Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 11, No. 2, pp. 239–255, April 2015

APPROXIMATION ALGORITHM FOR A MIXED BINARY QUADRATICALLY CONSTRAINED QUADRATIC PROGRAMMING PROBLEM

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Abstract: This paper proposes a semidefinite programming (SDP) relaxation based technique for a NP-hard mixed binary quadratically constrained quadratic programs (MBQCQP) and analyzes its approximation performance. The problem is to find two minimum norm vectors in N-dimensional real or complex Euclidean space, such that M out of $2M$ concave quadratic functions are satisfied. By employing a special randomized rounding procedure, we show that approximation ratio is upper bounded by $\mathcal{O}(M^4)$ in the real case and by $\mathcal{O}(M^3)$ in the complex case, which are both independent of problem dimensions.

Key words: *nonconvex quadratic constrained quadratic programming, semidefinite program relaxation, approximation bound, NP-hard*

Mathematics Subject Classification: *90C22, 90C20, 90C59*

1 Introduction

In this paper, we study a special mixed binary quadratically constrained quadratic programming (MBQCQP) problem. It is well-known that semidefinite programming (SDP) relaxation is an effective way to relax a nonconvex QCQP problem. Our focus is to investigate approximation bound of the SDP relaxation for the following MBQCQP problem:

$$\begin{array}{ll} \min\limits_{\mathbf{w}_1,\mathbf{w}_2\in\mathbb{F}^N,\boldsymbol{\beta}} & \|\mathbf{w}_1\|^2+\|\mathbf{w}_2\|^2 \\ \text{s.t.} & \mathbf{w}_1^H\mathbf{H}_i\mathbf{w}_1\geq\beta_i,\ i=1,\cdots,M \\ & \mathbf{w}_2^H\mathbf{H}_i\mathbf{w}_2\geq 1-\beta_i,\ i=1,\cdots,M \\ & \beta_i\in\{0,1\},\ i=1,\cdots,M, \end{array} \tag{1.1}$$

where $\mathbb{F}$ is either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$; $\boldsymbol{\beta}:=(\beta_1,\cdots,\beta_M)^T$; $\mathbf{H}_i$ $(i=1,\cdots,M)$ are $N\times N$ real symmetric or complex Hermitian positive semidefinite matrices, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{F}^N$ and $M>0$ is a given integer. We note that problem (1.1) is in general NP-hard, which will be proved in Section 2 in this paper. Notice that the problem (1.1) can be easily solved either when $N=1$ or $M=1$. Hence, we shall assume that $N\geq 2$ and $M\geq 2$ in the rest of the paper.

*The author's research was supported by the China NSF under the grant 11101261.

The problem is motivated by its application in wireless communication. Consider the network with a single base station (BS) and M users. Let $\mathcal{M} = \{1, \cdots, M\}$ denote the set of all users. Suppose there are two orthogonal time slots, and the objective is to properly assign the users into these two time slots as well as to multicast the desired signal to *each* time slot. Suppose the BS uses the linear transmit beams $\mathbf{w}_1, \ \mathbf{w}_2 \in \mathbb{C}^{N\times 1}$ to transmit the signals in time slot 1 and 2, respectively. Let $\bar{\mathbf{h}}_i \in \mathbb{C}^{N\times 1}$ denote the complex channel coefficient between the BS and user i, and let n_i denote the environmental noise power at the receiver of user i. Suppose that $\mathbf{h}_i$ and n_i remain unchanged during the considered time slots. Then the signal to noise ratio (SNR) at the receiver of each user $i \in \mathcal{M}$ in time slot q can be expressed as

$$\mathrm{SNR}_i = \frac{|\mathbf{w}_q^T \bar{\mathbf{h}}_i|^2}{n_i}, i \in \mathcal{M}, q = \{1, 2\}. \tag{1.2}$$

Let γ_i denote the SNR level prescribed by user i. Then the Quality of Service (QoS) constraint for a user i that is served in slot q can be expressed as the constraint on its received SNR level: $\frac{|\mathbf{w}_q \bar{\mathbf{h}}_i|^2}{n_i} \geq \gamma_i$. For notational simplicity, let $\mathbf{h}_i \triangleq \frac{\bar{\mathbf{h}}_i}{\sqrt{n_i \gamma_i}}$ to be user i's normalized channel. We are interested in assigning each user to either one of the available time slots, as well as to perform multicasting within each time slot to the selected users. Mathematically, it can be formulated as a special case of (1.1) with $\mathbf{H}_i = \mathbf{h}_i \mathbf{h}_i^T$.

For solving nonconvex QCQP problems with continuous variables, there are many classical results on the quality bounds of SDP relaxation [1, 2, 7, 16, 18, 11, 26, 27, 14, 17]. However, most of them can only deal with the inequality constraints, which can not include the discrete constraints as a special case.

For those QCQPs with only discrete constraints, there are also many existing results. In the seminal work by [9], a special QCQP problem with discrete variables only, i.e., the max cut problem, is investigated and the ratio of the optimal value of SDP relaxation over that of the original problem is bounded below by $0.87856\ldots$. Other related results can be found in [25, 8].

Although SDP relaxation technique has been quite successful in solving continuous QCQP or discrete QCQP, so far little is known about the effectiveness of applying it for MBQCQP, except for two recent results [13, 24]. In those two papers, the authors study MBQCQP models that differ substantially from that considered in this work. In [13], the authors consider a quadratic maximization problem with a *single* cardinality constraint on the binary variables. In [24], the authors consider an MBQCQP problem where the objective is to find a minimum norm vector under some mixed binary concave quadratic constraints and a single cardinality constraint on the binary variables. In the current work, we consider a different problem where *no* cardinality constraint for the binary variables is involved. As a result, the techniques and analysis developed in [13, 24] can not be applied directly. Except for using SDP relaxation, some alternative approaches to MBQCQP have been recently proposed in [4, 5, 20, 21, 3]. More detailed reviews of recent progress on related problems can be found in the excellent surveys [6, 12, 15].

The paper is organized as follows. In Sections 2, we show that problem (1.1) is NP-hard in general. We develop SDP relaxation for solving the MBQCQP problem (1.1) in Section 3. Based on an optimal solution of the relaxed problem, an approximation solution for the original problem is generated by a special randomization procedure. Moreover, we analyze the quality of such approximation solutions by deriving bounds on the approximation ratios between the optimal solution of the MBQCQP problem and its corresponding SDP relaxation. In Section 4, some numerical results are shown. Some conclusions are listed in the last section.

Notations. For a symmetric matrix $\mathbf{X}$, $\mathbf{X} \succ 0$ ($\mathbf{X} \succeq 0$) signifies that $\mathbf{X}$ is positive (semi)-definite. We use $\mathrm{Tr}[\mathbf{X}]$ and $\mathbf{X}[i,j]$ to denote the trace and the (i,j)th element of a matrix $\mathbf{X}$, respectively. For a vector $\mathbf{x}$, we use $\|\mathbf{x}\|$ to denote its Euclidean norm, and use $\mathbf{x}[i]$ or $\mathbf{x}_i$ to denote its ith element. Also, we use $\mathbb{R}^{N\times M}$ and $\mathbb{C}^{N\times M}$ to denote the set of real and complex $N\times M$ matrices, and use $\mathbb{S}^N$ and $\mathbb{S}^N_+$ to denote the set of $N\times N$ hermitian and hermitian positive semi-definite matrices, respectively. Finally, we use the superscript H and T to denote the complex Hermitian transpose and transpose of a matrix or a vector respectively.

2 NP-hardness

In this section, we show that the problem (1.1) is NP-hard in general. We only consider the real case of $\mathbb{F}=\mathbb{R}$, since the complex case can be handled similarly as in [16, 27]. To this end, we consider a reduction from the NP-complete equal partition problem: Given positive integers $a_1, a_2, \cdots, a_n$, decide whether there exists a subset $\mathcal{I}$ of $\{1,\cdots,n\}$ such that

$$\sum_{i\in\mathcal{I}} a_i = \sum_{i\notin\mathcal{I}} a_i. \tag{2.1}$$

Note that, by a linear transformation if necessary, the problem

$$\begin{aligned} \min_{\mathbf{x},\mathbf{y}\in\mathbb{F}^N} \quad & \mathbf{x}^H\mathbf{Q}\mathbf{x}+\mathbf{y}^H\mathbf{Q}\mathbf{y} \\ \text{s.t.} \quad & |\mathbf{x}_i|\ge 1, \ \text{or } |\mathbf{y}_i|\ge 1, \ i=1,\cdots,n. \end{aligned} \tag{2.2}$$

is a special case of (1.1), where $\mathbf{Q}\succ 0$. In the following, we will prove that (2.2) is NP-hard by showing that a subset $\mathcal{I}$ satisfying (2.1) exists if and only if the optimization problem (2.2) has a minimum value of n. Let $\mathbf{a}=(a_1,\cdots,a_n)^T$ and set $\mathbf{Q}=\mathbf{a}\mathbf{a}^T+\mathbf{I}$ where $\mathbf{I}$ is the identity matrix. The optimal value of the problem (2.2) is n, if and only if there exists $\mathbf{x}$ and $\mathbf{y}$, which satisfy

$$\mathbf{a}^T\mathbf{x}=0,\ \mathbf{a}^T\mathbf{y}=0,\ \mathbf{x}_i\in\{-1,1,0\},\ \mathbf{y}_i\in\{-1,1,0\},\ |\mathbf{x}_i|+|\mathbf{y}_i|=1,\ i=1,\cdots,n.$$

By setting $\mathbf{z}=\mathbf{x}+\mathbf{y}$, which is then binary, i.e., $\mathbf{z}_i=1$ or -1, and $\mathbf{a}^T\mathbf{z}=0$. Let $\mathcal{I}=\{i \mid z_i=1\}$, and $\mathcal{J}$ be its complement set. The above suggests that we will have a partition of $\mathbf{a}$, such that

$$\sum_{i\in\mathcal{I}} a_i = \sum_{i\in\mathcal{J}} a_i.$$

The proof is completed.

3 Algorithm and Approximation Bound Analysis

3.1 The Proposed Algorithm

Introducing a homogenizing constant ℓ, and transform $\{\beta_i\}$ to the domain $\{-1,1\}$, and denote the corresponding new variables by $\{\alpha_i\}$, then the problem is equivalent to

$$\begin{aligned}
\min \quad & \|\mathbf{w}_1\|^2 + \|\mathbf{w}_2\|^2 \\
\text{s.t.} \quad & \mathbf{w}_1^H \mathbf{H}_i \mathbf{w}_1 + \frac{1}{4}(\alpha_i - \ell)^2 \geq 1 \; i = 1, \cdots, M \\
& \mathbf{w}_2^H \mathbf{H}_i \mathbf{w}_2 + \frac{1}{4}(\alpha_i + \ell)^2 \geq 1, \; i = 1, \cdots, M \\
& \alpha_i \in \{-1, 1\}, \; i = 1, \cdots, M, \; \ell \in \{-1, 1\}.
\end{aligned}$$

Let $\mathbf{e}_i \in \mathbb{R}^{(M+1)\times 1}$ denote the elementary vector with all zeros except its ith component, which takes the value 1. Let $\mathbf{e} \in \mathbb{R}^{(M+1)\times 1}$ denote the all 1 vector and denote

$$\begin{aligned}
\mathbf{C}_{i,1} &= \frac{1}{4}\mathbf{e}_i\mathbf{e}_i^T + \frac{1}{4}\mathbf{e}_{M+1}\mathbf{e}_{M+1}^T - \frac{1}{4}\mathbf{e}_i\mathbf{e}_{M+1}^T - \frac{1}{4}\mathbf{e}_{M+1}\mathbf{e}_i^T, \\
\mathbf{A}_{i,1} &= \begin{bmatrix} \mathbf{C}_{i,1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbb{C}^{(M+1+2N)\times(M+1+2N)}, \\
\mathbf{C}_{i,2} &= \frac{1}{4}\mathbf{e}_{M+i}\mathbf{e}_{M+i}^T + \frac{1}{4}\mathbf{e}_{M+1}\mathbf{e}_{M+1}^T + \frac{1}{4}\mathbf{e}_{M+i}\mathbf{e}_{M+1}^T + \frac{1}{4}\mathbf{e}_{M+1}\mathbf{e}_{M+i}^T, \\
\mathbf{A}_{i,2} &= \begin{bmatrix} \mathbf{C}_{i,2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_i \end{bmatrix} \in \mathbb{C}^{(M+1+2N)\times(M+1+2N)}.
\end{aligned}$$

Let $\mathbf{x}^{(1)} = [\boldsymbol{\alpha}^T, \ell]^T$, $\mathbf{x}^{(2)} = \mathbf{w}_1$, $\mathbf{x}^{(3)} = \mathbf{w}_2$ and $\mathbf{x} = [\mathbf{x}^{(1)T}, \mathbf{x}^{(2)T}, \mathbf{x}^{(3)T}]^T$. Using these notations, the problem can be written compactly as

$$\begin{aligned}
\min_{\mathbf{x}} \quad & \|\mathbf{x}^{(2)}\|^2 + \|\mathbf{x}^{(3)}\|^2 \\
\text{s.t.} \quad & \mathbf{x}^H \mathbf{A}_{i,1} \mathbf{x} \geq 1, \; i = 1, \cdots, M \\
& \mathbf{x}^H \mathbf{A}_{i,2} \mathbf{x} \geq 1, \; i = 1, \cdots, M \\
& (\mathbf{x}[i])^2 = 1, \; i = 1, \cdots, M+1.
\end{aligned} \tag{P}$$

Let v^{P} be the optimal value to this original problem. For a feasible solution $\mathbf{x}$, let $v^{\mathrm{P}}(\mathbf{x})$ denote the objective value of problem (P). We consider an SDP relaxation of the problem (P), expressed as follows

$$\begin{aligned}
\min_{\mathbf{X}} \quad & \mathrm{Tr}[\mathbf{X}^{(2)}] + \mathrm{Tr}[\mathbf{X}^{(3)}] \\
\text{s.t.} \quad & \mathrm{Tr}[\mathbf{A}_{i,1}\mathbf{X}] \geq 1, i = 1, \cdots, M \\
& \mathrm{Tr}[\mathbf{A}_{i,2}\mathbf{X}] \geq 1, i = 1, \cdots, M \\
& \mathrm{diag}(\mathbf{X}^{(1)}) = \mathbf{e} \\
& \mathbf{X} \in \mathbb{S}_+^{(M+1+2N)\times(M+1+2N)}.
\end{aligned} \tag{SDP}$$

In the above formulation, $\mathbf{X}^{(1)} \in \mathbb{C}^{(M+1)\times(M+1)}$, $\mathbf{X}^{(2)}, \mathbf{X}^{(3)} \in \mathbb{C}^{N\times N}$ are the diagonal blocks of the matrix $\mathbf{X}$. Let $\mathbf{X}^*$ denote the optimal solution to this SDP problem, and let $v^{\mathrm{SDP}}(\mathbf{X}^*)$ denote its optimal solution. Because the off diagonal blocks of $\mathbf{X}$ does not appear in either the objective or the constraints of (SDP), without loss of generality, we can consider $\mathbf{X}^*$ as a block diagonal matrix of the following form

$$\mathbf{X}^* = \begin{bmatrix} \mathbf{X}^{*(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}^{*(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{X}^{*(3)} \end{bmatrix}. \tag{3.1}$$

Moreover, it can be easily concluded that $\mathbf{X}^{*(1)}$ is a real symmetric positive semi-definite matrix.

Once we obtain the solution $\mathbf{X}^*$, we need to convert it into a feasible solution for the problem (P). The randomization procedure listed in Table 1 is proposed for such purpose.

In essence, this procedure consists of two parts. The first part includes Steps S1 and S2 in Table 1, in which we generate three random vectors $\boldsymbol{\xi}^{(1)}$, $\boldsymbol{\xi}^{(2)}$, $\boldsymbol{\xi}^{(3)}$ from the normal distributions, and obtain the binary variable $\mathbf{x}^{(1)}$ from $\mathbf{X}^{(1)*}$ by using the simple sgn function to do the rounding for $\boldsymbol{\xi}^{(1)}$. The second part includes Steps S3 in Table 1, where we use similar scaling to the two continuous variables $\boldsymbol{\xi}^{(2)}$ and $\boldsymbol{\xi}^{(3)}$, to get $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$. In summary, by using the randomization procedure in Table 1, we obtain a feasible solution $\mathbf{x} = [\mathbf{x}^{(1)T}, \mathbf{x}^{(2)T}, \mathbf{x}^{(3)T}]^T$ to the problem (P).

Table 1: The Randomization Procedure

S0: Solve the SDP relaxation problem (SDP) to get $(\mathbf{X}^{(1)*}, \mathbf{X}^{(2)*}, \mathbf{X}^{(3)*})$;

S1: Generate independent random vectors $\boldsymbol{\xi}^{(1)} \in \mathbb{R}^n$, $\boldsymbol{\xi}^{(2)} \in \mathbb{F}^n$, $\boldsymbol{\xi}^{(3)} \in \mathbb{F}^n$ from the normal distribution $\mathcal{N}(\mathbf{0}, \mathbf{X}^{(\mathbf{1})*})$, $\mathcal{N}_{\mathbb{F}}(\mathbf{0}, \mathbf{X}^{(\mathbf{2})*})$, $\mathcal{N}_{\mathbb{F}}(\mathbf{0}, \mathbf{X}^{(\mathbf{3})*})$ respectively;

S2: Let $\mathbf{x}^{(1)} = \mathrm{sgn}(\boldsymbol{\xi}^{(1)})$ with $\mathrm{sgn}(x) = 1$ when $x \geq 0$ and $\mathrm{sgn}(x) = -1$ otherwise.

S3: Let $\mathbf{x}^{(2)} = t\boldsymbol{\xi}^{(2)}$, $\mathbf{x}^{(3)} = p\boldsymbol{\xi}^{(3)}$ with

$$t = \sqrt{\max_{i=1,\cdots M} \left\{ \frac{1-\left(\mathbf{x}^{(1)}\right)^T \mathbf{C}_{i,1}\mathbf{x}^{(1)}}{\left(\boldsymbol{\xi}^{(2)}\right)^H \mathbf{H}_i \boldsymbol{\xi}^{(2)}} \right\}};$$

$$p = \sqrt{\max_{i=1,\cdots M} \left\{ \frac{1-\left(\mathbf{x}^{(1)}\right)^T \mathbf{C}_{i,2}\mathbf{x}^{(1)}}{\left(\boldsymbol{\xi}^{(3)}\right)^H \mathbf{H}_i \boldsymbol{\xi}^{(3)}} \right\}};$$

S4: Let $\mathbf{x} = [\mathbf{x}^{(1)T}, \mathbf{x}^{(2)T}, \mathbf{x}^{(3)T}]^T$;

3.2 Analysis of the Approximation Ratio

Note that after a single execution of the algorithm, a feasible solution $\mathbf{x}$ for problem (P) is obtained and $v^{\mathrm{P}}(\mathbf{x}) = (t)^2\|\boldsymbol{\xi}^{(2)}\|^2 + (p)^2\|\boldsymbol{\xi}^{(3)}\|^2$. In the following, we aim to evaluate the quality of such solution. In particular, we would like to find a constant $\mu \geq 1$ such that

$$v^{\mathrm{P}}(\mathbf{x}) \leq \mu v^{\mathrm{SDP}}(\mathbf{X}^*).$$

By using the fact that such generated solution is feasible for problem (P), we have $v^{\mathrm{P}} \leq v^{\mathrm{P}}(\mathbf{x})$, which further implies that the same μ is an upper bound of the SDP relaxation performance, i.e.,

$$v^{\mathrm{P}} \leq \mu v^{\mathrm{SDP}}(\mathbf{X}^*). \tag{3.2}$$

The constant μ will be referred to as the *approximation ratio*. Before we give the key theorem, we need the following two lemmas.

Lemma 3.1. *Let* $r_i = \min\{\mathit{Rank}(\mathbf{H}_i), \mathit{Rank}(\mathbf{X}^{*(2)})\}$ *and* γ *be any given positive scalar, after a single execution of the randomization procedure listed in Table 1, we have*

$$P(t^2 \leq \gamma) \geq 1 - \sum_{i=1}^{M}(1 - P_i(\gamma, \Delta)), \tag{3.3}$$

where Δ *is an arbitrary constant satisfying that* $0 < \Delta < 1$ *and* $P_i(\gamma, \Delta) := \min\{P_{i,1}, P_2, P_{i,3}\}$ *and if* $\mathbb{F} = \mathbb{R}$

$$P_{i,1} = 1 - \max\left\{\sqrt{\frac{2}{\gamma}},\ \frac{2(r_i - 1)}{\pi - 2} \cdot \frac{2}{\gamma}\right\},$$
$$P_2 = \frac{1}{2} - \frac{1}{\pi}\arcsin(-1 + \Delta)$$
$$P_{i,3} = \frac{1}{2}\left[\left(1 - \max\left\{\sqrt{\frac{2}{\gamma\Delta}},\ \frac{2(r_i - 1)}{\pi - 2} \cdot \frac{2}{\gamma\Delta}\right\}\right) + 1\right],$$

otherwise if $\mathbb{F} = \mathbb{C}$,

$$P_{i,1} = 1 - \max\left\{\frac{4}{3} \cdot \frac{2}{\gamma},\ 16(r_i - 1)^2 \cdot \frac{4}{\gamma^2}\right\},$$
$$P_2 = \frac{1}{2} - \frac{1}{\pi}\arcsin(-1 + \Delta),$$
$$P_{i,3} = \frac{1}{2}\left[\left(1 - \max\left\{\frac{4}{3} \cdot \frac{2}{\gamma\Delta},\ 16(r_i - 1)^2 \cdot \frac{4}{\gamma^2\Delta^2}\right\}\right) + 1\right].$$

Proof. Define $t_i = \sqrt{\frac{1-\left(\mathbf{x}^{(1)}\right)^T \mathbf{C}_{i,1}\mathbf{x}^{(1)}}{\left(\boldsymbol{\xi}^{(2)}\right)^H \mathbf{H}_i \boldsymbol{\xi}^{(2)}}}$. We have the following probabilistic estimate

$$\begin{aligned} P(t^2 \le \gamma) &= 1 - P(t^2 \ge \gamma) = 1 - P(t_i^2 \ge \gamma, \exists\, i = 1, \cdots, M) \\ &\ge 1 - \sum_{i=1}^{M} P(t_i^2 \ge \gamma) = 1 - \sum_{i=1}^{M}(1 - P(t_i^2 \le \gamma)). \end{aligned} \tag{3.4}$$

Furthermore, we have the following estimates for $P(t_i^2 \le \gamma)$. Firstly, by the formula of total probability, we have

$$\begin{aligned} &P(t_i^2 \le \gamma) \\ =&P\left(t_i^2 \le \gamma \mid \mathbf{x}^{(1)}[i]\mathbf{x}^{(1)}[M+1] = -1\right) P\left(\mathbf{x}^{(1)}[i]\mathbf{x}^{(1)}[M+1] = -1\right) \\ &+ P\left(t_i^2 \le \gamma \mid \mathbf{x}^{(1)}[i]\mathbf{x}^{(1)}[M+1] = 1\right) P\left(\mathbf{x}^{(1)}[i]\mathbf{x}^{(1)}[M+1] = 1\right). \end{aligned} \tag{3.5}$$

Note that if $\mathbf{x}^{(1)}[i]\mathbf{x}^{(1)}[M{+}1] = -1$, then $\left(\mathbf{x}^{(1)}\right)^T \mathbf{C}_{i,1}\mathbf{x}^{(1)} = 1$ and thus $t_i = 0$. Consequently, we have

$$P\left(t_i^2 \le \gamma \mid \mathbf{x}^{(1)}[i]\mathbf{x}^{(1)}[M+1] = -1\right) = 1. \tag{3.6}$$

Since $\mathbf{X}^*$ is semidefinite and $\mathbf{X}^*[i,i] = 1,\ i = 1, \cdots, M+1$, we have $-1 \le \mathbf{X}^{(1)*}[i, M+1] \le 1$. Moreover, if $\boldsymbol{\xi}^{(1)} \sim \mathcal{N}(\mathbf{0}, \mathbf{X}^{(1)*})$ and $\mathbf{x}^{(1)} = \operatorname{sgn}(\boldsymbol{\xi}^{(1)})$, a well known result is that $E(\mathbf{x}^{(1)}[i]\mathbf{x}^{(1)}[M+1]) = \frac{2}{\pi}\arcsin(\mathbf{X}^{(1)*}[i, M+1])$ (see [9, Lemma 1.2]), and thus

$$\mathrm{P}\left(\mathbf{x}^{(1)}[i]\mathbf{x}^{(1)}[M+1] = -1\right) = \frac{1}{2} - \frac{1}{\pi}\arcsin(\mathbf{X}^{(1)*}[i, M+1]), \tag{3.7}$$
$$\mathrm{P}\left(\mathbf{x}^{(1)}[i]\mathbf{x}^{(1)}[M+1] = 1\right) = \frac{1}{2} + \frac{1}{\pi}\arcsin(\mathbf{X}^{(1)*}[i, M+1]). \tag{3.8}$$

By plugging in (3.6), (3.7) and (3.8) into (3.5), we have that

$$\begin{aligned}\mathrm{P}(t_i^2 \le \gamma) =& \left(\frac{1}{2} + \frac{1}{\pi} \arcsin(\mathbf{X}^{(1)*}[i, M+1])\right) \cdot \mathrm{P}\left(t_i^2 \le \gamma \mid \mathbf{x}^{(1)}[i]\mathbf{x}^{(1)}[M+1] = 1\right) \\ &+ \left(\frac{1}{2} - \frac{1}{\pi} \arcsin(\mathbf{X}^{(1)*}[i, M+1])\right). \end{aligned} \tag{3.9}$$

In the sequel, given a constant Δ with $0 < \Delta < 1$, we analyze the right hand side probability in (3.9) in the following three cases.

Case (I): $0 \le \mathbf{X}^{(1)*}[i, M+1] \le 1$. By the fact that any probability is less than or equal to 1, we obtain that

$$\begin{aligned}\mathrm{P}(t_i^2 \le \gamma) \ge& \left(\frac{1}{2} + \frac{1}{\pi} \arcsin(\mathbf{X}^{(1)*}[i, M+1])\right) \cdot \mathrm{P}\left(t_i^2 \le \gamma \mid \mathbf{x}^{(1)}[i]\mathbf{x}^{(1)}[M+1] = 1\right) \\ &+ \left(\frac{1}{2} - \frac{1}{\pi} \arcsin(\mathbf{X}^{(1)*}[i, M+1])\right) \cdot \mathrm{P}\left(t_i^2 \le \gamma \mid \mathbf{x}^{(1)}[i]\mathbf{x}^{(1)}[M+1] = 1\right) \\ =& \mathrm{P}\left(t_i^2 \le \gamma \mid \mathbf{x}^{(1)}[i]\mathbf{x}^{(1)}[M+1] = 1\right). \end{aligned} \tag{3.10}$$

Since $\mathrm{Tr}[\mathbf{A}_i\mathbf{X}^*] \ge 1$, we have $\frac{1}{2} - \frac{1}{2}\mathbf{X}^{(1)*}[i, M+1] + \mathrm{Tr}[\mathbf{H}_i\mathbf{X}^{*(2)}] \ge 1$, and thus $\mathrm{Tr}[\mathbf{H}_i\mathbf{X}^{*(2)}] \ge \frac{1}{2} + \frac{1}{2}\mathbf{X}^{(1)*}[i, M+1] \ge \frac{1}{2}$ in this case. Moreover, from the definition of t_i and by simple computation, we have that

$$\begin{aligned} &\mathrm{P}\left(t_i^2 \le \gamma \mid \mathbf{x}^{(1)}[i]\mathbf{x}^{(1)}[M+1] = 1\right) \\ =& \mathrm{P}\left(\gamma(\boldsymbol{\xi}^{(2)})^H \mathbf{H}_i \boldsymbol{\xi}^{(2)} \ge 1 \;\Big|\; \mathbf{x}^{(1)}[i]\mathbf{x}^{(1)}[M+1] = 1\right) \\ =& \mathrm{P}\left(\gamma(\boldsymbol{\xi}^{(2)})^H \mathbf{H}_i \boldsymbol{\xi}^{(2)} \ge 1\right) \qquad (3.11) \\ \ge& \mathrm{P}\left(\gamma(\boldsymbol{\xi}^{(2)})^H \mathbf{H}_i \boldsymbol{\xi}^{(2)} \ge 2\mathrm{Tr}[\mathbf{H}_i\mathbf{X}^{*(2)}]\right) \\ \ge& P_{i,1}, \qquad (3.12) \end{aligned}$$

where the second equality is due to the independence of $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$, and the last inequality is from [16, Lemma 1 and 3] and $P_{i,1}$ is defined in the lemma.

Case (II): $-1 \le \mathbf{X}^{(1)*}[i, M+1] \le -1 + \Delta$. From (3.9) and the nonnegativity of the probability, we have

$$\begin{aligned}\mathrm{P}(t_i^2 \le \gamma) \ge& \frac{1}{2} - \frac{1}{\pi} \arcsin(\mathbf{X}^{(1)*}[i, M+1]) \\ \ge& \frac{1}{2} - \frac{1}{\pi} \arcsin(-1+\Delta) = \mathrm{P}_2, \end{aligned} \tag{3.13}$$

where we have used the monotonicity of the function $\arcsin(\cdot)$.

Case (III): $-1 + \Delta \le \mathbf{X}^{(1)*}[i, M+1] \le 0$. Since $\mathrm{Tr}[\mathbf{A}_i\mathbf{X}^*] \ge 1$, in this case, we have $\frac{1}{2} - \frac{1}{2}\mathbf{X}^{(1)*}[i, M+1] + \mathrm{Tr}[\mathbf{H}_i\mathbf{X}^{*(2)}] \ge 1$, thus $\mathrm{Tr}[\mathbf{H}_i\mathbf{X}^{*(2)}] \ge \frac{1}{2} + \frac{1}{2}\mathbf{X}^{(1)*}[i, M+1] \ge \frac{\Delta}{2}$. From (3.9) and (3.11), we have

$$\begin{aligned} &\mathrm{P}\left(t_i^2 \le \gamma\right) \\ =& \mathrm{P}\left(\gamma(\boldsymbol{\xi}^{(2)})^H \mathbf{H}_i \boldsymbol{\xi}^{(2)} \ge 1\right) \cdot \left(\frac{1}{2} + \frac{1}{\pi} \arcsin(\mathbf{X}^{(1)*}[i, M+1])\right) \\ &+ \left(\frac{1}{2} - \frac{1}{\pi} \arcsin(\mathbf{X}^{(1)*}[i, M+1])\right). \end{aligned} \tag{3.14}$$

Due to the fact that $-1+\Delta \leq \mathbf{X}^{(1)*}[i, M+1] \leq 0$, $\mathrm{Tr}[\mathbf{H}_i \mathbf{X}^{*(2)}] \geq \frac{\Delta}{2}$, and from (3.14), we have

$$
\begin{aligned}
&\mathrm{P}\left(t_i^2 \leq \gamma\right)\\
\geq &\mathrm{P}\left(\gamma(\boldsymbol{\xi}^{(2)})^H \mathbf{H}_i \boldsymbol{\xi}^{(2)} \geq \frac{2}{\Delta}\mathrm{Tr}[\mathbf{H}_i \mathbf{X}^{*(2)}]\right) \cdot \left(\frac{1}{2} + \frac{1}{\pi}\arcsin(\mathbf{X}^{(1)*}[i, M+1])\right)\\
&+ \left(\frac{1}{2} - \frac{1}{\pi}\arcsin(\mathbf{X}^{(1)*}[i, M+1])\right)\\
= &\frac{1}{2}\left[\mathrm{P}\left(\gamma(\boldsymbol{\xi}^{(2)})^H \mathbf{H}_i \boldsymbol{\xi}^{(2)} \geq \frac{2}{\Delta}\mathrm{Tr}[\mathbf{H}_i \mathbf{X}^{*(2)}]\right) + 1\right]\\
&- \frac{1}{\pi}\arcsin(\mathbf{X}^{(1)*}[i, M+1])\left[1 - \mathrm{P}\left(\gamma(\boldsymbol{\xi}^{(2)})^H \mathbf{H}_i \boldsymbol{\xi}^{(2)} \geq \frac{2}{\Delta}\mathrm{Tr}[\mathbf{H}_i \mathbf{X}^{*(2)}]\right)\right]\\
\geq &\frac{1}{2}\left[\mathrm{P}\left(\gamma(\boldsymbol{\xi}^{(2)})^H \mathbf{H}_i \boldsymbol{\xi}^{(2)} \geq \frac{2}{\Delta}\mathrm{Tr}[\mathbf{H}_i \mathbf{X}^{*(2)}]\right) + 1\right]\\
\geq &P_{i,3},
\end{aligned}
\tag{3.15}
$$

where the second last inequality uses the fact that $\arcsin(x) \leq 0$ when $x \leq 0$, and the last inequality is from [16, Lemma 1 and 3].

Combining (3.12), (3.13) and (3.15), we have proved that for each $i = 1, \cdots, M$,

$$
\mathrm{P}\left(t_i^2 \leq \gamma\right) \geq P_i(\gamma, \Delta) = \min\left\{P_{i,1}, P_2, P_{i,3}\right\}. \tag{3.16}
$$

The proof is completed by plugging in the above result into (3.4). □

Lemma 3.2. *Let* $\bar{r}_i = \min\{Rank(\mathbf{H}_i), Rank(\mathbf{X}^{*(3)})\}$ *and* γ *be any given positive scalar, after a single execution of the randomization procedure listed in Table 1, we have*

$$
P(p^2 \leq \gamma) \geq 1 - \sum_{i=1}^{M}(1 - \bar{P}_i(\gamma, \Delta)), \tag{3.17}
$$

where Δ *is an arbitrary constant satisfying that* $0 < \Delta < 1$ *and* $\bar{P}_i(\gamma, \Delta)$ $(i = 1, \cdots, M)$ *are defined the same as* $P_i(\gamma, \Delta)$ *in Lemma 3.1 except that all* r_i *are replaced by* $\bar{r}_i$.

Proof. Define $p_i = \sqrt{\frac{1-\left(\mathbf{x}^{(1)}\right)^T \mathbf{C}_{i,2}\mathbf{x}^{(1)}}{\left(\boldsymbol{\xi}^{(3)}\right)^H \mathbf{H}_i \boldsymbol{\xi}^{(3)}}}$. Similar to the proof of (3.4), we have the following probabilistic estimate

$$
P(p^2 \leq \gamma) \geq 1 - \sum_{i=1}^{M}(1 - P(p_i^2 \leq \gamma)). \tag{3.18}
$$

Next, we bound $P(p_i^2 \leq \gamma)$ as follows. By the formula of total probability, we first have that

$$
\begin{aligned}
&P(t_i^2 \leq \gamma)\\
= &P\left(p_i^2 \leq \gamma \mid \mathbf{x}^{(1)}[i]\mathbf{x}^{(1)}[M+1] = -1\right) P\left(\mathbf{x}^{(1)}[i]\mathbf{x}^{(1)}[M+1] = -1\right)\\
&+ P\left(p_i^2 \leq \gamma \mid \mathbf{x}^{(1)}[i]\mathbf{x}^{(1)}[M+1] = 1\right) P\left(\mathbf{x}^{(1)}[i]\mathbf{x}^{(1)}[M+1] = 1\right).
\end{aligned}
\tag{3.19}
$$

Note that if $\mathbf{x}^{(1)}[i]\mathbf{x}^{(1)}[M+1] = 1$, then $\left(\mathbf{x}^{(1)}\right)^T \mathbf{C}_{i,2}\mathbf{x}^{(1)} = 1$ and thus $p_i = 0$,

$$P\left(p_i^2 \leq \gamma \mid \mathbf{x}^{(1)}[i]\mathbf{x}^{(1)}[M+1] = 1\right) = 1.$$

Consequently, by (3.7) and (3.8), we have

$$\begin{aligned}\mathrm{P}(p_i^2 \leq \gamma) =& \left(\frac{1}{2} - \frac{1}{\pi}\arcsin(\mathbf{X}^{(1)*}[i, M+1])\right) \cdot \mathrm{P}\left(p_i^2 \leq \gamma \mid \mathbf{x}^{(1)}[i]\mathbf{x}^{(1)}[M+1] = -1\right) \\ &+ \left(\frac{1}{2} + \frac{1}{\pi}\arcsin(\mathbf{X}^{(1)*}[i, M+1])\right). \qquad (3.20)\end{aligned}$$

By the definition of p_i, we have that

$$\mathrm{P}\left(p_i^2 \leq \gamma \mid \mathbf{x}^{(1)}[i]\mathbf{x}^{(1)}[M+1] = -1\right) = P\left(\gamma(\boldsymbol{\xi}^{(3)})^H \mathbf{H}_i \boldsymbol{\xi}^{(3)} \geq 1\right). \qquad (3.21)$$

Combining (3.20) with (3.21), we have

$$\begin{aligned}\mathrm{P}(p_i^2 \leq \gamma) =& \left(\frac{1}{2} - \frac{1}{\pi}\arcsin(\mathbf{X}^{(1)*}[i, M+1])\right) \cdot P\left(\gamma(\boldsymbol{\xi}^{(3)})^H \mathbf{H}_i \boldsymbol{\xi}^{(3)} \geq 1\right) \\ &+ \left(\frac{1}{2} + \frac{1}{\pi}\arcsin(\mathbf{X}^{(1)*}[i, M+1])\right). \qquad (3.22)\end{aligned}$$

Moreover, by $\mathrm{Tr}[\mathbf{A}_{i,2}\mathbf{X}^*] \geq 1$, we have that

$$\mathrm{Tr}[\mathbf{H}_i \mathbf{X}^{*(3)}] \geq 1 - \frac{1}{2} - \frac{1}{2}\mathbf{X}^{(1)}[i, M+1]. \qquad (3.23)$$

Fix a $\Delta \in (0, 1)$. Let us bound the probability in (3.22) in the following three cases.

Case (I): Suppose $\mathbf{X}^{*(1)}[i, M+1] \leq 0$. From (3.23), we have that $\mathrm{Tr}[\mathbf{H}_i \mathbf{X}^{*(3)}] \geq \frac{1}{2}$. Consequently, we obtain from (3.22) that

$$\begin{aligned}P(p_i^2 \leq \gamma) &\geq P\left(\gamma(\boldsymbol{\xi}^{(3)})^H \mathbf{H}_i \boldsymbol{\xi}^{(3)} \geq 1\right) \\ &\geq P\left(\gamma(\boldsymbol{\xi}^{(3)})^H \mathbf{H}_i \boldsymbol{\xi}^{(3)} \geq 2\mathrm{Tr}[\mathbf{H}_i \mathbf{X}^{*(3)}]\right) \geq \bar{P}_{i,1}, \qquad (3.24)\end{aligned}$$

where the last inequality is from[16, Lemma 1 and 3] and $\bar{P}_{i,1}$ is defined the same with $P_{i,1}$ in Lemma 3.1 except that r_i is replaced by $\bar{r}_i$.

Case (II): Suppose $1 - \Delta \leq \mathbf{X}^{*(1)}[i, M+1] \leq 1$. By (3.22), it can be easily concluded that

$$\begin{aligned}\mathrm{P}(p_i^2 \leq \gamma) &\geq \frac{1}{2} + \frac{1}{\pi}\arcsin(\mathbf{X}^{(1)*}[i, M+1]) \\ &\geq \frac{1}{2} + \frac{1}{\pi}\arcsin(1 - \Delta) \\ &= \frac{1}{2} - \frac{1}{\pi}\arcsin(\Delta - 1) = P_2, \qquad (3.25)\end{aligned}$$

where the second inequality is due to the monotonicity of the function $\arcsin(\cdot)$, and P_2 is defined the same as in Lemma 3.1.

Case (III): Suppose $0 \leq \mathbf{X}^{*(1)}[i, M+1] \leq 1 - \Delta$. In this case, (3.23) implies that

$$\mathrm{Tr}[\mathbf{H}_i \mathbf{X}^{*(3)}] \geq \frac{\Delta}{2}.$$

Utilizing this inequality and by (3.22), we have

$$\begin{aligned}
\mathrm{P}(p_i^2 \leq \gamma) \geq & \left(\frac{1}{2} - \frac{1}{\pi}\arcsin(\mathbf{X}^{(1)*}[i, M+1])\right) \cdot P\left(\gamma(\boldsymbol{\xi}^{(3)})^H \mathbf{H}_i \boldsymbol{\xi}^{(3)} \geq \frac{2}{\Delta}\mathrm{Tr}[\mathbf{H}_i \mathbf{X}^{*(3)}]\right) \\
& + \left(\frac{1}{2} + \frac{1}{\pi}\arcsin(\mathbf{X}^{(1)*}[i, M+1])\right) \\
= & \frac{1}{2}\left(1 + P\left(\gamma(\boldsymbol{\xi}^{(3)})^H \mathbf{H}_i \boldsymbol{\xi}^{(3)} \geq \frac{2}{\Delta}\mathrm{Tr}[\mathbf{H}_i \mathbf{X}^{*(3)}]\right)\right) + \frac{1}{\pi}\arcsin(\mathbf{X}^{*(3)}[i, M+1]) \\
& \times \left(1 - P\left(\gamma(\boldsymbol{\xi}^{(3)})^H \mathbf{H}_i \boldsymbol{\xi}^{(3)} \geq \frac{2}{\Delta}\mathrm{Tr}[\mathbf{H}_i \mathbf{X}^{*(3)}]\right)\right) \\
\geq & \frac{1}{2}\left(1 + P\left(\gamma(\boldsymbol{\xi}^{(3)})^H \mathbf{H}_i \boldsymbol{\xi}^{(3)} \geq \frac{2}{\Delta}\mathrm{Tr}[\mathbf{H}_i \mathbf{X}^{*(3)}]\right)\right) \geq \bar{P}_{i,3},
\end{aligned} \tag{3.26}$$

where the second to the last inequality is due to the fact that $\arcsin(\mathbf{X}^{(1)*}[i, M+1]) \geq 0$ when $\mathbf{X}^{*(1)}[i, M+1] \geq 0$, and the last inequality is from[16, Lemma 1 and 3] and $\bar{P}_{i,3}$ is the same with $P_{i,3}$ in Lemma 3.1 except that r_i is replaced by $\bar{r}_i$.

Combining (3.24), (3.25) and (3.26), we have proved that for each $i = 1, \cdots, M$,

$$\mathrm{P}\left(p_i^2 \leq \gamma\right) \geq \bar{P}_i(\gamma, \Delta) = \min\left\{\bar{P}_{i,1}, P_2, \bar{P}_{i,3}\right\}. \tag{3.27}$$

The proof is completed by plugging in the above result into (3.18). □

Theorem 3.3. *There exists a constant $\mu > 0$ such that*

$$\mathrm{P}\left(v^{\mathrm{P}} \leq \mu v^{\mathrm{SDP}}(\mathbf{X}^*)\right) \geq \frac{1}{12}, \tag{3.28}$$

where $\mu = \frac{18M^2}{\sin^2\frac{\pi}{6M}}$ when $\mathbb{F} = \mathbb{R}$, and $\mu = \frac{32M}{\sin^2\frac{\pi}{6M}}$ when $\mathbb{F} = \mathbb{C}$.

Proof. Note that we have the following series of inequalities

$$\begin{aligned}
& \mathrm{P}\left(v^{\mathrm{P}} \leq \mu v^{\mathrm{SDP}}(\mathbf{X}^*)\right) \\
= & \mathrm{P}\left(\|\mathbf{x}^{(2)}\|^2 + \|\mathbf{x}^{(3)}\|^2 \leq \mu(\mathrm{Tr}[\mathbf{X}^{*(2)}] + \mathrm{Tr}[\mathbf{X}^{*(3)}])\right) \\
\geq & \mathrm{P}\left(\|\mathbf{x}^{(2)}\|^2 \leq \mu\mathrm{Tr}[\mathbf{X}^{*(2)}], \|\mathbf{x}^{(3)}\|^2 \leq \mu\mathrm{Tr}[\mathbf{X}^{*(3)}]\right) \\
\geq & \mathrm{P}\left(t^2 \leq \gamma, p^2 \leq \gamma, \|\boldsymbol{\xi}^{(2)}\|^2 \leq \frac{\mu}{\gamma}\mathrm{Tr}[\mathbf{X}^{*(2)}], \|\boldsymbol{\xi}^{(3)}\|^2 \leq \frac{\mu}{\gamma}\mathrm{Tr}[\mathbf{X}^{*(3)}]\right) \\
\geq & 1 - \mathrm{P}\left(t^2 \leq \gamma\right) - \mathrm{P}\left(p^2 \leq \gamma\right) - \mathrm{P}\left(\|\boldsymbol{\xi}^{(2)}\|^2 \geq \frac{\mu}{\gamma}\mathrm{Tr}[\mathbf{X}^{*(2)}]\right) - \mathrm{P}\left(\|\boldsymbol{\xi}^{(3)}\|^2 \geq \frac{\mu}{\gamma}\mathrm{Tr}[\mathbf{X}^{*(3)}]\right) \\
\geq & 1 - \mathrm{P}\left(t^2 \leq \gamma\right) - \mathrm{P}\left(p^2 \leq \gamma\right) - \frac{2\gamma}{\mu},
\end{aligned} \tag{3.29}$$

where the last inequality is due to Markov's inequality. By Lemma 3.1 and Lemma 3.2, we obtain from (3.29) that

$$\mathrm{P}\left(v^{\mathrm{P}} \leq \mu v^{\mathrm{SDP}}(\mathbf{X}^*)\right) \geq 1 - \sum_{i=1}^{M}(1 - P_i(\gamma, \Delta)) - \sum_{i=1}^{M}\left(1 - \bar{P}_i(\gamma, \Delta)\right) - \frac{2\gamma}{\mu}. \tag{3.30}$$

By applying a suitable rank reduction procedure if necessary, we can assume that the rank r_1^* (r_2^*) of optimal SDP solution $\mathbf{X}^{(2)*}$ ($\mathbf{X}^{(3)*}$) satisfies $r_i^*(r_i^*+1)/2 \leq M$ $(i=1,2)$; cf., [19, 23]. Thus, r_i in Lemma 3.1 and $\bar{r}_i$ in Lemma 3.2 satisfy that $r_i, \bar{r}_i < \sqrt{2M}$. Let

$$P_0(\gamma,\Delta) = \min\left\{\hat{P}_1, P_2, \hat{P}_3\right\}$$

with $\hat{P}_1$ and $\hat{P}_3$ being defined the same with $P_{i,1}$ and $P_{i,3}$ in Lemma 3.1 except r_i replaced by $\sqrt{2M}$. Thus, we have

$$P_i(\gamma,\Delta) \geq P_0(\gamma,\Delta), \tag{3.31}$$

$$\bar{P}_i(\gamma,\Delta) \geq P_0(\gamma,\Delta). \tag{3.32}$$

By (3.30), (3.31) and (3.32), we can easily obtain that

$$\mathrm{P}\left(v^{\mathrm{P}} \leq \mu v^{\mathrm{SDP}}(\mathbf{X}^*)\right) \geq 1 - 2M\left(1 - P_0(\gamma,\Delta)\right) - \frac{2\gamma}{\mu}. \tag{3.33}$$

The Real Case. When $\mathbb{F}=\mathbb{R}$, let $\Delta = 2\sin^2\frac{\pi}{6M}$, $\gamma = \frac{9M^2}{2\Delta}$ and

$$\mu = 8\gamma = \frac{18M^2}{\sin^2\frac{\pi}{6M}}. \tag{3.34}$$

It can be easily checked that $\hat{\mathrm{P}}_1 = 1 - \frac{2\sqrt{\Delta}}{3M}$, $\mathrm{P}_2 = \hat{\mathrm{P}}_3 = 1 - \frac{1}{3M}$ in this case. When $M \geq 2$, we have $\Delta < \frac{1}{4}$, thus for each i, we can have that

$$P_0(\gamma,\Delta) = 1 - \frac{1}{3M}.$$

By plugging in these results into (3.33), and set $\sigma = \frac{1}{12}$, we obtain that

$$\mathrm{P}\left(v^{\mathrm{P}} \leq \mu v^{\mathrm{SDP}}(\mathbf{X}^*)\right) \geq 1 - \frac{2}{3} - \frac{1}{4} = \frac{1}{12}. \tag{3.35}$$

The Complex Case. When $\mathbb{F}=\mathbb{C}$, let $\Delta = 2\sin^2\frac{\pi}{6M}$, $\gamma = \frac{4M}{\Delta}$ and

$$\mu = 8\gamma = \frac{32M}{\sin^2\frac{\pi}{6M}}. \tag{3.36}$$

It can be easily checked that $\hat{\mathrm{P}}_1 = 1 - \frac{2\Delta}{3M}$, $\mathrm{P}_2 = \hat{\mathrm{P}}_3 = 1 - \frac{1}{3M}$ in this case. Obviously, when $M \geq 1$, we have $\Delta < \frac{1}{2}$, thus for each i, we can have that

$$P_0(\gamma,\Delta) = 1 - \frac{1}{3M}.$$

By plugging these results into (3.33), and set $\sigma = \frac{1}{12}$, we obtain that

$$\mathrm{P}\left(v^{\mathrm{P}} \leq \mu v^{\mathrm{SDP}}(\mathbf{X}^*)\right) \geq 1 - \frac{2}{3} - \frac{1}{4} = \frac{1}{12}.$$

By combining both the real and the complex cases, the proof is completed. □

Remark. In the proof of Theorem 3.3, there may exist some other possible choices of Δ and γ. However, it seems that by changing Δ and γ it is not possible to obtain better ratio in terms of the order of M. Therefore, we do not choose to do further optimization over the constants.

Through Theorem 3.3, we have proved the approximation ratio for the proposed algorithm is upper bounded by a constant μ which is in the order of M^4 (resp. M^3) when $\mathbb{F}=\mathbb{R}$ (resp. $\mathbb{F}=\mathbb{C}$). The ratios are independent of problem dimension N in both cases.

Table 2: Mean and standard deviation of the approximation ratio over 100 independent realizations of real Gaussian i.i.d. $\mathbf{h}_i$ $(i = 1, \cdots, M)$, when $\mathbb{F} = \mathbb{R}$.

M	N	min	mean	max	Std	μ
$M = 5$	$N = 4$	1.0000	1.0604	1.2884	0.0064	4.1185×10^4
	$N = 8$	1.0000	1.1185	1.3729	0.0093	
	$N = 12$	1.0000	1.1258	1.3921	0.0103	
$M = 10$	$N = 4$	1.0000	1.3161	1.9852	0.0406	6.5716×10^5
	$N = 8$	1.0001	1.5676	2.2997	0.0567	
	$N = 12$	1.2036	1.6702	2.2181	0.0491	
$M = 15$	$N = 4$	1.0774	1.6814	3.2398	0.1553	3.3252×10^6
	$N = 8$	1.4986	2.3431	3.6177	0.2298	
	$N = 12$	1.6155	2.6286	4.4366	0.2865	

4 Numerical Experiments

In this section we perform numerical study for the proposed algorithms. Throughout, we test the proposed procedure listed in Table 1 for (P) with different choices of M and N. We generate the data matrix $\mathbf{H}_i$ by using $\mathbf{H}_i = \mathbf{h}_i \mathbf{h}_i^H$ $(i = 1, \cdots, M)$, with randomly generated vectors $\mathbf{h}_i$. For each given $\mathbf{H}_i$, we call the correspongding test problem a realization. The SDP relaxation problems are all solved by CVX [10], and the optimal objective value for the SDPs are denoted by $v_{\text{SDP}}^{\min}$.

The sample of $\boldsymbol{\xi}^{(1)}$ in Step S1 in Table 1 are repeated by $T_1 = 100$ independent trials, and for a given $\boldsymbol{\xi}^{(1)}$, $T_2 = 100$ independent couple samples of $\boldsymbol{\xi}^{(2)}$ and $\boldsymbol{\xi}^{(3)}$ are generated. For a given realization, the best solution over these samples is recorded.

The solutions generated by kth trial are denoted by $(\mathbf{x}^{(1)})^k$, $(\mathbf{x}^{(2)})^k$ and $(\mathbf{x}^{(3)})^k$. Let

$$v_{\text{UBQP}}^{\min} := \min_{k_1 = 1, \cdots, T_1 \times T_2} \|(\mathbf{x}^{(2)})^k\|^2 + \|(\mathbf{x}^{(3)})^k\|^2.$$

It is clear that $v_{\text{UBQP}}^{\min} \geq v^{\text{P}}$, as a result, $v_{\text{UBQP}}^{\min}/v_{\text{SDP}}^{\min}$ is an upper bound of the true approximation ratio (which is difficult to obtain in polynomial time).

Table 2 shows the minimum value (min), the average value (mean), the maximum value (max) and the standard deviation (Std) of $v_{\text{UBQP}}^{\min}/v_{\text{SDP}}^{\min}$ over 100 independent realizations of i.i.d. real-valued $\mathbf{h}_i$, $(i = 1, \cdots, M)$ from the standard normal distribution, for several combinations of M and N. The last column in Table 2 shows the theoretic approximation bound given in Theorem 3.3. Table 3 shows the corresponding results of $v_{\text{UBQP}}^{\min}/v_{\text{SDP}}^{\min}$ for $\mathbb{F} = \mathbb{C}$. We can see that the results in both tables are significantly better than what is predicted by our worst-case analysis. In all test examples, the average values of $v_{\text{UBQP}}^{\min}/v_{\text{SDP}}^{\min}$ are lower than 2.7 (resp. lower than 2.1) when $\mathbb{F} = \mathbb{R}$ (resp. when $\mathbb{F} = \mathbb{C}$). Moreover, the minimum values of $v_{\text{UBQP}}^{\min}/v_{\text{SDP}}^{\min}$ often equal to 1, which means that for these cases the optimal solutions are obtained by our algorithm.

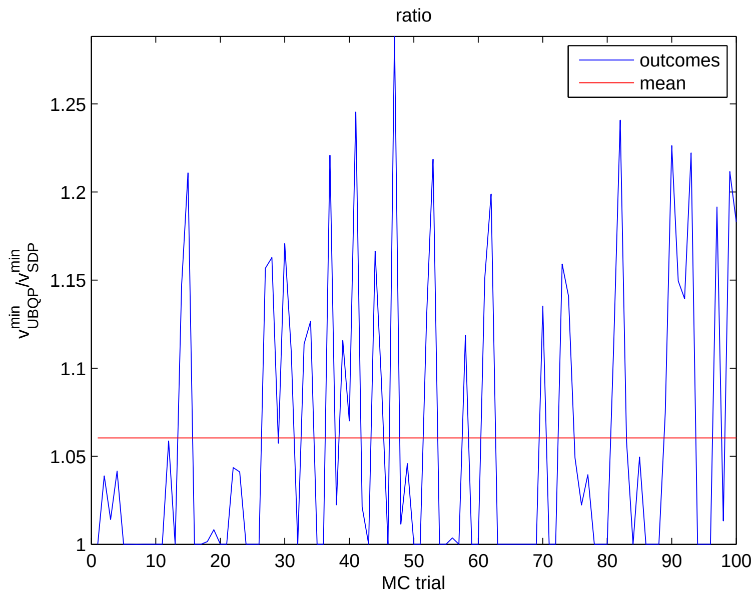
Figure 1 plots $v_{\text{UBQP}}^{\min}/v_{\text{SDP}}^{\min}$ for 100 independent realizations of i.i.d. real valued Gaussian $\mathbf{h}_i$ $(i = 1, \cdots, M)$ for $M = 5$ and $N = 4$. Figure 2 shows the corresponding histogram. Figure 3 and Figure 4 show the corresponding results for i.i.d complex-valued circular Gaussian $\mathbf{h}_i$ $(i = 1, \cdots, M)$. The mean of the upper bound $v_{\text{UBQP}}^{\min}/v_{\text{SDP}}^{\min}$ are lower in the complex case.

Moreover, our numerical results also collaborate well with our theoretic analysis. First, the upper bound of the approximation ratio is independent of the dimension of $\mathbf{w}$: the results

Table 3: Mean and standard deviation of upper bound ratio over 100 independent realizations of real Gaussian i.i.d. $\mathbf{h}_i$ $(i = 1, \cdots, M)$, when $\mathbb{F} = \mathbb{C}$.

M	N	min	mean	max	Std	μ
$M = 5$	$N = 4$	1.0000	1.0254	1.3166	0.0039	1.4644×10^4
	$N = 8$	1.0000	1.0291	1.3153	0.0042	
	$N = 12$	1.0000	1.0310	1.3460	0.0060	
$M = 10$	$N = 4$	1.0000	1.4371	1.9028	0.0365	1.1682×10^5
	$N = 8$	1.0000	1.5951	1.9944	0.0484	
	$N = 12$	1.0001	1.6106	1.9939	0.0459	
$M = 15$	$N = 4$	1.2597	1.7009	2.1490	0.0376	3.9410×10^5
	$N = 8$	1.4668	2.0380	2.5418	0.0447	
	$N = 12$	1.6498	2.0657	2.5358	0.0377	

Figure 1: Upper bound on $v_{\mathrm{QP}}^{\min}/v_{\mathrm{SDP}}^{\min}$ for $M = 5$, $N = 4$, 100 realizations of real Gaussian i.i.d. $\mathbf{h}_i$ for $i = 1, \cdots, M$ in the real case.



(maximum value) vary only slightly for different choices of N in both real and complex cases, except the case that $M = 15$ and $N = 4$. Second, from Table 2 and Table 3, it can be shown that for fixed N, the maximum value and the average value of $v_{\mathrm{UBQP}}^{\min}/v_{\mathrm{SDP}}^{\min}$ over 100 independent trials grow as M increases in all test examples except the case that $M = 5$ and $N = 4$ in Table 3. It corresponds to the result in Theorem 3.3. Moreover, the approximation ratios reported in Table 2 and 3 are significantly better than the corresponding theoretic bounds.

The above empirical analysis complements our theoretic worst-case analysis of the performance of SDP relaxation for the special MBQCQP problem considered herein.

Figure 2: Histogram of the outcomes in Figure 1.

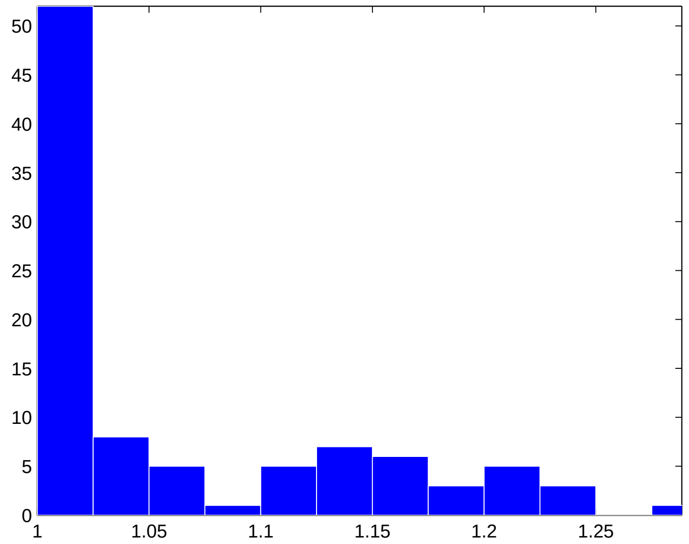
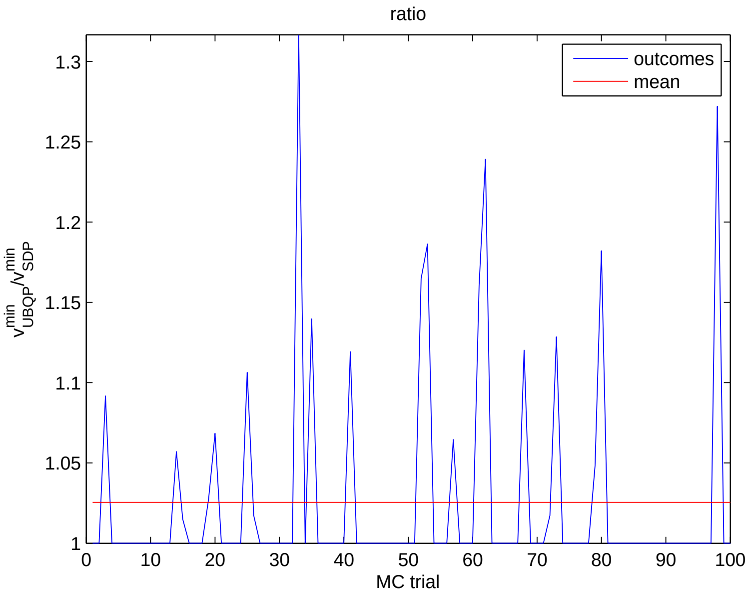


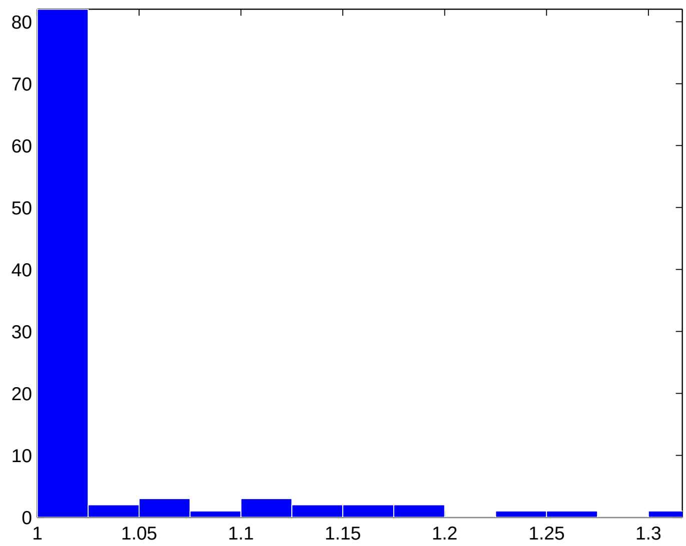
Figure 3: Upper bound on $v_{\mathrm{QP}}^{\min}/v_{\mathrm{SDP}}^{\min}$ for $M = 5$, $N = 4$, 100 realizations of real Gaussian i.i.d. $\mathbf{h}_i$ for $i = 1, \cdots, M$ in the complex case.



$\boxed{5}$ Conclusion

This paper studies the quality bounds of SDP relaxations for a special QCQP problem with mixed binary and continuous variables, for which the NP-hardness is also proved. Our

Figure 4: Histogram of the outcomes in Figure 3.



analysis is motivated by its important emerging applications in transmit beamforming for joint physical layer multicasting and scheduling in wireless networks. Our theoretic analysis provides the upper bound for the ratio between the SDP relaxation and the MBQCQP problem, and for optimization variables defined over either real or complex field. Numerical results show that our algorithm performs much better than the theoretic predicted result.

Acknowledgement

The authors are grateful for the anonymous reviewers' useful comments. The authors are also very grateful for Professor Shuzhong Zhang's helpful comment to improve this paper.

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Manuscript received 16 July 2014
revised 4 December 2014, 29 December 2014
accepted for publication 30 December 2014

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