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COPOSITIVITY-BASED APPROXIMATIONS FOR MIXED-INTEGER FRACTIONAL QUADRATIC OPTIMIZATION*

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Abstract: We propose a copositive reformulation of the mixed-integer fractional quadratic problem (MIFQP) under general linear constraints. This problem class arises naturally in many applications, e.g., for optimizing communication or social networks, or studying game theory problems arising from genetics. It includes several APX-hard subclasses: the maximum cut problem, the k-densest subgraph problem and several of its variants, or the ternary fractional quadratic optimization problem (TFQP). Problems of this type arise when modelling density clustering problems with two voting options plus the possibility of an abstention, which is a criterion-based graph tri-partitioning problem. This paper adds to the rich evidence for the versatility of copositive optimization approaches, and hints at possible novel approximation strategies combining continuous and discrete optimization techniques in the domain of (fractional) polynomial optimization.

Key words: *copositive optimization, fractional quadratic optimization, mixed-integer polynomial optimization*

Mathematics Subject Classification: *90C32, 90C11, 90C22, 90C20, 49M29, 65K05*

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1 Introduction

1.1 A short motivation

Given two quadratic functions $f(\mathsf{x}) = \mathsf{x}^\top \mathsf{A}\mathsf{x} + 2\mathsf{a}^\top \mathsf{x} + \alpha$ and $g(\mathsf{x}) = \mathsf{x}^\top \mathsf{B}\mathsf{x} + 2\mathsf{b}^\top \mathsf{x} + \beta$, consider the *binary fractional quadratic optimization problem (BFQP)*

$$\tau^*_{\text{BFQP}} := \min \left\{ \frac{f(\mathsf{x})}{g(\mathsf{x})} : \mathsf{x} \in \{0,1\}^n \right\} \quad . \tag{1.1}$$

Problems of this kind emerge, e.g., in the densest subgraph problem if A is the adjacency matrix of an undirected graph, $\mathsf{B} = \mathsf{I}_n$, and both f and g are homogeneous (and $\mathsf{x} = \mathsf{o}$ is excluded, e.g., by adding the constraint $\sum_i x_i \geq 1$). While the densest subgraph can be found in polynomial time by Goldberg's reduction to a maximum-flow problem [13], constraints on the minimum/maximum order of the subgraph renders the problem hard: if the order is fixed to k, the problem gets APX-hard (no Polynomial-Time Approximation

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Scheme (PTAS) exists unless `P = NP`) [16], and under similar restrictions which appear for instance in some genome analysis applications, it is NP-complete [27].

In this paper we consider the more general case of a mixed-integer fractional QP (MIFQP), and using a binary expansion of the integer variables, we describe how it can be reformulated as a mixed-binary fractional problem, for which a specific novel copositive reformulation is presented. The less studied case of fractional ternary problems is also specifically addressed. Real-world applications of fractional binary problems include: noise modelling for constrained nearfield array optimization [1]; efficient computation of the Pareto boundary in wireless communication channels [15]; and calculation of the quadratic coordination ratio in game theory [18], to cite just a few. Possible applications of ternary fractional quadratic problems in the context of graph tri-partitioning problems are related to clustering, voting systems with abstention, and other social media challenges.

A *copositive optimization problem (COP)* amounts to linear optimization over the intersection of an affine matrix subspace with the matrix cone of co(mpletely)positive matrices. For surveys on this nowadays highly popular conic optimization problem which pushes all hardness into the conic membership constraint, one may consult [4, 12].

It is worth mentioning that the COP approach was first applied to the Standard Quadratic Problem (StQP), which encodes, for instance, the Maximum-Clique-Problem. In contrast to the classes mentioned above, the class of StQPs admits a PTAS [5]. On the other hand, observe that by shifting $\{0,1\}$ to $\{-1,1\}$, then considering homogeneous f and g, now with $-\mathsf{A}$ the Laplace matrix of an undirected graph and $\mathsf{B} = \mathsf{I}_n$, the denominator is constant and we arrive at a quadratic optimization formulation of the Maximum-Cut-Problem which also is known as APX-hard [20].

The observations in this paper add to the rich evidence for high versatility of copositive optimization approaches, and thereby offer novel approximation strategies combining continuous and discrete optimization techniques. This statement is reinforced if we consider that,

- any polynomial of degree $d \geq 2$ can be written as an affine-linear function of new variables replacing the monomials, with additional quadratic constraints $z = xy$, see e.g. [28] for an early explicit account;
- any binary constraint $x \in \{0,1\}$ can be replaced with the quadratic $x(1-x) = 0$;

so, in principle, any mixed-integer fractional polynomial optimization problem can be recast as a fractional quadratic one, introducing additional variables and constraints. Although any fractional polynomial optimization problem

$$\min \left\{ \frac{f(\mathsf{x})}{g(\mathsf{x})} : p_i(\mathsf{x}) = 0\,,\ \mathsf{x} \in \mathbb{R}^n \right\}\,,$$

where f, g, p_i are polynomials of degree $\leq d$, can be written as a (purely) polynomial optimization problem in $n+1$ variables with degree $\leq d+1$ and one more polynomial constraint $f(\mathsf{x}) - tg(\mathsf{x}) = 0$, this generic approach may not be practical for all purposes as it ignores somehow the specific structure of the problem at the outset. So to reformulate the problem in the fractional form seems more adequate in some cases.

Let us note in passing that also the general copositive optimization problem is of course a polynomial optimization problem; in fact, a very popular approach in approximation hierarchies employs squared variables to get rid of positivity constraints, and can be traced back to the seminal works of Lasserre [17] and Parrilo [21, 22]. A very recent development involving copositive tensor reformulations of general polynomial optimization problems is provided in the elucidating article [25].

$\boxed{1.2}$ Notation and preliminaries

Given two integers L and U with $L \leq U$, we denote by $[L:U] = \{L, \ldots, U\}$ the integer range between L and U, including them. Vectors are denoted by lowercase boldface letters (e.g., o is the zero vector) and matrices by uppercase letters (e.g., O is the zero matrix, or I_n the $n \times n$ identity matrix, the columns of which are denoted by $\mathsf{e}_1, \ldots, \mathsf{e}_n$). $\mathbb{N}$ denotes the set of nonnegative integers, $\mathbb{R}^n$ denotes n-dimensional Euclidean space and $\mathbb{R}^n_+$ the positive orthant therein with $\mathsf{e} = \sum_i \mathsf{e}_i = [1, \ldots, 1]^\top \in \mathbb{R}^n_+$ where $^\top$ denotes transposition of a matrix or a vector. Given two (possibly rectangular) matrices A and B, we denote by $\mathsf{A} \otimes \mathsf{B}$ their Kronecker or tensor product. Given a $k \times d$ matrix C, we denote by $\ker_+ \mathsf{C} = \left\{ \mathsf{x} \in \mathbb{R}^d_+ : \mathsf{Cx} = \mathsf{o} \right\}$. For instance, if $\mathsf{C} = [1,1,1] \otimes \mathsf{I}_n$, we easily see that

$$\ker_+ \mathsf{C} = \{\mathsf{o}\} \ . \tag{1.2}$$

Recall that a polyhedron of the form $\left\{ \mathsf{x} \in \mathbb{R}^d_+ : \mathsf{Cx} = \mathsf{c} \right\}$ is bounded if and only if condition (1.2) is satisfied, and we will assume this at places in the sequel. The Frobenius inner product of two matrices A and B in

$$\mathcal{M}_d = \left\{ \mathsf{A} \text{ is a } d \times d \text{ matrix} : \mathsf{A}^\top = \mathsf{A} \right\}$$

is represented by $\mathsf{A} \bullet \mathsf{B} = \operatorname{trace}(\mathsf{AB})$.

The cone of *copositive matrices* is given by

$$\mathcal{C}_d = \left\{ \mathsf{S} \in \mathcal{M}_d : \mathsf{x}^\top \mathsf{Sx} \geq 0 \text{ for all } \mathsf{x} \in \mathbb{R}^d_+ \right\} \ .$$

So a symmetric matrix S is copositive if and only if S generates a quadratic form which takes no negative value over $\mathbb{R}^d_+$. Later on, we will need a generalization of this concept: a matrix $\mathsf{S} \in \mathcal{M}_d$ is said to be *$\ker_+\mathsf{C}$-copositive*, if

$$\mathsf{x}^\top \mathsf{Sx} \geq 0 \quad \text{for all } \mathsf{x} \in \ker_+ \mathsf{C} \, ,$$

in other words, if $\mathsf{x} \in \mathbb{R}^d_+$ and $\mathsf{Cx} = \mathsf{o}$ together imply $\mathsf{x}^\top \mathsf{Sx} \geq 0$.

Given a general closed, pointed convex cone $\mathcal{K} \subseteq \mathcal{M}_d$ its dual cone is given by

$$\mathcal{K}^* = \{ \mathsf{S} \in \mathcal{M}_d : \mathsf{S} \bullet \mathsf{Z} \geq 0 \text{ for all } \mathsf{Z} \in \mathcal{K} \}.$$

With respect to this duality, the dual cone of the copositive matrices is the cone of completely positive matrices

$$\mathcal{C}^*_d = \left\{ \mathsf{Y} \in \mathcal{M}_d : \mathsf{Y} = \mathsf{FF}^\top, \ \mathsf{F} \text{ is a } d \times k \text{ matrix with no negative entries} \right\} \ .$$

Both cones are intractable as testing membership is NP-hard [10, 19], but as relaxations we can resort to approximations of these.

Let $\mathcal{P}_d \subset \mathcal{M}_d$ be the cone of symmetric positive-semidefinite (psd) $d \times d$ matrices and $\mathcal{N}_d \subset \mathcal{M}_d$ be the cone of nonnegative symmetric matrices. The matrix cone $\mathcal{D}_d = \mathcal{P}_d \cap \mathcal{N}_d$ is called the cone of *doubly nonnegative matrices.* Its dual cone is $\mathcal{D}^*_d = \mathcal{P}_d + \mathcal{N}_d$, the cone of *nonnegative decomposable matrices.* It is evident that $\mathcal{D}^*_d$ provides an approximation of the copositive cone $\mathcal{C}_d$ in the sense of $\mathcal{D}^*_d \subseteq \mathcal{C}_d$. Since $\mathcal{P}_d$ and $\mathcal{N}_d$ are self-dual cones, we have

$$\mathcal{C}^*_d \subseteq \mathcal{D}_d = (\mathcal{P}_d + \mathcal{N}_d)^* = \mathcal{P}_d \cap \mathcal{N}_d \, ,$$

so the doubly nonnegative cone provides an outer approximation of the completely positive cone.

The following optimization problems form a pair of primal-dual conic optimization problems:

$$\inf \{\mathsf{A}_0 \bullet \mathsf{X} : \mathsf{A}_i \bullet \mathsf{X} = b_i\,,\ 1 \le i \le m\,,\, \mathsf{X} \in \mathcal{K}^*\} \tag{1.3}$$

and

$$\sup \left\{ \mathsf{b}^\top \mathsf{y} : \mathsf{y} \in \mathbb{R}^m\,,\, \mathsf{A}_0 - \sum_{i=1}^m y_i \mathsf{A}_i \in \mathcal{K} \right\}\,. \tag{1.4}$$

Possible copositive relaxations are obtained for $\mathcal{K} = \mathcal{D}_d^* = \mathcal{P}_d + \mathcal{N}_d$ and $\mathcal{K}^* = \mathcal{D}_d = \mathcal{P}_d \cap \mathcal{N}_d$. It is clear that better approximations for $\mathcal{C}_d^*$ and $\mathcal{C}_d$ would lead to better relaxations, as for instance SDP- or LP-based approximation hierarchies $(\mathcal{K}_d^r)_{r\in\mathbb{N}}$. However, checking membership of $\mathcal{K}_d^r$ in any such hierarchy usually involves psd matrices of order d^{r+1}, rendering these approximations computationally intractable for large d and even moderately sized r. This is why in general $\mathcal{D}_d^*$ and $\mathcal{D}_d$ are, by far, the most popular copositive relaxations employed. In fact, most (but not all) hierarchies, start with $\mathcal{K}_d^0 = \mathcal{D}_d^*$; see, e.g. [5, 11, 23, 24, 29] or [6] for a concise survey.

2 Copositive Formulation of Linearly Constrained, Mixed-Integer Fractional QPs

We now investigate the more general case of a *mixed-integer fractional QP (MIFQP)* in the variables x_i where the objective is of the same form as in (1.1) and subject to more general linear constraints of the form $\widehat{\mathsf{C}}\mathsf{x} = \widehat{\mathsf{c}}$ with an $m \times n$ matrix $\widehat{\mathsf{C}}$ and $\widehat{\mathsf{c}} \in \mathbb{R}^m$. To be more precise, we now discuss the linearly constrained, mixed-integer optimization problem

$$\tau_{\mathrm{MI}}^* := \inf \left\{ \frac{f(\mathsf{x})}{g(\mathsf{x})} : \mathsf{x} \in \mathbb{R}_+^n,\, \widehat{\mathsf{C}}\mathsf{x} = \widehat{\mathsf{c}},\ x_i \in [L_i : U_i] \text{ for all } i \in I \right\} \tag{2.1}$$

where $L_i \le U_i$ are integers for all $i \in I$ and I collects the indices of all integer variables. If $I = \emptyset$, we have a continuous fractional FQP which was treated by a copositive approach in [2]. In this case the feasible set is bounded if and only if condition (1.2) holds, and if we ensure that the denominator never vanishes on this set (to render the objective well defined), τ^* is attained and inf can be replaced by min.

In his influential paper [9], Burer introduced a copositive reformulation for mixed-binary QPs (without fractional objective) which we can employ also here as follows (the extension from the binary to the general integer case was kindly suggested by a referee): consider the binary expansion of $x_i,\ i \in I$ by additional variables $z_i^{(j)} \in \{0, 1\},\ j \in [0 : \ell_i]$, where $\ell_i = \lfloor \log_2(U_i - L_i) \rfloor$,

$$\text{namely} \quad x_i = L_i + \sum_{j=0}^{\ell_i} z_i^{(j)} 2^j\,,\ i \in I\,. \tag{2.2}$$

Next we replace the vector x by $\mathsf{v} \in \mathbb{R}^d$ with $d = n + \sum\limits_{i \in I} \ell_i$ coordinates equalling either x_k, all $k \in [1 : n] \setminus I$, or $z_i^{(j)}$, all $(i, j) \in B$, where

$$B := \bigcup_{i \in I} \{i\} \times [0 : \ell_i]\,. \tag{2.3}$$

Now, using (2.2), we rewrite all linear constraints $\widehat{\mathsf{C}}\mathsf{x} = \widehat{\mathsf{c}}$ into a new linear system $\mathsf{C}\mathsf{v} = \mathsf{c}$ of m equations in the v variables. From a numerical point of view, this may cause difficulties if $U_i - L_i$ is large, in which case a decomposition approach and/or spatial branching may be used in a preprocessing phase.

Anyhow, we now may and do assume in the sequel for ease of exposition that all integer variables are binary: $v_i \in \{0,1\}$ for all $i \in B$, and therefore arrive at

$$\tau^*_{\rm bin} := \inf \left\{ \frac{f(\mathsf{v})}{g(\mathsf{v})} : \mathsf{v} \in \mathbb{R}^d_+ ,\ \mathsf{Cv} = \mathsf{c}\,,\ v_i \in \{0,1\} \text{ for all } i \in B \right\} . \tag{2.4}$$

Further, notice that we can homogenize a general quadratic constraint $\mathsf{v}^\top \mathsf{Qv} + \mathsf{q}^\top \mathsf{v} + \gamma$ by increasing the dimension of the problem by one, considering new variables $\mathsf{w} = [1, \mathsf{v}^\top]^\top$ and the form

$$\overline{\mathsf{Q}} = \begin{bmatrix} \gamma & \mathsf{q}^\top \\ \mathsf{q} & \mathsf{Q} \end{bmatrix} \in \mathcal{M}_{d+1} \tag{2.5}$$

as well as

$$\mathsf{Y} = \mathsf{ww}^\top = \begin{bmatrix} 1 & \mathsf{v}^\top \\ \mathsf{v} & \mathsf{vv}^\top \end{bmatrix} \in \mathcal{C}^*_{d+1} \,. \tag{2.6}$$

Indeed, we have

$$\mathsf{v}^\top \mathsf{Qv} + \mathsf{q}^\top \mathsf{v} + \gamma = \overline{\mathsf{Q}} \bullet \mathsf{Y}\,. \tag{2.7}$$

Applying this to problem data $(\mathsf{Q}, \mathsf{q}, \gamma) = (\mathsf{A}, \mathsf{a}, \alpha)$ and $(\mathsf{Q}, \mathsf{q}, \gamma) = (\mathsf{B}, \mathsf{b}, \beta)$ occurring in f and g, we have $f(\mathsf{v}) = \overline{\mathsf{A}} \bullet \mathsf{Y}$ and $g(\mathsf{v}) = \overline{\mathsf{B}} \bullet \mathsf{Y}$. Now observe that the constraint $\mathsf{Cv} = \mathsf{c}$ is equivalent to

$$\overline{\mathsf{C_c}} \bullet \mathsf{Y} = \|\mathsf{Cv} - \mathsf{c}\|^2 = 0 \quad \text{where} \quad \overline{\mathsf{C_c}} = [-\mathsf{c}|\mathsf{C}]^\top [-\mathsf{c}|\mathsf{C}] \in \mathcal{P}_{d+1}\,, \tag{2.8}$$

since with Y as defined in (2.6) we have $\|\mathsf{Cv} - \mathsf{c}\|^2 = \overline{\mathsf{C_c}} \bullet \mathsf{Y}$. Recall that with the definition of (2.6), the constraints $Y_{0i} = Y_{ii}$ ensure that $v_i = v_i^2$ for all $i \in B$, which in turn is equivalent to $v_i \in \{0,1\}$, so that we arrive at

$$\tau^*_{\rm rk\,1} := \inf \left\{ \frac{\overline{\mathsf{A}} \bullet \mathsf{Y}}{\overline{\mathsf{B}} \bullet \mathsf{Y}} : \overline{\mathsf{C}}_\mathsf{c} \bullet \mathsf{Y} = 0,\ Y_{0i} - Y_{ii} = 0\,,\ \text{all } i \in B\,,\ \mathsf{Y} \in \mathcal{C}^{*,\mathrm{rk}\,1}_{d+1} \right\} , \tag{2.9}$$

where $\mathcal{C}^{*,\mathrm{rk}\,1}_d$ denotes the (non-convex, not closed) subcone of all completely positive $d \times d$ matrices Y of rank one.

In [9] an assumption regarding the linear portion of the problem is made, and it is often referred in the literature by *Burer's key condition*; see also the additional discussions in [7, 25]. It requires for a given problem with binary variables v_i with $i \in B$, that the linear constraints imply $v_i \in [0,\ 1]$ for $i \in B$. This condition can be assumed without loss of generality, since the introduction of constraints $v_i + s_i = 1$ and slack variables $s_i \geq 0$ for $i \in B$ suffices. For our problem this assumption is the following:

Burer's key condition [9, (1)]:

$$\mathsf{Cv} = \mathsf{c} \text{ and } \mathsf{v} \geq \mathsf{o} \text{ implies } v_i \in [0,1] \text{ for all } i \in B. \tag{2.10}$$

We now discuss conditions which guarantee that the objective functions in (2.1) and (2.4) are well behaved in the sense that the denominator does not vanish over the set

$$F := \left\{ \mathsf{v} \in \mathbb{R}^d_+ : \mathsf{Cv} = \mathsf{c}\,,\ v_i \in [0,1] \text{ for all } i \in B \right\}$$

which is the natural continuous relaxation of the feasible set in (2.4). For simplicity, we only consider the case $g(\mathsf{v}) > 0$ for all $\mathsf{v} \in F$. To this end, we pose the assumption that $\overline{\mathsf{B}}$ be strictly $(\ker_+ \overline{\mathsf{C}}_\mathsf{c})$-copositive:

$$\mathsf{w}^\top \overline{\mathsf{B}} \mathsf{w} > 0 \quad \text{if} \quad \overline{\mathsf{C}}_\mathsf{c} \mathsf{w} = \mathsf{o} \quad \text{and} \quad \mathsf{w} \in \mathbb{R}^{d+1}_+ \setminus \{\mathsf{o}\}\ . \tag{2.11}$$

Proposition 2.1. *Suppose that (1.2), i.e., $ker_+\mathsf{C} = \{\mathsf{o}\}$, and the key condition (2.10) hold. Then the condition (2.11) holds if and only if $g(\mathsf{v}) > 0$ for all $\mathsf{v} \in F$.*

Proof. First note that $\overline{\mathsf{C}}_\mathsf{c}$ is psd. Hence $\overline{\mathsf{C}}_\mathsf{c}\mathsf{w} = \mathsf{o}$ if and only if $\mathsf{w}^\top\overline{\mathsf{C}}_\mathsf{c}\mathsf{w} = 0$. Next, for any $\mathsf{v} \in F$, put $\mathsf{w} = [1, \mathsf{v}^\top]^\top \in \mathbb{R}^{d+1}_+$. Then $\mathsf{w}^\top\overline{\mathsf{C}}_\mathsf{c}\mathsf{w} = \|\mathsf{C}\mathsf{v} - \mathsf{c}\|^2 = 0$ as well as $\mathsf{w}^\top\overline{\mathsf{B}}\mathsf{w} = g(\mathsf{v})$. So (2.11) implies $g(\mathsf{v}) > 0$ for all $\mathsf{v} \in F$. To establish the reverse implication, assume that $\mathsf{w} = [\eta, \mathsf{v}^\top]^\top \in \mathbb{R}^{d+1}_+ \setminus \{\mathsf{o}\}$ satisfies $\mathsf{C}\mathsf{v} - \eta\mathsf{c} = \overline{\mathsf{C}}_\mathsf{c}\mathsf{w} = \mathsf{o}$. If $\eta = 0$, then this implied $\mathsf{C}\mathsf{v} = \mathsf{o}$, yielding the absurd $\mathsf{v} = \mathsf{o}$. Hence $\eta > 0$, and dividing by it, we get $\overline{\mathsf{C}}_\mathsf{c}\overline{\mathsf{w}} = \mathsf{o}$ for

$$\overline{\mathsf{w}} = \frac{1}{\eta}\,\mathsf{w} = [1, \overline{\mathsf{v}}^\top]^\top\,,$$

with $\mathsf{v} \in \mathbb{R}^d_+$ satisfying $\mathsf{C}\mathsf{v} = \mathsf{c}$ and $v_i \in [0,1]$ for all $i \in B$ by (2.10), so $\mathsf{v} \in F$. Hence we arrive at

$$\mathsf{w}^\top\overline{\mathsf{B}}\mathsf{w} = \eta^2\overline{\mathsf{w}}^\top\overline{\mathsf{B}}\overline{\mathsf{w}} = \eta^2 g(\mathsf{v}) > 0\,,$$

as required. □

Remark 1. As one referee kindly pointed out, condition (2.11) can be ignored in the purely binary case $B = [1:d]$ if we have $g(\mathsf{v}) > 0$ (or if g does not change sign) on $\{0,1\}^d$. Indeed, then for M large enough, the function $\tilde{g}(\mathsf{v}) = g(\mathsf{v}) + M\sum_{i=1}^d (v_i^2 - v_i) > 0$ even for all $\mathsf{v} \in \mathbb{R}^d$ (recall that $\mathsf{o} \in \{0,1\}^d$) and $\tilde{g}(\mathsf{v}) = g(\mathsf{v})$ for all $\mathsf{v} \in \{0,1\}^d$. So, by Proposition 2.1, condition (2.11) holds if g is replaced with $\tilde{g}$, and even without needing the other conditions (1.2) and (2.10).

The next lemma specifies sufficient conditions which guarantee that the matrices Y in (2.9) must have a strictly positive entry $Y_{00} > 0$ in the upper left corner, that the denominator in above objective is indeed strictly positive, even relaxing the condition rank $\mathsf{Y} = 1$ to $\mathsf{Y} \neq \mathsf{O}$ which will be important later on.

Lemma 2.2. *Consider any $\mathsf{Y} \in \mathcal{C}^*_{d+1} \setminus \{\mathsf{O}\}$ such that $\overline{\mathsf{C}}_\mathsf{c} \bullet \mathsf{Y} = 0$. Under condition (1.2), we have $Y_{00} > 0$. Furthermore, conditions (1.2) and (2.11) ensure that $\overline{\mathsf{B}} \bullet \mathsf{Y} > 0$.*

Proof. First we treat the case $\mathsf{Y} \in \mathcal{C}^{*,\mathrm{rk}\,1}_{d+1}$. Let $\mathsf{z}^\top = [\eta, \mathsf{w}^\top] \in \mathbb{R}^{d+1}_+ \setminus \{\mathsf{o}\}$ be such that $\mathsf{Y} = \mathsf{z}\mathsf{z}^\top$. Again, $\overline{\mathsf{C}}_\mathsf{c}$ is psd, so that $\overline{\mathsf{C}}_\mathsf{c} \bullet \mathsf{Y} = 0$ is equivalent to $\overline{\mathsf{C}}_\mathsf{c}\mathsf{z} = \mathsf{o}$ which can be written as $\mathsf{C}\mathsf{w} - \eta\mathsf{c} = \mathsf{o}$. Now assume that $\eta = 0$; then we would arrive at $\mathsf{C}\mathsf{w} = \mathsf{o}$ and $\mathsf{w} \in \mathbb{R}^d_+$, which, by assumption (1.2), would entail $\mathsf{w} = \mathsf{o}$ or $\mathsf{z} = \mathsf{o}$ which is absurd by assumption rank $\mathsf{Y} = 1$. Hence $\eta > 0$ and also $Y_{00} = \eta^2 > 0$, as asserted. Now define $\mathsf{v} := \frac{1}{\eta}\mathsf{w} \in \mathbb{R}^d_+$. We see $\mathsf{C}\mathsf{v} - \mathsf{c} = \frac{1}{\eta}(\mathsf{C}\mathsf{w} - \eta\mathsf{c}) = \mathsf{o}$. Further, the constraints $Y_{0i} = Y_{ii}$ boil down to $v_i = v_i^2$ for all $i \in B$, so that v is (2.1)-feasible, and $\overline{\mathsf{B}} \bullet \mathsf{Y} = \eta^2[1, \mathsf{v}^\top]\overline{\mathsf{B}}[1, \mathsf{v}^\top]^\top = \eta^2 g(\mathsf{v}) > 0$. Now consider a completely positive matrix $\mathsf{Y} \neq \mathsf{O}$ of arbitrary rank and its cp decomposition $\mathsf{Y} = \sum_j \mathsf{Y}^{(j)}$ where $\mathsf{Y}^{(j)} = \mathsf{z}_j\mathsf{z}_j^\top$ with $\mathsf{z}_j \in \mathbb{R}^{d+1}_+ \setminus \{\mathsf{o}\}$ for all j. Since $\overline{\mathsf{C}}_\mathsf{c}$ is psd, we know $\overline{\mathsf{C}}_\mathsf{c} \bullet \mathsf{Y}^{(j)} = 0$, so preceding arguments lead to $Y^{(j)}_{00} > 0$ and likewise to $\overline{\mathsf{B}} \bullet \mathsf{Y}^{(j)} > 0$ for all j. The result then follows by linearity. □

The next step is to relax the rank-one constraint so that we obtain the fractional copositive optimization problem

$$\tau^*_{\mathrm{cop}} := \min\left\{\frac{\overline{\mathsf{A}} \bullet \mathsf{Y}}{\overline{\mathsf{B}} \bullet \mathsf{Y}} : \overline{\mathsf{C}}_\mathsf{c} \bullet \mathsf{Y} = 0,\, Y_{0i} - Y_{ii} = 0\,,\ \text{all } i \in B\,, \mathsf{Y} \in \mathcal{C}^*_{d+1} \setminus \{\mathsf{O}\}\right\}\,. \tag{2.12}$$

Now observe that (2.12) only involves homogeneous linear constraints so that we can imitate the approach in [2]: indeed, it is straightforward to see that we can reformulate (2.12) into a linear copositive optimization problem

$$\tau^*_{\mathrm{cop}} = \min_{\mathsf{Y}\in\mathcal{C}^*_{d+1}} \left\{ \overline{\mathsf{A}} \bullet \mathsf{Y} : \overline{\mathsf{B}} \bullet \mathsf{Y} = 1\,,\ \overline{\mathsf{C}} \bullet \mathsf{Y} = 0,\ Y_{0i} - Y_{ii} = 0\,,\ \text{all } i \in B \right\}\,, \tag{2.13}$$

which may be more familiar, also because now the feasible set is the intersection of a linear subspace with a closed convex cone. The conic dual program to (2.13) reads

$$\tau^*_{\mathrm{dual}} = \sup \left\{ \lambda : \overline{\mathsf{A}} - \lambda\overline{\mathsf{B}} + \mathsf{M}(\mu,\mathsf{d}) \in \mathcal{C}_{d+1}\,,\ \lambda,\mu \in \mathbb{R}\,,\ \mathsf{d} \in \mathbb{R}^B \right\} \tag{2.14}$$

where $\mathsf{M} : \mathbb{R}\times\mathbb{R}^B \to \mathcal{M}_{d+1}$ is a linear map corresponding to the dualization of all linear constraints:

$$\mathsf{M}(\mu,\mathsf{d}) := \mu\overline{\mathsf{C}}_{\mathsf{c}} + \sum_{i\in B} d_i(\mathsf{e}_i\mathsf{e}_0^\top + \mathsf{e}_0\mathsf{e}_i^\top - \mathsf{e}_i\mathsf{e}_i^\top)\,. \tag{2.15}$$

Theorem 2.3. *Without any assumptions (except that the objective in (2.4) is well defined because $g(\mathsf{v}) > 0$ holds on the feasible set), we have*

$$\tau^*_{\mathrm{dual}} \le \tau^*_{\mathrm{cop}} \le \tau^*_{\mathrm{bin}} \tag{2.16}$$

and so we also obtain copositivity-based bounds on the optimal value τ^* of any MIFQP (2.1): for any $(\lambda,\mu,\mathsf{d}) \in \mathbb{R}^{b+2}$ such that $\overline{\mathsf{A}} - \lambda\overline{\mathsf{B}} + \mathsf{M}(\mu,\mathsf{d})$ is copositive, we have $\mu \le \tau^*$.

Proof. The leftmost inequality is simply weak conic duality. The rightmost one is an almost immediate observation. Indeed, if v is any feasible solution with $g(\mathsf{v}) > 0$ to the problem (2.4) which can be written in homogeneous form as explained above, then with $\mathsf{w}^\top = [1,\mathsf{v}^\top]$ evidently $\mathsf{Y} = \frac{1}{g(\mathsf{v})}\,\mathsf{w}\mathsf{w}^\top$ is feasible to problem (2.13), and $\frac{f(\mathsf{v})}{g(\mathsf{v})} = \overline{\mathsf{A}} \bullet \mathsf{Y}$. □

The copositive relaxation is in general tighter than other popular relaxations like the Lagrangian (SDP) relaxation, and even at the lowest approximation level ($\mathcal{K}^0_d = \mathcal{D}^*_d$), a tightening is still possible; for details the reader is referred to [8].

Note that $\overline{\mathsf{C}}_{\mathsf{c}}$ is psd (but singular) so that $\overline{\mathsf{C}}_{\mathsf{c}} \bullet \mathsf{Y} = 0$ cannot be satisfied for any Y interior to $\mathcal{C}_{d+1}$. Hence Slater's condition is always violated for (2.13) by construction, and dual attainability is an open issue, while primal attainability holds under conditions (2.11) and (1.2), as will be explained below. This is the reason why we did not use the infimum in above formulation.

A very natural question is the following: when do the inequalities in (2.16) become equalities ? We answer this in two separate parts: first we deal with conic duality, and then with the question when the copositive relaxation is indeed a reformulation.

Theorem 2.4. *Suppose that (2.11) holds. Then the duality gap is zero:*

$$\tau^*_{\mathrm{dual}} = \tau^*_{\mathrm{cop}}\,,$$

and primal attainability holds for problem (2.13).

Proof. Problem (2.14) is strictly feasible as its feasible set contains that of the dual copositive problem for the basic FQP without binarity constraints [2, Thm.2], corresponding to the case $\mathsf{M}(0,\mathsf{o}) = \mathsf{O}$ in (2.15). The result follows from general principles in convex optimization as Slater's condition for (2.14) is satisfied. □

Finally, we establish equivalence of both problems, (2.13) and (2.1). We already established one inequality (2.16). The proof of the reverse inequality is more difficult, and combines Burer's technique [9] with the arguments in the proofs of [2, Lem.2, Thm.1].

Theorem 2.5. *Suppose that (1.2), (2.10) and (2.11) hold. Then the copositive relaxation is in fact a reformulation of (2.4) or (2.1):*

$$\tau^*_{\mathrm{dual}} = \tau^*_{\mathrm{cop}} = \tau^*_{\mathrm{bin}} = \tau^*\,,$$

and there is an optimal solution $\mathsf{Y}^* \in \mathcal{C}^{*,\mathrm{rk}\,1}_{d+1}$ *to problem (2.13), which encodes an optimal solution* v^* *to problem (2.4) as follows: if* $\mathsf{Y}^* = [\eta, \mathsf{w}^\top]^\top [\eta, \mathsf{w}^\top]$*, then* ($\eta > 0$ *and)* $\mathsf{v}^* = \frac{1}{\eta}\mathsf{w}$. *So we have primal attainability for both problems (2.13) and (2.4).*

Proof. Let $\overline{\mathsf{Y}}$ be an optimal solution to problem (2.13), which exists due to Theorem 2.4. We decompose $\overline{\mathsf{Y}} = \sum_j \mathsf{z}_j \mathsf{z}_j^\top$ with $\mathsf{z}_j \in \mathbb{R}^{d+1}_+ \setminus \{\mathsf{o}\}$ for all j and observe that $\overline{\mathsf{C}}_{\mathsf{c}} \bullet \overline{\mathsf{Y}} = 0$ implies $\overline{\mathsf{C}}_{\mathsf{c}} \mathsf{z}_j = \mathsf{o}$, since $\overline{\mathsf{C}}_{\mathsf{c}}$ is psd. So, by assumption (2.11), we know $\lambda_j := \mathsf{z}_j^\top \overline{\mathsf{B}} \mathsf{z}_j > 0$, and define $\overline{\mathsf{w}}_j := \frac{1}{\sqrt{\mathsf{z}_j^\top \overline{\mathsf{B}} \mathsf{z}_j}} \mathsf{z}_j \in \mathbb{R}^{d+1}_+ \setminus \{\mathsf{o}\}$. By construction, we have $\overline{\mathsf{w}}_j^\top \overline{\mathsf{B}}\, \overline{\mathsf{w}}_j = 1$ and $\overline{\mathsf{C}}_{\mathsf{c}} \mathsf{w}_j = \mathsf{o}$. As in Lemma 2.2 we know $\eta_j > 0$, and can define $\mathsf{w}_j^\top := \frac{1}{\eta_j} \overline{\mathsf{w}}_j^\top = [1, \mathsf{v}_j^\top]$, so that $\mathsf{v}_j \in \mathbb{R}^d_+$ solves $\mathsf{C}\mathsf{v}_j = \mathsf{c}$. Hence, by (2.10), we have

$$(\mathsf{e}_i^\top \mathsf{v}_j) \in [0,1] \quad \text{for all} \quad i \in B\,, \text{ all } j. \tag{2.17}$$

But the condition $\overline{Y}_{0i} - \overline{Y}_{ii} = 0$ for $i \in B$ implies

$$\sum_j \eta_j^2 (\mathsf{e}_i^\top \mathsf{v}_j)[1 - (\mathsf{e}_i^\top \mathsf{v}_j)] = 0 \text{ for all } i \in B\,,$$

and since every term in above sum is non-negative by (2.17), every term must be zero which means that $\mathsf{e}_i^\top \mathsf{v}_j \in \{0,1\}$ for all j and all $i \in B$. Hence all v_j are (2.4)-feasible. Moreover, introducing $\mathsf{Y}^{(j)} = \overline{\mathsf{w}}_j \overline{\mathsf{w}}_j^\top$, we have

$$\frac{f(\mathsf{v}_j)}{g(\mathsf{v}_j)} = \frac{\overline{\mathsf{A}} \bullet (\mathsf{w}_j \mathsf{w}_j^\top)}{\overline{\mathsf{B}} \bullet (\mathsf{w}_j \mathsf{w}_j^\top)} = \frac{\overline{\mathsf{A}} \bullet (\overline{\mathsf{w}}_j \overline{\mathsf{w}}_j^\top)}{\overline{\mathsf{B}} \bullet (\overline{\mathsf{w}}_j \overline{\mathsf{w}}_j^\top)} = \overline{\mathsf{A}} \bullet \mathsf{Y}^{(j)}\,.$$

Further we see $\overline{\mathsf{Y}} = \sum_j \lambda_j \overline{\mathsf{w}}_j \overline{\mathsf{w}}_j{}^\top = \sum_j \lambda_j \mathsf{Y}^{(j)}$ where $\sum_j \lambda_j = \overline{\mathsf{B}} \bullet \overline{\mathsf{Y}} = 1$. Finally, we observe that all $\mathsf{Y}^{(j)}$ are also feasible to problem (2.13). Indeed, homogeneity of the only constraints ignored up to now, namely $Y^{(j)}_{0i} = Y^{(j)}_{ii}$, means that it is sufficient to check this on the matrix $\mathsf{w}_j \mathsf{w}_j^\top$ which also satisfies it because $\mathsf{e}_i^\top \mathsf{v}_j \in \{0,1\}$ holds for all $i \in B$ and all j. Basic convexity arguments and optimality of $\overline{\mathsf{Y}}$ now yield $\frac{f(\mathsf{v}_j)}{g(\mathsf{v}_j)} = \overline{\mathsf{A}} \bullet \mathsf{Y}^{(j)} = \overline{\mathsf{A}} \bullet \overline{\mathsf{Y}}$, and all v_j are optimal solutions to (2.4); Y^* can be chosen as one of the $\mathsf{Y}^{(j)}$. □

3 Ternary Fractional Quadratic Problems

Consider again an undirected graph, the adjacency matrix A describing a very elementary discrete similarity measure among the vertices ($a_{ij} = 1$ if there is an edge $\{i,j\}$ and $a_{ij} = 0$ else), and suppose we are interested in partitioning the vertex set into three classes – two "decided" (yes/no for individual i or $x_i \in \{-1,1\}$), but allowing for "undecidedness" or an outside option ($x_i = 0$). We exclude the case where all are undecided, i.e., require $\mathsf{x} \neq \mathsf{o}$.

If the fraction of undecided individuals is small, we may treat these as outliers and rely on the usual clustering methods, for instance by dominant set clustering using A, as done in Pattern Recognition, see, e.g. [26]. But, for instance, in mature democracies or other situations, the amount of undecided agents can be considerably large. Then it makes sense to search for the highest *density* of all decided individuals rather than considering their total absolute connectivity.

In other words, considering an edge within the cluster as good (good intraclass coherence) and those between clusters as bad (good separation between clusters) while ignoring relations to and among the undecided, we may consider maximizing the quadratic form $\mathsf{x}^\top \mathsf{A}\mathsf{x}$ of a ternary vector $\mathsf{x} \in \mathbb{T}^n$ (with $\mathbb{T} = \{-1, 0, 1\}$) and A is again the adjacency matrix of the underlying graph. Indeed, this form gives a positive contribution $a_{ij}x_ix_j = 1$ if $x_ix_j = 1$ but a negative one $a_{ij}x_ix_j = -1$ if $x_ix_j = -1$ along an edge $\{i, j\}$. This way, we may even think, more generally, of non-uniform edge weights $a_{ij} \geq 0$ modeling relation/similarity strength (zero if $\{i, j\}$ is no edge in the graph). To get a density measure within the "decided" classes, the denominator may simply count all "decided" individuals, which means putting $\mathsf{B} = \mathsf{I}_n$, or a weighted count ($\mathsf{B} = \operatorname{Diag} \mathsf{d}$ with $d_i > 0$ according to the "importance" of individual i) and again considering homogenous $f(\mathsf{x}) = -\mathsf{x}^\top \mathsf{A}\mathsf{x}$ and $g(\mathsf{x}) = \mathsf{x}^\top \mathsf{B}\mathsf{x}$. Observe that "(un)decided" is just an analogy, and this individual property (whether $x_i = -1$ or zero or one) is merely determined by the data (A, B), as always in a clustering method. In a voting context, it may be used as a projection/forecast if previous information can be distilled into (A, B). This way, we end up in a *ternary* (purely integer) fractional quadratic problem (TFQP) of the form

$$\tau^*_{\mathbb{T}^\blacklozenge} := \min \left\{ \frac{f(\mathsf{x})}{g(\mathsf{x})} : \mathsf{x} \in \mathbb{T}^n \setminus \{\mathsf{o}\} \right\} \quad \text{with } \mathbb{T} = \{-1, 0, 1\} \ . \tag{3.1}$$

Similar graph tri-partitioning problems occur, e.g., in PageRank for Folksonomy in social media/Web2.0 [14]. Despite of the applicability hinted at above, among many more possible, these problems are less studied than their binary counterpart.

Of course, we could treat them as special cases of mixed-integer FQP, but this would most likely incur loss of efficiency. Instead, we will now directly show that every such problem can be written as a mixed-binary fractional QP under some simple linear constraints. In particular, we will show how to treat the constraint $\mathsf{x} \neq \mathsf{o}$ which is ignored in [3] where an $n^{1/3}$-approximation algorithm for (3.1) is suggested.

If the objective functions f and g are inhomogenous as in the general case, we need not exclude $\mathsf{x} = \mathsf{o}$ a priori, and end up with a general form of TFQP:

$$\tau^*_{\mathbb{T}} := \min \left\{ \frac{f(\mathsf{x})}{g(\mathsf{x})} : \mathsf{x} \in \mathbb{T}^n \right\} . \tag{3.2}$$

Both problems can be written as MBFQPs:

Theorem 3.1. *Consider problems (3.2) and (3.1), and let* $\mathsf{C} := \begin{bmatrix} \mathsf{I}_n & \mathsf{I}_n & \mathsf{I}_n \end{bmatrix}$ *as well as* $\mathsf{C}^\blacklozenge := \begin{bmatrix} \mathsf{I}_n & \mathsf{I}_n & \mathsf{I}_n & \mathsf{o} \\ \mathsf{o}^\top & \mathsf{e}^\top & \mathsf{e}^\top & -1 \end{bmatrix}$. *Then we have*

$$\mathbb{T}^n = \left\{ \mathsf{y} - \mathsf{z} : \mathsf{C}\mathsf{v} = \mathsf{e} \, , \, \mathsf{v} = [\mathsf{u}^\top, \mathsf{y}^\top, \mathsf{z}^\top]^\top \in \mathbb{R}^{3n}_+ \, , \, [\mathsf{y}^\top, \mathsf{z}^\top]^\top \in \{0, 1\}^{2n} \right\} , \tag{3.3}$$

and any $\mathsf{v} \in \mathbb{R}^{3n}_+$ *such that* $\mathsf{C}\mathsf{v} = \mathsf{e}$ *satisfies* $\mathsf{v} \in [0, 1]^{3n}$*; furthermore,*

$$\mathbb{T}^n \setminus \{\mathsf{o}\} = \left\{ \mathsf{y} - \mathsf{z} : \mathsf{C}^\blacklozenge \mathsf{v} = \mathsf{e}, \mathsf{v} = [\mathsf{u}^\top, \mathsf{y}^\top, \mathsf{z}^\top, \nu]^\top \in \mathbb{R}^{3n+1}_+, [\mathsf{y}^\top, \mathsf{z}^\top]^\top \in \{0, 1\}^{2n} \right\} , \tag{3.4}$$

and also any $\mathsf{v} \in \mathbb{R}_+^{3n+1}$ *such that* $\mathsf{C}^{\blacklozenge}\mathsf{v} = \mathsf{e}$ *satisfies* $[\mathsf{y}^\top, \mathsf{z}^\top] \in [0,1]^{2n}$.

Proof. Indeed, $\mathsf{x} \in \mathbb{T}^n$ if and only if there are $\{\mathsf{y}, \mathsf{z}\} \subset \{0,1\}^n$ such that $\mathsf{y} + \mathsf{z} \leq \mathsf{e}$ and $\mathsf{x} = \mathsf{y} - \mathsf{z}$. We introduce the slack variables $\mathsf{u} = \mathsf{e} - \mathsf{y} - \mathsf{z} \geq \mathsf{o}$ and form the vector $\mathsf{v} = [\mathsf{u}^\top, \mathsf{y}^\top, \mathsf{z}^\top]^\top \in \mathbb{R}_+^{3n}$. Then $\mathsf{C}\mathsf{v} = \mathsf{e}$ reflects exactly above construction. Moreover, we can write $\mathsf{C}\mathsf{v} = \mathsf{u} + \mathsf{y} + \mathsf{z}$ and hence $0 \leq v_i \leq 1$ follows for all i. This establishes (a). To complete the proof for (b), observe that the last row of $\mathsf{C}^{\blacklozenge}\mathsf{v} = \mathsf{e}$ reads $\mathsf{e}^\top(\mathsf{y} + \mathsf{z}) \geq 1$. But binarity of y and z together with the constraints $y_i + z_i \leq 1$ and $x_i = y_i - z_i$ imply $|x_i| = y_i + z_i$, so that $\sum_{i=1}^n |x_i| = \mathsf{e}^\top(\mathsf{y} + \mathsf{z}) \geq 1$ which renders $\mathsf{x} \neq \mathsf{o}$. The last assertion follows since the first n rows of $\mathsf{C}^{\blacklozenge}\mathsf{v} = \mathsf{e}$ exactly copy the relation $\mathsf{C}\mathsf{v} = \mathsf{e}$ from case (a). □

Corollary 3.2. *Any TFQP (3.2) and likewise any problem (3.1) can be written as a linearly constrained MBFQP, with the key condition (2.10) satisfied.*

Positivity conditions for the denominator $g(\mathsf{x})$ in terms of this formulation can be dealt with as in the general MBFQP case: to this end, we employ the special 3×3 matrix

$$\mathsf{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix},$$

and introduce the notation $\widehat{\mathsf{Q}} = \mathsf{D} \otimes \mathsf{Q}$ for any $n \times n$ matrix Q. If, as in Theorem 3.1 above, $\mathsf{x} = \mathsf{y} - \mathsf{z}$ and $\mathsf{u} = \mathsf{e} - \mathsf{y} - \mathsf{z}$ as well as $\mathsf{v}^\top = [\mathsf{u}^\top, \mathsf{y}^\top, \mathsf{z}^\top]$, then $\mathsf{x}^\top\mathsf{Q}\mathsf{x} = \mathsf{v}^\top\widehat{\mathsf{Q}}\mathsf{v}$ holds. Similarly, for any $\mathsf{q} \in \mathbb{R}^n$ we see that $\mathsf{v}^\top\widehat{\mathsf{q}} = \mathsf{x}^\top\mathsf{q}$ if we put $\widehat{\mathsf{q}} = [\mathsf{o}^\top, \mathsf{q}^\top, -\mathsf{q}^\top]^\top \in \mathbb{R}^{3n}$. Furthermore, to homogenize the problems, we proceed as in (2.5), (2.6) and (2.7), here with:

$$\overline{\mathsf{Q}} = \begin{bmatrix} \gamma & \widehat{\mathsf{q}}^\top \\ \widehat{\mathsf{q}} & \widehat{\mathsf{Q}} \end{bmatrix}. \tag{3.5}$$

Again we observe that, using (2.8), the constraint $\mathsf{C}\mathsf{v} = \mathsf{e}$ is equivalent to

$$\overline{\mathsf{C}}_{\mathsf{e}} \bullet \mathsf{Y} = \|\mathsf{C}\mathsf{v} - \mathsf{e}\|^2 = 0 \quad \text{where} \quad \overline{\mathsf{C}}_{\mathsf{e}} = [-\mathsf{e}|\mathsf{C}]^\top[-\mathsf{e}|\mathsf{C}] \in \mathcal{P}^{3n+1}.$$

So the assumption that $\overline{\mathsf{B}}$ be strictly $(\ker_+\overline{\mathsf{C}}_{\mathsf{e}})$-copositive,

$$\mathsf{w}^\top\overline{\mathsf{B}}\mathsf{w} > 0 \quad \text{if} \quad \overline{\mathsf{C}}_{\mathsf{e}}\mathsf{w} = \mathsf{o} \quad \text{and} \quad \mathsf{w} \in \mathbb{R}_+^{3n+1} \setminus \{\mathsf{o}\}, \tag{3.6}$$

ensures that $g(\mathsf{x}) > 0$ for all $\mathsf{x} \in [-1,1]^n$, and thus the quotient objective remains well defined over $\operatorname{conv}\mathbb{T}^n = [-1,1]^n$ (which in particular implies $\beta > 0$ so that only inhomogenous denominators can be covered; unfortunately also the approach of Remark 1 cannot help here). However, if (3.6) is satisfied, we will obtain again a copositive reformulation of (3.2) as detailed in Theorem 3.3 below.

In any case, Theorem 2.3 remains valid, so that we can discuss tighter dual bounds for TFQP via copositivity. The dual copositive programs read as follows:

$$\tau^*_{\mathrm{T,dual}} := \sup\left\{\lambda : \overline{\mathsf{M}}(\lambda, \mu, \mathsf{d}) \in \mathcal{C}_{3n+1},\ \lambda, \mu \in \mathbb{R},\ \mathsf{d} \in \mathbb{R}^{2n}\right\}. \tag{3.7}$$

is the dual of

$$\tau^*_{\mathrm{T,cop}} := \min_{\mathsf{Y} \in \mathcal{C}^*_{3n+1}} \left\{\overline{\mathsf{A}} \bullet \mathsf{Y} : \overline{\mathsf{B}} \bullet \mathsf{Y} = 1, \overline{\mathsf{C}}_{\mathsf{e}} \bullet \mathsf{Y} = 0, Y_{0i} - Y_{ii} = 0,\ \text{all } i \in B\right\}, \tag{3.8}$$

and both represent a copositive relaxation of (3.2); while the primal-dual pair

$$\tau^*_{\mathrm{T}^\blacklozenge,\mathrm{cop}} := \min_{\mathsf{Y}\in\mathcal{C}^*_{3n+2}} \left\{ \mathsf{A}^\blacklozenge \bullet \mathsf{Y} : \mathsf{B}^\blacklozenge \bullet \mathsf{Y} = 1, \overline{\mathsf{C}_\mathsf{e}^\blacklozenge} \bullet \mathsf{Y} = 0, Y_{0i} - Y_{ii} = 0, \text{ all } i \in B \right\}, \tag{3.9}$$

and

$$\tau^*_{\mathrm{T}^\blacklozenge,\mathrm{dual}} := \sup \left\{ \lambda : \mathsf{M}^\blacklozenge(\lambda,\mu,\mathsf{d}) \in \mathcal{C}_{3n+2}, \lambda,\mu \in \mathbb{R}, \mathsf{d} \in \mathbb{R}^{2n+1} \right\}, \tag{3.10}$$

both represent a copositive relaxation of (3.1). Here $\mathsf{A}^\blacklozenge$ and $\mathsf{B}^\blacklozenge$ are $\overline{\mathsf{A}}$ and $\overline{\mathsf{B}}$, respectively, enlarged by a zero column and row, and $\overline{\mathsf{M}}(\lambda,\mu,\mathsf{d}) =$

$$\begin{bmatrix} \alpha - \lambda\beta + n\mu & -\mu\mathsf{e}^\top & (\mathsf{d}_1 - \mu\mathsf{e} + \mathsf{a} - \lambda\mathsf{b})^\top & (\mathsf{d}_2 - \mu\mathsf{e} - \mathsf{a} + \lambda\mathsf{b})^\top \\ -\mu\mathsf{e} & \mu\mathsf{I}_n & \mu\mathsf{I}_n & \mu\mathsf{I}_n \\ \mathsf{d}_1 - \mu\mathsf{e} + \mathsf{a} - \lambda\mathsf{b} & \mu\mathsf{I}_n & \mathsf{A} + \lambda\mathsf{B} + \mathrm{Diag}\,(\mu\mathsf{e} - \mathsf{d}_1) & \mu\mathsf{I}_n - \mathsf{A} - \lambda\mathsf{B} \\ \mathsf{d}_2 - \mu\mathsf{e} - \mathsf{a} + \lambda\mathsf{b} & \mu\mathsf{I}_n & \mu\mathsf{I}_n - \mathsf{A} - \lambda\mathsf{B} & \mathsf{A} + \lambda\mathsf{B} + \mathrm{Diag}\,(\mu\mathsf{e} - \mathsf{d}_2) \end{bmatrix}$$

with $\mathsf{d}^\top = [\mathsf{d}_1^\top, \mathsf{d}_2^\top]$ and $\mathsf{d}_i \in \mathbb{R}^n$, $i \in \{1,2\}$, as well as $\mathsf{M}^\blacklozenge(\lambda,\mu,\mathsf{d}) =$

$$\begin{bmatrix} \alpha - \lambda\beta + (n+1)\mu & -\mu\mathsf{e}^\top & (\mathsf{d}_1 - 2\mu\mathsf{e} + \mathsf{a} - \lambda\mathsf{b})^\top & (\mathsf{d}_2 - 2\mu\mathsf{e} - \mathsf{a} + \lambda\mathsf{b})^\top & \mu \\ -\mu\mathsf{e} & \mu\mathsf{I}_n & \mu\mathsf{I}_n & \mu\mathsf{I}_n & \mathsf{o} \\ \mathsf{d}_1 - 2\mu\mathsf{e} + \mathsf{a} - \lambda\mathsf{b} & \mu\mathsf{I}_n & \mathsf{A} + \lambda\mathsf{B} + \mathrm{Diag}\,(\mu\mathsf{e} - \mathsf{d}_1) + \mu\mathsf{e}\mathsf{e}^\top & \mu(\mathsf{I}_n + \mathsf{e}\mathsf{e}^\top) - \mathsf{A} - \lambda\mathsf{B} & -\mu\mathsf{e} \\ \mathsf{d}_2 - 2\mu\mathsf{e} - \mathsf{a} + \lambda\mathsf{b} & \mu\mathsf{I}_n & \mu(\mathsf{I}_n + \mathsf{e}\mathsf{e}^\top) - \mathsf{A} - \lambda\mathsf{B} & \mathsf{A} + \lambda\mathsf{B} + \mathrm{Diag}\,(\mu\mathsf{e} - \mathsf{d}_2) + \mu\mathsf{e}\mathsf{e}^\top & -\mu\mathsf{e} \\ \mu & \mathsf{o}^\top & -\mu\mathsf{e}^\top & -\mu\mathsf{e}^\top & \mu \end{bmatrix}$$

Both formulae are straightforward calculations, for the former we need

$$\overline{\mathsf{C}_\mathsf{e}} = \begin{bmatrix} n & -\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \otimes \mathsf{e}^\top \\ -\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \otimes \mathsf{e} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \otimes \mathsf{I}_n \end{bmatrix}$$

and for the latter the observation

$$\overline{\mathsf{C}_\mathsf{e}^\blacklozenge} = \begin{bmatrix} n+1 & -\mathsf{e}^\top & -2\mathsf{e}^\top & -2\mathsf{e}^\top & 1 \\ -\mathsf{e} & \mathsf{I}_n & \mathsf{I}_n & \mathsf{I}_n & \mathsf{o} \\ -2\mathsf{e} & \mathsf{I}_n & \mathsf{I}_n + \mathsf{e}\mathsf{e}^\top & \mathsf{I}_n + \mathsf{e}\mathsf{e}^\top & -\mathsf{e} \\ -2\mathsf{e} & \mathsf{I}_n & \mathsf{I}_n + \mathsf{e}\mathsf{e}^\top & \mathsf{I}_n + \mathsf{e}\mathsf{e}^\top & -\mathsf{e} \\ 1 & \mathsf{o}^\top & -\mathsf{e}^\top & -\mathsf{e}^\top & 1 \end{bmatrix}.$$

Theorem 3.3. *Consider the two TFQPs (3.2) and (3.1) with their copositive relaxations (3.8), (3.7) and (3.9), (3.10), respectively. Then*

$$\tau^*_{\mathrm{T},\mathrm{dual}} = \tau^*_{\mathrm{T},\mathrm{cop}} \le \tau^*_{\mathrm{T}} \quad \text{and} \quad \tau^*_{\mathrm{T}^\blacklozenge,\mathrm{dual}} = \tau^*_{\mathrm{T}^\blacklozenge,\mathrm{cop}} \le \tau^*_{\mathrm{T}^\blacklozenge}.$$

Furthermore, if (3.6) holds, i.e., if $g(\mathsf{x}) > 0$ for all $\mathsf{x} \in [-1,1]^n$, then

$$\tau^*_{\mathrm{T},\mathrm{dual}} = \tau^*_{\mathrm{T},\mathrm{cop}} = \tau^*_{\mathrm{T}}.$$

Proof. By Theorem 3.1, we see that $B = \{n+1, \dots, 3n\}$ and that (2.10) is automatically satisfied. We also have condition (1.2) for both C and $\mathsf{C}^\blacklozenge$, so the duality gap is zero in both cases. The last assertion is another application of Theorem 2.5. □

$\boxed{4}$ Conclusions

We presented copositive formulations and relaxations for mixed-integer fractional quadratic optimization problems under linear constraints, specifying sufficient conditions for zero duality gap and tightness of the relaxation. Even if the resulting copositive problems are intractable by themselves, usual approximation hierarchies can be used to obtain tighter bounds than SDP-based ones. We hope that empirical evidence hinting at the significance of this improvement, also with real-world data from various applications which we here merely sketched, can be collected in the near future.

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