Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 11, No. 2, pp. 257–278, April 2015

APPROXIMATION BOUND ANALYSIS FOR THE STANDARD MULTI-QUADRATIC OPTIMIZATION PROBLEM

Chen Ling*, Xinzhen Zhang† and Liqun Qi‡

Abstract: A Standard Multi-Quadratic Optimization Problem (StMQP) is studied in this paper, which consists of minimizing a multi-quadratic form over the Cartesian product of several simplices. Several lower bounding techniques for StMQP ranging from very simple and cheap ones to more complex constructions are presented. The main tools employed here are Semidefinite Programming (SDP), decomposition of the objective function into an appropriate form related to quadratic functions, and optimization of a multi-linear function over the Cartesian product of spheres. Especially, the approximation solution and relative approximation ratio of the considered problem are studied. Finally, a related bi-linear copositive programming reformulation is presented.

Key words: *standard multi-quadratic optimization, semidefinite programming relaxation, approximation bound, approximation solution*

Mathematics Subject Classification: *90C26, 90C59*

1 Introduction

In this paper, we consider an optimization problem of the following form

$$\begin{array}{ll} \min & p_{\mathcal{A}}\left(x^{(1)},\ldots,x^{(d)}\right) := \sum\limits_{i_1,j_1=1}^{n_1}\cdots\sum\limits_{i_d,j_d=1}^{n_d} a_{i_1j_1\ldots i_dj_d}x_{i_1}^{(1)}x_{j_1}^{(1)}\cdots x_{i_d}^{(d)}x_{j_d}^{(d)} \\ \text{s.t.} & \left(x^{(1)},\ldots,x^{(d)}\right)\in\Delta_{n_1}\times\cdots\times\Delta_{n_d}, \end{array} \tag{1.1}$$

where

$$\Delta_k := \left\{x\in\Re_+^k \mid \sum_{i=1}^k x_i = 1\right\}$$

is the standard simplex and $\Re_+^k = \left\{x\in\Re^k \mid x\geq 0\right\}$ denotes the non-negative orthant in k-dimensional Euclidean space $\Re^k$. Here, $\mathcal{A}=(a_{i_1j_1\ldots i_dj_d})$ is a $2d$-th order $(n_1\times n_1\times\cdots\times n_d\times n_d)$-dimensional real partially symmetric tensor, that is, for any fixed k $(1\leq k\leq d)$ and indices $\{i_1,j_1,\cdots,i_{k-1},j_{k-1},i_{k+1},j_{k+1},\cdots,i_d,j_d\}$, the matrix $M_k=(a_{i_1j_1\ldots i_dj_d})_{1\leq i_k,j_k\leq n_k}\in$

*This author is supported by the National Natural Science Foundation of China (Grant No. 11171083 and 11301123), and the Zhejiang Provincial Natural Science Foundation of China (Grant No. LZ14A010003).

†The corresponding author. This author is supported by the National Natural Science Foundation of China (Grant No. 11101303, 11471242 and 11171180).

‡This author is supported by the Hong Kong Research Grant Council (Grant No. PolyU 502510, 502111, 501212 and 501913).

$\Re^{n_k\times n_k}$ is symmetric. It is easy to see that for any chosen $k \in \{1,\ldots,d\}$, the objective function in (1.1) can be written briefly as

$$p_{\mathcal{A}}\left(x^{(1)},\ldots,x^{(d)}\right) = B^{(k)} \bullet \left(x^{(k)}(x^{(k)})^\top\right).$$

Here,

$$B^{(k)} = \left(\sum_{i_1,j_1=1}^{n_1}\cdots\sum_{i_{k-1},j_{k-1}=1}^{n_{k-1}}\sum_{i_{k+1},j_{k+1}=1}^{n_{k+1}}\cdots \sum_{i_d,j_d=1}^{n_d} a_{i_1j_1\ldots i_dj_d}x_{i_1}^{(1)}x_{j_1}^{(1)}\cdots x_{i_{k-1}}^{(k-1)}x_{j_{k-1}}^{(k-1)}x_{i_{k+1}}^{(k+1)}x_{j_{k+1}}^{(k+1)}\cdots x_{i_d}^{(d)}x_{j_d}^{(d)}\right)$$

is an $n_k \times n_k$ symmetric matrix and $X \bullet Y$ stands for the usual matrix inner product, i.e., $X \bullet Y = \mathrm{tr}(X^\top Y)$. From the continuity of $p_{\mathcal{A}}$ and the compactness of $\Delta_{n_1}\times\cdots\times\Delta_{n_d}$, it is clear that the optimal value of (1.1), denoted by $p_{\mathcal{A}}^{\min}$, is attainable. Without loss of generality, we assume that $2 \le n_1 \le n_2 \le \cdots \le n_d$.

In case that all $a_{i_1j_1\ldots i_dj_d}$ are independent of the indices $i_2, j_2, \ldots, i_d, j_d$, i.e., $a_{i_1j_1\ldots i_dj_d} = b_{i_1j_1}$ for every $i_k, j_k = 1,\ldots,n_k$ and $k = 1,\ldots,d$, then the problem (1.1) can be reduced to the following Standard Quadratic Optimization Problem (StQP)

$$\min\left\{\sum_{i,k=1}^{n_1} b_{ik}x_ix_k \mid x\in\Delta_{n_1}\right\}, \tag{1.2}$$

which is known to be NP-hard, but has a polynomial time approximation scheme (PTAS) [6]. On the other hand, if we fix any $(d-1)$ vectors in $\{x^{(1)},\cdots,x^{(d)}\}$, then we also obtain a standard quadratic optimization problem. Due to this reason, (1.1) is called a *Standard Multi-Quadratic Optimization Problem* (StMQP). StQPs were well studied, and not only occur frequently as subproblems in escape procedures for general quadratic optimization but also have manifold direct applications, e.g., in portfolio selection and in maximum weight clique problem for undirected graphs. For details, see, e.g., [2, 5, 23, 24, 28] and references therein. Furthermore, for portfolio selection problems with two groups of securities whose investment decisions influence each other, a generalized mean-variance model can be expressed as a standard bi-quadratic optimization problem (StBQP) [7]. In [7], some optimality conditions of StBQP were studied. To solve it, StBQP was reformulated as an unconstrained bi-quartic problem. Furthermore, a numerical algorithm was proposed based on the reformulated problem. As a further generalization, for portfolio selection problems with d groups of securities whose investment decisions influence each other, the related mean-variance model can be similarly formulated as StMQP.

Note that StBQP is different from bi-quadratic optimization problems over unit spheres in [22, 34]. StBQP comes from the mean-variance model in portfolio selection problems, however the latter problem arises from the strong ellipticity condition problem in solid mechanics and the entanglement problem in quantum physics; see [7, 11, 15, 16, 20, 29, 30, 33] and the references therein.

Since (1.1) is NP-hard, efficient algorithms to find the approximation solutions and bounds are very interesting. An important criterion for exact or approximation solution methods is the availability, which induces to propose a number of bounds for StQP, see, e.g., [1, 4, 6, 8, 9, 14, 26, 27]. However, for StBQP and StMQP, the approximation bound is still not clear until now. Motivated by this, we focus on approximation bounds of (1.1) in this paper. The main tools employed here are Semidefinite Programming (SDP), decomposition of the objective function into an appropriate form related to quadratic functions, and optimization of a multi-linear function over the Cartesian product of spheres.

Here are some notations. $\Re^n$ stands for the usual Euclidean space of real vectors of length n, which is equipped with the standard inner product and the Euclidean norm denoted by $\|\cdot\|$. The i-th coordinate of a vector x is denoted by x_i, and the i-th column vector of the identity matrix $I_n = (\delta_{ij})_{1\le i,j\le n}$ in $\Re^{n\times n}$ is denoted by e_n^i. We denote by $\mathcal{S}^n$ the space of symmetric matrices, denote by $J_n \in \mathcal{S}^n$ with all entries being 1. For any $2d$-th order $(n_1 \times n_1 \times \cdots \times n_d \times n_d)$-dimensional real tensor $\mathcal{A}$, $a_{i_1 i_1 \dots i_d i_d}$ $(i_k = 1, \dots, n_k, k = 1, \dots, d)$ are called its subdiagonal entries, and $i_1 i_1 \dots i_d i_d$ are called the corresponding subdiagonal subscript. We denote by $\mathcal{I}$ the $2d$-th order $(n_1 \times n_1 \times \cdots \times n_d \times n_d)$-dimensional tensor whose entries are zero except for subdiagonal entries being 1, and denote by $\mathcal{E}$ the tensor of all one in $\Re^{n_1\times n_1\times\cdots\times n_d\times n_d}$. Diag($\mathcal{A}$) denotes the tensor which has the same subdiagonal entries to $\mathcal{A}$ and off-subdiagonal entries 0. For any two tensors $\mathcal{A}$ and $\mathcal{B}$ of the same structure, we denote $\mathcal{A} + \mathcal{B} = (a_{i_1 j_1 \dots i_d j_d} + b_{i_1 j_1 \dots i_d j_d})$. For the given matrices $B^{(k)} = \left(b^{(k)}_{i_k j_k}\right) \in \Re^{n_k \times n_k}$ $(k = 1, \dots, d)$, we denote by $\mathcal{C} = B^{(1)} \otimes \cdots \otimes B^{(d)}$ the $2d$-th order $(n_1 \times n_1 \times \cdots \times n_d \times n_d)$-dimensional real tensor with

$$c_{i_1 j_1 \cdots i_d j_d} = \prod_{k=1}^{d} b^{(k)}_{i_k j_k}.$$

For every $i_k, j_k = 1, \dots, n_k$ and $k = 1, \dots, d$, the matrix $e^{i_k}_{n_k}\left(e^{j_k}_{n_k}\right)^\top + e^{j_k}_{n_k}\left(e^{i_k}_{n_k}\right)^\top$ is denoted by $E^{(k)}(i_k j_k)$. For any subscript set $IJ = \{i_1, j_1, i_2, j_2, \dots, i_d, j_d\}$ with $i_k, j_k = 1, \dots, n_k$ and $k = 1, \dots, d$, we further denote

$$\mathcal{E}^{IJ} = \frac{1}{2^d} E^{(1)}(i_1 j_1) \otimes \cdots \otimes E^{(d)}(i_d j_d). \tag{1.3}$$

The rest of this paper is organized as follows. In Section 2, we present three methods to establish cheap closed-form lower bounds for (1.1), which are generalizations of those in [9]. By using standard quadratic programming, in Section 3 we further estimate the lower bounds of StBQP. In Section 4, we present a method for solving approximation solution of StBQP. Finally, a bi-linear copositive programming problem related to StBQPs is discussed in Section 5.

2 Cheap Closed-Form Lower Bounds

In this section, we discuss the basic properties of $p^{\min}_{\mathcal{A}}$. Based upon these properties, we present some lower bounds for $p^{\min}_{\mathcal{A}}$, which have simple closed-form representations and can be computed efficiently. Immediately, we can verify that $p^{\min}_{\mathcal{A}}$ is shift-equivariant with respect to $\mathcal{E}$, i.e., $p^{\min}_{\mathcal{A}+t\mathcal{E}} = p^{\min}_{\mathcal{A}} + t$ for all $t \in \Re$. Hence, without loss of generality, in this section we assume that all entries of $\mathcal{A}$ are positive.

Let $\mathfrak{S}$ be the set of all $2d$-th order $(n_1\times n_1\times\cdots\times n_d\times n_d)$-dimensional partially symmetric tensors, and $\mathcal{P} \subseteq \mathfrak{S}$ be a convex cone, i.e., $\alpha\mathcal{A} + \beta\mathcal{B} \in \mathcal{P}$ whenever $\mathcal{A}, \mathcal{B} \in \mathcal{P}$ and $\alpha, \beta \in \Re_+$. On $\mathfrak{S}$, we may introduce a partial order related with the given convex cone $\mathcal{P}$. In particular, if $\mathcal{P} \subseteq \mathfrak{S}$ is the convex cone consisting of all nonnegative tensors, then the corresponding order " $\ge$ ", is defined by componentwise inequalities, i.e., for $\mathcal{A}, \mathcal{B} \in \mathcal{P}$, $\mathcal{A} \ge \mathcal{B}$ if and only if $\mathcal{A} - \mathcal{B} \in \mathcal{P}$.

Another frequently used partial order defined on $\mathfrak{S}$ is Löwner partial order " $\succeq$ ", which is induced by the cone $\mathcal{P}$, consisting of all positive semidefinite tensors. Here,

$$\mathcal{P} := \left\{ \mathcal{A} \in \mathfrak{S} \mid p_{\mathcal{A}}(x^{(1)}, \ldots, x^{(d)}) \geq 0, \ \ \forall\, x^{(1)} \in \Re^{n_1}, \ldots, x^{(d)} \in \Re^{n_d} \right\}.$$

In other words, for $\mathcal{A}, \mathcal{B} \in \mathfrak{S}$, we write $\mathcal{A} \succeq \mathcal{B}$ whenever $\mathcal{A} - \mathcal{B} \in \mathcal{P}$. It is straightforward to verify that the minimum $p_{\mathcal{A}}^{\min}$ is isotone with respect to both the standard and Löwner partial orders. That is, if $\mathcal{A} \geq \mathcal{B}$ or $\mathcal{A} \succeq \mathcal{B}$, then $p_{\mathcal{A}}^{\min} \geq p_{\mathcal{B}}^{\min}$.

It is easy to see that every subdiagonal entry $a_{i_1 i_1 \ldots i_d i_d}$ of $\mathcal{A}$ is an upper bound for $p_{\mathcal{A}}^{\min}$, since $e_{n_k}^{i_k} \in \Delta_{n_k}$ and $p_{\mathcal{A}}\left(e_{n_1}^{i_1}, \ldots, e_{n_d}^{i_d}\right) = a_{i_1 i_1 \ldots i_d i_d}$. On the other hand, in the case $\mathcal{A} = \mathrm{Diag}(\mathcal{A})$, problem (1.1) reduces to the following optimization problem

$$\begin{array}{ll} \min & \displaystyle\sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} a_{i_1 i_1 \ldots i_d i_d} \left(x_{i_1}^{(1)}\right)^2 \cdots \left(x_{i_d}^{(d)}\right)^2 \\ \text{s.t.} & (x^{(1)}, \ldots, x^{(d)}) \in \Delta_{n_1} \times \cdots \times \Delta_{n_d}, \end{array}$$

which can be relaxed as

$$\begin{array}{ll} \min & z^\top C z \\ \text{s.t.} & z \in \Delta_{n_1 \cdots n_d}. \end{array} \tag{2.1}$$

Here, C is a $\prod_{k=1}^d n_k \times \prod_{k=1}^d n_k$ diagonal matrix with the diagonal entries being $a_{i_1 i_1 \ldots i_d i_d}$ ($i_k = 1, 2, \ldots, n_k, k = 1, 2, \ldots, d$). Note that (2.1) is a strictly convex program since $a_{i_1 i_1 \ldots i_d i_d} > 0$ for $i_k = 1, 2, \ldots, n_k$ and $k = 1, 2, \ldots, d$. From the first-order optimality condition, we obtain a lower bound for $p_{\mathcal{A}}^{\min}$

$$p_{\mathcal{A}}^{z} = \left[\sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} a_{i_1 i_1 \ldots i_d i_d}^{-1} \right]^{-1}.$$

Consequently, by shift-equivariance, one immediately knows that for tensor $\mathcal{A}$ with off-subdiagonal entries being $\bar{a} \leq \min\limits_{1 \leq i_1 \leq n_1, \ldots, 1 \leq i_d \leq n_d} a_{i_1 i_1 \ldots i_d i_d}$,

$$p_{\mathcal{A}}^{z} := \bar{a} + \left[\sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} (a_{i_1 i_1 \ldots i_d i_d} - \bar{a})^{-1} \right]^{-1}$$

is a lower bound of $p_{\mathcal{A}}^{\min}$. Furthermore, if there exist $b_{i_1}^{(1)} > 0, \ldots, b_{i_d}^{(d)} > 0$ such that $a_{i_1 i_1 \ldots i_d i_d} - \bar{a} = b_{i_1}^{(1)} \cdots b_{i_d}^{(d)}$ for $i_k = 1, \ldots, n_k$ and $k = 1, \ldots, d$, we then have

$$p_{\mathcal{A}}^{\min} = \bar{a} + \left[\sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} (a_{i_1 i_1 \ldots i_d i_d} - \bar{a})^{-1} \right]^{-1}. \tag{2.2}$$

Based on these preliminary observations, we are ready to derive our first class of lower bounds. To this end, we need the following lemma.

Lemma 2.1. *Consider problem (1.1) with tensor $\mathcal{A} \in \mathfrak{S}$, whose off-subdiagonal entries are all $\bar{a}$ satisfying*

$$\bar{a} < \min_{1 \leq i_1 \leq n_1, \ldots, 1 \leq i_d \leq n_d} a_{i_1 i_1 \ldots i_d i_d}.$$

For any off-subdiagonal subscript $IJ = \{i_1, j_1, \ldots, i_d, j_d\}$, we have $p_{\mathcal{A}}^{\min} < p_{\mathcal{A}'}^{\min}$, where $\mathcal{A}' = \mathcal{A} + \tau \mathcal{E}^{IJ}$ and $\tau > 0$.

Proof. By shift-equivariance, without loss of generality, we assume that $\bar{a} = 0$. Let $(\bar{x}^{(1)}, \ldots, \bar{x}^{(d)}) \in \Delta_{n_1} \times \cdots \times \Delta_{n_d}$ such that $p^{\min}_{\mathcal{A}} = p_{\mathcal{A}}(\bar{x}^{(1)}, \ldots, \bar{x}^{(d)})$. It is clear that for every $i_1 = 1, \ldots, n_1$, it holds

$$\bar{b}_{i_1} := \sum_{i_2=1}^{n_2} \cdots \sum_{i_d=1}^{n_d} a_{i_1 i_1 \ldots i_d i_d} \left(\bar{x}^{(2)}_{i_2}\right)^2 \cdots \left(\bar{x}^{(d)}_{i_d}\right)^2 > 0,$$

since $(\bar{x}^{(2)}, \ldots, \bar{x}^{(d)}) \in \Delta_{n_2} \times \cdots \times \Delta_{n_d}$ and $a_{i_1 i_1 \ldots i_d i_d} > 0$ for $i_k = 1, \ldots, n_k$ and $k = 1, \ldots, d$. Consequently, $\bar{x}^{(1)}$ is the unique solution of the following strictly convex program

$$\begin{array}{ll} \min & \sum_{i_1=1}^{n_1} \bar{b}_{i_1} x^2_{i_1} \\ \text{s.t.} & x \in \Delta_{n_1}. \end{array} \tag{2.3}$$

From the first-order optimality condition, we know that

$$\bar{x}^{(1)}_{i_1} = \frac{\bar{b}^{-1}_{i_1}}{\sum_{i_1=1}^{n_1} \bar{b}^{-1}_{i_1}}, \quad i = 1, 2, \cdots, n_1$$

which implies that $\bar{x}^{(1)} > 0$. Similarly, we have that $\bar{x}^{(k)} > 0$ for $k = 2, \ldots, d$.

Let $\left(\hat{x}^{(1)}, \ldots, \hat{x}^{(d)}\right) \in \Delta_{n_1} \times \cdots \times \Delta_{n_d}$ such that

$$p^{\min}_{\mathcal{A}'} = p_{\mathcal{A}'}\left(\hat{x}^{(1)}, \ldots, \hat{x}^{(d)}\right) = p_{\mathcal{A}}\left(\hat{x}^{(1)}, \ldots, \hat{x}^{(d)}\right) + \tau \hat{x}^{(1)}_{i_1} \hat{x}^{(1)}_{j_1} \cdots \hat{x}^{(d)}_{i_d} \hat{x}^{(d)}_{j_d}.$$

If $\hat{x}^{(1)}_{i_1} \hat{x}^{(1)}_{j_1} \cdots \hat{x}^{(d)}_{i_d} \hat{x}^{(d)}_{j_d} = 0$, then $\left(\hat{x}^{(1)}, \ldots, \hat{x}^{(d)}\right)$ is not an optimal solution of (1.1). Hence,

$$p^{\min}_{\mathcal{A}'} = p_{\mathcal{A}}\left(\hat{x}^{(1)}, \ldots, \hat{x}^{(d)}\right) > p^{\min}_{\mathcal{A}}.$$

If $\hat{x}^{(1)}_{i_1} \hat{x}^{(1)}_{j_1} \cdots \hat{x}^{(d)}_{i_d} \hat{x}^{(d)}_{j_d} > 0$, then

$$p^{\min}_{\mathcal{A}'} = p_{\mathcal{A}}\left(\hat{x}^{(1)}, \ldots, \hat{x}^{(d)}\right) + \tau \hat{x}^{(1)}_{i_1} \hat{x}^{(1)}_{j_1} \cdots \hat{x}^{(d)}_{i_d} \hat{x}^{(d)}_{j_d} \geq p^{\min}_{\mathcal{A}} + \tau \hat{x}^{(1)}_{i_1} \hat{x}^{(1)}_{j_1} \cdots \hat{x}^{(d)}_{i_d} \hat{x}^{(d)}_{j_d} > p^{\min}_{\mathcal{A}}.$$

We obtain the desired result and complete the proof. □

Let

$$p^{\text{ref}}_{\mathcal{A}} = p^0_{\mathcal{A}} + \left[\sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} \left(a_{i_1 i_1 \ldots i_d i_d} - p^0_{\mathcal{A}}\right)^{-1}\right]^{-1}, \tag{2.4}$$

where

$$p^0_{\mathcal{A}} = \min_{1 \leq i_1, j_1 \leq n_1, \ldots, 1 \leq i_d, j_d \leq n_d} a_{i_1 j_1 \ldots i_d j_d}.$$

Theorem 2.2. *For problem (1.1) with $\mathcal{A} \in \mathfrak{S}$, we have*

$$p^0_{\mathcal{A}} \leq p^{\text{ref}}_{\mathcal{A}} \leq p^{\min}_{\mathcal{A}}.$$

The equality $p^0_{\mathcal{A}} = p^{\text{ref}}_{\mathcal{A}}$ holds if and only if a minimum entry of $\mathcal{A}$ is located on the subdiagonal, which leads to $p^0_{\mathcal{A}} = p^{\min}_{\mathcal{A}}$. Furthermore, it holds that all the off-subdiagonal entries of $\mathcal{A}$ are equal to the value $p^0_{\mathcal{A}}$ provided $p^{\text{ref}}_{\mathcal{A}} = p^{\min}_{\mathcal{A}}$. Conversely, if all the off-subdiagonal entries of $\mathcal{A}$ are equal to the value $\bar{a}$ satisfying $\bar{a} < \min_{1 \leq i_1 \leq n_1, \ldots, 1 \leq i_d \leq n_d} a_{i_1 i_1 \ldots i_d i_d}$ and there exist some positive vectors $(b^{(k)}_1, \ldots, b^{(k)}_{n_k})^\top \in \Re^{n_k}$ $(k = 1, \ldots, d)$ such that $a_{i_1 i_1 \ldots i_d i_d} = b^{(1)}_{i_1} \cdots b^{(d)}_{i_d} + \bar{a}$ for $i_k = 1, \ldots, n_k$ and $k = 1, \ldots, d$, then $p^{\text{ref}}_{\mathcal{A}} = p^{\min}_{\mathcal{A}}$.

Proof. The inequality, $p^0_{\mathcal{A}} \le p^{\text{ref}}_{\mathcal{A}}$, is trivial. Let

$$\mathcal{A}' = \text{Diag}(\mathcal{A}) + p^0_{\mathcal{A}}(\mathcal{E} - \mathcal{I}),$$

then $p^{\text{ref}}_{\mathcal{A}} \le p^{\min}_{\mathcal{A}'}$. Consequently, the inequality $p^{\text{ref}}_{\mathcal{A}} \le p^{\min}_{\mathcal{A}}$ follows from the isotonicity of $p^{\min}_{\mathcal{A}}$ and the tensor inequality $\mathcal{A}' \le \mathcal{A}$. Next we characterize the cases where one of the cheap bounds is exact. If $a_{\bar{i}_1\bar{i}_1\ldots\bar{i}_d\bar{i}_d} = \min_{1\le i_1,j_1\le n_1,\ldots,1\le i_d,j_d\le n_d} a_{i_1j_1\ldots i_dj_d}$ for some indices $\bar{i}_1,\ldots,\bar{i}_d$, then $a_{\bar{i}_1\bar{i}_1\ldots\bar{i}_d\bar{i}_d} = p^0_{\mathcal{A}} \le p^{\min}_{\mathcal{A}} \le a_{\bar{i}_1\bar{i}_1\ldots\bar{i}_d\bar{i}_d}$ because every subdiagonal entry $a_{i_1i_1\ldots i_di_d}$ of $\mathcal{A}$ is an upper bound for $p^{\min}_{\mathcal{A}}$. Consequently, it holds that $p^0_{\mathcal{A}} = p^{\min}_{\mathcal{A}}$, which implies that $p^0_{\mathcal{A}} = p^{\text{ref}}_{\mathcal{A}}$. Conversely, if $p^0_{\mathcal{A}} = p^{\text{ref}}_{\mathcal{A}}$, then

$$\left[\sum_{i_1=1}^{n_1}\cdots\sum_{i_d=1}^{n_d}\left(a_{i_1i_1\ldots i_di_d} - p^0_{\mathcal{A}}\right)^{-1}\right]^{-1} = 0.$$

So that $a_{\bar{i}_1\bar{i}_1\ldots\bar{i}_d\bar{i}_d} = p^0_{\mathcal{A}} = p^{\min}_{\mathcal{A}}$ for some indices $\bar{i}_1,\ldots,\bar{i}_d$.

Next, we assume that $p^{\text{ref}}_{\mathcal{A}} = p^{\min}_{\mathcal{A}}$, then $p^{\text{ref}}_{\mathcal{A}} \le p^{\min}_{\mathcal{A}'}$. Hence, from the fact that $p^{\text{ref}}_{\mathcal{A}} = p^z_{\mathcal{A}'} \le p^{\min}_{\mathcal{A}'} \le p^{\min}_{\mathcal{A}}$, it holds that $p^{\min}_{\mathcal{A}'} = p^{\min}_{\mathcal{A}}$. If there exists an off-subdiagonal entry $a_{\bar{i}_1\bar{j}_1\ldots\bar{i}_d\bar{j}_d}$ of $\mathcal{A}$ such that $a_{\bar{i}_1\bar{j}_1\ldots\bar{i}_d\bar{j}_d} > p^0_{\mathcal{A}}$, then the tensor $\mathcal{A}'' = \mathcal{A}' + (a_{\bar{i}_1\bar{j}_1\ldots\bar{i}_d\bar{j}_d} - p^0_{\mathcal{A}})\bar{\mathcal{E}}$ satisfies $\mathcal{A}' \le \mathcal{A}'' \le \mathcal{A}$, where $\bar{\mathcal{E}}$ is defined by (1.3) with $\bar{I}\bar{J} = \{\bar{i}_1,\bar{j}_1,\ldots,\bar{i}_d,\bar{j}_d\}$. By Lemma 2.1, $p^{\min}_{\mathcal{A}'} < p^{\min}_{\mathcal{A}''}$. This contradicts the equality $p^{\min}_{\mathcal{A}'} = p^{\min}_{\mathcal{A}}$. Hence, all off-subdiagonal entries of $\mathcal{A}$ must coincide with $p^0_{\mathcal{A}}$.

Assume now that all off-subdiagonal entries of $\mathcal{A}$ are equal to $\bar{a}$ satisfying $\bar{a} < \min_{1\le i_1\le n_1,\ldots,1\le i_d\le n_d} a_{i_1i_1\ldots i_di_d}$, or equivalently $p^0_{\mathcal{A}} < p^{\text{ref}}_{\mathcal{A}}$. Then $p^{\text{ref}}_{\mathcal{A}} = p^{\min}_{\mathcal{A}}$ follows immediately from the closed-form expression (2.2). We complete the proof. □

From (2.4), it is easy to see that

$$p^0_{\mathcal{A}} + \tfrac{1}{n_1\cdots n_d}\left[\min_{1\le i_1\le n_1,\ldots,1\le i_d\le n_d} a_{i_1i_1\ldots i_di_d} - p^0_{\mathcal{A}}\right] \le p^{\text{ref}}_{\mathcal{A}}$$
$$\le p^0_{\mathcal{A}} + \tfrac{1}{n_1\cdots n_d}\left[\max_{1\le i_1\le n_1,\ldots,1\le i_d\le n_d} a_{i_1i_1\ldots i_di_d} - p^0_{\mathcal{A}}\right],$$

which implies that $p^0_{\mathcal{A}}$ and $p^{\text{ref}}_{\mathcal{A}}$ tend to coincide when $\max\{n_1,\ldots,n_d\}$ increases. In particular, when all the subdiagonal entries are equal, it holds that

$$p^{\text{ref}}_{\mathcal{A}} = p^0_{\mathcal{A}} + \frac{1}{n_1\cdots n_d}\left(a_{i_1i_1\ldots i_di_d} - p^0_{\mathcal{A}}\right).$$

Theorem 2.3. *Consider the standard multi-quadratic optimization problem (1.1) with $\mathcal{A} \in \mathfrak{S}$. We have*

$$p^0_{\mathcal{A}} \le p^{xy}_{\mathcal{A}} \le p^{\min}_{\mathcal{A}},$$

where

$$p^{xy}_{\mathcal{A}} := \prod_{k=1}^{d}\left[\sqrt[d]{p^0_{\mathcal{A}}} + \left(\sum_{i_k=1}^{n_k}\left(c^{(k)}_{i_ki_k} - \sqrt[d]{p^0_{\mathcal{A}}}\right)^{-1}\right)^{-1}\right],$$

and

$$c^{(k)}_{i_ki_k} = \min_{1\le i_1,j_1\le n_1,\ldots,1\le i_{k-1},j_{k-1}\le n_{k-1},1\le i_{k+1},j_{k+1}\le n_{k+1},\ldots,1\le i_d,j_d\le n_d} \sqrt[d]{a_{i_1j_1\ldots i_{k-1}j_{k-1}i_ki_ki_{k+1}j_{k+1}\ldots i_dj_d}}.$$

Moreover, the equality $p^0_{\mathcal{A}} = p^{xy}_{\mathcal{A}}$ *holds if and only if for every* $k = 1,\dots,d$, *there exists* $i_k \in \{1,\dots,n_k\}$ *such that* $c^{(k)}_{i_k i_k} = \sqrt[d]{p^0_{\mathcal{A}}}$.

Proof. The first inequality, $p^0_{\mathcal{A}} \le p^{xy}_{\mathcal{A}}$, is trivial, because $\sqrt[d]{p^0_{\mathcal{A}}} \le c^{(k)}_{i_k i_k}$ for $i_k = 1,\dots,n_k$ and $k = 1,\dots,d$. Now we prove the second inequality. Let $C^{(k)} \in \mathcal{S}^{n_k}$ $(k = 1,\cdots,d)$ with its entries

$$c^{(k)}_{i_k j_k} = \min_{1\le i_1,j_1\le n_1,\dots,1\le i_{k-1},j_{k-1}\le n_{k-1},1\le i_{k+1},j_{k+1}\le n_{k+1},\dots,1\le i_d,j_d\le n_d} \sqrt[d]{a_{i_1j_1\dots i_{k-1}j_{k-1}i_kj_ki_{k+1}j_{k+1}\dots i_dj_d}}$$

and $\mathcal{C} = (c_{i_1j_1\dots i_dj_d}) \in \mathfrak{S}$ with $c_{i_1j_1\dots i_dj_d} = \prod_{k=1}^d c^{(k)}_{i_kj_k}$. It follows that $\mathcal{C} \le \mathcal{A}$, which implies that $p_{\mathcal{C}}\left(x^{(1)},\dots,x^{(d)}\right) \le p_{\mathcal{A}}\left(x^{(1)},\dots,x^{(d)}\right)$ for any $\left(x^{(1)},\dots,x^{(d)}\right) \in \Delta_{n_1}\times\cdots\times\Delta_{n_d}$. Moreover, from the special structure of $\mathcal{C}$, it holds that $p_{\mathcal{C}}\left(x^{(1)},\dots,x^{(d)}\right) = \prod_{k=1}^d \left(x^{(k)}\right)^\top C^{(k)}x^{(k)}$. Now we consider the following StQP

$$\begin{cases} \min & (x^{(k)})^\top C^{(k)}x^{(k)} \\ \text{s.t.} & x^{(k)} \in \Delta_{n_k}. \end{cases} \tag{2.5}$$

It is easy to see that $\sqrt[d]{p^0_{\mathcal{A}}} = \min_{1\le i_k,j_k\le n_k} c^{(k)}_{i_kj_k}$. Consequently, by Theorem 2 in [9], we know that $\sqrt[d]{p^0_{\mathcal{A}}} + \left[\sum_{i_k=1}^{n_k}\left(c^{(k)}_{i_ki_k} - \sqrt[d]{p^0_{\mathcal{A}}}\right)^{-1}\right]^{-1}$ is a lower bound of the optimal value of (2.5). Hence, the second inequality holds.

If $p^0_{\mathcal{A}} = p^{xy}_{\mathcal{A}}$, then

$$\left[\sum_{i_k=1}^{n_k}\left(c^{(k)}_{i_ii_k} - \sqrt[d]{p^0_{\mathcal{A}}}\right)^{-1}\right]^{-1} = 0, \quad k = 1,\cdots,d.$$

Hence, $c^{(k)}_{i_ki_k} = \sqrt[d]{p^0_{\mathcal{A}}}$ for some index $i_k \in \{1,\dots,n_k\}$. Conversely, if for every $k = 1,\dots,d$, there exists $i_k \in \{1,\dots,n_k\}$ such that $c^{(k)}_{i_ki_k} = \sqrt[d]{p^0_{\mathcal{A}}}$, we may prove that $p^0_{\mathcal{A}} = p^{xy}_{\mathcal{A}}$ by using the same way as above. We complete the proof of the theorem. □

Now we state and prove the last theorem in this section. For the sake of simplicity, we only study the case $d = 2$. In fact, the obtained result can be extended to the case where $d \ge 3$. We denote $a^{(1)}_{i_1j_1} = \min\{a_{i_1j_1i_2j_2} \mid 1 \le i_2,j_2 \le n_2\}$ for $i_1,j_1 = 1,\dots,n_1$, $a^{(2)}_{i_2j_2} = \min\{a_{i_1j_1i_2j_2} \mid 1 \le i_1,j_1 \le n_1\}$ for $i_2,j_2 = 1,\dots,n_2$, and

$$p^{(1)}_{i_1j_1} = a^{(1)}_{i_1j_1} + \left[\sum_{i_2=1}^{n_2}\left(a_{i_1j_1i_2i_2} - a^{(1)}_{i_1j_1}\right)^{-1}\right]^{-1}, \quad p^{(2)}_{i_2j_2} = a^{(2)}_{i_2j_2} + \left[\sum_{i_1=1}^{n_1}\left(a_{i_1i_1i_2j_2} - a^{(2)}_{i_2j_2}\right)^{-1}\right]^{-1}. \tag{2.6}$$

Furthermore, we denote by $B^{(i_1j_1)} \in \mathcal{S}^{n_2}$ with its entries being $a_{i_1j_1i_2j_2}$ for $i_1,j_1 = 1,\dots,n_1$, and denote by $C^{(i_2j_2)} \in \mathcal{S}^{n_1}$ with the entries being $a_{i_1j_1i_2j_2}$ for $i_2,j_2 = 1,\dots,n_2$.

Theorem 2.4. *Consider the standard multi-quadratic optimization problem (1.1) with* $d = 2$ *and* $\mathcal{A} \in \mathfrak{S}$. *It holds that* $p^{\text{ref}}_{\mathcal{A}} \le p^{ab}_{\mathcal{A}} \le p^{\min}_{\mathcal{A}}$, *where*

$$p^{ab}_{\mathcal{A}} := \max\left\{p^{(1)}_{\mathcal{A}} + \left[\sum_{i_1=1}^{n_1}\left(p^{(1)}_{i_1i_1} - p^{(1)}_{\mathcal{A}}\right)^{-1}\right]^{-1}, \; p^{(2)}_{\mathcal{A}} + \left[\sum_{i_2=1}^{n_2}\left(p^{(2)}_{i_2i_2} - p^{(2)}_{\mathcal{A}}\right)^{-1}\right]^{-1}\right\}$$

with $p_{\mathcal{A}}^{(1)} = \min\left\{p_{i_1j_1}^{(1)} \mid 1 \le i_1, j_1 \le n_1\right\}$ *and* $p_{\mathcal{A}}^{(2)} = \min\left\{p_{i_2,j_2}^{(2)} \mid 1 \le i_2, j_2 \le n_2\right\}$.

Proof. We only prove that

$$p_{\mathcal{A}}^{\text{ref}} \le p_{\mathcal{A}}^{(1)} + \left[\sum_{i_1=1}^{n_1}\left(p_{i_1i_1}^{(1)} - p_{\mathcal{A}}^{(1)}\right)^{-1}\right]^{-1} \le p_{\mathcal{A}}^{\min}. \tag{2.7}$$

The proof of

$$p_{\mathcal{A}}^{\text{ref}} \le p_{\mathcal{A}}^{(2)} + \left[\sum_{i_2=1}^{n_2}\left(p_{i_2i_2}^{(2)} - p_{\mathcal{A}}^{(2)}\right)^{-1}\right]^{-1} \le p_{\mathcal{A}}^{\min}$$

is similar. We now consider

$$\text{StQP}(i_1, j_1): \quad \min\left\{(x^{(2)})^\top B^{(i_1j_1)}x^{(2)} \mid x^{(2)} \in \Delta_{n_2}\right\}. \tag{2.8}$$

By Theorem 2 in [9], we know that

$$p_{i_1j_1}^{(1)} = a_{i_1j_1}^{(1)} + \left[\sum_{i_2=1}^{n_2}\left(a_{i_1j_1i_2i_2} - a_{i_1j_1}^{(1)}\right)^{-1}\right]^{-1}$$

is a lower bound of $\text{StQP}(i_1, j_1)$, which implies that $p_{i_1j_1}^{(1)} \le \left(x^{(2)}\right)^\top B^{(i_1j_1)}x^{(2)}$ for any $x^{(2)} \in \Delta_{n_2}$. Hence,

$$\left(x^{(1)}\right)^\top P^{(1)}x^{(1)} \le p_{\mathcal{A}}(x^{(1)}, x^{(2)}) \tag{2.9}$$

for $\left(x^{(1)}, x^{(2)}\right) \in \Delta_{n_1} \times \Delta_{n_2}$, where $P^{(1)} = \left(p_{i_1j_1}^{(1)}\right)_{1\le i_1,j_1\le n_1} \in \mathcal{S}^{n_1}$. By Theorem 2 in [9] again, we know that for any $x^{(1)} \in \Delta_{n_1}$,

$$p_{\mathcal{A}}^{(1)} + \left[\sum_{i_1=1}^{n_1}\left(p_{i_1i_1}^{(1)} - p_{\mathcal{A}}^{(1)}\right)^{-1}\right]^{-1} \le \left(x^{(1)}\right)^\top P^{(1)}x^{(1)},$$

which implies, together with (2.9), that the right inequality in (2.7) holds.

By (2.6), it is obvious that $p_{i_1j_1}^{(1)} \ge a_{i_1j_1}^{(1)}$ for every (i_1, j_1), which implies that $p_{\mathcal{A}}^{(1)} \ge p_{\mathcal{A}}^0$, where $p_{\mathcal{A}}^0$ is denoted in (2.4). Furthermore, since $f(x) := x + \left[\sum_{i_1=1}^{n_1}\left(p_{i_1i_1}^{(1)} - x\right)^{-1}\right]^{-1}$ is monotone increasing on $\bigcap_{i_1=1}^{n_1}\left(-\infty, p_{i_1i_1}^{(1)}\right]$, we have

$$p_{\mathcal{A}}^{(1)} + \left[\sum_{i_1=1}^{n_1}\left(p_{i_1i_1}^{(1)} - p_{\mathcal{A}}^{(1)}\right)^{-1}\right]^{-1} \ge p_{\mathcal{A}}^0 + \left[\sum_{i_1=1}^{n_1}\left(p_{i_1i_1}^{(1)} - p_{\mathcal{A}}^0\right)^{-1}\right]^{-1}. \tag{2.10}$$

On the other hand, we have that for every $i_1 = 1, \ldots, n_1$,

$$\begin{aligned} p_{i_1i_1}^{(1)} - p_{\mathcal{A}}^0 &= a_{i_1i_1}^{(1)} + \left[\sum_{i_2=1}^{n_2}\left(a_{i_1i_1i_2i_2} - a_{i_1i_1}^{(1)}\right)^{-1}\right]^{-1} - p_{\mathcal{A}}^0 \\ &\ge p_{\mathcal{A}}^0 + \left[\sum_{i_2=1}^{n_2}\left(a_{i_1i_1i_2i_2} - p_{\mathcal{A}}^0\right)^{-1}\right]^{-1} - p_{\mathcal{A}}^0 \\ &= \left[\sum_{i_2=1}^{n_2}\left(a_{i_1i_1i_2i_2} - p_{\mathcal{A}}^0\right)^{-1}\right]^{-1}, \end{aligned}$$

which implies

$$\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\left(a_{i_1i_1i_2i_2}-p^0_{\mathcal{A}}\right)^{-1} \geq \sum_{i_1=1}^{n_1}\left(p^{(1)}_{i_1i_1}-p^0_{\mathcal{A}}\right)^{-1}. \tag{2.11}$$

By (2.10) and (2.11), we know that the left inequality in (2.7) holds. We complete the proof. □

From Theorems 2.2 and 2.3, we know that $p^{\text{ref}}_{\mathcal{A}} = p^0_{\mathcal{A}}$ implies $p^{xy}_{\mathcal{A}} = p^0_{\mathcal{A}}$. However, the following examples show that $p^{xy}_{\mathcal{A}} = p^0_{\mathcal{A}}$ does not imply $p^{\text{ref}}_{\mathcal{A}} = p^0_{\mathcal{A}}$, $p^{\text{ref}}_{\mathcal{A}}$ and $p^{xy}_{\mathcal{A}}$ are incomparable.

Example 2.5. Consider problem (1.1) with $d = n_1 = n_2 = 2$. Let $a_{1112} = a_{1121} = a_{1222} = a_{2122} = 1$, and other entries be 4. In this case, it is easy to see that

$$C^{(1)} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad C^{(2)} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Consequently, it follows that $p^0_{\mathcal{A}} = p^{xy}_{\mathcal{A}} = 1$ but $p^{\text{ref}}_{\mathcal{A}} = 1.75$. On the other hand, by a direct computation, we may obtain $p^{ab}_{\mathcal{A}} = 2$.

Example 2.6. Consider problem (1.1) with $d = n_1 = n_2 = 2$. Let $a_{1212} = a_{2112} = a_{1221} = a_{2121} = 1$, and other entries be 4. In this case, it is easy to see that

$$C^{(1)} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad C^{(2)} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Consequently, it follows that $p^0_{\mathcal{A}} = 1 < p^{\text{ref}}_{\mathcal{A}} = 1.75 < p^{xy}_{\mathcal{A}} = 2.25$. On the other hand, by a direct computation, we obtain $p^{(1)}_{\mathcal{A}} = p^{(2)}_{\mathcal{A}} = 2.5$, and hence $p^{ab}_{\mathcal{A}} = 3.25$.

3 Bounds Based Upon Standard Quadratic Programming

In this section, we further present some methods to find lower bounds on optimal value of (1.1). For the sake of simplicity, we only discuss the case that $d = 2$ in (1.1), of the following form

$$\begin{array}{ll} \min & p_{\mathcal{A}}(x,y) := \sum\limits_{i,j=1}^{n}\sum\limits_{k,l=1}^{m} a_{ijkl}x_ix_jy_ky_l \\ \text{s.t.} & (x,y) \in \Delta_n \times \Delta_m, \end{array} \tag{3.1}$$

which is also studied in [7]. We first present two results on the lower bounds of (3.1).

Theorem 3.1. *Let $b_{ij} = \min\limits_{1\leq k\leq m} a_{ijkk}$ for $i,j = 1,\ldots,n$ and $c_{kl} = \min\limits_{1\leq i\leq n} a_{iikl}$ for $k,l = 1,\ldots,m$. Then, we have*

$$p^0_{\mathcal{A}} + \max\left\{\frac{1}{m}\left(v^{\min}_B - p^0_{\mathcal{A}}\right), \frac{1}{n}\left(v^{\min}_C - p^0_{\mathcal{A}}\right)\right\} \leq p^{\min}_{\mathcal{A}}, \tag{3.2}$$

where $v^{\min}_B$ and $v^{\min}_C$ are the optimal values of the following standard quadratic optimization problems

$$\left\{\begin{array}{ll} \min\limits_{x\in\Re^n} & \sum\limits_{i,j=1}^{n} b_{ij}x_ix_j \\ \text{s.t.} & x \in \Delta_n \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{ll} \min\limits_{y\in\Re^m} & \sum\limits_{k,l=1}^{n} c_{kl}y_ky_l \\ \text{s.t.} & y \in \Delta_m, \end{array}\right.$$

respectively.

Proof. It suffices to show that $p^0_{\mathcal{A}} + \frac{1}{m}\left(v^{\min}_B - p^0_{\mathcal{A}}\right) \le p^{\min}_{\mathcal{A}}$ holds. By shift-equivariance, we have that for $x \in \Delta_n$ and $y \in \Delta_m$,

$$\begin{aligned} p_{\mathcal{A}}(x,y) &= p^0_{\mathcal{A}} + \sum_{i,j=1}^{n}\sum_{k,l=1}^{m}(a_{ijkl} - p^0_{\mathcal{A}})x_ix_jy_ky_l \\ &\ge p^0_{\mathcal{A}} + \sum_{i,j=1}^{n}\sum_{k=1}^{m}(a_{ijkk} - p^0_{\mathcal{A}})x_ix_jy_k^2 \\ &\ge p^0_{\mathcal{A}} + \sum_{k=1}^{m} y_k^2 \sum_{i,j=1}^{n}(b_{ij} - p^0_{\mathcal{A}})x_ix_j \\ &\ge p^0_{\mathcal{A}} + \frac{1}{m}\sum_{i,j=1}^{n}(b_{ij} - p^0_{\mathcal{A}})x_ix_j \\ &= p^0_{\mathcal{A}} + \frac{1}{m}\left(\sum_{i,j=1}^{n} b_{ij}x_ix_j - p^0_{\mathcal{A}}\right), \end{aligned}$$

where the second inequality comes from the fact $b_{ij} \le a_{ijkk}$ for every $k = 1, \ldots, m$, and the last inequality is due to the fact that $\sum_{k=1}^m y_k^2 \ge 1/m$ for any $y \in \Delta_m$. By this, it follows that $p^0_{\mathcal{A}} + \frac{1}{m}\left(v^{\min}_B - p^0_{\mathcal{A}}\right) \le p^{\min}_{\mathcal{A}}$ holds. We complete the proof of the theorem. □

Theorem 3.2. *Let* $g_{ij} = t_{ij} \min_{1\le k,l\le m} a_{ijkl}$ *with* $t_{ij} \in (0,1)$ *and* $h_{kl} = \min_{1\le i,j\le n}(a_{ijkl} - g_{ij})$. *Then, we have*

$$v^{\min}_G + v^{\min}_H \le p^{\min}_{\mathcal{A}},$$

where $v^{\min}_G$ *and* $v^{\min}_H$ *are the global minimums of the following standard quadratic optimization problems*

$$\begin{cases} \min_{x\in\Re^n} & \sum_{i,j=1}^{n} g_{ij}x_ix_j \\ \text{s.t.} & x \in \Delta_n \end{cases} \quad \text{and} \quad \begin{cases} \min_{y\in\Re^m} & \sum_{k,l=1}^{m} h_{kl}y_ky_l \\ \text{s.t.} & y \in \Delta_m, \end{cases}$$

respectively.

Proof. Since $0 < t_{ij} < 1$, it follows that $g_{ij} > 0$ and $h_{kl} > 0$ for every $i, j = 1, \ldots, n$ and $k, l = 1, \ldots, m$. Moreover, it is easy to see that for $i, j = 1, \ldots, n$ and $k, l = 1, \ldots, m$, $a_{ijkl} = a_{ijkl} - g_{ij} + g_{ij} \ge h_{kl} + g_{ij}$. Consequently, we know that for $x \in \Delta_n$ and $y \in \Delta_m$,

$$p_{\mathcal{A}}(x,y) \ge \sum_{i,j=1}^{n} g_{ij}x_ix_j + \sum_{k,l=1}^{m} h_{kl}y_ky_l.$$

Based upon this, we obtain the desired result and complete the proof. □

Remark 3.3. Note that the standard quadratic optimization problems in Theorems 3.1 and 3.2 are also NP-hard themselves. However, based upon Theorems 3.1 and 3.2, we may obtain some lower bounds for $p^{\min}_{\mathcal{A}}$ by solving approximately the corresponding quadratic problems in polynomial time.

Example 3.4. Consider Example 2.5. We first estimate its lower bound for $p^{\min}_{\mathcal{A}}$ by the method presented in Theorem 3.1. It is easy to know that the two matrices mentioned in Theorem 3.1 are

$$B = C = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

Based upon this, we can obtain $v_B^{\min} = v_C^{\min} = 2.5$. Hence, by Theorem 3.1, we know that $1.75 \le p_{\mathcal{A}}^{\min}$ since $p_{\mathcal{A}}^0 = 1$ and $m = n = 2$.

We continue to estimate $p_{\mathcal{A}}^{\min}$ by Theorem 3.2. We take $t_{11} = t_{12} = t_{21} = t_{22} = 0.25$. Then, we have

$$G = (g_{ij}) = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad H = (h_{kl}) = \frac{3}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Based upon this, it is easy to know that $v_G^{\min} = 0.25$ and $v_H^{\min} = 0.75$, which implies $1 \le p_{\mathcal{A}}^{\min}$ by Theorem 3.2.

Example 3.5. Consider Example 2.6. We first estimate its lower bound for $p_{\mathcal{A}}^{\min}$ by Theorem 3.1. It is easy to know that the two matrices mentioned in Theorem 3.1 are

$$B := (b_{ij}) = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \quad \text{and} \quad C := (c_{kl}) = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix},$$

respectively. Based upon this, it follows that $v_B^{\min} = v_C^{\min} = 4$. Hence, we obtain $2.5 \le p_{\mathcal{A}}^{\min}$, which is larger than $p_{\mathcal{A}}^{\mathrm{ref}}$ and $p_{\mathcal{A}}^{xy}$ obtained in Example 2.6. We continue to estimate the lower bound for $p_{\mathcal{A}}^{\min}$ in Example 2.6 by using the result obtained in Theorem 3.2. We take $t_{11} = t_{22} = 1/4$ and $t_{12} = t_{21} = 1/2$. By a simple computation, it is easy to see that

$$G := (g_{ij}) = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \quad \text{and} \quad H := (h_{kl}) = \begin{bmatrix} 3 & 1/2 \\ 1/2 & 3 \end{bmatrix}.$$

Consequently, we obtain $v_G^{\min} = 0.75$ and $v_H^{\min} = 1.75$. It follows that $2.5 \le p_{\mathcal{A}}^{\min}$.

Example 3.6. Consider the case where $n_1 = d = l = 2$ and $n_2 = 3$. Let $a_{1112} = a_{1121} = a_{2213} = a_{2231} = 1$ and other entries be 2.

In this case, it is obvious that $p_{\mathcal{A}}^0 = 1$. Consequently, it is easy to see that $p_{\mathcal{A}}^{\mathrm{ref}} = 7/6$ by Theorem 2.2. By a simple computation, we know that $p_{\mathcal{A}}^{xy} = (2 + \sqrt{2})/3$, by Theorem 2.3. At the same time, by Theorem 2.4, we have that $p_{\mathcal{A}}^{ab} = 4/3$. On the other hand, by Theorem 3.1, we can obtain $1.5 \le p_{\mathcal{A}}^{\min}$. Finally, we estimate the lower bound for $p_{\mathcal{A}}^{\min}$ by Theorem 3.2. Take $t_{11} = t_{12} = t_{21} = t_{22} = 0.5$. It is easy to see that

$$G = \frac{1}{2}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad H = \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}.$$

Hence, we can obtain that $v_G^{\min} = 0.5$ and $v_H^{\min} = 0.75$, which implies $1.25 \le p_{\mathcal{A}}^{\min}$ by Theorem 3.2.

Example 3.7. Consider the case that $\mathcal{A} = (a_{ijkl})$, where $i, j = 1, 2, \ldots, n$, $k, l = 1, 2, \ldots, m$ and

$$a_{ijkl} = \begin{cases} a, & \text{if } i = j \text{ and } k = l, \\ b, & \text{otherwise.} \end{cases}$$

It is obvious that $p_{\mathcal{A}}^{\min} = a$ when $a = b$.

When $a < b$, it is obvious that $p_{\mathcal{A}}^0 = a$. Furthermore, we can obtain a tight lower bound a of $p_{\mathcal{A}}^{\min}$, i.e, $p_{\mathcal{A}}^{\min} = a$, by all the methods in Theorems 2.2-2.4, 3.1 and Theorem 3.2 with $t_{ij} = a/b$ $(i, j = 1, 2, \ldots, n)$.

When $a > b$, by a direct computation, we obtain a tight lower bound $b + \frac{a-b}{n_1 n_2}$ of $p_{\mathcal{A}}^{\min}$, i.e., $p_{\mathcal{A}}^{\min} = b + \frac{a-b}{nm}$, by the methods in Theorems 2.2, 2.4 and 3.1. On the other hand, we can obtain another lower bound b of $p_{\mathcal{A}}^{\min}$ by Theorem 2.3 and Theorem 3.2 with $t_{ij} = b/a$ $(i, j = 1, 2, \ldots, n)$.

The above examples also show that the approximation bounds presented in Sections 2 and 3 are incomparable in general.

We now consider the following optimization problem

$$p_Z^* := \begin{array}{ll} \min\limits_{Z \in \Re^{n\times m}} & \mathcal{A} \bullet (Z \otimes Z) \\ \text{s.t.} & J_{n,m} \bullet Z = 1, \\ & Z \geq 0, \ \text{rank}(Z) = 1, \end{array} \tag{3.3}$$

where $\mathcal{A} = (a_{ijkl})$ is a fourth order $(n \times n \times m \times m)$-dimensional real partially symmetric tensor, $J_{n,m}$ is the $n \times m$ matrix of all ones in $\Re^{n\times m}$.

The following statement characterizes the relationship between (3.1) and (3.3).

Proposition 3.8. *The problems (3.1) and (3.3) are equivalent.*

Proof. Since $Z = xy^\top$ is a feasible solution of (3.3) for every $(x, y) \in \Delta_n \times \Delta_m$, it is clear that $p_Z^* \leq p_{\mathcal{A}}^{\min}$. On the other hand, let $\bar{Z}$ be an optimal solution of (3.3), then there exist $\bar{x} \in \Re^n$ and $\bar{y} \in \Re^m$ such that $\bar{Z} = \bar{x}\bar{y}^\top$. It is clear that $(\sum_{i=1}^n \bar{x}_i)(\sum_{j=1}^m \bar{y}_j) = J_{n,m} \bullet \bar{Z} = 1$. Denote $x^* = \bar{x}/\sum_{i=1}^n \bar{x}_i$ and $y^* = \bar{y}/\sum_{j=1}^m \bar{y}_j$. Since $x^*{y^*}^\top = \bar{Z} \geq 0$, it is easy to see that $\bar{x}_i(\sum_{j=1}^m \bar{y}_j) \geq 0$ for $i = 1, 2, \ldots, n$ and $\bar{y}_j(\sum_{i=1}^n \bar{x}_i) \geq 0$ for $j = 1, 2, \ldots, m$. Consequently, we know that $x^* \geq 0$ and $y^* \geq 0$. Hence, $(x^*, y^*) \in \Delta_n \times \Delta_m$. Furthermore, we can verify that $\mathcal{A} \bullet (x^*{y^*}^\top \otimes x^*{y^*}^\top) = p_Z^*$, which implies $p_{\mathcal{A}}^{\min} \leq p_Z^*$. The desired result is obtained. ☐

By removing the rank constraint in (3.3), we have the following relaxation of problem (3.1):

$$\begin{array}{ll} \min\limits_{Z \in \Re^{n\times m}} & \mathcal{A} \bullet (Z \otimes Z) \\ \text{s.t.} & J_{n,m} \bullet Z = 1, \\ & Z \geq 0. \end{array} \tag{3.4}$$

Let $z = \text{vec}(Z)$, where the operator "vec" is defined as

$$\text{vec}(Z) = (Z_{11}, \ldots, Z_{n1}, \ldots, Z_{1m}, \ldots, Z_{nm})^\top.$$

Then $z \in \Delta_{mn}$ and (3.4) can be rewritten as the following standard quadratic programming

$$v_{\text{sqp}}^* := \begin{array}{ll} \min\limits_{z \in \Re^{mn}} & z^\top \bar{A} z \\ \text{s.t.} & z \in \Delta_{mn}, \end{array} \tag{3.5}$$

where $\bar{A} = (\bar{a}_{ij}) \in \mathcal{S}^{mn}$, generated by rearranging the entries of $\mathcal{A}$. It is clear that $v_{\text{sqp}}^* \leq p_{\mathcal{A}}^{\min}$ and the smallest element in $\bar{A}$ is exactly $p_{\mathcal{A}}^0$. Let z^* be an optimal solution of (3.5) . We pack z^* back into an $n \times m$ matrix $\bar{Z} = \text{mat}(z^*)$ by columns, i.e., the n elements in the j-th column of $\bar{Z}$ consist of $z_{(j-1)n+1}^*, \ldots, z_{jn}^*$ of z^*, $j = 1, \ldots, m$. If $\text{rank}(\bar{Z}) = 1$, then by the proof of Proposition 3.8, there exist $\bar{x} \in \Delta_n$ and $\bar{y} \in \Delta_m$ such that $\bar{Z} = \bar{x}\bar{y}^\top$, which implies that $(\bar{x}, \bar{y})$ is an optimal solution of (3.1). However, since $\bar{A}$ is indefinite in general, (3.5) is an NP-hard problem with mn dimension variables, which will bring higher computational cost when n or m is very large. In [27], Nowak proposed a method for solving the lower bound p_z^{nb} of the global minimum of (3.5), which is based upon the following semidefinite programming (SDP)

$$\begin{array}{ll} \min\limits_{G \in \mathcal{S}^{mn}} & J_{mn} \bullet (\bar{A} - G) \\ \text{s.t.} & G \leq \bar{A}, \Phi(G) \succeq 0, \\ & \text{diag}(G) = \text{diag}(\bar{A}) \end{array} \tag{3.6}$$

and the following quadratic optimization problem

$$\begin{aligned} v_{G^*} := \quad & \min \quad z^\top G^* z \\ & \text{s.t.} \quad z \in \Delta_{mn}. \end{aligned} \tag{3.7}$$

Here $\Phi : \Re^{mn\times mn} \to \Re^{(mn-1)\times(mn-1)}$ is the linear map defined by

$$\Phi(G)_{i,j} := G_{i,j} - G_{mn,i} - G_{mn,j} + G_{mn,mn}, \quad i,j = 1,\dots,mn-1,$$

and G^* is an optimal solution of (3.6). Since $\Phi(G^*) \succeq 0$, by Lemma 2 in [27], we know that (3.7) is a convex optimization problem, which can be solved in polynomial time. Moreover, if $z^\top \bar{A} z$ is concave on the edges of the simplex Δ_{mn}, then $G^* = (g^*_{ij})_{1\le i,j\le mn}$ with

$$g^*_{ij} = \frac{1}{2}(\bar{a}_{ii} + \bar{a}_{jj}), \quad i,j = 1,\dots,mn,$$

and v_{G^*} is exact, see [27] for details.

4 Approximation Solutions and Relative Approximation Ratio

In this section, we present a polynomial time approximation algorithm for solving (3.1), which can be extended to the case where $d \ge 3$. As mentioned in Section 2, it holds that $p^{\min}_{\mathcal{A}+t\mathcal{E}} = p^{\min}_{\mathcal{A}} + t$ for all $t \in \Re$. Hence, in this section we assume that all entries of $\mathcal{A}$ are negative. We first introduce the following definition to characterize the quality measure of approximation ratio.

Definition 4.1. Let $1 \ge \varepsilon > 0$ and $\mathfrak{A}$ be an approximation algorithm for (3.1). We say $\mathfrak{A}$ is a ε-approximation algorithm for (3.1) if for any instance of (3.1) the algorithm $\mathfrak{A}$ returns a feasible pair $(\bar{x}, \bar{y})$ of (3.1) such that

$$p_{\mathcal{A}}(\bar{x},\bar{y}) - p^{\min}_{\mathcal{A}} \le (1-\varepsilon)(p^{\max}_{\mathcal{A}} - p^{\min}_{\mathcal{A}}).$$

Here, $p^{\min}_{\mathcal{A}}$ (resp., $p^{\max}_{\mathcal{A}}$) is the minimum (resp., maximum) value of the objective in (3.1).

To get rid of the sign constraints $x \ge 0$ and $y \ge 0$ in (3.1), motivated by [10], we replace the variables x_i and y_j with z_i^2 and w_j^2, respectively. Then the conditions $\sum_{i=1}^n x_i = 1$ and $\sum_{j=1}^m y_j = 1$ become to $\|z\|^2 = 1$ and $\|w\|^2 = 1$, respectively. Therefore, the considered problem (3.1) can be equivalently written as

$$\begin{aligned} \min \quad & g(z,w) := \sum_{i,j=1}^{n} \sum_{k,l=1}^{m} a_{ijkl} z_i^2 z_j^2 w_k^2 w_l^2 \\ \text{s.t.} \quad & \|z\|^2 = 1, \|w\|^2 = 1,\ (z,w) \in \Re^n \times \Re^m. \end{aligned} \tag{4.1}$$

Furthermore, by introducing a suitable 8-th order $(n\times n\times n\times n\times m\times m\times m\times m)$-dimensional tensor $\bar{\mathcal{A}} = (\bar{a}_{i_1i_2i_3i_4j_1j_2j_3j_4})$, the above problem can be rewritten as

$$\begin{aligned} \min \quad & g(z,w) := \sum_{1\le i_1,\dots,i_4\le n} \sum_{1\le j_1,\dots,j_4\le m} \bar{a}_{i_1i_2i_3i_4j_1j_2j_3j_4} z_{i_1} z_{i_2} z_{i_3} z_{i_4} w_{j_1} w_{j_2} w_{j_3} w_{j_4} \\ \text{s.t.} \quad & \|z\|^2 = 1, \|w\|^2 = 1,\ (z,w) \in \Re^n \times \Re^m, \end{aligned} \tag{4.2}$$

one of whose relaxations is

$$\begin{aligned} \min \quad & F(z^{(1)},\dots,z^{(4)},w^{(1)},\dots,w^{(4)}) \\ & := \sum_{1\le i_1,\dots,i_4\le n} \sum_{1\le j_1,\dots,j_4\le m} \bar{a}_{i_1i_2i_3i_4j_1j_2j_3j_4} z^{(1)}_{i_1} z^{(2)}_{i_2} z^{(3)}_{i_3} z^{(4)}_{i_4} w^{(1)}_{j_1} w^{(2)}_{j_2} w^{(3)}_{j_3} w^{(4)}_{j_4} \\ \text{s.t.} \quad & \|z^{(k)}\|^2 = 1, \|w^{(k)}\|^2 = 1,\ (z^{(k)},w^{(k)}) \in \Re^n \times \Re^m,\ k = 1,\dots,4. \end{aligned} \tag{4.3}$$

In the sequel of this section, we study the approximation bound of (3.1), based upon the obtained approximation solution of (4.3). To this end, we first consider the following general multi-linear function

$$F(z^{(1)},\dots,z^{(r)},w^{(1)},\dots,w^{(l)}) = \sum_{\substack{1\le i_1,\dots,i_r\le n \\ 1\le j_1,\dots,j_l\le m}} \hat{a}_{i_1\dots i_r j_1\dots j_l} z^{(1)}_{i_1}\cdots z^{(r)}_{i_r} w^{(1)}_{j_1}\cdots w^{(l)}_{j_l}, \tag{4.4}$$

where $z^{(k)} \in \Re^n$ $(k = 1,2,\dots,r)$ and $w^{(s)} \in \Re^m$ $(s = 1,2,\dots,l)$. In the shorthand notation we shall denote $\hat{\mathcal{A}} = (\hat{a}_{i_1\dots i_r j_1\dots j_l})$ to be a $(r+l)$-th order tensor. We call the tensor $\hat{\mathcal{A}}$ partially symmetric with respect to the subscript set $\{i_1,i_2,\dots,i_r\}$ (resp., $\{j_1,j_2,\dots,j_l\}$), if $\hat{a}_{i_1\dots i_r j_1\dots j_l}$ is invariant under all permutations of $\{i_1,i_2,\dots,i_r\}$ (resp., $\{j_1,j_2,\dots,j_l\}$). Closely related to the tensor $\hat{\mathcal{A}}$ is a general homogeneous polynomial function $f(z,w) := F(z,\dots,z,w,\dots,w)$, where $z \in \Re^n$ and $w \in \Re^m$. As any quadratic function uniquely determines a symmetric matrix, a given homogeneous polynomial function $f(z,w)$ of degree $(r+l)$ also uniquely determines a partially symmetric tensor form. For F in (4.4) and the related homogeneous polynomial function f, we have the following lemma, which is a special case of Lemma 2.3.3 of [21] and is an extension of Proposition 5 in [32].

Lemma 4.2. *Suppose that $z^{(1)},\dots,z^{(r)} \in \Re^n$, $w^{(1)},\dots,w^{(l)} \in \Re^m$, and that $\xi_1,\dots,\xi_r$ and $\zeta_1,\dots,\zeta_l$ are i.i.d. random variables, each takes values 1 and -1 with equal probability 1/2. For any partially symmetric multi-linear function $F(z^{(1)},\dots,z^{(r)},w^{(1)},\dots,w^{(l)})$ and function $f(z,w) = F(z,\dots,z,w,\dots,w)$, it holds that*

$$\mathrm{E}\left[\prod_{i=1}^r \xi_i \prod_{j=1}^l \zeta_j f\left(\sum_{k=1}^r \xi_k z^{(k)}, \sum_{s=1}^l \zeta_s w^{(s)}\right)\right] = (r!)(l!)F(z^{(1)},\dots,z^{(r)},w^{(1)},\dots,w^{(l)}).$$

By applying Lemma 4.2 to the special case where $r = l = 4$, i.e., (4.2) and (4.3), we have the following theorem.

Theorem 4.3. *Problem (3.1) admits a polynomial-time approximation algorithm with relative approximation ratio ε, i.e. there exists a feasible solution $(\hat{x},\hat{y})$ of (3.1) such that*

$$p_{\mathcal{A}}(\hat{x},\hat{y}) - p_{\mathcal{A}}^{\min} \le (1-\varepsilon)(p_{\mathcal{A}}^{\max} - p_{\mathcal{A}}^{\min}), \tag{4.5}$$

where $\varepsilon = \Omega\left(\frac{\log m \log n}{mn}\phi(n,m)\right)$ with

$$\phi(n,m) = \begin{cases} \frac{\log n}{n}, & \text{if } n \le m \\ \frac{\log m}{m}, & \text{otherwise.} \end{cases}$$

Proof. In order to obtain (4.5), we only prove that $p_{\mathcal{A}}^{\max} - p_{\mathcal{A}}(\hat{x},\hat{y}) \ge \varepsilon(p_{\mathcal{A}}^{\max} - p_{\mathcal{A}}^{\min})$. Denote $\mathbf{z} = (z,z,z,z), \mathbf{z}^{(1,4)} = (z^{(1)},z^{(2)},z^{(3)},z^{(4)}) \in \Re^n \times \Re^n \times \Re^n \times \Re^n$ and $\mathbf{w} = (w,w,w,w), \mathbf{w}^{(1,4)} = (w^{(1)},w^{(2)},w^{(3)},w^{(4)}) \in \Re^m \times \Re^m \times \Re^m \times \Re^m$. Denote $H(\mathbf{z},\mathbf{w})$ to be the partially symmetric form with respect to the function $h(z,w) := \|z\|^4\|w\|^4$. Pick any fixed (z^0,w^0) with $\|z^0\| = \|w^0\| = 1$, and consider the following problem

$$\begin{aligned} p_H^{\max} = \max \quad & g(z^0,w^0)H(\mathbf{z}^{(1,4)},\mathbf{w}^{(1,4)}) - F(\mathbf{z}^{(1,4)},\mathbf{w}^{(1,4)}) \\ \text{s.t.} \quad & \|z^{(k)}\| = 1, \|w^{(k)}\| = 1, \\ & z^{(k)} \in \Re^n, w^{(k)} \in \Re^m,\ k = 1,\dots,4. \end{aligned}$$

It is clear that $p_H^{\max} \geq 0$. By Theorem 4 in [32], we can obtain a solution $(\hat{\mathbf{z}}^{(1,4)}, \hat{\mathbf{w}}^{(1,4)})$ in polynomial time such that

$$g(z^0, w^0)H(\hat{\mathbf{z}}^{(1,4)}, \hat{\mathbf{w}}^{(1,4)}) - F(\hat{\mathbf{z}}^{(1,4)}, \hat{\mathbf{w}}^{(1,4)}) \geq \varepsilon_1 p_H^{\max}, \tag{4.6}$$

where $\varepsilon_1 = \Omega\left(\frac{\log m \log n}{mn}\phi(n,m)\right)$.

If $p_{\mathcal{A}}^{\max} - g(z^0, w^0) > (\varepsilon_1/4)(p_{\mathcal{A}}^{\max} - p_{\mathcal{A}}^{\min})$, then we have $p_{\mathcal{A}}^{\max} - g(z^0, w^0) > \varepsilon(p_{\mathcal{A}}^{\max} - p_{\mathcal{A}}^{\min})$ since $\varepsilon_1/4 > \varepsilon$. Consequently, by letting $\hat{x} = \left((z_1^0)^2, \ldots, (z_n^0)^2\right)^\top$ and $\hat{y} = \left((w_1^0)^2, \ldots, (w_m^0)^2\right)^\top$, we have obtained the desired result. Now we suppose that

$$p_{\mathcal{A}}^{\max} - g(z^0, w^0) \leq (\varepsilon_1/4)(p_{\mathcal{A}}^{\max} - p_{\mathcal{A}}^{\min}). \tag{4.7}$$

Notice that $|H(\hat{\mathbf{z}}^{(1,4)}, \hat{\mathbf{w}}^{(1,4)})| \leq 1$, by (4.6), we have

$$\begin{aligned}
p_{\mathcal{A}}^{\max} H(\hat{\mathbf{z}}^{(1,4)}, \hat{\mathbf{w}}^{(1,4)}) - F(\hat{\mathbf{z}}^{(1,4)}, \hat{\mathbf{w}}^{(1,4)}) \;=\;& g(z^0, w^0)H(\hat{\mathbf{z}}^{(1,4)}, \hat{\mathbf{w}}^{(1,4)}) - F(\hat{\mathbf{z}}^{(1,4)}, \hat{\mathbf{w}}^{(1,4)}) \\
&+ (p_{\mathcal{A}}^{\max} - g(z^0, w^0))H(\hat{\mathbf{z}}^{(1,4)}, \hat{\mathbf{w}}^{(1,4)}) \\
&\geq \varepsilon_1 p_H^{\max} - (p_{\mathcal{A}}^{\max} - g(z^0, w^0)) \\
&\geq \varepsilon_1 (g(z^0, w^0) - p_{\mathcal{A}}^{\min}) - (\varepsilon_1/4)(p_{\mathcal{A}}^{\max} - p_{\mathcal{A}}^{\min}) \\
&\geq (\varepsilon_1(1 - \varepsilon_1/4) - \varepsilon_1/4)(p_{\mathcal{A}}^{\max} - p_{\mathcal{A}}^{\min}) \\
&\geq (\varepsilon_1/2)(p_{\mathcal{A}}^{\max} - p_{\mathcal{A}}^{\min}),
\end{aligned}$$

where the second inequality comes from (4.7) and the fact $p_H^{\min} \geq g(z_0, w_0) - p_{\mathcal{A}}^{\min}$, the third inequality is due to (4.7). On the other hand, it holds that

$$\begin{aligned}
&(4!)^2[p_{\mathcal{A}}^{\max} H(\hat{\mathbf{z}}^{(1,4)}, \hat{\mathbf{w}}^{(1,4)}) - F(\hat{\mathbf{z}}^{(1,4)}, \hat{\mathbf{w}}^{(1,4)})] \\
&= E\left[\prod_{i=1}^4(\xi_i\zeta_i)\left(p_{\mathcal{A}}^{\max} h(\hat{Z}\xi, \hat{W}\zeta) - g(\hat{Z}\xi, \hat{W}\zeta)\right)\right] \\
&= E\left[p_{\mathcal{A}}^{\max} h(\hat{Z}\xi, \hat{W}\zeta) - g(\hat{Z}\xi, \hat{W}\zeta) | \prod_{i=1}^4(\xi_i\zeta_i) = 1\right] \mathrm{Prob}\left\{\prod_{i=1}^4(\xi_i\zeta_i) = 1\right\} \\
&\quad - E\left[p_{\mathcal{A}}^{\max} h(\hat{Z}\xi, \hat{W}\zeta) - g(\hat{Z}\xi, \hat{W}\zeta) | \prod_{i=1}^4(\xi_i\zeta_i) = -1\right] \mathrm{Prob}\left\{\prod_{i=1}^4(\xi_i\zeta_i) = -1\right\} \\
&\leq \frac{1}{2} E\left[p_{\mathcal{A}}^{\max} h(\hat{Z}\xi, \hat{W}\zeta) - g(\hat{Z}\xi, \hat{W}\zeta) | \prod_{i=1}^4(\xi_i\zeta_i) = 1\right],
\end{aligned}$$

where $\hat{Z} = \left[\hat{z}^{(1)}, \hat{z}^{(2)}, \hat{z}^{(3)}, \hat{z}^{(4)}\right]$, $\hat{W} = \left[\hat{w}^{(1)}, \hat{w}^{(2)}, \hat{w}^{(3)}, \hat{w}^{(4)}\right]$, $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)^\top$ and $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)^\top$. Thus there exists two vectors $\beta = (\beta_1, \ldots, \beta_4)$, $\gamma = (\gamma_1, \ldots, \gamma_4)$ with $\beta_i^2 = \gamma_i^2 = 1$ $(i = 1, \ldots, 4)$ and $\prod_{i=1}^4 \beta_i\gamma_i = 1$, such that

$$\frac{1}{2}\left(p_{\mathcal{A}}^{\max} h(\hat{Z}\beta, \hat{W}\gamma) - g(\hat{Z}\beta, \hat{W}\gamma)\right) \geq (4!)^2(\varepsilon_1/2)(p_{\mathcal{A}}^{\max} - p_{\mathcal{A}}^{\min}).$$

Letting $\hat{z} = \hat{Z}\beta/\|\hat{Z}\beta\|$ and $\hat{w} = \hat{W}\gamma/\|\hat{W}\gamma\|$, we have

$$p_{\mathcal{A}}^{\max} - g(\hat{z}, \hat{w}) \geq \frac{(4!)^2\varepsilon_1(p_{\mathcal{A}}^{\max} - p_{\mathcal{A}}^{\min})}{\|\hat{Z}\beta\|^4\|\hat{W}\gamma\|^4} \geq \varepsilon(p_{\mathcal{A}}^{\max} - p_{\mathcal{A}}^{\min}),$$

where the second inequality comes from the fact that $\|\hat{Z}\beta\| \leq 4$ and $\|\hat{W}\gamma\| \leq 4$. Here $\varepsilon := \frac{(4!)^2}{4^8}\varepsilon_1$. By taking $\hat{x} = (\hat{z}_1^2, \ldots, \hat{z}_n^2)^\top$ and $\hat{y} = (\hat{w}_1^2, \ldots, \hat{w}_m^2)^\top$, we obtain the desired result under the assumption (4.7). We complete the proof. □

5 The Related Bi-Linear Copositive Programming

In this section, we still consider the problem (3.1). Recall that $M \in \mathcal{S}^n$ is said to be copositive (more precisely, $\Re^n_+$-copositive), if $x^\top Mx \geq 0$ for any $x \in \Re^n_+$. Denote by $\mathcal{K}_n$ the cone of all $n \times n$ copositive matrices, i.e.,

$$\mathcal{K}_n = \left\{M \in \mathcal{S}^n \mid x^\top Mx \geq 0 \ \text{ for any } \ x \in \Re^n_+\right\},$$

and denote by $\mathcal{K}_n^*$ the cone of all completely positive matrices, i.e.,

$$\mathcal{K}_n^* = \left\{N \in \mathcal{S}^n \mid N = \sum_{i=1}^{k} x^i (x^i)^\top \ \text{ for some positive integer } k \text{ and } x^i \in \Re^n_+, i = 1, \ldots, k\right\}.$$

It is well known that $\mathcal{K}_n^*$ is the dual of the copositive cone $\mathcal{K}_n$ (which justifies the notation $\mathcal{K}_n^*$) with respect to the matrix inner product.

Consider the following conic bi-linear programming

$$\begin{array}{cl} \min\limits_{X\in\mathcal{K}_n^*, Y\in\mathcal{K}_m^*} & (\mathcal{A}X) \bullet Y \\ \text{s.t.} & J_n \bullet X = 1, \\ & J_m \bullet Y = 1, \end{array} \tag{5.1}$$

where $\mathcal{A}X \in \mathcal{S}^m$ with $(\mathcal{A}X)_{kl} = \sum\limits_{i,j=1}^{n} a_{ijkl}X_{ij}$ $(k, l = 1, \ldots m)$. Denoted by $v_{\min}^{\text{cop}}$ the optimal value of (5.1). It follows that $v_{\min}^{\text{cop}} \leq p_{\mathcal{A}}^{\min}$ since $\{xx^\top \mid x \in \Delta_n\} \subseteq \{X \in \mathcal{K}_n \mid J_n \bullet X = 1\}$. However, (5.1) is not a relaxation but indeed an exact reformulation of problem (3.1), which is shown in the following theorem.

Theorem 5.1. *The bi-quadratic optimization (3.1) and bi-linear copositive programming (5.1) are equivalent, that is, (3.1) and (5.1) have the same optimal value and one optimal solution pair of (3.1) can be obtained from the optimal solution pair of (5.1).*

Proof. Let (X^*, Y^*) be an optimal solution pair of (5.1) with the objective value $v_{\min}^{\text{cop}}$. By the definition of $\mathcal{K}_n^*$, there exists a positive integer r_1 such that $X^* = \sum_{i=1}^{r_1} x^{(i)}(x^{(i)})^\top$ with $x^{(i)} \in \Re^n_+\backslash\{0\}$. Let $\lambda_i = \left(\sum_{k=1}^n x_k^{(i)}\right)^2$. Then $\lambda_i > 0$ for $i = 1, 2, \ldots, r_1$, as well as $\sum_{i=1}^{r_1} \lambda_i = 1$ since $J_n \bullet X^* = 1$. Moreover, it is easy to see that $X^* = \sum_{i=1}^{r_1} \lambda_i \bar{x}^{(i)}(\bar{x}^{(i)})^\top$, where $\bar{x}^{(i)} = x^{(i)}/\sum_{k=1}^n x_k^{(i)} \in \Delta_n$. Since $v_{\min}^{\text{cop}} = (Y^*\mathcal{A}) \bullet X^*$, where $Y^*\mathcal{A} \in \mathcal{S}^n$ with $(Y^*\mathcal{A})_{ij} = \sum\limits_{k,l=1}^{m} a_{ijkl}Y_{kl}^*$ $(i, j = 1, \ldots n)$, it follows that $v_{\min}^{\text{cop}} = \sum_{i=1}^{r_1} \lambda_i (Y^*\mathcal{A}) \bullet \bar{x}^{(i)}(\bar{x}^{(i)})^\top$, which implies that there must exist an index, say 1, such that

$$(Y^*\mathcal{A}) \bullet \bar{x}^{(1)}(\bar{x}^{(1)})^\top \leq v_{\min}^{\text{cop}}. \tag{5.2}$$

Similarly, there exists a positive integer r_2 such that $Y^* = \sum_{j=1}^{r_2} \mu_j \bar{y}^{(j)}(\bar{y}^{(j)})^\top$, where $\bar{y}^{(j)} \in \Delta_m$ and $\mu_j > 0$ for $j = 1, 2, \ldots, r_2$ satisfying $\sum_{j=1}^{r_2} \mu_j = 1$. By (5.2), it holds that $(\mathcal{A}\bar{x}^{(1)}(\bar{x}^{(1)})^\top) \bullet Y^* = (Y^*\mathcal{A}) \bullet \bar{x}^{(1)}(\bar{x}^{(1)})^\top \leq v_{\min}^{\text{cop}}$. Continue the above process on Y^*. There must be an index, say 1, such that

$$(\mathcal{A}\bar{x}^{(1)}(\bar{x}^{(1)})^\top) \bullet \bar{y}^{(1)}(\bar{y}^{(1)})^\top \leq v_{\min}^{\text{cop}}. \tag{5.3}$$

On the other hand, since $\bar{x}^{(1)} \in \Delta_n$ and $\bar{y}^{(1)} \in \Delta_m$, it is clear that

$$p_{\mathcal{A}}^{\min} \leq (\mathcal{A}\bar{x}^{(1)}(\bar{x}^{(1)})^\top) \bullet \bar{y}^{(1)}(\bar{y}^{(1)})^\top,$$

which implies, together with (5.3) and the fact that $v_{\min}^{\text{cop}} \leq p_{\mathcal{A}}^{\min}$, that $p_{\mathcal{A}}^{\min} = (\mathcal{A}\bar{x}^{(1)}(\bar{x}^{(1)})^\top) \bullet \bar{y}^{(1)}(\bar{y}^{(1)})^\top = v_{\min}^{\text{cop}}$. We obtain the desired result and complete the proof. □

Remark 5.2. (1) By Lemma 5 in [3], we know that the extremal points of the feasible set of (5.1) are exactly $(xx^\top, yy^\top)$, the rank-one matrix pairs based on vectors $x \in \Delta_n$ and $y \in \Delta_m$. Theorem 5.1 shows that a solution of the problem (5.1) is attained at an extremal point of the feasible set.

(2) Copositive programming is a useful tool in dealing with all sorts of optimization problems. However, it also has very complex structures in terms of computational solvability, since checking whether a symmetric matrix belongs to $\mathcal{K}_n$ or not (the strong membership problem for $\mathcal{K}_n$) is co-NP-complete [25]. Due to the facts that the objective function in (5.1) is bilinear and the strong membership problem for the completely positive cone is NP-hard [13], we know that (5.1) is also NP-hard. Consequently, by Theorem 5.1, there exists no polynomial time algorithm for solving (3.1).

To obtain the lower bounds of (5.1), we consider two aspects mentioned earlier. We first focus on approximating $\mathcal{K}_n^*$ and $\mathcal{K}_m^*$.

Let us denote by $\mathcal{S}_+^n$ the cone of all $n \times n$ positive semidefinite symmetric matrices, and denote by $\mathcal{N}_n$ the cone of all nonnegative symmetric matrices. It is well known that the nonnegative and semidefinite cones are self-dual, i.e., $(\mathcal{S}_+^n)^* = \mathcal{S}_+^n$ and $\mathcal{N}_n^* = \mathcal{N}_n$. Recall that $\mathcal{C}_n = \mathcal{S}_+^n + \mathcal{N}_n$ is a (zero-order) inner approximation [6] of the copositive cone: $\mathcal{C}_n \subseteq \mathcal{K}_n$, with $\mathcal{C}_n = \mathcal{K}_n$ if and only if $n \leq 4$, see [12]. Hence, we have

$$\mathcal{K}_n^* \subseteq \mathcal{C}_n^* = (\mathcal{S}_+^n + \mathcal{N}_n)^* = \mathcal{S}_+^n \cap \mathcal{N}_n.$$

Consider the following positive semidefinite programming

$$\begin{array}{rll} v_{\min}^{\text{SN}} := & \min\limits_{X,Y} & (\mathcal{A}X) \bullet Y \\ & \text{s.t.} & J_n \bullet X = 1, X \in \mathcal{S}_+^n \cap \mathcal{N}_n, \\ & & J_m \bullet Y = 1, Y \in \mathcal{S}_+^m \cap \mathcal{N}_m, \end{array} \tag{5.4}$$

which is a relaxation of (5.1). It is clear that $v_{\min}^{\text{SN}} \leq v_{\min}^{\text{cop}} = p_{\mathcal{A}}^{\min}$.

Furthermore, we choose two finite sets $\{P_1, \ldots, P_s\}$ and $\{Q_1, \ldots, Q_t\}$ from $\mathcal{K}_n$ and $\mathcal{K}_m$ respectively, and consider two generated cones

$$\mathcal{D}_n = \left\{ \sum_{i=1}^s a_i P_i \mid a_i \in \Re_+ \right\} \text{ and } \mathcal{D}_m = \left\{ \sum_{j=1}^t b_j Q_j \mid b_j \in \Re_+ \right\}.$$

It is easy to see that $\mathcal{C}_n \subseteq \mathcal{C}_n + \mathcal{D}_n \subseteq \mathcal{K}_n$ and $\mathcal{C}_m \subseteq \mathcal{C}_m + \mathcal{D}_m \subseteq \mathcal{K}_m$, since $\mathcal{K}_n, \mathcal{K}_m$ are convex and $0 \in \mathcal{D}_n, 0 \in \mathcal{D}_m$. Hence, problem

$$\begin{array}{rll} v_{\text{sdp}}^{\mathcal{D}} := & \min\limits_{X,Y} & (\mathcal{A}X) \bullet Y \\ & \text{s.t.} & J_n \bullet X = 1, \\ & & X \in \mathcal{S}_+^n \cap \mathcal{N}_n \cap \mathcal{D}_n^*, \\ & & J_m \bullet Y = 1, \\ & & Y \in \mathcal{S}_+^m \cap \mathcal{N}_m \cap \mathcal{D}_m^*, \end{array} \tag{5.5}$$

can be used to improve bound $v_{\min}^{\rm SN}$. That is, $v_{\min}^{\rm SN} \le v_{\rm sdp}^{\mathcal{D}}$ and $v_{\rm sdp}^{\mathcal{D}}$ is a valid lower bound for $p_{\mathcal{A}}^{\min}$. From the structure of $\mathcal{D}_n$ and $\mathcal{D}_m$, it follows that (5.5) can be rewritten as

$$\begin{array}{rll} v_{\rm sdp}^{\mathcal{D}} := & \min\limits_{X,Y} & (\mathcal{A}X)\bullet Y \\ & \text{s.t.} & J_n \bullet X = 1, \\ & & P_i \bullet X \ge 0,\ \forall\ i=1,\ldots,s, \\ & & X \in \mathcal{S}_+^n \cap \mathcal{N}_n, \\ & & J_m \bullet Y = 1, \\ & & Q_j \bullet Y \ge 0,\ \forall\ j=1,\ldots,t, \\ & & Y \in \mathcal{S}_+^m \cap \mathcal{N}_m, \end{array} \tag{5.6}$$

which is a special type of conic optimization problem, but is also NP-hard itself.

Let $C = (c_{kl})_{1\le k,l\le m} \in \mathcal{S}^m$ with $c_{kl} = \min\limits_{1\le i,j\le n}\{a_{ijkl}\}$. Then, it is ready to see that $\mathcal{A}X \ge C$ for any X satisfying $J_n \bullet X = 1$ and $X \in \mathcal{N}_n$. We consider the following SDP problem

$$\begin{array}{rll} v_{\rm sdp}^{Y\mathcal{D}} := & \min\limits_{Y} & C\bullet Y \\ & \text{s.t.} & J_m \bullet Y = 1, \\ & & Q_j \bullet Y \ge 0,\ \forall\ j=1,\ldots,t, \\ & & Y \in \mathcal{S}_+^m \cap \mathcal{N}_m, \end{array}$$

which can be solved in polynomial time. It is clear that $v_{\rm sdp}^{Y\mathcal{D}} \le v_{\rm sdp}^{\mathcal{D}}$. Similarly, we have that $v_{\rm sdp}^{X\mathcal{D}} \le v_{\rm sdp}^{\mathcal{D}}$, where $v_{\rm sdp}^{X\mathcal{D}}$ is the optimal value of the following problem

$$\begin{array}{ll} \min\limits_{X} & D\bullet X \\ \text{s.t.} & J_n \bullet X = 1, \\ & P_i \bullet X \ge 0,\ \forall\ i=1,\ldots,s, \\ & X \in \mathcal{S}_+^n \cap \mathcal{N}_n, \end{array}$$

and $D = (d_{ij})_{1\le i,j\le n} \in \mathcal{S}^n$ with $d_{ij} = \min\limits_{1\le k,l\le m}\{a_{ijkl}\}$. Therefore, it holds that

$$\max\left\{v_{\rm sdp}^{X\mathcal{D}}, v_{\rm sdp}^{Y\mathcal{D}}\right\} \le v_{\rm sdp}^{\mathcal{D}}.$$

Furthermore, from the fact that $C \ge p_{\mathcal{A}}^0 J_m$, we know that $C\bullet Y \ge p_{\mathcal{A}}^0$ for any $Y \in \mathcal{N}_m$ satisfying $J_m \bullet Y = 1$, which implies that $v_{\rm sdp}^{Y\mathcal{D}} \ge p_{\mathcal{A}}^0$. Similarly, it holds that $v_{\rm sdp}^{X\mathcal{D}} \ge p_{\mathcal{A}}^0$. Therefore, we have

$$p_{\mathcal{A}}^0 \le \min\left\{v_{\rm sdp}^{X\mathcal{D}}, v_{\rm sdp}^{Y\mathcal{D}}\right\}.$$

On the other hand, we notice that relaxation problem (3.5) can be reformulated as the following instance of a linear optimization problem over the cone $\mathcal{K}_{nm}^*$ [3]:

$$\begin{array}{rll} \nu(Q) = & \min & Q\bullet Z \\ & \text{s.t.} & J_{nm}\bullet Z = 1, \\ & & Z \in \mathcal{K}_{nm}^*. \end{array} \tag{5.7}$$

Let us define

$$\Theta_{nm}^r = \left\{ z \in \mathbb{N}^{nm} \mid \sum_{i=1}^{nm} z_i = r+2 \right\}, \quad r = 0,1,2,\ldots,$$

where $\mathbb{N}^{nm}$ denotes the set of nm-dimensional nonnegative integer vectors. Define

$$\mathfrak{O}_r = \left\{ \sum_{z\in\Theta_{nm}^r} \beta_z \left(zz^\top - \mathrm{Diag}(z)\right) \mid \beta_z \ge 0 \ \text{ for all } z \in \Theta_{nm}^r \right\}, \quad r = 0,1,2,\ldots.$$

In [17], it was established that $\mathcal{K}_{nm}^* \subseteq \cdots \subseteq \mathfrak{D}_1 \subseteq \mathfrak{D}_0$ and $\mathcal{K}_{nm}^* = \bigcap_{r=0}^{\infty} \mathfrak{D}_r$. We consider

$$l_r(Q) = \begin{array}{ll} \min & Q \bullet Z \\ \text{s.t.} & J_{nm} \bullet Z = 1, \\ & Z \in \mathfrak{D}_r, \end{array} \tag{5.8}$$

which can be solved in polynomial time, for each fixed $r = 0, 1, 2, \ldots$. It is obvious that $l_r(Q) \le \nu(Q) \le p_{\mathcal{A}}^{\min}$ for any $r = 0, 1, \ldots$, as $\nu(Q) \le p_{\mathcal{A}}^{\min}$ and $l_r(Q)$ is a lower bound on $\nu(Q)$ for every $r = 0, 1, \ldots$, see [6, 31, 35] for details.

It is worth pointing out that, how to choose appropriate matrices $P_i (i = 1, \ldots, s)$ and Q_j $(j = 1, \ldots, t)$ in (5.5) is very important to obtain better lower bounds of (1.1). Furthermore, after the choice of P_i and Q_j, how to obtain an approximation solution of (5.6) is also very interesting. These remain as topics for further research.

6 Final Remarks

In this paper, some preliminary lower bounds for the optimal value of StMQP are presented. Furthermore, a relative approximation ratio and a bi-linear copositive optimization reformulation for StBQP are also studied. Based on these, it is hopeful to design suitable approximation algorithms for the problem (1.1) and study the related approximation ratio properties. Indeed, it is well known that StQP has a PTAS [6]. For fixed degree polynomial optimization over the simplex, De Klerk, Laurent and Parrilo [18] considered sequences of hierarchical lower and upper bounds and showed a PTAS. More recently, by using the properties of Bernstein approximation on the simplex, a new proof of a PTAS for fixed-degree polynomial optimization over the simplex was presented in [19]. These naturally raise the question— whether the same holds for StMQP. However, the appearance of Cartesian product of several simplices results in that designing a PTAS for the considered problem becomes a more complex task, which also differs from the problems considered in [6,18]. We will focus on studying PTAS for StMQP in another paper.

Acknowledgments

We thank the editor and two anonymous referees for their valuable comments and suggestions, which are much helpful to improve the quality of the paper.

References

[1] K. Anstreicher and S. Burer, D.C. Versus copositive bounds for standard QP, *J. Global Optim.* 33 (2005) 299–312.

[2] I.M. Bomze, On standard quadratic optimization problems, *J. Global Optim.* 13 (1998) 369–387.

[3] I.M. Bomze, M. Dür, E. de Klerk, C. Roos, A.J. Quist and T. Terlaky, On copositive programming and standard quadratic optimization problems, *J. Global Optim.* 18 (2000) 301–320.

[4] I.M. Bomze, Branch-and-bound approaches to standard quadratic optimization problems, *J. Global Optim.* 22 (2002) 17–37.

[5] I.M. Bomze, M. Budinich, P. Pardalos and M. Pelillo, The maximum clique problem, in *Handbook of Combinatorial Optimization (supp. A)*, D.-Z. Du, P.M. Pardalos (eds.), Kluwer, Dordrecht, 1999, pp. 1–74. de Klerk, E., Quist, A., Roos, C., Terlaky,

[6] I.M. Bomze and E. de Klerk, Solving standard quadratic optimization problems via linear, semidefinite and copositive programming, *J. Global Optim.* 24 (2002) 163–185.

[7] I.M. Bomze, C. Ling, L. Qi and X.Z. Zhang, Standard bi-quadratic optimization problems and unconstrained polynomial reformulations, *J. Global Optim.* 52 (2012) 663–687.

[8] I.M. Bomze and M. Locatelli, Undominated d.c. decompositions of quadratic functions and applications to branch-and-bound approaches, *Comput. Optim. Appl.* 28 (2004) 227–245.

[9] I.M. Bomze, M. Locatelli and F. Tardella, New and old bounds for standard quadratic optimization: dominance, equivalence and incomparability, *Math. Program.* 115 (2008) 31–64.

[10] I.M. Bomze and L. Palagi, Quartic formulation of standard quadratic optimization problems, *J. Global Optim.* 32 (2005) 181–205.

[11] G. Dahl, J.M. Leinaas, J. Myrheim and E. Ovrum, A tensor product matrix aproximation problem in quantum physics, *Linear Algebra Appl.* 420 (2007) 711–725.

[12] P.H. Diananda, On non-negative forms in real variables some or all of which are non-negative, *Proc. Cambridge Philos. Soc.* 58 (1962) 17–25.

[13] P.J.C. Dickinson and L. Gijben, On the computational complexity of membership problems for the completely positive cone and its dual, *Comput. Optim. Appl.* 57 (2014) 403–415.

[14] M. Dür, A class of problems where dual bounds beat underestimation bounds, *J. Global Optim.* 22 (2002) 49–57.

[15] A. Einstein, B. Podolsky and N. Rosen, Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.* 47 (1935) 777–780.

[16] D. Han, H. H. Dai and L. Qi, Conditions for strong ellipticity of anisotropic elastic materials, *J. Elasticity* 97 (2009) 1–13.

[17] E. de Klerk and D. V. Pasechnik, Approximation of the stability number of a graph via copositive programming, *SIAM J. Optim.* 12 (2002) 875–892.

[18] E. de Klerk, M. Laurent and P.A. Parrilo, A PTAS for the minimization of polynomials of fixed degree over the simplex, *Theoret. Comput. Sci.* 361 (2006) 210–225.

[19] E. de Klerk, M. Laurent and Z. Sun, An alternative proof of a PTAS for fixed-degree polynomial optimization over the simplex, *Math. Program.* DOI 10.1007/s10107-014-0825-6.

[20] J.K. Knowles and E. Sternberg, On the ellipticity of the equations of the equations for finite elastostatics for a special material, *J. Elasticity.* 5 (1975) 341–361.

[21] Z. Li, S. He and S. Zhang, *Approximation Methods for Polynomial Optimization: Models, Algorithms, and Applications*, Springer, New York, 2012.

[22] C. Ling, J. Nie, L. Qi and Y. Ye, Bi-quadratic optimization over unit spheres and semidefinite programming relaxations, *SIAM J. Optim.* 20 (2009) 1286–1310.

[23] H.M. Markowitz, Portfolio selection, *The Journal of Finance* 7 (1952) 77–91.

[24] H.M. Markowitz, The general mean-variance portfolio selection problem, in *Mathematical Models in Finance*, S.D. Howison, F.P. Kelly and P. Wilmott (eds.), London, 1995, pp. 93–99.

[25] K.G. Murty and S N. Kabadi, Some NP-complete problems in quadratic and nonlinear programming, *Math. Program.* 39 (1987) 117–129.

[26] Y.E. Nesterov, Global quadratic optimization on the sets with simplex structure, Discussion paper 9915, CORE. Catholic University of Louvain, Belgium 1999.

[27] I. Nowak, A new semidefinite programming bound for indefinite quadratic forms over a simplex, *J. Global Optim.* 14 (1999) 357–364.

[28] J.S. Pang A new and efficient algorithm for a class of portfolio selection problems, *Oper. Res.* 8 (1980) 754–767.

[29] L. Qi, H.H. Dai and D. Han, Conditions for strong ellipticity and M-eigenvalues, *Front. Math. China* 4 (2009), 349–364.

[30] P. Rosakis, Ellipticity and deformations with discontinuous deformation gradients in finite elastostatics, *Arch. Ration. Mech. Anal.* 109 (1990) 1–37.

[31] G. Sagol and E.A. Yildirim, Analysis of copositive optimization based bounds on standard quadratic optimization, Technical Report, Department of Industrial Engineering, Koc University, Sariyer, Istanbul, Turkey 2013.

[32] A.M.C. So, Deterministic approximation algorithms for sphere constrained homogeneous polynomial optimization problems, *Math. Program.* 129 (2011) 357–382.

[33] Y. Wang and M. Aron, A reformulation of the strong ellipticity conditions for unconstrained hyperelastic media, *J. Elasticity.* 44 (1996) 89–96.

[34] Y. Wang, L. Qi and X. Zhang, A practical method for computing the largest M-eigenvalue of a fourth-order partially symmetric tensor, *Numer. Linear Algebra Appl.* 16 (2009) 589–601.

[35] E.A. Yildirim, On the accuracy of uniform polyhedral approximations of the copositive cone, *Optim. Methods Softw.* 27 (2012) 155–173.

Manuscript received 6 January 2014
revised 10 September 2014, 30 November 2014
accepted for publication 8 December 2014

CHEN LING
School of Science, Hangzhou Dianzi University
Hangzhou, 310018, China
E-mail address: macling@hdu.edu.cn

Xinzhen Zhang
Department of Mathematics, School of Science
Tianjin University, Tianjin, 300072, China
E-mail address: xzzhang@tju.edu.cn

Liqun Qi
Department of Applied Mathematics
The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong
E-mail address: maqilq@polyu.edu.hk