t_n(x_n - \bar{x}), \nu_n^1) \leqq f_1^\circ(\bar{x}, d) < 0, \tag{4.8}$$

$$0 \leqq \overline{\lim_{n\to\infty}} f_k^\circ(t_n(x_n - \bar{x}), \nu_n^k) \leqq f_k^\circ(\bar{x}, d) = 0,$$

i.e.,

$$\overline{\lim_{n\to\infty}} f_k^\circ(t_n(x_n - \bar{x}), \nu_n^k) = f_k^\circ(\bar{x}, d) = 0. \tag{4.9}$$

By (4.8), fix $r < 0$ such that

$$f_1^\circ(\bar{x}, d) < r < 0.$$

Then again from (4.8) that there exists a positive integer N_0 such that

$$-f_1^\circ(t_n(x_n - \bar{x}), \nu_n^1) > -r > 0.$$

Hence from (4.7), it follows that for any $k \in \Gamma_n$ and $n > N$,

$$\begin{aligned} 0 < \frac{f_k(\bar{x}) - f_k(x_n)}{f_1(x_n) - f_1(\bar{x})} &\leqq \frac{f_k^\circ(t_n(x_n - \bar{x}), \nu_n^k)}{-f_1^\circ(t_n(x_n - \bar{x}), \nu_n^1)} \\ &\leqq \frac{f_k^\circ(t_n(x_n - \bar{x}), \nu_n^k)}{-r}. \end{aligned} \tag{4.10}$$

Therefore, from (4.9)-(4.10), we have

$$0 \leqq \lim_{n\to\infty} \frac{f_k(\bar{x}) - f_k(x_n)}{f_1(x_n) - f_1(\bar{x})} \leqq \frac{1}{-r} \overline{\lim_{n\to\infty}} f_k^\circ(t_n(x_n - \bar{x}), \nu_n^k) = 0,$$

i.e.,

$$\lim_{n\to\infty} \frac{f_1(x_n) - f_1(\bar{x})}{f_k(\bar{x}) - f_k(x_n)} = +\infty,$$

which contradicts to Geoffrion properly efficiency. □

Theorem 4.2. *Let $\bar{x} \in S$. Assume that (EGARC) holds at $\bar{x}$ and for each $i \in I$, the cones*

$$D_i = \operatorname{coneconv}\left(\bigcup_{j \neq i} \partial_c f_j(\bar{x})\right) + \operatorname{coneconv}\left(\bigcup_{j \in J(\bar{x})} \partial_c g_j(\bar{x})\right)$$

are closed. Furthermore, if $\bar{x} \in S$ is a Geoffrion properly efficient solution to (P), then there exist $\lambda \in \mathbb{R}^l$ and $u \in \mathbb{R}^m$ such that

$$0 \in \sum_{i \in I} \lambda_i \partial_c f_i(\bar{x}) + \sum_{j=1}^{m} u_j \partial_c g_j(\bar{x}), \tag{4.11}$$

$$u_j g_j(\bar{x}) = 0, j = 1, 2, \cdots, m, \lambda > 0, u \geqq 0. \tag{4.12}$$

Proof. Let $\bar{x}$ be a Geoffrion properly efficient solution to (P). Then from Theorem 4.1, we have for each $i \in I$, none of the l systems

$$\begin{cases} f_i^\circ(\bar{x}, d) < 0, \\ f_k^\circ(\bar{x}, d) \leqq 0, \forall k \in I \backslash \{i\}, \\ g_j^\circ(\bar{x}, d) \leqq 0, j \in J(\bar{x}), \end{cases}$$

has a solution $d \in \mathbb{R}^n$. By generalized Tucker alternative Theorem in [8] and Lemma 5.2.7 in [15], there exist $\lambda \in \mathbb{R}^l, \lambda_i > 0$ and $u_j \geqq 0, j \in J(\bar{x})$ such that

$$0 \in \sum_{i \in I} \lambda_i \partial_c f_i(\bar{x}) + \sum_{j \in J(\bar{x})} u_j \partial_c g_j(\bar{x}).$$

By setting $u_j = 0, j \notin J(\bar{x})$, we arrive at (4.11)-(4.12). □

Remark 4.3. Theorem 4.1 and Theorem 4.2 generalize Theorem 4.3 and Theorem 4.4 in [2] to the nonsmooth case, respectively.

Example 4.4. Consider the following problem:

$$\min f(x) = (f_1(x), f_2(x))$$

$$\text{s.t. } \; g_i(x) \leqq 0, i = 1, 2, x \in \mathbb{R}^2,$$

where $f_1(x_1, x_2) = x_1$, $f_2(x_1, x_2) = |x_2|$, $g_1(x_1, x_2) = 1 - e^{x_1}$, $g_2(x_1, x_2) = -x_2^3$. Let $\bar{x} = (0, 0)$. It is clear that $\bar{x}$ is a Geoffrion properly efficient solution and

$$\begin{aligned} M^1(\bar{x}) &= \{x \in \mathbb{R}^2 | x_1 = 0\}, M^2(\bar{x}) = \{x \in \mathbb{R}^2 | x_1 \geqq 0, x_2 = 0\}; \\ T(M^1(\bar{x}), \bar{x}) &= \{d \in \mathbb{R}^2 | d_1 = 0\}, T(M^2(\bar{x}), \bar{x}) = \{d \in \mathbb{R}^2 | d_1 \geqq 0, d_2 = 0\}; \\ L(M(\bar{x}), \bar{x}) &= \{(0, 0)\} = \bigcap_{i=1}^{2} T(M^i(\bar{x}), \bar{x}). \end{aligned}$$

That is, (EGARC) holds at $\bar{x}$. Furthermore,

$$\begin{aligned} D_1 &= \text{coneconv} \partial_c f_2(\bar{x}) + \text{coneconv} \left(\bigcup_{j=1}^{2} \partial_c g_j(\bar{x}) \right) = \{x \in \mathbb{R}^2 | x_1 \leqq 0\} \text{ and} \\ D_2 &= \text{coneconv} \partial_c f_1(\bar{x}) + \text{coneconv} \left(\bigcup_{j=1}^{2} \partial_c g_j(\bar{x}) \right) = \{x \in \mathbb{R}^2 | x_2 = 0\}, \end{aligned}$$

are closed. Hence, strong Kuhn-Tucker conditions hold for $\lambda_1 = \lambda_2 = u_1 = u_2 = 1$.

Remark 4.5. The closedness of $D_i (i \in I)$ could not be removed in Theorem 4.2.

Example 4.6. Consider the following problem:

$$\min f(x) = (f_1(x), f_2(x))$$
$$\text{s.t. } g(x) \leqq 0, x \in \mathbb{R}^2,$$

where $f_1(x_1, x_2) = x_1$, $f_2(x_1, x_2) = x_1$, $B = \{x \in \mathbb{R}^2 | x_1^2 + (x_2 + 1)^2 \leqq 1, x_1 \leqq 0\}$ and g be the support function of the compact convex set B, i.e.,

$$g(x) = \sup_{b \in B} \langle b, x \rangle, x \in \mathbb{R}^2.$$

Let $\bar{x} = (0, 0)$. We can verify that $\bar{x} = (0, 0)$ is a Geoffrion properly efficient solution. Clearly, $S = \mathbb{R}_+^2$. $g^\circ(\bar{x}, \nu) = g(\nu)$ and $\partial g(\bar{x}) = B$, since g is convex and positively homogeneous. Furthermore,

$$M^1(\bar{x}) = M^2(\bar{x}) = \{x \in \mathbb{R}^2 | x_1 = 0, x_2 \geqq 0\},$$
$$T(M^1(\bar{x}), \bar{x}) = T(M^2(\bar{x}), \bar{x}) = \{d \in \mathbb{R}^2 | d_1 = 0, d_2 \geqq 0\},$$
$$L(M(\bar{x}), \bar{x}) = \{d \in \mathbb{R}^2 | d_1 = 0, d_2 \geqq 0\} = \bigcap_{i=1}^{2} L(M^i(\bar{x}), \bar{x}).$$

That is, (EGARC) holds at $\bar{x} = (0, 0)$. But the cones

$$D_1 = D_2 = \{x \in \mathbb{R}^2 | x_1 \geqq 0, x_2 = 0\} \cup \{x \in \mathbb{R}^2 | x_2 < 0\}$$

are not closed. Now, we can verify that the strong Kuhn-Tucker conditions do not hold.

Remark 4.7. Burachik and Rizvi proposed a Guignard regularity condition (GRC) for a differentiable multiobjective optimization problem and then obtained weak Kuhn-Tucker condition of efficient solution in [2]. However, in the sense of Clarke derivative and Clarke differential, we could not expect that the corresponding results are true. The following example shows that Theorem 4.1 in [2] may not be true in the sense of Clarke derivative.

Example 4.8. Consider the following problem:

$$\min f(x) = (f_1(x), f_2(x))$$
$$\text{s.t. } g(x) \leqq 0, x \in \mathbb{R}^2,$$

where $f_1(x_1, x_2) = -x_1$, $f_2(x_1, x_2) = 2|x_2| - x_1$, $g(x_1, x_2) = (x_1 + x_2)(x_1 - x_2)$. Let $\bar{x} = (0, 0)$. We can verify that $\bar{x}$ is an efficient solution to problem (P). It is clear that

$$f_1^\circ(\bar{x}, d) = -d_1, f_2^\circ(\bar{x}, d) = 2|d_2| - d_1, g^\circ(\bar{x}, d) = 0, \text{ for all } d \in \mathbb{R}^2,$$
$$M^1(\bar{x}) = \{x \in \mathbb{R}^2 | -x_1 \leqq 0\} \cap X, \text{clconv} T(M^1(\bar{x}), \bar{x}) = \{d \in \mathbb{R}^2 | d_1 \geqq 0\},$$
$$L(M^1(\bar{x}), \bar{x}) = \{d \in \mathbb{R}^2 | d_1 \geqq 0\}.$$

In consequence, (GGRC) is satisfied for $i = 1$. However, the following system

$$\begin{cases} f_i^\circ(\bar{x}, d) < 0, i \in I, \\ g^\circ(\bar{x}, d) \leqq 0, \end{cases}$$

has a solution $d = (1, 0)$.

5 Concluding Remarks

We generalize the regularity conditions in [2] to the nonsmooth case in terms of Clarke derivative and investigate the relations with other regularity conditions. We also derive a strong Kuhn-Tucker necessary optimality condition. It is meaningful that how to propose weaker regularity conditions than (EGARC) to ensure strong Kuhn-Tucker optimality conditions.

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Ke Quan Zhao
College of Mathematics Science, Chongqing Normal University
Chongqing 401331, China
E-mail address: kequanz@163.com