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AN EXISTENCE THEOREM FOR A VARIATIONAL RELATION PROBLEM WITH APPLICATIONS TO MINIMAX INEQUALITIES

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Abstract: Variational relation problems, a recent concept introduced by Luc, are general models for a large class of problems in optimization and nonlinear analysis. In this paper we establish an existence theorem for the solution of the following variational relation problem: Find $x^* \in X$ such that $(x^*, y) \in R$ for every $y \in Y$, where X is a nonempty convex subset of a vector topological space, Y is a compact convex subset of a Hausdorff topological vector space and R is a relation between the elements of the two sets. As applications, we obtain an intersection theorem, a fixed point theorem and several minimax inequalities.

Key words: *variational relation problem, almost convex set-valued mapping, maximal element, minimax inequality, saddle point*

Mathematics Subject Classification: *47H10, 49J53*

1 Introduction and preliminaries

In 2008, Luc [26] introduced and studied an abstract problem in variational analysis, called by him, variational relation problem. If X and Y are two given sets, a nonempty subset R of $X \times Y$ determines a relation between the elements of X and Y in a natural manner: we say that x is in relation R with y, if $(x, y) \in R$. When Y is a parameter set, then R is called a variational relation. Luc's approach offers a unifying treatment for a wide class of problems in diverse fields of pure and applied mathematics and for this reason, in the last years, the variational relation problems have attracted increasingly the attention of researchers. Various types of variational relation problems or systems of variational relation problems have been investigated in many recent papers (see [1, 6, 8–10, 18, 20–24, 27, 32, 34]). In these papers, as for other mathematical models, the main focus has been on sufficient conditions for the existence of solutions.

Having in mind the well-known Ky Fan's section theorem ([14]) the following problem, studied in this paper, arises in a natural manner:

Find $x^* \in X$ such that $(x^*, y) \in R$ for every $y \in Y$,

where X is a convex subset of a vector topological space, Y is a compact convex subset of a Hausdorff topological vector space and R is a nonempty subset of $X \times Y$.

To the best of our knowledge, though very simple and natural, so far, this variational relation problem has been studied just in the case when $X = Y$ (see [7] and [32]). Nevertheless, it can be regarded as a particular case of some variational relation problems studied in various recent papers. For exemplification let us compare our problem to that studied by Luc in [26].

If X, Y and Z are nonempty sets, $S_1 : X \rightrightarrows X$, $S_2 : X \rightrightarrows Y$, $T : X \times Y \rightrightarrows Z$ are set-valued mappings and R is a relation between elements of the sets X, Y and Z, Luc's variational relation problem can be formulated as follows:

Find $x^* \in X$ such that $x^* \in S_1(x^*)$, and $(x^*, y, z) \in R$ holds for every $y \in S_2(x^*)$ and $z \in T(x^*, y)$.

One can observe that if $S_1 = 1_X$ (the identity map on X), $S_2(x) = Y$ for all $x \in X$, and the variable z is missing, Luc's problem reduces to the one studied in this paper. Thus, existence theorems for solutions can be derived from more general results, but we will not use this method. For bringing our investigations closer to some recent research, we could compare our problem with another one, recently studied in [12] and [20]:

Find $x^* \in X$ such that $(x^*, y) \in R$ for every $y \in T(x^*)$,
where T is a set-valued mapping from X to Y.

Though, more general as our problem, the above problem is studied in the mention papers only in the case when R is a linear relation (this means that R, regarded as a subset of $X \times Y$, is defined by a system of linear inequalities).

As we shall see in the obtained applications, for special relations R, a solution x^* of the above problem can be seen as a common point for the family of values of a set-valued mapping, a maximal element, a solution for a generalized Ky Fan inequality, a saddle point or others.

Further we fix the used terminology and some notations. A set-valued mapping $F : X \rightrightarrows Y$ is a function from a set X into the power set 2^Y, that is a function with the values $F(x) \subseteq Y$ for $x \in X$. To a set-valued mapping $F : X \rightrightarrows Y$ we associate two other set- valued mappings $F^c : X \rightrightarrows Y$ (called the complementary of F) and $F^- : Y \rightrightarrows X$ (called the lower inverse of F) defined by $F^c(x) = Y \setminus F(x)$, and respectively $F^-(y) = \{x \in X : y \in F(x)\}$. The values of F^- are called the fibers of F. As usual, the set $\mathrm{Gr}(F) = \{(x, y) \in X \times Y : y \in F(x)\}$ is called the graph of F. For $A \subseteq X$, set $F(A) = \bigcup_{x \in A} F(x)$.

If X and Y are topological spaces, a set-valued mapping $F : X \rightrightarrows Y$ is said to be: (i) lower semicontinuous, if for every open subset G of Y the set $\{x \in X : F(x) \cap G \neq \emptyset\}$ is open; (ii) upper semicontinuous, if for every open subset G of Y the set $\{x \in X : F(x) \subseteq G\}$ is open.

Let U and V be two vector spaces and $X \subseteq U$, $Y \subseteq V$ be two convex sets. A set-valued mapping $F : X \rightrightarrows Y$ is said to be convex, if $\lambda F(x_1) + (1-\lambda)F(x_2) \subseteq F(\lambda x_1 + (1-\lambda)x_2)$, for every $x_1, x_2 \in X$ and every $\lambda \in [0, 1]$. Clearly, a set-valued mapping $F : X \rightrightarrows Y$ is convex if and only if its graph is a convex subset of $X \times Y$.

The rest of the paper is structured as follows. In Section 2, by means of a lower semicontinuous set-valued mapping, we establish an existence theorem for the solution of the considered variational relation problem. Further, our main result is reformulated as an intersection theorem, from which we then obtain a fixed point theorem for a composed set-valued mapping. In Section 3, as applications, we obtain several minimax theorems.

2 Main result

Theorem 2.1. *Let X be a nonempty convex set in a topological vector space, Y be a compact convex set in a Hausdorff topological vector space and R be a nonempty subset of $X \times Y$. Assume that:*

(i) $\{(x,y) \in X \times Y : (x,y) \notin R\}$ *is convex;*

(ii) *for each $y \in Y$, the set $\{x \in X : (x,y) \in R\}$ is closed in X;*

(iii) *there exists a lower semicontinuous set-valued mapping with nonempty convex values $F : Y \rightrightarrows X$ such that $Gr(F^{-}) \subseteq R$.*

Then, there exists $x^* \in X$ such that $(x^*, y) \in R$ for every $y \in Y$.

Proof. Suppose that for each $x \in X$ there exists $y \in Y$ such that $(x,y) \notin R$. Then X is covered by the sets

$$G_y = \{x \in X : (x,y) \notin R\}.$$

Since F has nonempty values, $Y = F^{-}(X)$. Then, $Y = F^{-}(\bigcup_{y \in Y} G_y) = \bigcup_{y \in Y} F^{-}(G_y)$. Denote by

$$H_y = F^{-}(G_y) = \{u \in Y : \exists x \in X \text{ such that } u \in F^{-}(x) \text{ and } (x,y) \notin R\}.$$

By (ii) and (iii) we infer that the sets H_y are open in Y. Since Y is compact it can be covered by a finite number of subsets $\{H_{y_1}, \ldots, H_{y_n}\}$. Consider a continuous partition of unity $\{\alpha_1, \ldots, \alpha_n\}$ associated to this finite covering. Let $K := \text{co}\{y_1, \ldots, y_n\}$ and $L := \text{span}(K)$. Consider on L the topology induced by the one on Y, then one obtains the (unique) separated locally convex topology on L.

Define the following function:

$$p : K \to K, \; p(y) = \sum_{i=1}^{n} \alpha_i(y) y_i.$$

The function p is a continuous self mapping of the compact convex K and then, by Brouwer's fixed point theorem, there exists $\bar{y} \in K$ such that

$$\bar{y} = p(\bar{y}) = \sum_{i=1}^{n} \alpha_i(\bar{y}) y_i.$$

Let $I = \{i \in \{1, \ldots, n\} : \alpha_i(\bar{y}) > 0\}$. Then $\bar{y} = \sum_{i \in I} \alpha_i(\bar{y}) y_i$. For every $i \in I$, $\bar{y} \in H_{y_i}$, hence there exists $x_i \in X$ such that $\bar{y} \in F^{-}(x_i)$ and $(x_i, y_i) \notin R$. Set $\bar{x} = \sum_{i \in I} \alpha_i(\bar{y}) x_i$. By (i), it follows that $(\bar{x}, \bar{y}) \notin R$. As, $\{x_i : i \in I\} \subseteq F(\bar{y})$ and $F(\bar{y})$ is a convex set, $\bar{x} \in F(\bar{y})$. Hence $(\bar{x}, \bar{y}) \in$ Gr (F^{-}), and in view of (iii), $(\bar{x}, \bar{y}) \in R$. Thus we have arrived at a contradiction. □

Remark 2.2. Theorem 2.1 has the same conclusion as Theorem 2.1 in [32], but the hypotheses in the mentioned theorem are more restrictive. First, there the relation R is between elements of the same set X. Then, the set X is assumed to have the fixed point property but, as Park showed in [30], this condition is redundant. Finally, instead of condition (iii), there appears an abstract condition, which in Park's terminology [29], means that, in a suitable Φ_A-space, the set-valued mapping $T : X \rightrightarrows X$ defined by $T(x) = \{u \in X : (u, x) \in R\}$ is a KKM-map.

When $Y = X$ and F is the identity map on Y, Theorem 2.1 reduces to the following corollary:

Corollary 2.3. *Let Y be a compact convex set in a Hausdorff topological vector space and $R(x, y)$ be a relation linking elements $x, y \in Y$. Assume that:*

(i) $\{(x, y) \in Y \times Y : (x, y) \notin R\}$ *is convex;*

(ii) *for each $y \in Y$, the set $\{x \in Y : (x, y) \in R\}$ is closed in Y;*

(iii) *for each $y \in Y$, $(y, y) \in R$.*

Then, there exists $x^* \in Y$ such that $(x^*, y) \in R$ holds for every $y \in Y$.

As any continuous function $f : Y \to X$, regarded as a set-valued mapping, is lower semicontinuous, with nonempty convex values, we infer that:

Corollary 2.4. *The conclusion of Theorem 2.1 remains true if condition (iii) is replaced with the following:*

(iii)$'$ there exists a continuous function $f : Y \to X$ such that $(f(y), y) \in R$ for all $y \in Y$.

Theorem 2.1 can be reformulated as an intersection theorem as follows.

Theorem 2.5. *Let X be a nonempty convex set in a topological vector space, Y be a compact convex set in a Hausdorff topological vector space and $T : Y \rightrightarrows X$ be a set-valued mapping. Assume that the following conditions are satisfied:*

(i) *the values of T are closed in X and the set-valued mapping T^c is convex;*

(ii) *there exists a lower semicontinuous set-valued mapping with nonempty convex values $F : Y \rightrightarrows X$ such that $F(y) \subseteq T(y)$ for all $y \in Y$.*

Then, $\bigcap_{y \in Y} T(y) \neq \emptyset$.

Proof. We apply Theorem 2.1 when the relation R is defined by

$$(x, y) \in R \text{ holds iff } x \in T(y).$$

Since T^c is a convex set-valued mapping, its graph is a convex set, so condition (i) in Theorem 2.1 is satisfied. As T is closed-valued, condition (ii) in Theorem 2.1 is fulfilled. If $(x, y) \in \mathrm{Gr}(F^-)$, then $x \in F(y) \subseteq T(y)$, hence $(x, y) \in R$. It follows that condition (iii) in Theorem 2.1 holds.

Applying Theorem 2.1 we get a point $x^* \in X$ satisfying $x^* \in \bigcap_{y \in Y} T(y)$. □

As we already said, Theorems 2.1, and 2.5 are actually equivalent. If the assumptions of Theorem 2.1 are satisfied, its conclusion follows applying Theorem 2.5 to the set-valued mapping $T : Y \rightrightarrows X$ defined by $T(y) = \{x \in X : (x, y) \in R\}$.

Theorem 2.6. *Let X be a nonempty convex set in a topological vector space, Y be a compact convex set in a Hausdorff topological vector space and $S : X \rightrightarrows Y$, $F : Y \rightrightarrows X$ be two set-valued mappings. Assume that the following conditions are satisfied:*

(i) *S has nonempty values and the set-valued mapping S^{-} is convex and has open values in X;*

(ii) *F is lower semicontinuous with nonempty convex values.*

Then, the set-valued mapping $S \circ F$ has a fixed point,that is, there exists $\bar{y} \in Y$ such that $\bar{y} \in S(F(\bar{y}))$.

Proof. Suppose that for all $y \in Y$, $y \notin S(F(y))$. This means that for every $y \in Y$, $F(y) \subseteq X \setminus S^{-}(y)$. Consider the set-valued mapping $T : Y \rightrightarrows X$ defined by $T(y) := X \setminus S^{-}(y)$ (that is, $T = (S^{-})^c$). One can easily check that condition (i) implies the condition similarly noted in Theorem 2.5. By Theorem 2.5, there exists $x^* \in \bigcap_{y \in Y} (X \setminus S^{-}(y))$. Then, $S(x^*) = \emptyset$; a contradiction. □

Remark 2.7. Since the set-valued mapping S has open fibers, one can easily prove that it is lower semicontinuous. Consequently, the set-valued mapping $S \circ F$ in Theorem 2.6 is lower semicontinuous as a compsition of lower semicontinuous mappings. In contrast to upper semicontinuous set-valued mappings, for which there exists an important literature dealing with the existence of fixed points, for lower semicontinuous set-valued mappings, fixed point theorems are much less numerous (see, for instance, [19, 31, 33, 35]). In general, their proofs rely on the existence of a continuous selection for a lower semicontinuous mapping and on Brouwer or Tychonoff fixed point theorem. Such an argument does not work in the case of Theorem 2.6.

3 Minimax theorems

Recall that a real-valued function g defined on a topological space X is lower (resp. upper) semicontinuous if the set $\{x \in X : g(x) \leq r\}$ (resp. $\{x \in X : g(x) \geq r\}$) is closed for each $r \in \mathbf{R}$. If X is a convex set in a vector space, then g is quasiconvex (resp. quasiconcave) if $\{x \in X : g(x) \leq r\}$ (resp. $\{x \in X : g(x) \geq r\}$) is a convex set for every $r \in \mathbf{R}$.

Theorem 3.1. *Let X be a nonempty convex set in a topological vector space and Y be a compact convex set in a Hausdorff topological vector space. Let $f : X \times Y \to \mathbf{R}$ be a real function and $F : Y \rightrightarrows X$ be a set-valued mapping satisfying the following conditions:*

(i) *f is quasiconcave on $X \times Y$;*

(ii) *for each $y \in Y$, the function $x \to f(x, y)$ is lower semicontinuous on X;*

(iii) *F is lower semicontinuous with nonempty convex values.*

Then,

$$\inf_{x \in X} \sup_{y \in Y} f(x, y) \leq \sup_{(y,x) \in \, Gr(F)} f(x, y).$$

Proof. We may assume that

$$r := \sup_{(y,x)\in \operatorname{Gr}(F)} f(x,y) < \infty.$$

We show that the assumptions of Theorem 2.1 are satisfied, if the relation R is defined by

$$(x,y) \in R \text{ iff } f(x,y) \leq r.$$

As the function f is quasiconcave, the set $\{(x,y) \in X \times Y : f(x,y) > r\}$ is convex. By (ii), for each $y \in Y$, the set $\{x \in X : f(x,y) \leq r\}$ is closed in X. If $(x,y) \in \operatorname{Gr}(F^{-})$, taking into consideration the the choice of r, it follows that $f(x,y) \leq r$, hence $(x,y) \in R$. According to Theorem 3.1, there exists $x^* \in X$ such that $\sup_{y\in Y} f(x^*,y) \leq r$. Consequently,

$$\inf_{x\in X} \sup_{y\in Y} f(x,y) \leq \sup_{y\in Y} f(x^*,y) \leq r = \sup_{(y,x)\in \operatorname{Gr}(F)} f(x,y).$$

□

Remark 3.2. The origin of the above corollary goes back to the famous Ky Fan minimax inequality [15]. The conclusion of our theorem is the same as that of Theorem 1 in [17]. But the assumptions are different, because in the mentioned theorem the set-valued mapping F is assumed upper semicontinuous with nonempty closed convex values and the function f is quasiconcave only in the first variable.

The following numerical example provides a case when Theorem 3.1 is applicable, while Theorem 1 in [17] is not.

Example 3.3 ([17]). Let $X =]-2,2[$ and $Y = [-1,1]$. Let $f :]-2,2[\times[-1,1] \to \mathbf{R}$ and $F : [-1,1] \rightrightarrows]-2,2[$ be defined by

$$f(x,y) = 1 + xy - x^2 - y^2, \ F(y) =]x-1, x+1[.$$

One can easily verify that all requirements of Theorem 3.1 are satisfied. The set-valued mapping F is not upper semicontinuous (since the set $\{y \in Y : F(y) \subseteq]-11[\} = \{0\}$ is not open in X) and its values are not closed. Hence, Theorem 1 in [17] can not be applied. By direct checking, one can see that

$$\inf_{x\in X} \sup_{y\in Y} f(x,y) = -2 \leq 1 = \sup_{(y,x)\in \operatorname{Gr}(F)} f(x,y).$$

For X, Y topological spaces a function $g : X \times Y \to \mathbb{R}$ is said to be marginally upper semicontinuos in the second variable [5] if for every open subset G of X the function $y \to \inf_{x\in G} g(x,y)$ is upper semicontnuous on Y. Obviously, every function upper semicontinuous in y is marginally upper semicontinuous in y. The example given in [11, p. 249] shows that the converse is not true.

Theorem 3.4. *Let X be a nonempty convex set in a topological vector space, Y be a compact convex set in a Hausdorff topological vector space and $f, g : X \times Y \to \mathbf{R}$ be two functions satisfying the following conditions:*

(i) *$f(x,y) \leq g(x,y)$, for each $(x,y) \in X \times Y$;*

(ii) *f is quasiconcave on $X \times Y$ and for each $y \in Y$ the function $x \to f(x,y)$ is lower semicontinuous on X.*

(iii) *g is marginally upper semicontinuos in the second variable and for each $x \in X$ the function $y \to g(x,y)$ is quasiconvex on X.*

Then, $\inf_{x\in X} \sup_{y\in Y} f(x,y) \le \sup_{y\in Y} \inf_{x\in X} g(x,y)$.

Proof. Without loss of generality, assume that $s := \sup_{y\in Y} \inf_{x\in X} g(x,y) < \infty$. For $\varepsilon > 0$ arbitrarily fixed, we intend to apply Theorem 2.1, when the relation R and the set-valued mapping $F : Y \rightrightarrows X$ are defined as follows:

$$(x,y) \in R \text{ if } f(x,y) \le s + \varepsilon,$$

$$F(y) = \{x \in X : g(x,y) < s + \varepsilon\}.$$

By (ii), it readily follows that the first two requirements of Theorem 2.1 are fulfilled. For each $y \in Y$, since $\inf_{x\in X} g(x,y) < s+\varepsilon$, there exists $x \in X$ such that $x \in F(y)$, hence $F(y) \ne \emptyset$. Let $x_1, x_2 \in F(y)$. As g is quasiconvex in x, for each $\lambda \in [0,1]$,

$$g(\lambda x_1 + (1-\lambda)x_2, y) \le \max\{g(x_1,y), g(x_2,y)\} < s + \varepsilon,$$

hence $\lambda x_1 + (1-\lambda)x_2 \in F(y)$. Consequently F has convex values.

For each open subset G of X, $F^{-}(G) = \{y \in Y : \inf_{x\in G} g(x,y) < s + \varepsilon\}$ is open in Y, because g is marginally upper semicontinuous in the second variable. Consequently, F is lower semicontinuous. By (i), it follows that $f(x,y) \le s + \varepsilon$, for every $y \in Y$ and $x \in F(y)$, hence $\mathrm{Gr}\ F^{-} \subseteq R$.

From Theorem 2.1, there is $x^* \in X$ such that $f(x^*,y) \le s + \varepsilon$ for every $y \in Y$. Then

$$\inf_{x\in X} \sup_{y\in Y} f(x,y) \le \sup_{y\in Y} f(x^*,y) \le \sup_{y\in Y} \inf_{x\in X} g(x,y) + \varepsilon.$$

Clearly this implies the desired inequality. □

Remark 3.5. The first minimax inequality for two functions f and g, satisfying $f \le g$ on $X \times Y$, has been obtained by Ky Fan [13] in 1964, motivated by Nash equilibrium and the theory of non-cooperative games. Though the conclusion is the same, Theorem 2.5 differs from Fan's result, as well as from other two-function minimax inequalities established in [3, 4, 16], [25, 28], by requirements imposed in hypothesis and by techniques of proof. For instance, if we compare Theorem 2.5 with Theorem 1 in [25], we note first that some assumptions are the same in the two theorems ($f \le g$, f is lower semicontinuous in the first variable, g is quasiconvex in the first variable). But, in the mentioned theorem, f is assumed quasiconcave only in the second variable (a condition weaker than the one from Theorem 2.5) and g is assumed upper semcontinuous in the variable y (a condition stronger than the corresponding condition in Theorem 2.5). It is worth also noting that in Theorem 3.2 in [3], Theorem 5.2 in [4] and Theorem 1 in [16], the minimax inequality is establish assuming the existence of one or two real-valued functions which majorize the function f and minorize the function g.

Recall that $(x^*, y^*) \in X \times Y$ is a saddle point for a real function f defined on $X \times Y$ if $f(x^*,y) \le f(x,y^*)$ for all $x \in X$, $y \in Y$. This concept has been extended in [2] as follows:

Definition 3.6. Given two functions $f, g : X \times Y \to \mathbf{R}$, we say that a point $(x^*, y^*) \in X \times Y$ is a saddle point for the pair (f, g) if $f(x^*, y) \leq g(x, y^*)$ for all $x \in X, y \in Y$.

Remark 3.7. It is clear that in the case $f = g$ the concept above reduces to the classical concept of saddle point. It is also worth mentioning that the existence of a saddle point for the pair (f, g) implies the inequality $\inf_{x\in X} \sup_{y\in Y} f(x, y) \leq \sup_{y\in Y} \inf_{x\in X} g(x, y)$.

Example 3.8. Let $X = \mathbf{R}$, $Y = [0, 1]$ and for each $(x, y) \in X \times Y$ let $f(x, y) = (x+1)(y-1)$ and $g(x, y) = e^{xy} - x - y$. For each $x \in X$ and $y \in Y$ we have

$$f(0, y) = y - 1 \leq 0 \leq e^x - x - 1 = g(x, 1),$$

hence $(0, 1)$ is a saddle point for the pair (f, g).

Theorem 3.9. *Let X and Y be nonempty compact convex sets in two Hausdorff topological vector spaces and $f, g : X \times Y \to \mathbf{R}$ be two functions satisfying the following conditions:*

(i) *f is concave and g is convex on $X \times Y$;*

(ii) *for each $y \in Y$, the function $x \to f(x, y)$ is lower semicontinuous on X and for each $x \in X$ the function $y \to g(x, y)$ is upper semicontinuous on Y;*

(iii) *there exists a lower semicontinuous set-valued mapping with nonempty convex values $F : X \times Y \rightrightarrows X \times Y$ such that for each $(x, y) \in X \times Y$ and any $(u, v) \in F(x, y)$, $f(x, v) \leq g(u, y)$.*

Then, the pair (f, g) has a saddle point.

Proof. Let the relation R defined on $(X \times Y) \times (X \times Y)$ as follows:

$$((x, y), (u, v)) \in R \text{ iff } f(x, v) \leq g(u, y).$$

The set $M := \{((x, y), (u, v)) \in (X\times Y)\times(X\times Y) : ((x, y), (u, v)) \notin R\}$ is convex, that is, condition (i) in Theorem 2.1 is satisfied. Indeed, if $((x_1, y_1), (u_1, v_1)), ((x_2, y_2), (u_2, v_2)) \in M$, then $f(x_1, v_1) > g(u_1, y_1)$, $f(x_2, v_2) > g(u_2, y_2)$. Taking into account (i), for every $\lambda \in [0, 1]$ we have

$$f(\lambda(x_1, v_1) + (1 - \lambda)(x_2, v_2)) \geq \lambda f(x_1, v_1) + (1 - \lambda)f(x_2, v_2) > \lambda g(u_1, y_1) + (1 - \lambda)g(u_2, y_2)$$
$$\geq g(\lambda(u_1, y_1) + (1 - \lambda)(u_2, y_2)).$$

Therefore $\lambda((x_1, y_1), (u_1, v_1)) + (1 - \lambda)((x_2, y_2), (u_2, v_2)) \in M$, hence M is convex.

We show that for each $(u, v) \in X \times Y$ the set $N := \{(x, y) \in X \times Y : ((x, y), (u, v)) \in R\}$ is closed. Let $\{(x_t, y_t)\}$ be a net in N converging to (x, y). Then for each index t, $f(x_t, v) \leq g(u, y_t)$. By (ii), we get

$$f(x, v) \leq \underline{\lim}\, f(x_t, v) \leq \overline{\lim}\, g(u, y_t) \leq g(u, y),$$

hence N is closed.

The existence of a saddle point (x^*, y^*) is now obtained applying Theorem 2.1. □

Note that, taking into account Remark 3.7, Theorem 3.9 supplies sufficient conditions, different from those of Theorem 3.1, under which the inequality $\inf_{x\in X} \sup_{y\in Y} f(x, y) \leq \sup_{y\in Y} \inf_{x\in X} g(x, y)$ holds.

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References

[1] R.P. Agarwal, M. Balaj and D. O'Regan, A unifying approach to variational relation problems, *J. Optim. Theory Appl.* 155 (2012), 417–429.

[2] R.P. Agarwal, M. Balaj and D. O'Regan, An intersection theorem for set-valued mappings, Appl. Math. 58 (2013), 269–278.

[3] M. Balaj, New minimax theorems, *Carpathian J. Math.* 24 (2008), 287-292.

[4] M. Balaj, Alternative principles and their applications, *J. Global Optim.* 50 (2011), 537–547.

[5] M. Balaj, New matching theorems and their applications, *Math. Commun.* 16 (2011), 369–381.

[6] M. Balaj, Three types of variational relation problems, *Taiwanese J. Math.* 17 (2013), 47-61.

[7] M. Balaj and L.J. Lin, Equivalent forms of a generalized KKM theorem and their applications, *Nonlinear Anal.* 73 (2010), 673-682.

[8] M. Balaj and L.J. Lin, Generalized variational relation problems with applications, *J. Optim. Theory Appl.* 148 (2011), 1–13.

[9] M. Balaj and L.J. Lin, Existence criteria for the solutions of two types of variational relation problems, *J. Optim. Theory Appl.* 156 (2013), 232–246.

[10] M. Balaj and D.T. Luc, On mixed variational relation problems, *Comput. Math. Appl.* 60 (2010), 2712–2722.

[11] R.C. Bassanezi and G.H. Greco, A minimax theorem for marginally u.s.c./l.s.c. functions, *Topol. Methods Nonlinear Anal.* 5 (1995) 249-253.

[12] A. Dhara and D.T. Luc, A solution method for linear variational relation problems, *J Glob Optim.* 59 (2014), 729–756.

[13] K. Fan, Sur un théorème minimax, *C. R. Acad. Sci. Paris* 259 (1964), 3925–3928.

[14] K. Fan, Ageneralization of Tychonoff's fixed point theorem, *Math. Ann.* 142 (1961), 305–310.

[15] K. Fan, A minimax inequality and applications, in: Inequalities, III (O. Shisha ed.), Academic Press, New York, 1972, pp. 103–113.

[16] A. Granas and F.C. Liu, Quelques théorèmes de minimax sans convexité, *C. R. Acad. Sci. Paris Sér. I Math.* 300 (1985), 347–350.

[17] C.W. Ha, On a minimax inequality of Ky Fan, *Proc. Amer. Math. Soc.* 99 (1987), 680–682.

[18] P.Q. Khanh and D.T. Luc, Stability of solutions in parametric variational relation problems, *Set-Valued Anal.* 16, (2008) 1015–1035.

[19] W.K. Kim, A fixed point theorem in a Hausdorff topological vector space, *Comment. Math. Univ. Carolin.* 36 (1995), 33–38.

[20] A. Latif and D.T. Luc, Variational relation problems: existence of solutions and fixed points of contraction mappings, *Fixed Point Theory Appl.* 2013, 2013:315.

[21] L.J. Lin, Variational relation problems and equivalent form of Generalized Fan-Browder fixed point theorems with applications, *J. Global Optim.* 53 (2012), 215–229.

[22] L.J. Lin and Q.H. Ansari, Systems of quasi-variational relations with applications, *Nonlinear Anal.* 72 (2010), 1210–1220.

[23] L.J. Lin, M. Balaj and Y.C. Ye, Quasi-variational relation problems and generalized Ekeland's variational principle with applications, *Optimization* 63 (2014), 1353–1365.

[24] L.J. Lin and S.Y. Wang, Simultaneous variational relation problems and related applications, *Comput. Math. Appl.* 58 (2009),1711–1721.

[25] F.C. Liu, A note on the von Neumann-Sion minimax principle, *Bull. Inst. Math. Acad. Sinica* 6 (1978), 517–524.

[26] D.T. Luc, An abstract problem in variational analysis, *J. Optim. Theory Appl.* 138 (2008), 65–76.

[27] D. T. Luc, E. Sarabi, A. Soubeyran, Existence of solutions in variational relation problems without convexity, J. Math. Anal. Appl., 364 (2010), 544–555.

[28] S. Park, A generalized minimax inequality related to admissible multimaps and its applications, *J. Korean Math. Soc.* 34 (1997), 719–730.

[29] S. Park, The KKM principle in abstract convex spaces: Equivalent formulations and applications, *Nonlinear Anal.* 73 (2010), 1028-1042.

[30] S. Park, Remarks on the paper "Variational relation problem without the KKM property with applications" by Y.J. Pu and Z. Yang, *Nonlinear Anal. Forum* 17 (2012), 43–51.

[31] M. Patriche, A new fixed point theorem and its applications in equilibrium theory, *Fixed Point Theory* 10 (2009), 159–171.

[32] Y.J. Pu and Z. Yang, Variational relation problem without the KKM property with applications, *J. Math. Anal. Appl.* 393 (2012), 256–264.

[33] X. Wu, A new fixed point theorem and its applications, *Proc. Amer. Math. Soc.* 125 (1997), 1779–1783.

[34] Z. Yang and Y.J. Pu, Generalized Knaster-Kuratowski-Mazurkiewicz theorem without convex hull, *J. Optim. Theory Appl.* 154 (2012), 17–29.

[35] X. Zheng, Approximate selection theorems and their applications, *J. Math. Anal. Appl.* 212 (1997), 88–97.

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