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# LARGE-UPDATE PRIMAL-DUAL INTERIOR-POINT ALGORITHM FOR SEMIDEFINITE OPTIMIZATION

Gyeong-Mi Cho*

**Abstract:** Large-update primal-dual interior-point algorithms are one of the most efficient methods in practice. In this paper we propose a new large-update primal-dual interior algorithm for semidefinite optimization based on a parametric kernel function which has a finite value at the boundary of the feasible region. New search directions and proximity measures are defined based on these functions. We show that the algorithm has $\mathcal{O}((\log q)^{\frac{3}{2}}\sqrt{n}(\log n)\log\frac{n}{\epsilon})$ complexity result, where $q$ is the parameter in the kernel function. This is the best known complexity result for such method.

**Key words:** *interior-point algorithm, kernel function, polynomial complexity, primal-dual method, semidefinite optimization*

**Mathematics Subject Classification:** *90C22, 90C51*

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## 1 Introduction

In this paper, we consider the standard form of semidefinite optimizatiom(SDO) problem as follows:

$$\min\{C \bullet X : A_i \bullet X = b_i,\ 1 \le i \le m,\ X \succeq 0\} \tag{1.1}$$

and its dual problem

$$\max\{b^T y : \sum_{i=1}^{m} y_i A_i + S = C,\ S \succeq 0\}, \tag{1.2}$$

where $C,\ A_i \in \mathbf{S}^n,\ 1 \le i \le m$, $b,\ y \in \mathbf{R}^m$, $C \bullet X = \mathbf{Tr}(C^T X)$ and $T$ denotes the transpose.

SDO problems have many applications in combinatorial optimization, approximation theory, system and control theory and mechanical and electrical engineering and contain linear and quadratic optimization as special cases. We refer the reader to [9] for theory, algorithms and applications of SDO.

Many researchers have proposed various versions of interior-point methods(IPMs). Ghami [3] generalized primal-dual IPMs for linear optimization(LO) to SDO and gave numerical results to show the influence of the choice of the kernel function on the computational behavior of the primal-dual algorithm for LO and concluded that the kernel function in [1] deserves further investigation and its performance seems quite promising. Ghami-Roos-Steihaug [4]

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proposed a primal-dual IPM for SDO based on a parameterized version of the kernel function in [1] with the parameter $p$ in the growth term and the best known complexity result is obtained when $p = 1$, i.e., the kernel function in [1].

The motivation of the paper is to study a finite kernel function, to generalize a finite kernel function defined in [1] and to define a new interior-point algorithm for SDO based on a new kernel function and give a complexity result of large-update method. We obtained the best known complexity result for all parameters in the kernel function.

The paper is organized as follows: In section 2, we give some definitions and fundamental properties. In section 3, we define a class of new kernel functions and its properties. In section 4, we propose a new primal-dual interior-point algorithm and show its complexity results.

Conventional notations : $\mathbf{R}^n$, $\mathbf{R}^n_+$ and $\mathbf{R}^n_{++}$ denote the set of real, nonnegative real and positive real vectors with $n$ components, respectively. $\mathbf{S}^n$, $\mathbf{S}^n_+$ and $\mathbf{S}^n_{++}$ denote the set of symmetric, symmetric positive semidefinite and symmetric positive definite $n \times n$ matrices, respectively. $\|\cdot\|$ denotes the Frobenius norm for matrices. For $A$, $B \in S^n$, $A \succeq B (A \succ B)$ means $A - B$ is symmetric positive semidefinite(positive definite). For $Q \in \mathbf{S}^n_{++}$, $Q^{1/2}$ stands for the symmetric square root of $Q$. For any $V \in \mathbf{S}^n$, we denote by $\lambda(V)$ the vector of eigenvalues of $V$ and $\Lambda := \mathrm{diag}(\lambda(V))$, the diagonal matrix from a vector $\lambda(V)$. $\mathbf{E}$ denotes an $n \times n$ identity matrix. For $a \in \mathbf{R}$, $\lceil a \rceil$ denotes the smallest integer greater than or equal to $a$. For $f(x)$, $g(x) : \mathbf{R}_{++} \to \mathbf{R}_{++}$, $f(x) = \mathcal{O}(g(x))$ if $f(x) \leq c_1 g(x)$ for some positive constant $c_1$ and $f(x) = \Theta(g(x))$ if $c_2 g(x) \leq f(x) \leq c_3 g(x)$ for some positive constants $c_2$ and $c_3$. $\log x$ stands for the natural logarithm of $x \in \mathbf{R}_{++}$.

## 2 Preliminaries

In this section, we give basic definitions and introduce generic IPMs for SDO. For $V \in S^n_{++}$, let $V := Q^T \mathrm{diag}(\lambda_1(V), \lambda_2(V), \cdots, \lambda_n(V))Q$ be the spectral decomposition. We generalize an analytic function $\psi(t) : \mathbf{R}_{++} \to \mathbf{R}_+$ to the matrix function $\psi(V) : \mathbf{S}^n_{++} \to \mathbf{S}^n_+$ as follows( [5]):

$$\begin{aligned} \psi(V) &:= Q^T \mathrm{diag}\left(\psi(\lambda_1(V)), \psi(\lambda_2(V)), \cdots, \psi(\lambda_n(V))\right) Q, \\ \psi'(V) &:= Q^T \mathrm{diag}\left(\psi'(\lambda_1(V)), \psi'(\lambda_2(V)), \cdots, \psi'(\lambda_n(V))\right) Q. \end{aligned} \tag{2.1}$$

In the following we define a matrix function.

**Definition 2.1.** A matrix $M(t)$ is said to be a matrix function if each entry of $M(t)$ is a function of $t$, i.e., $M(t) = [M_{ij}(t)]$, $1 \leq i, j \leq n$. $M(t)$ is said to be differentiable if $M_{i,j}$, $1 \leq i,\ j \leq n$, is differentiable.

Let $M(t) := [M_{ij}(t)]$ and $N(t) := [N_{ij}(t)]$, $1 \leq i, j \leq n$, be matrices of functions which are differentiable at $t$. Then we have

$$\begin{aligned} \tfrac{d}{dt} M(t) &= \left[\tfrac{d}{dt} M_{ij}(t)\right] := M'(t), \\ \tfrac{d}{dt} \mathbf{Tr}(M(t)) &= \mathbf{Tr}\left(M'(t)\right), \\ \tfrac{d}{dt} \mathbf{Tr}(\psi(M(t))) &= \mathbf{Tr}\left(\psi'(M(t)) M'(t)\right), \end{aligned}$$

where $\mathbf{Tr}$ denotes the trace.

Throughout the paper, we assume that the matrices $A_i$, $1 \leq i \leq m$, are linearly independent and SDO (1.1) and (1.2) satisfy the interior-point condition(IPC), i.e., there exists $X \in \mathcal{F}_P$, $S \in \mathcal{F}_D$ with $X \succ 0$, $S \succ 0$, where $\mathcal{F}_P$ and $\mathcal{F}_D$ denote the feasible sets of the problem

(1.1) and (1.2), respectively. Under these assumptions, finding an optimal solution of the problems (1.1) and (1.2) is equivalent to solving the following system:

$$A_i \bullet X = b_i,\ 1 \le i \le m,\ X \succeq 0,\ \sum_{i=1}^{m} y_i A_i + S = C,\ S \succeq 0,\ XS = 0. \tag{2.2}$$

The basic idea of primal-dual IPMs is to replace the complementarity condition of (2.2), i.e., $XS = 0$, by the parameterized equation $XS = \mu\mathbf{E}$ with $X, S \succ 0$ and $\mu > 0$. So we consider the following system: for $\mu > 0$,

$$A_i \bullet X = b_i,\ 1 \le i \le m,\ X \succ 0,\ \sum_{i=1}^{m} y_i A_i + S = C,\ S \succ 0,\ XS = \mu\mathbf{E}. \tag{2.3}$$

Under the assumptions, the system (2.3) has a unique solution $(X(\mu), y(\mu), S(\mu))$ for $\mu > 0$ and $(X(\mu), y(\mu), S(\mu))$ converges to the optimal solution of the problem (1.1) and (1.2) as $\mu$ goes to zero [9]. We call the set $\{(X(\mu), y(\mu), S(\mu)) \mid \mu > 0\}$ the central path of the problems (1.1) and (1.2). IPMs follow the central path approximately.

Let
$P := X^{1/2}(X^{1/2}SX^{1/2})^{-1/2}X^{1/2} = S^{-1/2}(S^{1/2}XS^{1/2})^{1/2}S^{-1/2}$, $D := P^{1/2}$.
Define

$$V := \frac{1}{\sqrt{\mu}} D^{-1}XD^{-1} = \frac{1}{\sqrt{\mu}} DSD = \frac{1}{\sqrt{\mu}} (D^{-1}XSD)^{1/2}. \tag{2.4}$$

Then $D \in S^n_{++}$ and $V \in S^n_{++}$. Define for $\mu > 0$ and $1 \le i \le m$,

$$\bar{A}_i := \frac{1}{\sqrt{\mu}} DA_iD, \quad D_X := \frac{1}{\sqrt{\mu}} D^{-1}\Delta X D^{-1}, \quad D_S := \frac{1}{\sqrt{\mu}} D\Delta S D. \tag{2.5}$$

Applying Newton's method and using the NT symmetrizing scheme [7], we obtain the following system:

$$\bar{A}_i \bullet D_X = 0,\ 1 \le i \le m,\ \sum_{i=1}^{m} \Delta y_i \bar{A}_i + D_S = 0,\ D_X + D_S = V^{-1} - V. \tag{2.6}$$

Note that the right-hand side $V^{-1} - V$ of the third equation of (2.6) is $-\psi_c'(V)$, where $\psi_c(t) := \frac{t^2-1}{2} - \log t$, the classical kernel function. The solution $(D_X, \Delta y, D_S)$ of the system (2.6) is called the scaled NT search direction. Using (2.5), we can find a unique search direction $(\Delta X, \Delta y, \Delta S) \in S^n \times \mathbf{R}^m \times S^n$.

The generic primal-dual interior-point algorithm for SDO works as follows: Assume that $\tau \ge 1$ and there is a strictly feasible point $(X, y, S)$ which is in a $\tau$-neighborhood of the given $\mu$-center [6]. We update $\mu$ to $\mu_+ := (1-\theta)\mu$, for some fixed $\theta \in (0, 1)$ and then solve the system (2.6) and (2.5) to obtain the NT search direction. The positivity condition for the new iterates is ensured with the right choice of the step size $\alpha$. This procedure is repeated until we find new iterates $(X_+, y_+, S_+)$ that belongs to a $\tau$-neighborhood of the $\mu_+$-center and then we let $\mu := \mu_+$ and $(X, y, S) := (X_+, y_+, S_+)$. We repeat the process until $n\mu < \varepsilon$.

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**Primal-Dual Algorithm for SDO**

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Input:

A threshold parameter $\tau \ge 1$;

an accuracy parameter $\varepsilon > 0$;
a fixed barrier update parameter $\theta,\ 0 < \theta < 1$;
a strictly feasible $(X^0, S^0)$ and $\mu^0 = 1$ such that $\Psi(X^0, S^0, \mu^0) \le \tau$.
**begin**
  $X := X^0;\quad S := S^0;\quad \mu := \mu^0;$
  while $n\mu \ge \varepsilon$ do
  **begin**
    $\mu := (1-\theta)\mu;$
    while $\Psi(X, S, \mu) > \tau$ do
    **begin**
      solve the system (2.6) for $\Delta X,\ \Delta y,\ \Delta S$;
      determine a step size $\alpha$;
      $X := X + \alpha\Delta X;$
      $S := S + \alpha\Delta S;$
      $y := y + \alpha\Delta y;$
    **end**
  **end**
**end**

## 3 Kernel Functions

In this section, we define a class of kernel functions which are generalized version of the barrier function in [1] and give its essential properties for complexity analysis. Each barrier function is defined by its univariate kernel function. We consider a function $\psi(t)$ as follows:

$$\psi(t) := \frac{(\log q)(t^2-1)}{2} + \frac{1}{\sigma}(q^{\sigma(1-t)} - 1),\ q \ge e,\ \sigma \ge 1,\ t > 0. \tag{3.1}$$

For $\psi(t)$ as in (3.1), the first three derivatives are as follows:

$$\begin{aligned} \psi'(t) &= (\log q)(t - q^{\sigma(1-t)}), \\ \psi''(t) &= (\log q)(1 + \sigma(\log q)q^{\sigma(1-t)}), \\ \psi^{(3)}(t) &= -\sigma^2(\log q)^3 q^{\sigma(1-t)}. \end{aligned} \tag{3.2}$$

From (3.1) and (3.2), we have for $q \ge e$, $\sigma \ge 1$ and $t > 0$,

$$\psi'(1) = \psi(1) = 0,\ \ \psi''(t) > \log q,\ \ \lim_{t\to 0^+}\psi(t) < \infty,\ \ \lim_{t\to\infty}\psi(t) = \infty. \tag{3.3}$$

**Lemma 3.1.** *Let $\psi(t)$ be defined as in* (3.1). *Then we have for $q \ge e$ and $\sigma \ge 1$,*
*(i)* $t\psi''(t) + \psi'(t) \ge 0,\ t > \frac{1}{\sigma(\log q)}$,
*(ii)* $t\psi''(t) - \psi'(t) \ge 0,\ t > 0$,
*(iii)* $\psi^{(3)}(t) < 0,\ t > 0$.

**Remark 3.2.** (Lemma 2.4 in [2]) If $\psi(t)$ satisfy *(ii)* and *(iii)* of Lemma 3.1, then $\psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi''(\beta t) > 0,\ \ t > 0,\ \ \beta > 1$.

**Lemma 3.3.** *For $\psi(t)$ as in (3.1),* $\frac{\log q}{2}(t-1)^2 \le \psi(t) \le \frac{1}{2\log q}(\psi'(t))^2$, $q \ge e$, $\sigma \ge 1$, $t > 0$.

**Lemma 3.4.** *Let $\varrho : [0, \infty) \to [1, \infty)$ be the inverse function of $\psi(t)$ for $t \geq 1$. Then* $\varrho(u) \leq 1 + \sqrt{\frac{2u}{\log q}}, \quad q \geq e, \quad u \geq 0.$

**Lemma 3.5.** *Let $\rho : [0, \infty) \to (0, 1]$ be the inverse function of $-\frac{1}{2}\psi'(t)$ for $0 < t \leq 1$. Then* $\rho(z) \geq \frac{\sigma(\log q) - \log(\frac{2z}{\log q} + 1)}{\sigma \log q}, \quad q > e, \quad \sigma \geq 1, \quad z \geq 0.$

## 4 Complexity Results

In this section, we define a new primal-dual interior-point algorithm for SDO and give the complexity of large-update methods.

First of all, we replace $V^{-1} - V$ of (2.6) by $-\psi'(V)$. This defines new search directions. For the complexity analysis, we follow the similar framework in [2].

Define $\Psi(V) : \mathbf{S}^n_{++} \to \mathbf{R}_+$ by

$$\Psi(V) := \mathbf{Tr}\,(\psi(V)) = \sum_{i=1}^{n} \psi(\lambda_i(V)). \tag{4.1}$$

Then $\Psi(V)$ is strictly convex with respect to $V \succ 0$ and vanishes at its global minimal point $V = E$ and $\Psi(E) = 0$. Since $D_X$ and $D_S$ are orthogonal, for $\mu > 0$,

$$\Psi(V) = 0 \;\Leftrightarrow\; V = \mathbf{E} \;\Leftrightarrow\; D_X = D_S = 0 \;\Leftrightarrow\; X = X(\mu),\ S = S(\mu).$$

Hence we can use $\Psi(V)$ as a proximity function to measure the distance between the current iteration and the corresponding $\mu$-center. For the analysis of the algorithm, we also define the norm-based proximity measure $\delta(V)$ as follows: For $V \in S^n_{++}$,

$$\delta(V) \;:=\; \frac{1}{2}\|\psi'(V)\| \;=\; \frac{1}{2}\sqrt{\sum_{i=1}^{n}(\psi'(\lambda_i(V)))^2} \;=\; \frac{1}{2}\|D_X + D_S\|. \tag{4.2}$$

In the following we compute upper bounds of proximity function during an outer iteration. Using (4.2) and Lemma 3.3, we have the following lemma.

**Lemma 4.1.** *Let $\delta(V)$ and $\Psi(V)$ be defined as in* (4.2) *and* (4.1)*, respectively. Then* $\delta(V) \geq \sqrt{\frac{\log q}{2}\Psi(V)}, \quad V \in \mathbf{S}^n_{++}, \quad q \geq e.$

**Lemma 4.2** (Theorem 3.2 in [2])**.** *Let $\varrho$ be defined as in* Lemma 3.4*. Then* $\Psi(\beta V) \leq n\psi\left(\beta\varrho\left(\frac{\Psi(V)}{n}\right)\right),\ V \in \mathbf{S}^n_{++},\ \beta \geq 1.$

Using Lemma 4.2, Lemma 3.3 and Lemma 3.4, we have the following theorem.

**Theorem 4.3.** *Let $0 < \theta < 1$ and $V_+ := \frac{V}{\sqrt{1-\theta}}$. If $\Psi(V) \leq \tau$, then we have for $q \geq e$,* $\Psi(V_+) \leq \frac{\log q}{2(1-\theta)}\left(\sqrt{n}\theta + \sqrt{\frac{2\tau}{\log q}}\right)^2.$

Define for $q > e$ and $0 < \theta < 1$, $\tilde{\Psi}_0 := \frac{\log q}{2(1-\theta)}\left(\sqrt{n}\theta + \sqrt{\frac{2\tau}{\log q}}\right)^2$. We will use $\tilde{\Psi}_0$ for the upper bounds of $\Psi(V)$.

**Remark 4.4.** For large-update method with $\tau = \mathcal{O}(n)$ and $\theta = \Theta(1)$, $\tilde{\Psi}_0 = \mathcal{O}((\log q)n)$.

Define $\lambda_1(V) := \min\{\lambda_i(V),\, 1 \le i \le n\}$ for $V \in \mathbf{S}^n_{++}$.

Now we compute a feasible step size $\alpha$ and the decrement of the proximity function during an inner iteration.

**Lemma 4.5** (Modification of Proposition 5.2.6 in [8])**.** *Let $V_1$ and $V_2 \in \mathbf{S}^n_{++}$ satisfying $\lambda_1(V_1),\ \lambda_1(V_2) \ge \frac{1}{\sigma(\log q)}$. Then we have*

$$\Psi\left(\left[V_1^{1/2} V_2 V_1^{1/2}\right]^{1/2}\right) \ \le\ \frac{1}{2}\left(\Psi(V_1) + \Psi(V_2)\right).$$

For fixed $\mu$, if we take a step size $\alpha$ along the search direction $(\Delta X, \Delta y, \Delta S)$, we obtain a new iteration $(X_+,\ y_+,\ S_+)$, where $X_+ := X + \alpha\Delta X,\ \ y_+ := y + \alpha\Delta y,\ \ S_+ := S + \alpha\Delta S,\ \ \alpha > 0$. From (2.4), we have $V_+ = \frac{1}{\sqrt{\mu}}\left(D^{-1}X_+S_+D\right)^{1/2}$. Hence, we have $V_+^2 = (V + \alpha D_X)(V + \alpha D_S)$. Since $V,\, D_X,\, D_S \in \mathbf{S}^n_{++}$, $V + \alpha D_X \in \mathbf{S}^n_{++}$ and $V + \alpha D_S \in \mathbf{S}^n_{++}$ for some $\alpha > 0$. Thus $V_+^2$ is similar to $(V + \alpha D_X)^{1/2}(V + \alpha D_S)(V + \alpha D_X)^{1/2}$. This implies that the eigenvalues of $V_+$ are the same as those of $\left((V + \alpha D_X)^{1/2}(V + \alpha D_S)(V + \alpha D_X)^{1/2}\right)^{1/2}$. From (4.1), we have

$$\Psi(V_+) = \Psi\Big(\left((V + \alpha D_X)^{1/2}(V + \alpha D_S)(V + \alpha D_X)^{1/2}\right)^{1/2}\Big).$$

By Lemma 4.5, we obtain

$$\Psi(V_+) \le \frac{1}{2}\left(\Psi(V + \alpha D_X) + \Psi(V + \alpha D_S)\right). \tag{4.3}$$

Define for $\alpha > 0$,

$$f(\alpha) := \Psi(V_+) - \Psi(V), \quad f_1(\alpha) := \frac{1}{2}\left(\Psi(V + \alpha D_X) + \Psi(V + \alpha D_S)\right) - \Psi(V).$$

From (4.3), $f(\alpha) \le f_1(\alpha)$ and $f(0) = f_1(0) = 0$.
Let

$$\tilde{\alpha} := \frac{1}{\psi''(\rho(2\delta))}. \tag{4.4}$$

Let $t := \rho(2\delta)$. Then $0 < t \le 1$ and $\tilde{\alpha} = \frac{1}{(\log q)(1 + \sigma(\log q)q^{\sigma(1-t)})}$. On the other hand, $-\frac{1}{2}\psi^{'}(t) = 2\delta$. By (3.2) and $0 < t \le 1$, $-(\log q)(t - q^{\sigma(1-t)}) = 4\delta$. $(\log q)q^{\sigma(1-t)} = 4\delta + (\log q)t \le 4\delta + \log q$. Thus

$$(\log q)(1 + \sigma(\log q)q^{\sigma(1-t)}) \le (\log q)(1 + \sigma(4\delta + \log q)).$$

Hence

$$\tilde{\alpha} \ge \frac{1}{(\log q)(1 + \sigma(4\delta + \log q))} \ge \frac{1}{7(\log q)^2\sigma\delta}.$$

Define the default step size $\bar{\alpha}$ as follows:

$$\bar{\alpha} := \frac{1}{7(\log q)^2\sigma\delta}. \tag{4.5}$$

**Lemma 4.6.** *Let $L \ge 8$ and $\Psi(V) \le L$. If $\sigma \ge 1 + 2\log(1 + L)$, $e \le q \le e^{2\left(\frac{L^2 + 2L}{1 + 2\log(1+L)} - L\right)}$, then $\lambda_i(V) \ge \frac{3}{2(\log q)\sigma}$, $1 \le i \le n$.*

*Proof.* Let $t := \lambda_1(V)$. If $t \geq 1$, then $t = \lambda_1(V) \geq 1 > \frac{3}{2(\log q)\sigma}$. Suppose that $t < 1$. Since $\Psi(V) \leq L$, $\psi(t) \leq L$, i.e., $\frac{(\log q)(t^2-1)}{2} + \frac{1}{\sigma}(q^{\sigma(1-t)} - 1) \leq L$. This implies that $\frac{1}{\sigma}(q^{\sigma(1-t)} - 1) \leq L + \frac{(\log q)(1-t^2)}{2} \leq L + \frac{\log q}{2}$, $q^{1-\sigma t} \leq \frac{1+\sigma(L+\frac{\log q}{2})}{q^{\sigma-1}}$. Let $g_1(\sigma) := \frac{1+\sigma(L+\frac{\log q}{2})}{q^{\sigma-1}}$. Then $g_1(\sigma)$ is monotone decreasing in $\sigma$. Since $\sigma \geq 1 + 2\log(1+L)$ by assumption,

$$\begin{aligned} q^{1-\sigma t} &\leq \frac{1+(1+2\log(1+L))(L+\frac{\log q}{2})}{q^{2\log(1+L)}} \\ &\leq \frac{1+(1+2\log(1+L))(L+\frac{\log q}{2})}{e^{2\log(1+L)}} \\ &= \frac{1+(1+2\log(1+L))(L+\frac{\log q}{2})}{(1+L)^2}. \end{aligned}$$

Let $g_2(L) := \frac{1+(1+2\log(1+L))(L+\frac{\log q}{2})}{(1+L)^2}$. Then $g_2(L)$ is monotone decreasing in $L$ and $q^{1-\sigma t} \leq g_2(L)$. This implies that $t \geq \frac{1}{\sigma(\log q)} \log \frac{q}{g_2(L)}$. Let $g_3(q, L) := \frac{q}{g_2(L)} := \frac{q(1+L)^2}{1+(1+2\log(1+L))(L+\frac{\log q}{2})}$. $g_3(q, L)$ is monotone increasing in $q$ and $L$, respectively. Since $q \geq e$ and $L \geq 8$, $g_3(q, L) \geq \frac{81e}{1+8.5(1+4\log 3)} \geq 4.6$. Hence $t \geq \frac{\log(4.6)}{\sigma(\log q)} \geq \frac{3}{2\sigma(\log q)}$. □

By Lemma 4.6 and Lemma 5.2 in [4], $\lambda_i(V + \alpha D_X) \geq \lambda_1(V) - 2\bar{\alpha}\delta \geq \frac{3}{2\sigma(\log q)} - \frac{2\delta}{7(\log q)^2\sigma\delta} \geq \frac{17}{14\sigma(\log q)} > \frac{1}{\sigma(\log q)}$, $1 \leq i \leq n$. By the same way, $\lambda_i(V + \alpha D_S) \geq \frac{1}{\sigma(\log q)}$, $1 \leq i \leq n$.

In the following we compute the complexity bound of large-update methods.

**Lemma 4.7** (Lemma 4.6 in [2]). *Let $\tilde{\alpha}$ be defined as in* (4.4). *Then $f(\alpha) \leq -\alpha\delta^2$, $\alpha \leq \tilde{\alpha}$.*

Using Lemma 4.7, we have the following lemma.

**Lemma 4.8.** *Let $\bar{\alpha}$ be defined as in (4.5). Then $f(\bar{\alpha}) \leq -\frac{\Psi^{\frac{1}{2}}}{7\sqrt{2}(\log q)^{\frac{3}{2}}\sigma}$.*

We denote the value of $\Psi$ after $\mu$-update as $\Psi_0$ and the subsequent values in the same outer iteration are denoted as $\Psi_l$, $l = 0, 1, 2, 3, \cdots, K$, where $K$ denotes the total number of inner iterations per an outer iteration. Then we have $\Psi_0 \leq \tilde{\Psi}_0$. Then we have $\Psi_{K-1} > \tau$ and $0 \leq \Psi_K \leq \tau$.
Using Proposition 1.3.2 in [8] and letting $L := \tilde{\Psi}_0$, we have the following theorem.

**Theorem 4.9.** *Let a SDO* (1.1) *and* (1.2) *be given and $0 < \theta < 1$ and $\tau \geq 1$. Let $L \geq 8$, $\Psi(V) \leq L$ and $\sigma = 1 + 2\log(1+L)$, $e \leq q \leq e^{2(\frac{L^2+2L}{1+2\log(1+L)} - L)}$. If there is a strictly feasible starting point $(X^0, S^0)$ s.t. $\Psi(X^0, S^0, \mu^0 := 1) \leq \tau$, then the total number of iterations required by the algorithm to have an approximate solution such that $n\mu < \varepsilon$ is bounded by $\left\lceil 14\sqrt{2}(\log q)^{\frac{3}{2}}\sigma\tilde{\Psi}_0^{\frac{1}{2}}\frac{1}{\theta}\log\frac{n}{\varepsilon}\right\rceil$.*

**Remark 4.10.** Using Remark 4.4, Theorem 4.9 and $\sigma = \mathcal{O}(\log n)$, we have $\mathcal{O}((\log q)^{\frac{3}{2}}\sqrt{n}(\log n)\log\frac{n}{\epsilon})$ iteration complexity for large-update method. This is the best known complexity result for such methods.

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Gyeong-Mi Cho
Department of Software Engineering, Dongseo University
Busan 617-716, South Korea
E-mail address: gcho@gdsu.dongseo.ac.kr