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ON A SURJECTIVITY-TYPE PROPERTY FOR MAXIMAL MONOTONE OPERATORS

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Abstract: In the present note we point out and rectify an incorrect result in Corollary 2.5 of Martínez-Legaz, Some generalizations of Rockafellar's surjectivity theorem (*Pac. J. Optim.*, 4 (2008), no. 3, 527–535). The revision of that result provides further insight on the extent to which the surjectivity theorem can be generalized and on the role played by the duality mapping.

Key words: *Surjectivity theorem, monotone operator, convex function*

Mathematics Subject Classification: *47H05, 47N10, 46T99*

1 Introduction

The well-known surjectivity theorem for maximal monotone operators was first proved by Minty [9] for monotone operators defined on Hilbert spaces and extended by Browder [3] and Rockafellar [13] to the more general case of a reflexive Banach space X (with the additional hypothesis that the norm of X and its polar norm on X^*, the topological dual space of X, are Gâteaux differentiable everywhere, except at the origin). This theorem states that a monotone operator $T : X \rightrightarrows X^*$ is maximal if and only if the range of the sum of T with the duality mapping is all of X^*, where the duality mapping $J : X \rightrightarrows X^*$ is the subdifferential of the function $j : X \to \mathbb{R}, \;\; x \mapsto \frac{1}{2}\|x\|^2$.

Several generalizations of the surjectivity theorem were obtained in the last four decades, mainly along three lines, namely: dismissing the assumptions on the norms and recasting the result in terms of the sum of the graphs of T and $-J$ [14, 16]; partially extending the result to nonreflexive Banach spaces [4, 7, 11]; replacing the duality mapping by more general maximal monotone operators satisfying appropriate conditions [8, 11].

In this note we will focus on the following generalization contained in [8] (see Section 2 below for an explanation of the notions of convex analysis that appear in the statement).

[8, Corollary 2.5] *Let X be a reflexive Banach space, $A : X \rightrightarrows X^*$ be a monotone operator and $f : X \to \mathbb{R}$ be a cofinite lower semicontinuous proper convex function. Then A is maximal monotone if and only if $\mathcal{G}(A) + \mathcal{G}(-\partial f) = X \times X^*$.*

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Unfortunately, the statement that the surjectivity-type condition $\mathcal{G}(A) + \mathcal{G}(-\partial f) = X \times X^*$ implies maximality of A is not true under the hypotheses of this corollary. The wrong conclusion is due to a slight but improper extension of the domain of a quantifier at the end of the proof. At first sight, one could hope to fix this problem quite easily, without introducing further assumptions, but this turns out not to be the case. Indeed, in Section 3 we provide a counterexample to the incorrect implication of [8, Corollary 2.5]. Moreover, we propose new formulations, adding sufficient hypotheses for this implication to hold, while refining the existing assumptions. In particular, the reflexivity assumption will be dropped, unless otherwise specified.

2 Notation

In the present note we will consider a Banach space X and denote by X^* its topological dual. For ease of notation, we will identify X with its image in the bidual X^{**} via the natural injection.

An operator $A : X \rightrightarrows X^*$ is a set-valued mapping and its domain, range and graph are respectively defined as

$$\mathcal{D}(T) = \{x \in X : \ T(x) \neq \emptyset\}, \qquad \mathcal{R}(T) = \{x^* \in X^* : \ \exists x \in X \ (x^* \in T(x))\};$$

$$\mathcal{G}(T) = \{(x, x^*) \in X \times X^* : \ x^* \in T(x)\}.$$

Given two points $(x, x^*), (y, y^*) \in X \times X^*$, we say that they are *monotonically related* if

$$\langle x - y, x^* - y^* \rangle \geq 0,$$

while (x, x^*) is monotonically related to a set $S \subseteq X \times X^*$ if it is monotonically related to any point of S. An operator $A : X \rightrightarrows X^*$ is *monotone* if any point of $\mathcal{G}(A)$ is monotonically related to $\mathcal{G}(A)$ and it is *maximal monotone* if any point $(x, x^*) \in X \times X^*$ that is monotonically related to $\mathcal{G}(A)$ belongs to $\mathcal{G}(A)$.

An important class of maximal monotone operators is given by the subdifferentials of lower semicontinuous proper convex functions. Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$, the *subdifferential* of f is the operator $\partial f : X \rightrightarrows X^*$ defined as

$$\partial f(x) = \begin{cases} \{x^* \in X^* : \quad f(y) \geq f(x) + \langle y - x, x^* \rangle, \ \forall y \in X\}, & \text{if} \quad f(x) \in \mathbb{R} \\ \emptyset, & \text{if} \quad f(x) \notin \mathbb{R}, \end{cases}$$

or, equivalently,

$$\partial f(x) = \begin{cases} \{x^* \in X^* : \quad f(x) + f^*(x^*) = \langle x, x^* \rangle\}, & \text{if} \quad f(x) \in \mathbb{R} \\ \emptyset, & \text{if} \quad f(x) \notin \mathbb{R}, \end{cases}$$

where $f^* : X^* \to \overline{\mathbb{R}}$ is the Fenchel conjugate of f, i.e. the function that maps each $x^* \in X^*$ to

$$f^*(x^*) = \sup_{x \in X} \{\langle x, x^* \rangle - f(x)\}.$$

It is easy to check that the subdifferential is a monotone operator. If f is lower semicontinuous proper convex, then ∂f is maximal monotone [6, 12, 15].

In the case when the proper convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ is Gâteaux differentiable at $x \in X$, we have $\partial f(x) = \{\nabla f(x)\}$, where $\nabla f(x) \in X^*$ is such that, for all $d \in X$,

$$\lim_{t \to 0} \frac{f(x + td) - f(x)}{t} = \langle d, \nabla f(x) \rangle.$$

A sufficient condition for the converse implication to hold, i.e. for the function f to be Gâteaux differentiable at x, given that $\partial f(x)$ is a singleton, is that f be continuous at x [19, Corollary 2.4.10].

Recall that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is cofinite if its Fenchel conjugate f^* is finite.

Finally, in the following section we will denote by $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ the function that maps each real number to its integer part.

3 A Counterexample and Some New Generalizations

We first show, by means of an example, that the surjectivity-type condition $\mathcal{G}(A)+\mathcal{G}(-\partial f) = X \times X^*$ does not imply maximality of the monotone operator A, in general.

Example 3.1. Consider the monotone operator $A : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$A(x) = \begin{cases} \{0\}, & x \in\,]-\infty, 0[\\ ]0, +\infty[, & x = 0. \end{cases}$$

Clearly A is monotone, though not maximal monotone, since $(0, 0)$ is monotonically related to $\mathcal{G}(A)$ but does not belong to it. Moreover, consider the piece-wise affine and continuous function that interpolates $j : \mathbb{R} \to \mathbb{R}$, $x \mapsto \frac{1}{2}x^2$, in its points with abscissa $z+\frac{1}{2}$ for all $z \in \mathbb{Z}$, namely $f : \mathbb{R} \to \mathbb{R}$ such that, for all $x \in \mathbb{R}$,

$$f(x) = x \left\lfloor x + \frac{1}{2} \right\rfloor - \frac{1}{2} \left\lfloor x + \frac{1}{2} \right\rfloor^2 + \frac{1}{8}.$$

It is easy to check that this function is continuous (hence, lower semicontinuous), proper, convex and finite-valued; it is also cofinite, since j is cofinite and $f \geq j$, so that $f^* \leq j^* < +\infty$.

We will prove that $\mathcal{G}(A) + \mathcal{G}(-\partial f) = \mathbb{R}^2$, though A is not maximal monotone. To this end, note that the negative of the subdifferential of f is

$$-\partial f(x) = \begin{cases} -\left\lfloor x + \frac{1}{2} \right\rfloor, & x + \frac{1}{2} \notin \mathbb{Z} \\ \left[-x - \frac{1}{2}, -x + \frac{1}{2}\right], & x + \frac{1}{2} \in \mathbb{Z}, \end{cases}$$

for all $x \in \mathbb{R}$. Given an arbitrary point $(w, w^*) \in \mathbb{R}^2$, we have to prove that $(w, w^*) \in \mathcal{G}(A) + \mathcal{G}(-\partial f)$. Consider three (non-disjoint) cases:

(a) if $w^* > -\left\lfloor w + \frac{1}{2} \right\rfloor$, then

$$(w, w^*) = \left(0, \left\lfloor w + \frac{1}{2} \right\rfloor + w^*\right) + \left(w, -\left\lfloor w + \frac{1}{2} \right\rfloor\right) \in \mathcal{G}(A) + \mathcal{G}(-\partial f);$$

(b) if $w < -\left(\lfloor w^* \rfloor + \frac{1}{2}\right)$, then

$$(w, w^*) = \left(w + \lfloor w^* \rfloor + \frac{1}{2}, 0\right) + \left(-\lfloor w^* \rfloor - \frac{1}{2}, w^*\right) \in \mathcal{G}(A) + \mathcal{G}(-\partial f);$$

(c) if $w^* = -\left\lfloor w + \frac{1}{2} \right\rfloor$, then

$$(w, w^*) = \left(w - \frac{1}{2} - \left\lfloor w + \frac{1}{2} \right\rfloor, 0\right) + \left(\left\lfloor w + \frac{1}{2} \right\rfloor + \frac{1}{2}, w^*\right) \in \mathcal{G}(A) + \mathcal{G}(-\partial f).$$

To conclude, we finally show that the preceding three cases cover all possible cases. Indeed, suppose by contradiction that $(w, w^*) \in \mathbb{R}^2$ does not satisfy any of (a)–(c), i.e.

$$w^* < -\left\lfloor w + \frac{1}{2} \right\rfloor \quad \textit{and} \quad w \geq -\left(\lfloor w^* \rfloor + \frac{1}{2} \right). \tag{3.1}$$

Then, taking the first inequality of (3.1) into account, we obtain

$$-\lfloor w^* \rfloor \geq -w^* > \left\lfloor w + \frac{1}{2} \right\rfloor,$$

that is

$$-\lfloor w^* \rfloor \geq \left\lfloor w + \frac{1}{2} \right\rfloor + 1,$$

from which

$$-\left(\lfloor w^* \rfloor + \frac{1}{2} \right) \geq \left\lfloor w + \frac{1}{2} \right\rfloor + \frac{1}{2} > w,$$

a contradiction to the second inequality (3.1). Then we conclude that $\mathcal{G}(A) + \mathcal{G}(-\partial f) = \mathbb{R}^2$.

In order for the incorrect implication in [8, Corollary 2.5] to hold, it is sufficient to assume that there exists $(p, p^*) \in X \times X^*$ such that

$$\partial f(p) = \{p^*\} \quad \text{and} \quad \partial f^*(p^*) \cap X = \{p\}. \tag{3.2}$$

Notice that these equalities automatically hold if f and f^* are Gâteaux differentiable at p and p^*, respectively. The proof of the implication in [8, Corollary 2.5] that we are considering uses the assumptions that X is reflexive and f is finite-valued and cofinite to prove that f is Gâteaux differentiable at p. Therefore, the hypotheses of [8, Corollary 2.5] yield the first equality of (3.2), by which they can thus be replaced; on the other hand, they have to be complemented by the second equation of (3.2) in order to prove that A is maximal monotone, given the surjectivity-type condition $\mathcal{G}(A) + \mathcal{G}(-\partial f) = X \times X^*$, as shown by the following result.

Proposition 3.2. *Let X be a Banach space, $A : X \rightrightarrows X^*$ be a monotone operator, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous proper convex function and $(p, p^*) \in X \times X^*$ be such that $\partial f(p) = \{p^*\}$ and $\partial f^*(p^*) \cap X = \{p\}$. If $\mathcal{G}(A) + \mathcal{G}(-\partial f) = X \times X^*$, then A is maximal monotone.*

Proof. The equality $\partial f(p) = \{p^*\}$ yields

$$\langle p, p^* \rangle - f^*(p^*) = f(p) > \langle p, y^* \rangle - f^*(y^*), \quad \forall y^* \in X^* \backslash \{p^*\},$$

from which follows that

$$f^*(y^*) > f^*(p^*) + \langle p, y^* - p^* \rangle, \quad \forall y^* \in X^* \backslash \{p^*\}. \tag{3.3}$$

In particular, (3.3) holds for all $(y, y^*) \in \mathcal{G}(\partial f) \backslash \{(p, p^*)\}$, since $y \neq p$ implies $y^* \neq p^*$, as otherwise

$$f^{**}(y) + f^*(p^*) = f(y) + f^*(p^*) = \langle y, p^* \rangle,$$

i.e. $y \in \partial f^*(p^*) \cap X = \{p\}$, a contradiction.

On the other hand, for all $(y, y^*) \in \mathcal{G}(\partial f)$, one has $(y^*, y) \in \mathcal{G}(\partial f^*)$; hence,

$$f^*(p^*) \geq f^*(y^*) + \langle y, p^* - y^* \rangle.$$

The preceding inequality, along with (3.3), yields

$$\langle p-y, p^*-y^*\rangle > 0, \quad \forall (y,y^*) \in \mathcal{G}(\partial f)\backslash\{(p,p^*)\}. \tag{3.4}$$

Hence, assuming that $\mathcal{G}(A)+\mathcal{G}(-\partial f) = X\times X^*$ and reasoning as in the proof of implication (3) $\Longrightarrow$ (1) of [8, Theorem 2.1] (see also [11, Remark 3.5]), we deduce that A is maximal monotone. Indeed, for all $(x,x^*) \in X \times X^*$, we have

$$(x+p, x^*-p^*) \in X \times X^* = \mathcal{G}(A) + \mathcal{G}(-\partial f).$$

As a consequence, there exist $(z,z^*) \in \mathcal{G}(A)$ and $(y,y^*) \in \mathcal{G}(\partial f)$ such that $(x+p, x^*-p^*) = (z,z^*)+(y,-y^*)$, i.e. $x-z = y-p$ and $x^*-z^* = p^*-y^*$. Thus, if (x,x^*) is monotonically related to $\mathcal{G}(A)$, we conclude that

$$0 \leq \langle x-z, x^*-z^*\rangle = -\langle y-p, y^*-p^*\rangle,$$

which, by (3.4), implies that $(y,y^*) = (p,p^*)$, so that $(x,x^*) = (z,z^*) \in \mathcal{G}(A)$. This proves that A is maximal monotone. □

Remark 3.3. (a) Under the hypotheses of the preceding proposition, we do not need to assume that X is reflexive to prove that A is maximal monotone. In [8] it is a blanket assumption and is employed in [8, Corollary 2.5] to prove that f is Gâteaux differentiable at p (furthermore, it is necessary to prove the opposite implication, which we are ignoring here). Notice that, assuming reflexivity of X in Proposition 3.2, the condition $\partial f^*(p^*) \cap X = \{p\}$ simply reads $\partial f^*(p^*) = \{p\}$.

(b) It is easy to check that the condition $\mathcal{G}(A) + \mathcal{G}(-\partial f) = X \times X^*$ is equivalent to $\mathcal{R}(A(\cdot+w) + \partial(f\circ s)) = X^*$ for all $w \in X$, denoting by $s : X \longrightarrow X$ the change of sign operator $s(x) = -x$. Notice that f is a lower semicontinuous proper convex function if and only if $f \circ s$ is; moreover, $\partial(f\circ s) = -\partial f(-\cdot)$ (as a consequence, $\partial(f\circ s)(x)$ does not coincide with $\partial f(s(x)) = \partial f(-x)$, in general) and $\partial(f\circ s)^* = -\partial f^*(-\cdot)$, hence $(p,p^*) \in X \times X^*$ satisfies the assumptions of Proposition 3.2 if and only if $\partial(f\circ s)(-p) = \{-p^*\}$ and $\partial(f\circ s)^*(-p^*) \cap X = \{-p\}$. Therefore, under the assumptions of Proposition 3.2, one also has that if $\mathcal{G}(A) + \mathcal{G}(-\partial(f\circ s)) = X \times X^*$ or, equivalently (as s is idempotent), if

$$\mathcal{R}(A(\cdot+w)+\partial f) = X^* \text{ for all } w \in X, \tag{3.5}$$

then A is maximal monotone. Using a similar argument one can also show that the assumption $\mathcal{G}(A) + \mathcal{G}(-\partial f) = X \times X^*$ in corollaries 3.4, 3.6 and 3.8 below can be replaced by the surjectivity condition (3.5).

Corollary 3.4. *Let X be a Banach space, $A : X \rightrightarrows X^*$ be a monotone operator and $f : X \to \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous proper convex function. Assume that $\mathcal{G}(A) + \mathcal{G}(-\partial f) = X \times X^*$, the subdifferential $\partial f(p)$ is a singleton for some $p \in X$ and one of the following conditions holds:*

(a) *∂f^* is single valued on $\mathcal{R}(\partial f)$;*

(b) *f is strictly convex.*

Then A is maximal monotone.

Proof. Let $p^* \in X^*$ be such that $\partial f(p) = \{p^*\}$.

(a) Suppose that ∂f^* is single valued on $\mathcal{R}(\partial f)$. Then $\partial f^*(p^*) = \{p\}$ and the result follows from Proposition 3.2.

(b) Suppose that f is strictly convex and, towards a contradiction, that there exists $y \in X\backslash\{p\}$ such that $p^* \in \partial f(y)$. Since f is strictly convex, then ∂f is strictly monotone [19, Theorem 2.4.4]. Hence,

$$0 < \langle y - p, p^* - p^* \rangle = 0,$$

which is absurd. Therefore, $\partial f^*(p^*) \cap X = \{p\}$ and the result follows again from Proposition 3.2. □

Remark 3.5. (a) Notice that $\mathcal{R}(\partial f) \subseteq \mathcal{D}(\partial f^*)$, but the inclusion may be strict in nonreflexive Banach spaces. Indeed, if $f := \frac{1}{2}\|\cdot\|^2$ then $\mathcal{D}(\partial f^*) = X^*$, and $\mathcal{R}(\partial f) = X^*$ if and only if X is reflexive (this assertion is equivalent to James' theorem [5, Theorem 5], because $\mathcal{R}(\partial f)$ is the set of continuous linear functionals $x^* \in X^*$ that attain their maxima over the closed unit ball of X); consequently, for any nonreflexive Banach space and $f := \frac{1}{2}\|\cdot\|^2$ the inclusion $\mathcal{R}(\partial f) \subseteq \mathcal{D}(\partial f^*)$ is strict. Thus, statement (a) above is not equivalent to requiring ∂f^* to be single valued on the whole of its domain.

(b) The two assumptions in statements (a) and (b) above are closely related, since single valuedness of the subdifferential of a lower semicontinuous proper convex function at a point is implied by Gâteaux differentiability at that point and strict convexity and differentiability are, in a certain sense, dual properties [2]. One can easily prove, for instance, that if f^{**} is strictly convex at every point in $\mathcal{D}(\partial f)$ (in the sense that $f^{**}((1-\lambda)x + \lambda x^{**}) < (1-\lambda) f(x) + \lambda f^{**}(x^{**})$ for all $x \in \mathcal{D}(\partial f)$, $x^{**} \in (\operatorname{dom} f^{**}) \setminus \{x\}$ and $\lambda \in]0,1[$) then ∂f^* is single valued on $\mathcal{R}(\partial f)$. Note that (a) and (b) in the preceding corollary can be replaced by the following slightly weaker assumptions, respectively: (a') ∂f^* is single valued if we take intersections of its images with X, i.e. the operator $T : \mathcal{R}(\partial f) \rightrightarrows X$ that maps $x^* \in \mathcal{R}(\partial f)$ to $T(x^*) = \partial f^*(x^*) \cap X$ is single valued; (b') f is strictly convex on $\mathcal{D}(\partial f)$, in the sense that, for all $\lambda \in]0,1[$ and all $x, y \in \mathcal{D}(\partial f)$, with $x \neq y$, $f((1-\lambda)x + \lambda y) < (1-\lambda)f(x) + \lambda f(y)$. In this case, (a') implies (b').

(c) If X is a weak Asplund space (in particular, if it is reflexive [17, Corollary 5]), the Gâteaux differentiability of f at some point, and hence the single valuedness of ∂f at that point, is automatically satisfied if dom f has a nonempty interior [10, Proposition 3.3].

As an immediate consequence of Proposition 3.2, we can also recover [11, Corollary 4.4 (b)], a refinement of [8, Proposition 2.9].

Corollary 3.6. *Let X be a Banach space, $A : X \rightrightarrows X^*$ be a monotone operator and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous proper convex function that admits a unique global minimizer p with $\partial f(p) = \{0_{X^*}\}$. If $\mathcal{G}(A) + \mathcal{G}(-\partial f) = X \times X^*$, then A is maximal monotone.*

Proof. Since p is the unique minimizer of f, then $0_{X^*} \notin \partial f(y)$ for any $y \in X\backslash\{p\}$, i.e. $\partial f^*(0_{X^*}) \cap X = \{p\}$. Hence, the result follows from Proposition 3.2. □

Remark 3.7. The equality $\partial f(p) = \{0_{X^*}\}$ is automatically satisfied if f is Gâteaux differentiable at its minimizer p. This latter assumption is used in [11, Corollary 4.4 (b)].

Our last result deals with functions defined on Banach spaces having the Radon-Nikodým property, a large class of spaces which includes all reflexive spaces and, among other nonreflexive spaces, ℓ_1. A Banach space X is said to have the Radon-Nikodým property if every nonempty bounded set $B \subseteq X$ is dentable (that is, for all $\epsilon > 0$ there exists an open half-space in X the intersection of which with B is nonempty and has a diameter smaller than ϵ). We refer to [18] for a detailed discussion on such spaces.

Corollary 3.8. *Let X be a Banach space having the Radon-Nikodým property, $A : X \rightrightarrows X^*$ be a monotone operator and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous proper convex function such that its associated subdifferential mapping ∂f is single-valued on $\mathcal{D}(\partial f)$ and its conjugate function $f^* : X \to \mathbb{R} \cup \{+\infty\}$ is continuous. If $\mathcal{G}(A) + \mathcal{G}(-\partial f) = X \times X^*$, then A is maximal monotone.*

Proof. Since f^* is weak*-lower semicontinuous, by [1, Theorem 1] it is Gâteaux differentiable at some point p^* of its domain with derivative p in the predual space X. Then, since the pair $(p, p^*) \in X \times X^*$ satisfies the assumptions of Proposition 3.2, from this proposition we conclude that A is maximal monotone. □

Remark 3.9. The single valuedness assumption on ∂f will automatically hold if f is Gâteaux differentiable on dom f. As for the continuity assumption on f^*, a sufficient condition for it to hold is the function f to be cofinite, that is, f^* to be finite valued. Indeed, in such a case f^* will be continuous, since it is weak*-lower semicontinuous and hence lower semicontinuous and every lower semicontinuous convex finite valued function is continuous. On the other hand, a sufficient (and necessary in the finite dimensional case) condition for f to be cofinite is it to be supercoercive [2, Theorem 3.4]:

$$\lim_{\|x\| \longrightarrow +\infty} \frac{f(x)}{\|x\|} = +\infty.$$

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References

[1] M. Bachir and A. Daniilidis, A dual characterisation of the Radon-Nikodým property, *Bull. Austral. Math. Soc.* 62 (2000) 379–387.

[2] H.H. Bauschke, J.M. Borwein and P.L. Combettes, Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces, *Commun. Contemp. Math.* 3 (2001) 615–647.

[3] F.E. Browder, Nonlinear maximal monotone operators in Banach space, *Math. Ann.* 175 (1968) 89–113.

[4] J.-P. Gossez, Opérateurs monotones non linéaires dans les espaces de Banach non réflexifs, *J. Math. Anal. Appl.* 34 (1971) 371–395.

[5] R.C. James, Characterizations of reflexivity, *Studia Math.* 23 (1963/1964) 205–216.

[6] M. Marques Alves and B.F. Svaiter, A new proof for maximal monotonicity of subdifferential operators, *J. Convex Anal.* 15 (2008) 345–348.

[7] M. Marques Alves and B.F. Svaiter, On the surjectivity properties of perturbations of maximal monotone operators in non-reflexive Banach spaces, *J. Convex Anal.* 18 (2011) 209–226.

[8] J.E. Martínez-Legaz, Some generalizations of Rockafellar's surjectivity theorem, *Pac. J. Optim.* 4 (2008) 527–535.

[9] G.J. Minty, A theorem on maximal monotonic sets in Hilbert space, *J. Math. Anal. Appl.* 11 (1967) 434–439.

[10] R.R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, Springer, Berlin, 1989.

[11] M. Rocco and J.E. Martínez-Legaz, On surjectivity results for maximal monotone operators of type (D), *J. Convex Anal.* 18 (2011), 545–576.

[12] R.T. Rockafellar, On the maximal monotonicity of subdifferential mappings, *Pacific J. Math.* 33 (1970) 209–216.

[13] R.T. Rockafellar, On the maximality of sums of nonlinear monotone operators, *Trans. Am. Math. Soc.* 149 (1970) 75–88.

[14] S. Simons, *Minimax and Monotonicity*, Springer, Berlin, 1998.

[15] S. Simons, A new proof of the maximal monotonicity of subdifferentials, *J. Convex Anal.* 16 (2009) 165–168.

[16] S. Simons and C. Zălinescu, A new proof for Rockafellar's characterization of maximal monotone operators, *Proc. Amer. Math. Soc.* 132 (2004) 2969–2972.

[17] S.L. Troyanski, On locally uniformly convex and differentiable norms in certain non-separable Banach spaces, *Studia Math.* 37 (1970/71) 173–180.

[18] J.J. Uhl, Jr., The Radon-Nikodým property: a point of view. *Vector space measures and applications*, Lecture Notes in Phys., 77, (Proc. Conf., Univ. Dublin, Dublin, 1977), I, pp. 442–451.

[19] C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific Publishing, River Edge, New Jersey, 2002.

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