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THE BOUNDS FOR THE BEST RANK-1 APPROXIMATION RATIO OF A FINITE DIMENSIONAL TENSOR SPACE*

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Abstract: One topic concerning the best rank-1 approximation of a tensor is the best rank-1 approximation ratio of a finite dimensional tensor space which was defined by Qi [L. Qi, The best rank-one approximation ratio of a tensor space, SIAM J. Matrix Anal. Appl., 32(2)(2011) 430-442]. In this paper we establish a relation between the best rank-1 approximation ratio of a finite dimensional tensor space and the maximal orthogonal rank of this tensor space. By using the maximal orthogonal rank of a tensor space, we provide an alternative proof on the lower bound for the best rank-1 approximation ratio of this tensor space. Furthermore, we acquire that the exact values of the best rank-1 approximation ratios of the real $2 \times n_1 \times n_2$ $(n_1 \leq n_2)$ tensor spaces are equal to $\frac{1}{\sqrt{2n_1}}$ for even n_1 and $\frac{1}{\sqrt{2n_1-1}}$ for odd n_1, respectively. Some upper bounds for the third order finite dimensional general and symmetric tensor spaces are also derived. For completeness, we show that the best rank-1 approximation ratio of the complex $2 \times 2 \times 2$ tensor space is larger than or equal to $\frac{1}{\sqrt{3}}$, illustrating that the exact value of the best rank-1 approximation ratio of a real tensor space tends to be different with the ratio of the complex tensor space with the same dimension.

Key words: *tensor approximation, orthogonal rank, best rank-1 approximation ratio bounds*

Mathematics Subject Classification: *15A18; 15A69*

1 Introduction

One important and basic studying subject relating to the tensor approximation is the best rank-1 approximation of a tensor under the Frobenius norm, which also can be formulated as the extreme Z-eigenvalue problem [11, 20], and the research on this topic takes an important position in current research [4, 8, 23, 24]. Through the best rank-1 approximation of a tensor, Qi [19] proposed the best rank-1 approximation ratio of a tensor space, which is defined as the maximum positive lower bound for the quotient of the best rank-one approximation of any tensor in the tensor space and the norm of that tensor. As stated in [19], the best rank-1 approximation ratio of a tensor space can provide a convergence rate for the greedy rank-1 update algorithm [6, 7, 16, 22]. Furthermore, in the numerical computation field and information science, one always wants to find a suitable low-rank approximation of a given tensor [1, 22]. Thus how to determine the number of rank-1 tensors is worth studying. Coincidentally, one can obtain a reasonable upper bound for the number of rank-1 tensors by using the best rank-1 approximation ratio. Hence, the research on estimating the exact

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value or the upper and lower bounds for the best rank-1 approximation ratio of a tensor space is also a worth studying topic.

However, as pointed out by [19], it is difficult to estimate the exact value, even the upper or lower bounds for the best rank-1 approximation ratio of a tensor space. Meanwhile, Qi [19] proposed four outstanding challenging questions relating to the best rank-1 approximation ratio, which compose the most important studies on the best rank-1 approximation ratio. Based on the result that the best symmetric rank-1 approximation of a symmetric tensor is its best rank-1 approximation [2, 3, 18, 24], Zhang et al. [24] obtained a positive lower bound for the best rank-1 approximation ratio of a symmetric tensor space. We illustrate the previous results on the lower and upper bounds for some classes of tensor space in Table 1.1.

In this paper, we will present the exact values of the best rank-1 approximation ratios of some special real tensor spaces and derive the upper bounds for the third order finite dimensional general and symmetric tensor spaces. Furthermore, based on the phenomenon that the best rank-1 approximation of a nonzero real tensor over the real field may not equal to its best rank-1 approximation over the complex field, we will show that the exact value of the best rank-1 approximation ratio of a real tensor space may not equal to the ratio of the complex tensor space with the same dimension. Meanwhile, some results on the best rank-1 approximation ratio of a complex tensor space are also given. Our contributions on the lower and upper bounds for some classes of tensor space are displayed in Table 1.2.

Table 1.1 Previous results on the lower/upper bounds for some classes of tensor space

tensor spaces	lower bounds	upper bounds
$\mathbb{V}_{\mathbb{R}}(m; n_1, \cdots, n_m)$	$1/\sqrt{n_1 \cdots n_{m-1}}$ [19]	1 [19]
$\mathrm{Sym}^2(\mathbb{R}^n)$	$1/\sqrt{n}$ [19]	$1/\sqrt{n}$ [19]
$\mathrm{Sym}^3(\mathbb{R}^n)$	$1/n$ [19]	$\sqrt{6/(n+5)}$ [19]
$\mathrm{Sym}^4(\mathbb{R}^n)$	$1/(n\sqrt{n})$ [24]	$\sqrt{3/(n^2+2n)}$ [19]

Table 1.2 Our contributions on the lower/upper bounds for some classes of tensor space

tensor spaces	lower bounds	upper bounds
$\mathbb{V}_{\mathbb{R}}(3; 2, n_1, n_2)$	$1/\sqrt{2n_1}$ (for even n_1)	$1/\sqrt{2n_1}$ (for even n_1)
	$1/\sqrt{2n_1 - 1}$ (for odd n_1)	$1/\sqrt{2n_1 - 1}$ (for odd n_1)
$\mathrm{Sym}^3(\mathbb{R}^n)$	$1/n$ [19]	$1/\sqrt{2n}$ (for even n)
	$1/n$ [19]	$1/\sqrt{2n-1}$ (for odd n)
$\mathbb{V}_{\mathbb{C}}(3; 2, 2, 2)$	$1/\sqrt{3}$	$1/\sqrt{2}$

Without loss of generality, we always assume that the tensors discussed in this paper are nonzero tensors and just consider the finite dimensional tensor space. The corresponding notations in the paper are arranged as follows : a tensor is denoted by an underlined letter (e.g. $\underline{A}$), while a scalar is denoted by a plain letter, and a matrix and a vector are denoted by bold letters (e.g. $\mathbf{A}$ and $\mathbf{a}$). Furthermore, we employ some notations in [19]. For example, we let $\mathbb{V}_{\mathbb{R}} \equiv \mathbb{V}_{\mathbb{R}}(m; n_1, \cdots, n_m) = \bigotimes_{j=1}^m \mathbb{R}^{n_j}$, $\mathbb{V}_{\mathbb{C}} \equiv \mathbb{V}_{\mathbb{C}}(m; n_1, \cdots, n_m) = \bigotimes_{j=1}^m \mathbb{C}^{n_j}$, and $\mathrm{Sym}^m(\mathbb{R}^n) = \bigotimes_{j=1}^m \mathbb{R}^{n_j}$ $(n_1 = \cdots = n_m = n)$ be the symmetric tensor spaces, where $n_1 \leq \cdots \leq n_m$.

The rest of this paper is organized as follows. In Section 2, we simply recall some definitions and important results which are needed for the subsequent sections. In Section 3, by using the maximal orthogonal rank of a tensor space, we present the main results of the paper. In Section 4, we discuss the best rank-1 approximation ratio of a complex $2 \times 2 \times 2$ tensor space. Finally, some conclusions are made in Section 5.

2 Preliminaries

This section is devoted to review some concepts and results which are needed for the following sections. Interested readers can refer to more tensor approximation knowledge in [12, 14].

For the finite dimensional tensor space, we denote $\underline{A} = (a_{i_1\cdots i_m}) \in \mathbb{V}_\mathbb{R}$ or $\mathbb{V}_\mathbb{C}$. The Frobenius norm of the tensor $\underline{A}$ is formulated as

$$||\underline{A}||_F = \sqrt{\langle \underline{A}, \underline{A}\rangle} = \left(\sum_{i_1=1}^{n_1} \cdots \sum_{i_m=1}^{n_m} a_{i_1\cdots i_m}\overline{a_{i_1\cdots i_m}}\right)^{1/2}.$$

Let

$$\mathcal{S}_\mathbb{R} = \{\underline{B} \in \mathbb{V}_\mathbb{R} | \underline{B} = \otimes_{j=1}^m \mathbf{v}^{(j)}, \mathbf{v}^{(j)} \in \mathbb{R}^{n_j}\}$$

and

$$\mathcal{S}_\mathbb{C} = \{\underline{B} \in \mathbb{V}_\mathbb{C} | \underline{B} = \otimes_{j=1}^m \mathbf{v}^{(j)}, \mathbf{v}^{(j)} \in \mathbb{C}^{n_j}\}$$

be the set of rank-1 tensors over the real field and complex field, respectively. Then, for $\underline{A} \in \mathbb{V}_\mathbb{R}$, the number

$$\sigma_\mathbb{R}(\underline{A}) = \sqrt{\max_{\underline{B}\in\mathcal{S}_\mathbb{R}, ||\underline{B}||_F=1} |\langle \underline{A}, \underline{B}\rangle|}, \tag{2.1}$$

is equal to Frobenius norm of the best rank-1 approximation of $\underline{A}$ over the real field [19]. Furthermore, it should be noted that $\sigma_\mathbb{R}(\underline{A})$ is well known in the literature to be the spectrum norm of the tensor $\underline{A}$ or the largest singular value of $\underline{A}$, which was first proposed by Lim [17]. Similarly, for $\underline{A} \in \mathbb{V}_\mathbb{C}$, the number

$$\sigma_\mathbb{C}(\underline{A}) = \sqrt{\max_{\underline{B}\in\mathcal{S}_\mathbb{C}, ||\underline{B}||_F=1} |\langle \underline{A}, \underline{B}\rangle|} \tag{2.2}$$

is equal to Frobenius norm of the best rank-1 approximation of $\underline{A}$ over the complex field.

Obviously, by equalities (2.1) and (2.2), one knows that if $\underline{A} \in \mathbb{V}_\mathbb{R}$, then it holds that $\sigma_\mathbb{R}(\underline{A}) \le \sigma_\mathbb{C}(\underline{A})$. Based on the preparation above and the definition of the best rank-1 approximation ratio of a tensor space [19], the best rank-1 approximation ratio of $\mathbb{V}_\mathbb{R}$ and $\mathbb{V}_\mathbb{C}$ can be expressed in the following way, respectively,

$$\mathrm{App}(\mathbb{V}_\mathbb{R}) = \max\left\{\mu : \mu \le \frac{\sigma_\mathbb{R}(\underline{A})}{||\underline{A}||_F}, \forall \underline{A} \in \mathbb{V}_\mathbb{R}, \underline{A} \neq \underline{0}\right\}, \tag{2.3}$$

and

$$\mathrm{App}(\mathbb{V}_\mathbb{C}) = \max\left\{\mu : \mu \le \frac{\sigma_\mathbb{C}(\underline{A})}{||\underline{A}||_F}, \forall \underline{A} \in \mathbb{V}_\mathbb{C}, \underline{A} \neq \underline{0}\right\}. \tag{2.4}$$

At the end of this section, we pay attention to the orthogonal rank of a tensor over the real field, which is given by Definition 2.1 [13].

Definition 2.1. The orthogonal rank of $\underline{A} = (a_{i_1\cdots i_m}) \in \mathbb{V}_\mathbb{R}$ over the real field is defined as the minimal r such that $\underline{A}$ can be expressed as the following form:

$$\underline{A} = \sum_{i=1}^r \sigma_i \underline{U}^{(i)},$$

where $\sigma_i > 0$ and all $\underline{U}^{(i)} \in \mathbb{V}_\mathbb{R}$ are rank-1 tensors such that $\langle \underline{U}^{(i)}, \underline{U}^{(j)}\rangle = 0$ $(i \neq j)$, $||\underline{U}^{(i)}||_F = 1$, for $i, j = 1, 2, \cdots, r$.

According to Definition 2.1, it is easy to prove that the orthogonal rank of a tensor over the real field keeps to be invariant under the multilinear orthogonal transformation [13]. Using the orthogonal rank, we present the definition for the maximal orthogonal rank of a tensor space.

Definition 2.2. The maximal orthogonal rank of a tensor space $\mathbb{V}_{\mathbb{R}}$ is defined as the minimal r such that for any tensor $\underline{A} \in \mathbb{V}_{\mathbb{R}}$, the orthogonal rank of $\underline{A}$ is less than or equal to r.

The orthogonal rank of a tensor over the complex field can be defined in a similar way and the orthogonal rank also keeps to be invariant under the multilinear unitary transformation [13]. For conciseness, we omit it.

3 Estimating the Best Rank-1 Approximation Ratio of a Real Tensor Space

This section consists of two subsections. The first subsection provides a short discussion on the orthogonal rank of a real tensor, which can be used to estimate the best rank-1 approximation ratio of a real tensor space, and presents a way to estimate the orthogonal rank of a given third order tensor. The second subsection is devoted to discuss the bounds for the best rank-1 approximation ratio, which is the main purpose of this section.

3.1 The orthogonal rank of a tensor

In this subsection, we illustrate the relation between the maximal orthogonal rank of a tensor space and the best rank-1 approximation ratio of the corresponding tensor space and present a new proof of the lower bounds for the best rank-1 approximation ratios.

Lemma 3.1. *Let the orthogonal rank of $\underline{A} \in \mathbb{V}_{\mathbb{R}}$ over the real field $\mathbb{R}$ be r. Then*

$$\frac{\sigma_{\mathbb{R}}(\underline{A})}{||\underline{A}||_F} \geq \frac{1}{\sqrt{r}}. \tag{3.1}$$

Proof. Let the orthogonal rank decomposition of the tensor $\underline{A}$ over the real field $\mathbb{R}$ be expressed in the following form:

$$\underline{A} = \underline{A}^{(1)} + \cdots + \underline{A}^{(r)},$$

where all the $\underline{A}^{(i)}s$ are rank-1 tensors such that $\langle \underline{A}^{(i)}, \underline{A}^{(j)} \rangle = 0$ $(i \neq j)$, for $1 \leq i, j \leq r$.

Then it holds that

$$||\underline{A}||_F^2 = \sum_{i=1}^{r} ||\underline{A}^{(i)}||_F^2, \text{ and } ||\underline{A} - \underline{A}^{(i)}||_F^2 = ||\underline{A}||_F^2 - ||\underline{A}^{(i)}||_F^2.$$

Without loss of generality, we assume $||\underline{A}^{(1)}||_F = \max\left\{||\underline{A}^{(i)}||_F | 1 \leq i \leq r\right\}$. According to the Property 3.1 of [15], we have that for any rank-1 tensor $\underline{B}$ it holds

$$||\underline{A} - \underline{B}||_F^2 \geq ||\underline{A}||_F^2 - \sigma_{\mathbb{R}}(\underline{A})^2.$$

Especially, let $\underline{B} = \underline{A}^{(1)}$, then we have

$$||\underline{A} - \underline{B}||_F^2 = ||\underline{A} - \underline{A}^{(1)}||_F^2 = ||\underline{A}||_F^2 - ||\underline{A}^{(1)}||_F^2 \geq ||\underline{A}||_F^2 - \sigma_{\mathbb{R}}(\underline{A})^2. \tag{3.2}$$

It follows from (3.2) that $\sigma_{\mathbb{R}}(\underline{A})^2 \geq ||\underline{A}^{(1)}||_F^2$. Therefore, we have

$$\frac{\sigma_{\mathbb{R}}(\underline{A})}{||\underline{A}||_F} \geq \frac{||\underline{A}^{(1)}||_F}{\sqrt{||\underline{A}^{(1)}||_F^2 + \cdots + ||\underline{A}^{(r)}||_F^2}} \geq \frac{||\underline{A}^{(1)}||_F}{\sqrt{r||\underline{A}^{(1)}||_F^2}} = \frac{1}{\sqrt{r}}.$$

The conclusion is obtained. □

By using Lemma 3.1, one knows that if the upper bound for the maximal orthogonal rank of a tensor space is acquired, then naturally, the lower bound for the best rank-1 approximation ratio of this tensor space can be obtained. Consequently, the problem of estimating the lower bound for the best rank-1 approximation ratio of a tensor space is transformed into the estimation of the upper bound for the maximal orthogonal rank of this tensor space. Furthermore, by (3.1), we get

$$r \geq \left\lfloor \left(\frac{1}{\sigma_{\mathbb{R}}(\underline{A})/||\underline{A}||_F} \right)^2 \right\rfloor, \tag{3.3}$$

where $\lfloor x \rfloor$ denotes the closest integer smaller than x.

Thus, the inequality (3.3) can be used to estimate the lower bound for the orthogonal rank of a tensor.

In what follows, we are to present a simple example to illustrate how to estimate the upper bound for the maximal orthogonal rank of a tensor space.

Example 3.2. Let $\underline{A} = (a_{i_1 i_2 i_3}) \in \mathbb{V}_{\mathbb{R}}(3; 2, 2, 3)$. Then by a simple computation, we have

$$\begin{aligned}
\underline{A} &= \left[\begin{array}{cc|cc|cc} a_{111} & a_{121} & a_{112} & a_{122} & a_{113} & a_{123} \\ a_{211} & a_{221} & a_{212} & a_{222} & a_{213} & a_{223} \end{array}\right] \\
&= \left[\begin{array}{cc|cc|cc} a_{111} & 0 & a_{112} & 0 & a_{113} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] + \left[\begin{array}{cc|cc|cc} 0 & a_{121} & 0 & a_{122} & 0 & a_{123} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\
&\quad + \left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ a_{211} & 0 & a_{212} & 0 & a_{213} & 0 \end{array}\right] + \left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{221} & 0 & a_{222} & 0 & a_{223} \end{array}\right] \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \left(a_{111} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_{112} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_{113} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \\
&\quad + \cdots + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \left(a_{221} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_{222} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_{223} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \\
&= \textstyle\sum_{i_1=1}^2 \sum_{i_2=1}^2 \mathbf{e}_{i_1,2} \otimes \mathbf{e}_{i_2,2} \otimes \left(\sum_{i_3=1}^3 a_{i_1 i_2 i_3} \mathbf{e}_{i_3,3} \right),
\end{aligned}$$

where $\mathbf{e}_{i_j,2}$ and $\mathbf{e}_{i_k,3}$ denote the standard vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

As a result, the maximal orthogonal rank of $\mathbb{V}_{\mathbb{R}}(3; 2, 2, 3)$ is less than or equal to 4. Hence, it holds that $\mathrm{App}(\mathbb{V}_{\mathbb{R}}(3; 2, 2, 3)) \geq \frac{1}{2}$.

With the aforementioned discussion, we provide an alternative proof for the following result, which is one of the important results in [19].

Theorem 3.3 ([19])**.** *Let $n_1 \leq \cdots \leq n_m$, and*

$$\mu = \frac{1}{\sqrt{n_1 \cdots n_{m-1}}}.$$

Then μ is a positive lower bound for $\mathrm{App}(\mathbb{V}_{\mathbb{R}})$.

Proof. Let $\underline{A} = (a_{i_1\cdots i_m}) \in \mathbb{V}_{\mathbb{R}}$. Then the tensor $\underline{A}$ can be expressed in the following form:

$$\underline{A} = \sum_{i_1=1}^{n_1} \cdots \sum_{i_{m-1}=1}^{n_{m-1}} \otimes_{k=1}^{m-1} \mathbf{e}_{i_k,n_k} \otimes \left(\sum_{i_m=1}^{n_m} a_{i_1 i_2 \cdots i_m} \mathbf{e}_{i_m,n_m} \right), \tag{3.4}$$

where $\mathbf{e}_{i_k,n_k} (1 \le i_k \le n_k)$ denote the standard vectors in $\mathbb{R}^{n_k}$ for $1 \le k \le m$.

It follows from (3.4) that the orthogonal rank of $\underline{A}$ over the real field is less than or equal to $n_1 \cdots n_{m-1}$. By the arbitrariness of $\underline{A}$, we get that the maximal orthogonal rank of $\mathbb{V}_{\mathbb{R}}$ is less than or equal to $n_1 \cdots n_{m-1}$. Then by using Lemma 3.1, we get the conclusion. □

At last in this subsection, we present a result relating to the upper bound for the orthogonal rank of a third order tensor.

Lemma 3.4. *Let $\underline{A} = [\mathbf{A}_1 | \cdots | \mathbf{A}_{n_3}] \in \mathbb{V}_{\mathbb{R}}(3; n_1, n_2, n_3)$. Suppose that the rank of the matrix $\mathbf{A}_i$ is equal to r_i, $1 \le i \le n_3$, respectively. Then the orthogonal rank of $\underline{A}$ over the real field is less than or equal to $\sum_{i=1}^{n_3} r_i$.*

Proof. The proof is straightforward. According to the assumptions that the rank of the matrix $\mathbf{A}_i$ is equal to r_i, by using the singular value decomposition of the matrix $\mathbf{A}_i$,

$$\mathbf{A}_i = \sigma_1^{(i)} \mathbf{u}_1^{(i)} (\mathbf{v}_1^{(i)})^T + \cdots + \sigma_{r_i}^{(i)} \mathbf{u}_{r_i}^{(i)} (\mathbf{v}_{r_i}^{(i)})^T, 1 \le i \le n_3,$$

we have that the tensor $\underline{A}$ can be expressed as the following form:

$$\underline{A} = \sum_{i=1}^{r_1} \sigma_i^{(1)} \mathbf{u}_i^{(1)} \otimes \mathbf{v}_i^{(1)} \otimes \mathbf{e}_{1,n_3} + \cdots + \sum_{i=1}^{r_{n_3}} \sigma_i^{(n_3)} \mathbf{u}_i^{(n_3)} \otimes \mathbf{v}_i^{(n_3)} \otimes \mathbf{e}_{n_3,n_3}, \tag{3.5}$$

where $\langle \mathbf{u}_j^{(i)}, \mathbf{u}_k^{(i)} \rangle = 0$, $\langle \mathbf{v}_j^{(i)}, \mathbf{v}_k^{(i)} \rangle = 0 (j \ne k)$, $||\mathbf{u}_j^{(i)}||_2 = ||\mathbf{v}_j^{(i)}||_2 = 1$, and $\sigma_j^{(i)} > 0$, $1 \le i \le n_3$.

Thus, by (3.5) we get that the orthogonal rank of $\underline{A}$ is less than or equal to $\sum_{i=1}^{n_3} r_i$. □

By Lemma 3.4, it is easy to see that one can obtain the upper bound for the orthogonal rank of a tensor through estimating the ranks of all slices of this tensor.

3.2 The bounds for the best rank-1 approximation ratio of a real tensor space

On the basis of the preceding subsection, in this subsection, we will present the exact values for the best rank-1 approximation ratio of some special real tensor spaces, and estimate the upper bounds for the best rank-1 approximation ratio of general real tensor spaces.

Lemma 3.5. $\mathrm{App}(\mathbb{V}_{\mathbb{R}}(3; 2, 2, 2)) = \frac{1}{2}$.

Proof. According to the results on the maximization of a multilinear function over spherical constraints [9], or the largest singular value of a tensor [17], is easy to evaluate that the largest singular value of the tensor

$$\underline{A} = [\mathbf{I}_2 | \mathbf{A}] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{array} \right] \tag{3.6}$$

over the real field is equal to 1, where $\mathbf{I}_2$ denotes the 2×2 identity matrix and

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{3.7}$$

Hence, it holds that $\mathrm{App}(\mathbb{V}_{\mathbb{R}}(3;2,2,2)) \le \frac{1}{\sqrt{1+1+1+1}} = \frac{1}{2}$. Furthermore, by using Theorem 3.3, we obtain that $\mathrm{App}(\mathbb{V}_{\mathbb{R}}(3;2,2,2)) \ge \frac{1}{2}$. Two inequalities together lead to the conclusion. $\square$

Remark 3.6. The largest singular value of a real tensor over the real field may be not equal to its largest singular value over the complex field.

It is easy to check that the tensor $\underline{A}$ defined by equality (3.6) can be formulated as

$$\underline{A} = \frac{1}{2}\begin{bmatrix} 1 \\ -i \end{bmatrix} \otimes \begin{bmatrix} 1 \\ i \end{bmatrix} \otimes \begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ i \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -i \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -i \end{bmatrix}. \tag{3.8}$$

Let

$$\mathbf{L} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}, \mathbf{M} = \mathbf{N} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}.$$

Then it holds that

$$(\mathbf{L}, \mathbf{M}, \mathbf{N}) \cdot \underline{A} = \left[\begin{array}{cc|cc} \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{array}\right]. \tag{3.9}$$

Since all three matrices $\mathbf{L}$, $\mathbf{M}$, and $\mathbf{N}$ are unitary matrices, it follows from (3.8) and (3.9) that the largest singular value of $\underline{A}$ over the complex field is equal to $\sqrt{2}$ [21]. That is $\sigma_{\mathbb{C}}(\underline{A}) = \sqrt{2}$, which is not equal to $\sigma_{\mathbb{R}}(\underline{A}) = 1$.

We then present the main results of this subsection as follows.

Theorem 3.7. *If $n_1 \le n_2$ and n_1 is even, then* $\mathrm{App}(\mathbb{V}_{\mathbb{R}}\ (3;2,n_1,n_2)) = \frac{1}{\sqrt{2n_1}}$.

Proof. Let $\mathbf{A}$ be defined by (3.7). Then, by a simple computation, we get that the largest singular value of the tensor

$$\underline{A} = \left[\begin{array}{cccc|ccc|ccccccc} \multicolumn{4}{c}{\overbrace{\hphantom{\mathbf{I}_2\ \mathbf{0}\ \cdots\ \mathbf{0}}}^{n_1\ \textbf{columns}}} & \multicolumn{3}{c}{\overbrace{\hphantom{\mathbf{0}\ \cdots\ \mathbf{0}}}^{(n_2-n_1)\ \textbf{columns}}} & & & & & & & \\ \mathbf{I}_2 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{A} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{A} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \mathbf{0} & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}_2 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A} & \mathbf{0} & \cdots & \mathbf{0} \end{array}\right] \in \mathbb{R}^{n_1 \times n_2 \times 2}$$

over the real field is equal to 1. Thus it holds that $\mathrm{App}(\mathbb{V}_{\mathbb{R}}(3;2,n_1,n_2)) \le \frac{1}{\sqrt{2n_1}}$. Noting the inequality $\mathrm{App}(\mathbb{V}_{\mathbb{R}}(3;2,n_1,n_2)) \ge \frac{1}{\sqrt{2n_1}}$, we have $\mathrm{App}(\mathbb{V}_{\mathbb{R}}(3;2,n_1,n_2)) = \frac{1}{\sqrt{2n_1}}$. $\square$

Corollary 3.8. *If $n_1 \le n_2$ and n_1 is even, then the maximal orthogonal rank of $\mathbb{V}_{\mathbb{R}}$ $(3;2,n_1,n_2)$ over the real field is equal to $2n_1$.*

Proof. Based on the proof of Theorem 3.7, we know that there exists a tensor $\underline{A}$ such that $\frac{\sigma_{\mathbb{R}}(\underline{A})}{||\underline{A}||_F} = \frac{1}{\sqrt{2n_1}}$. Thus, it follows from inequality (3.3) that the maximal orthogonal rank of $\mathbb{V}_{\mathbb{R}}$ $(3;2,n_1,n_2)$ over the real field is larger than or equal to $2n_1$. Furthermore, by (3.4), we get that the maximal orthogonal rank of $\mathbb{V}_{\mathbb{R}}$ $(3;2,n_1,n_2)$ over the real field is less than or equal to $2n_1$. Hence, the conclusion can be conducted. $\square$

Similar to the proof of Theorem 3.7, by adding a zeros vector and a stand vector ($\mathbf{e}_{n_1,n_2} \in \mathbb{R}^{n_2}$) in the previous construction, we can get

$$\mathrm{App}(\mathbb{V}_{\mathbb{R}}(3;2,n_1,n_2)) \leq \frac{1}{\sqrt{2n_1-1}}, \tag{3.10}$$

where n_1 is odd and $n_1 \leq n_2$.

Thus, the maximal orthogonal rank of $\mathbb{V}_{\mathbb{R}}$ $(3;2,n_1,n_2)$ over the real field is large than or equal to $2n_1-1$ when n_1 is odd. However, what are the exact values of the maximal orthogonal rank of $\mathrm{App}(\mathbb{V}_{\mathbb{R}}(3;2,n_1,n_2))$ over the real field and what are the exact values of $\mathrm{App}(\mathbb{V}_{\mathbb{R}}(3;2,n_1,n_2))$ for odd n_1? In what follows, we will discuss these two problems.

Lemma 3.9. *If $n \geq 3$ and n is odd, then the maximal orthogonal rank of $\mathbb{V}_{\mathbb{R}}$ $(3;2,n,n)$ over the real field is equal to $2n-1$.*

Proof. Based on the discussion above, we know that the maximal orthogonal rank of $\mathbb{V}_{\mathbb{R}}(3;2,n,n)$ over the real field is larger than or equal to $2n-1$. As a result, we just need to prove that the maximal orthogonal rank of $\mathbb{V}_{\mathbb{R}}(3;2,n,n)$ over the real field is less than or equal to $2n-1$.

Let $\underline{A} = [\mathbf{A}_1|\mathbf{A}_2] \in \mathbb{R}^{n\times n\times 2}$, where $\mathbf{A}_i \in \mathbb{R}^{n\times n}$ and $i=1,2$. If at least one of $\mathbf{A}_i$ is nonsingular, we get that at least one rank of $\mathbf{A}_i$ is less than or equal to $n-1$. Thus, by Lemma 3.4, we obtain that the orthogonal rank of $\underline{A}$ over the real field is less than or equal to $2n-1$. Thus to prove the final result, we only need to further discuss the case that both $\mathbf{A}_1$ and $\mathbf{A}_2$ are nonsingular.

Let $\mathbf{A}_1^{-1}$ denote the inverse of the matrix $\mathbf{A}_1$. Since n is an odd number, we have that the matrix $\mathbf{A}_1^{-1}\mathbf{A}_2$ has at least one real eigenvalues. Assume that λ is one real eigenvalue of $\mathbf{A}_1^{-1}\mathbf{A}_2$, then there exists a nonsingular matrix $\mathbf{P} \in \mathbb{V}_{\mathbb{R}}(2;n,n)$ such that

$$\mathbf{A}_1^{-1}\mathbf{A}_2 = \mathbf{P}^{-1}\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \lambda \end{bmatrix}\mathbf{P} = \mathbf{P}^{-1}\tilde{\mathbf{A}}\mathbf{P},$$

where $\mathbf{A} \in \mathbb{V}_{\mathbb{R}}(2;n-1,n-1)$, and

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \lambda \end{bmatrix}.$$

Then, by using the properties of the multilinear transformation, we have

$$(\mathbf{P},(\mathbf{P}^{-1})^T,\mathbf{I}_2)\cdot(\mathbf{A}_1^{-1},\mathbf{I}_n,\mathbf{I}_2)\cdot\underline{A} = [\mathbf{I}_n|\tilde{\mathbf{A}}].$$

Thus, it holds that

$$\underline{A} = (\mathbf{L},\mathbf{M},\mathbf{I}_2)\cdot[\mathbf{I}_n|\tilde{\mathbf{A}}] = [\mathbf{L}\mathbf{M}^T|\mathbf{L}\tilde{\mathbf{A}}\mathbf{M}^T], \tag{3.11}$$

where $\mathbf{L} = \mathbf{A}_1\mathbf{P}^{-1}$, $\mathbf{M} = \mathbf{P}^T$.

Let the QR decomposition of the matrices $\mathbf{L}$ and $\mathbf{M}$ be

$$\mathbf{L} = \mathbf{Q}^{(1)}\mathbf{R}^{(1)} = \mathbf{Q}^{(1)}\begin{bmatrix} r_{11}^{(1)} & r_{12}^{(1)} & \cdots & r_{1n}^{(1)} \\ 0 & r_{22}^{(1)} & \cdots & r_{2n}^{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & r_{nn}^{(1)} \end{bmatrix} = \mathbf{Q}^{(1)}\begin{bmatrix} \mathbf{R}_1^{(1)} & \mathbf{r}^{(1)} \\ \mathbf{0} & r_{nn}^{(1)} \end{bmatrix}, \tag{3.12}$$

and

$$\mathbf{M} = \mathbf{Q}^{(2)}\mathbf{R}^{(2)} = \mathbf{Q}^{(2)} \begin{bmatrix} r_{11}^{(2)} & 0 & \cdots & 0 \\ r_{21}^{(2)} & r_{22}^{(2)} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ r_{n1}^{(2)} & r_{n2}^{(2)} & \cdots & r_{nn}^{(2)} \end{bmatrix} = \mathbf{Q}^{(2)} \begin{bmatrix} \mathbf{R}_1^{(2)} & \mathbf{0} \\ \mathbf{r}^{(2)^T} & r_{nn}^{(2)} \end{bmatrix}, \tag{3.13}$$

respectively, where $\mathbf{r}^{(1)} = [r_{1n}^{(1)}, r_{2n}^{(1)}, \cdots, r_{n-1,n}^{(1)}]^T$, $\mathbf{r}^{(2)} = [r_{n1}^{(1)}, r_{n2}^{(1)}, \cdots, r_{n,n-1}^{(1)}]^T$, and

$$\mathbf{R}_1^{(1)} = \begin{bmatrix} r_{11}^{(1)} & r_{12}^{(1)} & \cdots & r_{1n-1}^{(1)} \\ 0 & r_{22}^{(1)} & \cdots & r_{2n-1}^{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & r_{n-1,n-1}^{(1)} \end{bmatrix}, \mathbf{R}_1^{(2)} = \begin{bmatrix} r_{11}^{(2)} & 0 & \cdots & 0 \\ r_{21}^{(2)} & r_{22}^{(2)} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ r_{n-1,1}^{(2)} & r_{n-1,2}^{(2)} & \cdots & r_{n-1,n-1}^{(2)} \end{bmatrix}.$$

Then by using (3.11), (3.12) and (3.13), we have

$$((\mathbf{Q}^{(1)})^T, (\mathbf{Q}^{(2)})^T, \mathbf{I}_2) \cdot \underline{A} = [\mathbf{R}^{(1)}(\mathbf{R}^{(2)})^T | \mathbf{R}^{(1)}\tilde{\mathbf{A}}(\mathbf{R}^{(2)})^T]. \tag{3.14}$$

Furthermore, by a simple computation, we get

$$\begin{aligned} \mathbf{R}^{(1)}(\mathbf{R}^{(2)})^T &= \begin{bmatrix} \mathbf{R}_1^{(1)} & \mathbf{r}^{(1)} \\ \mathbf{0} & r_{nn}^{(1)} \end{bmatrix} \begin{bmatrix} (\mathbf{R}_1^{(2)})^T & \mathbf{r}^{(2)} \\ \mathbf{0} & r_{nn}^{(2)} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{R}_1^{(1)}(\mathbf{R}_1^{(2)})^T & \mathbf{R}_1^{(1)}\mathbf{r}^{(2)} + r_{nn}^{(2)}\mathbf{r}^{(1)} \\ \mathbf{0} & r_{nn}^{(1)}r_{nn}^{(2)} \end{bmatrix}, \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \mathbf{R}^{(1)}\tilde{\mathbf{A}}(\mathbf{R}^{(2)})^T &= \begin{bmatrix} \mathbf{R}_1^{(1)} & \mathbf{r}^{(1)} \\ \mathbf{0} & r_{nn}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \lambda \end{bmatrix} \begin{bmatrix} (\mathbf{R}_1^{(2)})^T & \mathbf{r}^{(2)} \\ \mathbf{0} & r_{nn}^{(2)} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{R}_1^{(1)}\mathbf{A}(\mathbf{R}_1^{(2)})^T & \mathbf{R}_1^{(1)}\mathbf{A}\mathbf{r}^{(2)} + \lambda r_{nn}^{(2)}\mathbf{r}^{(1)} \\ \mathbf{0} & \lambda r_{nn}^{(1)}r_{nn}^{(2)} \end{bmatrix}. \end{aligned} \tag{3.16}$$

Take right hand sides of equalities (3.15) and (3.16) into (3.14), respectively, we get

$$\begin{aligned} &((\mathbf{Q}^{(1)})^T, (\mathbf{Q}^{(2)})^T, \mathbf{I}_2) \cdot \underline{A} \\ &= \left[\begin{array}{cc|cc} \mathbf{R}_1^{(1)}(\mathbf{R}_1^{(2)})^T & \mathbf{R}_1^{(1)}\mathbf{r}^{(2)} + r_{nn}^{(2)}\mathbf{r}^{(1)} & \mathbf{R}_1^{(1)}\mathbf{A}(\mathbf{R}_1^{(2)})^T & \mathbf{R}_1^{(1)}\mathbf{A}\mathbf{r}^{(2)} + \lambda r_{nn}^{(2)}\mathbf{r}^{(1)} \\ \mathbf{0} & 0 & \mathbf{0} & 0 \end{array}\right] \\ &\quad + \left[\begin{array}{cc|cc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & r_{nn}^{(1)}r_{nn}^{(2)} & \mathbf{0} & \lambda r_{nn}^{(1)}r_{nn}^{(2)} \end{array}\right]. \end{aligned} \tag{3.17}$$

It follows from Corollary 3.8 and (3.17) that the orthogonal rank of $\underline{A}$ over the real field is less than or equal to $2(n-1)+1 = 2n-1$. Thus the maximal orthogonal rank of $\mathbb{V}_{\mathbb{R}}(3; 2, n, n)$ over the real field is equal to $2n-1$. $\square$

Through the proof of Lemma 3.9, one can see that by using the multi-linear orthogonal transformation, a given tensor can be simplified, thus it is easy to formulate the tensor as the sum of some orthogonal rank-1 tensors. Similarly, the following conclusion can be obtained.

Corollary 3.10. *If $n_1 \le n_2$ and n_1 is odd, then the maximal orthogonal rank of $\mathbb{V}_{\mathbb{R}}(3;2,n_1,n_2)$ over the real field is equal to $2n_1-1$.*

Conversely, by using the maximal orthogonal rank of a real tensor space over the real field, one can get the exact value of the best rank-1 approximation ratio of a real tensor space.

Theorem 3.11. *If $n_1 \le n_2$ and n_1 is odd, then* $\mathrm{App}(\mathbb{V}_{\mathbb{R}}(3;2,n_1,n_2)) = \frac{1}{\sqrt{2n_1-1}}$.

Proof. The proof is straightforward. On the one hand it follows from Lemma 3.9 and Corollary 3.10 that $\mathrm{App}(\mathbb{V}_{\mathbb{R}}(3;2,n_1,n_2)) \ge \frac{1}{\sqrt{2n_1-1}}$, and on the other hand by (3.10) we get $\mathrm{App}(\mathbb{V}_{\mathbb{R}}(3;2,n_1,n_2)) \le \frac{1}{\sqrt{2n_1-1}}$. Hence, the conclusion is obtained. □

It should be noted that in general cases it is difficult to get the exact value of the best rank-1 approximation ratio of a tensor space.

Furthermore, by using Theorem 3.7 and Theorem 3.11, we have the following conclusion.

Corollary 3.12. *Let $2 \le n_1 \le n_2 \le n_3$. Then it holds that*

$$\mathrm{App}(\mathbb{V}_{\mathbb{R}}(3;n_1,n_2,n_3)) \le \begin{cases} \frac{1}{\sqrt{2n_2}}, & \text{for even } n_2, \\ \frac{1}{\sqrt{2n_2-1}}, & \text{for odd } n_2. \end{cases}$$

Proof. If n_2 is even, then by Theorem 3.7, we know that there exists a tensor $\underline{A} = [\mathbf{A}_1|\mathbf{A}_2] \in \mathbb{R}^{n_2\times n_3\times 2}$ such that $\frac{\sigma_{\mathbb{R}}(\underline{A})}{||\underline{A}||_F} = \frac{1}{\sqrt{2n_2}}$. Let

$$\hat{\underline{A}} = [\mathbf{A}_1|\mathbf{A}_2|\mathbf{0}|\cdots|\mathbf{0}] \in \mathbb{R}^{n_2\times n_3\times n_1}.$$

Then it holds $\frac{\sigma_{\mathbb{R}}(\hat{\underline{A}})}{||\hat{\underline{A}}||_F} = \frac{1}{\sqrt{2n_2}}$. Thus, we have $\mathrm{App}(\mathbb{V}_{\mathbb{R}}(3;n_1,n_2,n_3)) \le \frac{1}{\sqrt{2n_2}}$. In a similar way, for odd n_2, the conclusion $\mathrm{App}(\mathbb{V}_{\mathbb{R}}(3;n_1,n_2,n_3)) \le \frac{1}{\sqrt{2n_2-1}}$ can be proved. □

Especially, for the symmetric tensor spaces $\mathrm{Sym}^3(\mathbb{R}^n)$ $(n \ge 2)$, the above conclusion is also true.

Theorem 3.13.

$$\mathrm{App}(\mathrm{Sym}^3(\mathbb{R}^n)) \le \begin{cases} \frac{1}{\sqrt{2n}}, & \text{for even } n, \\ \frac{1}{\sqrt{2n-1}}, & \text{for odd } n. \end{cases} \tag{3.18}$$

Furthermore, if $n = 2$, then $\mathrm{App}(\mathrm{Sym}^3(\mathbb{R}^n)) = \frac{1}{2}$.

Proof. Let

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3.19}$$

It is obvious that the tensor

$$\underline{A}_2 = [\mathbf{A}_1|\mathbf{A}_2] = \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{array}\right]$$

is a tensor in $\mathrm{Sym}^3(\mathbb{R}^2)$. According to the result that the best symmetric rank-1 approximation of a symmetric tensor is its best rank-1 approximation [2, 3, 18, 24] and the results on the maximization of a multilinear function over spherical constraints [9], or the largest singular value of a tensor [17], we get that the Frobenius norm of the best symmetric rank-1 approximation of $\underline{A}_2$ over the real field is equal to 1. Thus, it holds that $\mathrm{App}(\mathrm{Sym}^3(\mathbb{R}^2)) \leq \frac{1}{2}$. Based on Theorem 3.3, we then have $\mathrm{App}(\mathrm{Sym}^3(\mathbb{R}^2)) = \frac{1}{2}$.

In a similar way, let

$$\underline{A}_3 = \left[\begin{array}{cc|cc|cc} \mathbf{A}_1 & \mathbf{0} & \mathbf{A}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & \mathbf{0} & 0 & \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

It is easy to check that the tensor $\underline{A}_3$ is a tensor in $\mathrm{Sym}^3(\mathbb{R}^3)$ and using the similar method above, we can get that the Frobenius norm of the best rank-1 approximation of $\underline{A}_3$ over the real field is equal to 1. Thus, it holds that $\mathrm{App}(\mathrm{Sym}^3(\mathbb{R}^3)) \leq \frac{1}{\sqrt{5}}$.

Recursively, for even n, let

$$\underline{A}_n = \left[\begin{array}{c|cccc|c|cccc|cccc} \overbrace{\begin{matrix} \mathbf{A}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{matrix}}^{n\ \textbf{columns}} & \begin{matrix} \mathbf{A}_2 \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{matrix} & \begin{matrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{matrix} & \begin{matrix} \cdots \\ \cdots \\ \cdots \\ \cdots \end{matrix} & \begin{matrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{matrix} & \cdots & \begin{matrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{matrix} & \begin{matrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{matrix} & \begin{matrix} \cdots \\ \cdots \\ \cdots \\ \cdots \end{matrix} & \begin{matrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{A}_1 \end{matrix} & \begin{matrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{matrix} & \begin{matrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{matrix} & \begin{matrix} \cdots \\ \cdots \\ \cdots \\ \cdots \end{matrix} & \begin{matrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{A}_2 \end{matrix} \end{array}\right].$$

Then $\underline{A}_n$ is a tensor in $\mathrm{Sym}^3(\mathbb{R}^n)$.

For odd n, let

$$\underline{A}_n = \left[\begin{array}{c|c|ccccc|ccccc|cccc} \overbrace{\begin{matrix} \mathbf{A}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & 0 \end{matrix}}^{n\ \textbf{columns}} & \cdots & \begin{matrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \end{matrix} & \begin{matrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \end{matrix} & \begin{matrix} \cdots \\ \cdots \\ \cdots \\ \cdots \end{matrix} & \begin{matrix} \mathbf{0} \\ \vdots \\ \mathbf{A}_1 \\ \mathbf{0} \end{matrix} & \begin{matrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ 0 \end{matrix} & \begin{matrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \end{matrix} & \begin{matrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \end{matrix} & \begin{matrix} \cdots \\ \cdots \\ \cdots \\ \cdots \end{matrix} & \begin{matrix} \mathbf{0} \\ \vdots \\ \mathbf{A}_2 \\ \mathbf{0} \end{matrix} & \begin{matrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ 0 \end{matrix} & \begin{matrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \end{matrix} & \begin{matrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \end{matrix} & \begin{matrix} \cdots \\ \cdots \\ \cdots \\ \cdots \end{matrix} & \begin{matrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ 1 \end{matrix} \end{array}\right].$$

Then $\underline{A}_n$ is a tensor in $\mathrm{Sym}^3(\mathbb{R}^n)$.

Thus, by using a similar method, the conclusion can be obtained. $\square$

Remark 3.14. The upper bounds given by (3.18) are smaller than the upper bounds $\sqrt{\frac{6}{n+5}}$ given by Qi [19], for $n \geq 2$.

$\boxed{4}$ Estimating the Best Rank-1 Approximation Ratio of a Complex $2 \times 2 \times 2$ Tensor Space

The goal of this section is to discuss the best rank-1 approximation ratio of a complex tensor space and illustrate that the exact value of the best rank-1 approximation ratio of a finite dimensional real tensor space may not equal the exact value of the best rank-1 approximation ratio of the complex tensor space with the same dimension.

Lemma 4.1. *Let $\underline{A} \in \mathbb{V}_{\mathbb{C}}(3; 2, 2, 2)$. Then the maximal orthogonal rank of $\underline{A}$ over the complex field is equal to* 3.

Proof. It is well know that the maximal rank of $\underline{A}$ over the complex field is equal to 3 [5]. Since the orthogonal rank of a tensor over a field is larger than or equal to its rank over this field, we can get that the maximal orthogonal rank of $\underline{A}$ over the complex field is large than or equal to 3. Hence, the rest of proof is to show that the maximal orthogonal rank of $\underline{A}$ over the complex field is less than or equal to 3.

According to the rank of the tensor $\underline{A}$, the proof can be divided into the following three cases.

(I) If $\underline{A}$ is a rank-1 tensor, obviously, the orthogonal rank of $\underline{A}$ over the complex field is equal to 1.

(II) If $\underline{A}$ is a rank-2 tensor, then there exist $\mathbf{a}_i \in \mathbb{C}^2$, $\mathbf{b}_i \in \mathbb{C}^2$ and $\mathbf{c}_i \in \mathbb{C}^2, (1 \leq i \leq 2)$ such that

$$\underline{A} = \mathbf{a}_1 \circ \mathbf{b}_1 \circ \mathbf{c}_1 + \mathbf{a}_2 \circ \mathbf{b}_2 \circ \mathbf{c}_2.$$

Let the QR factorization of the matrix $[\mathbf{a}_1, \mathbf{a}_2]$ be

$$[\mathbf{a}_1, \mathbf{a}_2] = \mathbf{Q}^{(1)}\mathbf{R}^{(1)} = \mathbf{Q}^{(1)} \begin{bmatrix} r_{11}^{(1)} & r_{12}^{(1)} \\ 0 & r_{22}^{(1)} \end{bmatrix}.$$

Then it holds that

$$\begin{aligned} ((\mathbf{Q}^{(1)})^*, \mathbf{I}_2, \mathbf{I}_2) \cdot \underline{A} &= \begin{bmatrix} r_{11}^{(1)} \\ 0 \end{bmatrix} \circ \mathbf{b}_1 \circ \mathbf{c}_1 + \begin{bmatrix} r_{12}^{(1)} \\ r_{22}^{(1)} \end{bmatrix} \circ \mathbf{b}_2 \circ \mathbf{c}_2 \\ &= \mathbf{e}_1 \circ (r_{11}^{(1)}\mathbf{b}_1 \circ \mathbf{c}_1 + r_{12}^{(1)}\mathbf{b}_2 \circ \mathbf{c}_2) + r_{22}^{(1)}\mathbf{e}_2 \circ \mathbf{b}_2 \circ \mathbf{c}_2. \end{aligned} \tag{4.1}$$

By using (4.1), it is easy to obtain that the orthogonal rank of $\underline{A}$ over the complex field is less than or equal to 3.

(III) If $\underline{A}$ is a rank-3 tensor, according to the canonical forms under the nonsingular transformation [5], then there exist nonsingular matrices $\mathbf{L}$, $\mathbf{M}$ and $\mathbf{N} \in \mathbb{C}^{2\times 2}$, such that

$$(\mathbf{L}, \mathbf{M}, \mathbf{N}) \cdot \underline{A} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right] = \mathbf{e}_1 \circ \mathbf{e}_1 \circ \mathbf{e}_1 + \mathbf{e}_2 \circ \mathbf{e}_2 \circ \mathbf{e}_1 + \mathbf{e}_1 \circ \mathbf{e}_2 \circ \mathbf{e}_2.$$

Thus, it holds that

$$\begin{aligned} \underline{A} &= (\mathbf{L}^{-1}, \mathbf{M}^{-1}, \mathbf{N}^{-1}) \cdot (\mathbf{e}_1 \circ \mathbf{e}_1 \circ \mathbf{e}_1 + \mathbf{e}_2 \circ \mathbf{e}_2 \circ \mathbf{e}_1 + \mathbf{e}_1 \circ \mathbf{e}_2 \circ \mathbf{e}_2) \\ &= \tilde{\mathbf{l}}_1 \circ \tilde{\mathbf{m}}_1 \circ \tilde{\mathbf{n}}_1 + \tilde{\mathbf{l}}_2 \circ \tilde{\mathbf{m}}_2 \circ \tilde{\mathbf{n}}_1 + \tilde{\mathbf{l}}_1 \circ \tilde{\mathbf{m}}_2 \circ \tilde{\mathbf{n}}_2, \end{aligned}$$

where $\mathbf{L}^{-1} = [\tilde{\mathbf{l}}_1, \tilde{\mathbf{l}}_2]$, $\mathbf{M}^{-1} = [\tilde{\mathbf{m}}_1, \tilde{\mathbf{m}}_2]$, and $\mathbf{N}^{-1} = [\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2]$.

Similar to the proof above, let the QR factorization of the matrix $\mathbf{L}^{-1}$ be

$$\mathbf{L}^{-1} = \mathbf{Q}^{(2)}\mathbf{R}^{(2)} = \mathbf{Q}^{(2)} \begin{bmatrix} r_{11}^{(2)} & r_{12}^{(2)} \\ 0 & r_{22}^{(2)} \end{bmatrix},$$

then

$$\begin{aligned} ((\mathbf{Q}^{(2)})^*, \mathbf{I}_2, \mathbf{I}_2) \cdot \underline{A} &= \begin{bmatrix} r_{11}^{(2)} \\ 0 \end{bmatrix} \circ \tilde{\mathbf{m}}_1 \circ \tilde{\mathbf{n}}_1 + \begin{bmatrix} r_{12}^{(2)} \\ r_{22}^{(2)} \end{bmatrix} \circ \tilde{\mathbf{m}}_2 \circ \tilde{\mathbf{n}}_1 + \begin{bmatrix} r_{11}^{(2)} \\ 0 \end{bmatrix} \circ \tilde{\mathbf{m}}_2 \circ \tilde{\mathbf{n}}_2 \\ &= \mathbf{e}_1 \circ (r_{11}^{(2)}\tilde{\mathbf{m}}_1 \circ \tilde{\mathbf{n}}_1 + r_{12}^{(2)}\tilde{\mathbf{m}}_2 \circ \tilde{\mathbf{n}}_1 + r_{22}^{(2)}\tilde{\mathbf{m}}_2 \circ \tilde{\mathbf{n}}_2) + r_{22}^{(2)}\mathbf{e}_2 \circ \tilde{\mathbf{m}}_2 \circ \tilde{\mathbf{n}}_1. \end{aligned}$$

Thus, the orthogonal rank of $\underline{A}$ over the complex field is less than or equal to 3.

Summarizing the aforementioned results, we can then obtain the conclusion. □

We can then easily obtain the following result based on Lemma 4.1.

Theorem 4.2. $\mathrm{App}(\mathbb{V}_{\mathbb{C}}(3; 2, 2, 2)) \geq \frac{1}{\sqrt{3}}$.

It follows from Lemma 3.5 and Theorem 4.2 that the exact value of the best rank-1 approximation ratio of a real tensor space may be different from the best rank-1 approximation ratio of the complex tensor space with the same dimension.

5 Conclusions

By using the maximal orthogonal rank and the method for computing the best rank-1 approximation of a tensor, we present the exact values of the real $2 \times n_1 \times n_2$ spaces and illustrate that the best rank-1 approximation ratio of a real tensor space may be different from the best rank-1 approximation ratio of the complex tensor space with the same dimension. However, estimating the best rank-1 approximation ratio of a general tensor space is still a challenging task. More attention will be paid on the estimation of the best rank-1 approximation of a general tensor space in the future.

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