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WELL-POSEDNESS FOR A SYSTEM OF GENERALIZED VECTOR QUASI-EQUILIBRIUM PROBLEMS*

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Abstract: In this paper, the notion of the generalized Tykhonov well-posedness for a system of generalized vector quasi-equilibrium problems with set-valued mappings is investigated. By using the gap functions of a system of generalized vector quasi-equilibrium problems, we establish the equivalent relationship between the generalized Tykhonov well-posedness of the systems of generalized vector quasi-equilibrium problems and that of the minimization problems. We also present some metric characterizations for the generalized Tykhonov well-posedness of the system of generalized vector quasi-equilibrium problems. The results in this paper extend some known results in the literature.

Key words: *system of generalized vector quasi-equilibrium problems, generalized Tykhonov well-posedness, approximating solution sequence, gap function*

Mathematics Subject Classification: *58E35, 49K40, 49J53*

1 Introduction

Well-posedness plays a crucial role in the stability theory for optimization problems, which guarantees that, for an approximating solution sequence, there exists a subsequence which converges to a solution. The study of well-posedness for scalar minimization problems started from Tykhonov [42] and Levitin and Polyak [26]. Since then, various notions of well-posedness for scalar minimization problems have been defined and studied (see, e.g., [10,13, 19,24,36,44] and the references therein). Recent studies on various notions of well-posedness for vector optimization problems can be found in [6, 9, 17, 18, 20, 33, 35]. It is worth noting that the recent study for various types of well-posedness have been generalized to variational inequalities [11, 12, 22, 30, 34], generalized variational inequalities [7, 21], quasi-variational inequalities [29], generalized quasi-variational inequalities [23], generalized vector variational inequalities [43], vector quasi-variational inequalities [31], equilibrium problems [32], vector equilibrium problems ([28, 38]), vector quasi-equilibrium problems [30], generalized vector quasi-equilibrium problems [27,40], system of vector quasi-equilibrium problems with single-valued maps [39] and many other problems.

There are some results on the gap functions or existence of solutions for several types of systems of generalized vector equilibrium problems with set-valued maps [15, 41], which contain as special cases the mathematical models in [6, 7, 9–13, 16–24, 26–36, 38–40, 42–44].

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However, so far there is no study on the well-posedness of any classes of systems of generalized vector quasi-equilibrium problems with set-valued maps. In this paper, we are interested in investigating generalized Tykhonov well-posedness for a system of generalized vector quasi-equilibrium problems with set-valued maps. The paper is organized as follows: In section 2, we present some preliminaries. In section 3, the lower semi-continuous property of the gap functions of the system of generalized vector quasi-equilibrium problems and the equivalent relationship between the generalized Tykhonov well-posednesses of the system of generalized vector quasi-equilibrium problems and that of the minimization problems are established. In section 4, some metric characterizations for the generalized Tykhonov well-posedness of the system of generalized vector quasi-equilibrium problems are obtained. The results in this paper extend some known results in [16, 27–29, 31, 39].

2 Preliminaries

Throughout this paper, let I be a countable index set and for each $i \in I$, let (E_i, d_i) be a metric space, X_i be a nonempty convex subset of E_i, and Y_i be a locally convex Hausdorff topological vector space. Let $E = \prod_{i\in I} E_i$, $X = \prod_{i\in I} X_i$, $E_{-i} = \prod_{j\in I\setminus i} E_i$ and $X_{-i} = \prod_{j\in I\setminus i} X_i$. For each fixed $i \in I$ and $x \in X$, we write $x = (x_i, x_{-i}) = (x_i)_{i\in I}$, where x_i and x_{-i} denote the projection of x onto X_i and X_{-i}, respectively. Let $d(x, y) = \sup_{i\in I} \frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)}$ for all $x, y \in E$. It is easy to verify that (E, d) is a metric space. For each $i \in I$, let $C_i : X \to 2^{Y_i}$ be a set-valued map such that for any $x \in X$, $C_i(x)$ is a proper, pointed, closed and convex cone in Y_i with nonempty interior $intC_i(x)$. For each $i \in I$, let $e_i : X \to Y_i$ be a continuous vector-valued map and satisfy that for any $x \in X$, $e_i(x) \in intC_i(x)$, $F_i : X \times X_i \to 2^{Y_i}$ and $A_i : X \to 2^{X_i}$ be set-valued maps. We consider the following system of generalized vector quasi-equilibrium problems with set-valued maps: find $\bar{x}$ in X such that for each $i \in I$,

$$\text{(SGVQEP)} \qquad \bar{x}_i \in A_i(\bar{x}) \text{ and } F_i(\bar{x}, y_i) \not\subset -intC_i(\bar{x}), \forall\ \ y_i \in A_i(\bar{x}).$$

It is worth mentioning that (SGVQEP) was introduced and studied by Huang, Li and Wu [15], Peng and Yang [41].

The following problems are special cases of the (SGVQEP).

(1) For each $i \in I$ and for all $x \in X$, if $A_i(x) \equiv X_i$, then the (SGVQEP) reduces to the system of generalized vector equilibrium problems with set-valued maps introduced and studied by Ansari, Schaible and Yao in [4].

(2) For each $i \in I$, if the set-valued map F_i is replaced by a vector-valued map $\varphi_i : X \times X_i \to Y_i$, then the (SGVQEP) reduces to a system of vector quasi-equilibrium problems (in short, SVQEP) which is to find $\bar{x} = (\bar{x}^i, \bar{x}_i)$ in X such that for each $i \in I$,

$$\bar{x}_i \in A_i(\bar{x}) \text{ and } \varphi_i(\bar{x}, y_i) \notin -intC_i(\bar{x}), \text{ for all } y_i \in A_i(\bar{x}).$$

For each $i \in I$, let $\varphi_i : X \to Y_i$ be a vector-valued map, and let $f_i(x, y_i) = \varphi_i(x^i, y_i) - \varphi_i(x)$, then the (SVQEP) is equivalent to the following Debreu type equilibrium problem for vector-valued maps (in short, Debreu VEP) which is to find $\bar{x} = (\bar{x}^i, \bar{x}_i)$ in X such that for each $i \in I$,

$$\bar{x}_i \in A_i(\bar{x}) \text{ and } \varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) \notin -intC_i(\bar{x}), \text{ for all } y_i \in A_i(\bar{x}).$$

The (SVQEP) and the (Debreu VEP) were introduced and studied by Ansari, Chan and Yang [2]. And if $A_i(x) \equiv X_i$ for each $i \in I$ and for all $x \in X$, then the (Debreu VEP) becomes the Nash equilibrium problem for vector-valued maps in [3].

(3) It is easy to see that the system of generalized vector quasi-variational-like inequality problems (in short, (SGVQVLIP) introduced by Peng [37], the system of generalized vector variational inequality problems (in short, SGVVIP) introduced by Ansari, Schaible and Yao in [4] and Allevi, Gnudi and Konnov [1] and the system of vector variational inequality problems (in short, SVVIP) considered by Ansari, Schaible and Yao in [3] are all special cases of the (SGVQEP).

It is worth noting that all of the above special cases of (SGVQEP) can not be handled by a single generalized vector quasi-equilibrium problems with set-valued maps.

We denote by Ω the set of solutions of (SGVQEP).

Let (P, d) be a metric space, $P_1 \subset P$ and $x \in P$. We denote by $d(x, P_1) = \inf\{d(x, p) : p \in P_1\}$ the distance function from a point $x \in P$ to the set P_1.

Definition 2.1. A sequence $\{x^n\} \subset X$ is called an approximating solution sequence of (SGVQEP) if there exists a sequence $\{\epsilon^n\} \subseteq \mathbf{R}_+ = \{r \in \mathbf{R} : r \geq 0\}$ with $\epsilon^n \to 0+$ such that for each $i \in I$,

$$d_i(x_i^n, A_i(x^n)) \leq \epsilon^n, \tag{2.1}$$

and

$$F_i(x^n, y_i) + \epsilon^n e_i(x^n) \not\subset -intC_i(x^n), \forall y_i \in A_i(x^n). \tag{2.2}$$

Definition 2.2. (SGVQEP) is said to be Tykhonov well-posed in the generalized sense (i.e., generalized Tykhonov well-posed) if $\Omega \neq \emptyset$ and for every approximating solution sequence $\{x^n\}$ for (SGVQEP), there exists a subsequence $\{x^{n_j}\}$ of $\{x^n\}$ and $\bar{x} \in \Omega$ such that $x^{n_j} \to \bar{x}$.

Remark 2.3. (i) Generalized Tykhonov well-posedness for (SGVQEP) implies that the solution set Ω is nonempty and compact.

(ii) If I is a singleton, then by Definitions 2.1 and 2.2, respectively, we recover Definitions 2.1 and 2.2 in [27]).

(iii) If for each $i \in I$, F_i is replaced by a single-valued map $f_i : X \times X_i \to Y_i$, then Definitions 2.1 and 2.2 reduce Definitions 2.1 and 2.2 in [39], respectively.

(iv) It is easy to see that Definition 2.2 extends and unifies the corresponding definitions of well-posedness in [16, 27–29, 31, 39].

Definition 2.4 ([5, 25])**.** Let Z_1, Z_2 be two metric spaces. A set-valued map F from Z_1 to 2^{Z_2} is

(i) closed on $Z_3 \subseteq Z_1$, if for any sequence $\{x_n\} \subseteq Z_3$ with $x_n \to x$ and $y_n \in F(x_n)$ with $y_n \to y$, one has $y \in F(x)$;

(ii) lower semicontinuous (*l.s.c.* in short) at $x \in Z_1$, if $\{x_n\} \subseteq Z_1, x_n \to x$, and $y \in F(x)$ imply that there exists a sequence $\{y_n\} \subseteq Z_2$ satisfying $y_n \to y$ such that $y_n \in F(x_n)$ for n sufficiently large. If F is *l.s.c.* at each point of Z_1, we say that F is *l.s.c.* on Z_1.

(iii) upper semicontinuous (*u.s.c.* in short) at $x \in Z_1$, if for any neighborhood V of $F(x)$, there exists a neighborhood U of x such that $F(z) \subseteq V, \forall z \in U$. If F is *u.s.c.* at each point of Z_1, we say that F is *u.s.c.* on Z_1.

(iv) continuous at $x \in Z_1$, if it is both *u.s.c.* and *l.s.c.* at x. If F is continuous at each point of Z_1, we say that F is continuous on Z_1.

Definition 2.5 ([8])**.** Let X and Y be two locally convex Hausdorff topological vector spaces, $C : X \to 2^Y$ a set-valued map such that, for any $x \in X$, $C(x)$ is proper, pointed, closed and convex cone in Y with nonempty interior $intC(x)$. Let $e : X \to Y$ be a continuous vector-valued map and satisfy that for any $x \in X$, $e(x) \in intC(x)$. The nonlinear scalarization function $\xi_e : X \times Y \to \mathbf{R}$ is defined as follows

$$\xi_e(x, y) =: \inf\{\lambda \in \mathbf{R} : y \in \lambda e(x) - C(x)\}.$$

3 Gap Functions of (SGVQEP)

In this section, the lower semi-continuous property of the gap functions of (SGVQEP) and the equivalent relationship of the generalized Tykhonov well-posedness of (SGVQEP) and that of minimization problems will be presented.

Definition 3.1. A map $\phi : X \to \mathbf{R} \cup \{+\infty\}$ is said to be a gap function for (SGVQEP), if
(i) $\phi(x) \geq 0, \forall x \in \bar{X}$, where $\bar{X} = \{x \in X : x_i \in A_i(x), i \in I\}$;
(ii) $\phi(x^*) = 0$ and $x^* \in \bar{X}$ if and only if $x^* \in \Omega$.

By similar arguments in the proof of Theorems 3.1 in [15], we can prove the following result:

Proposition 3.2. *Assume that for each $i \in I$,*

(i) *for each $x \in X$, the set-valued function $F_i(x, .)$ is compact-valued on X_i;*

(ii) *for each $x \in X$, $F_i(x, x_i) \subset -C_i(x)$;*

(iii) *the set-valued map $W_i : X \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus -intC_i(x)$ is upper semi-continuous.*

Then the function $\phi(x)$ is a gap function of (SGVQEP), where the function ϕ is defined as follows:

$$\phi(x) = \sup_{i \in I} \alpha(x, i), \forall x \in X, \tag{3.1}$$

and $\alpha(x, i) = \sup\limits_{y_i \in A_i(x)} \inf\limits_{z_i \in F_i(x, y_i)} \{-\xi_{e_i}(x, z_i)\}$.

Proof. For each $x \in \bar{X}$ and for each $i \in I$, $F_i(x, x_i) \subset -C_i(x)$ implies $z_i \in -C_i(x)$ for all $z_i \in F_i(x, x_i)$. It follows from Proposition 2.3 in [8] that $\xi_{e_i}(x, z_i) \leq 0$ for all $z_i \in F_i(x, x_i)$. So $\inf_{z_i \in F_i(x, x_i)}\{-\xi_{e_i}(x, z_i)\} \geq 0$.

Since $x_i \in A_i(x)$, for any $x \in X$ and $i \in I$,

$$\alpha(x, i) = \sup_{y_i \in A_i(x)} \inf_{z_i \in F_i(x, y_i)} \{-\xi_{e_i}(x, z_i)\} \geq 0,$$

and so

$$\phi(x) = \sup_{i \in I} \alpha(x, i) \geq 0, \forall x \in X. \tag{3.2}$$

If $\phi(\bar{x}) = 0$ and $\bar{x} \in \bar{X}$, then for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and $\inf\limits_{z_i \in F_i(\bar{x}, y_i)} \{-\xi_{e_i}(\bar{x}, z_i)\} \leq 0$ for all $y_i \in A_i(\bar{x})$. It follows (iii) and Theorem 2.1(i) in [8] that $\xi_{e_i}(\cdot, \cdot)$ is upper semi-continuous. Since for any $x \in X$, $F_i(x, .)$ is compact-valued map, for any $y_i \in A_i(\bar{x})$, there exists $z_i \in F_i(\bar{x}, y_i)$ such that $\xi_{e_i}(\bar{x}, z_i) \geq 0$. It follows from Proposition 2.3 in [8] that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and $F_i(\bar{x}, y_i) \not\subset -intC_i(\bar{x})$, $\forall y_i \in A_i(\bar{x})$, and thus, $\bar{x} \in \Omega$.

Conversely, if $\bar{x} \in \Omega$, then $\bar{x} \in X$ and for each $i \in I$,

$$\bar{x}_i \in A_i(\bar{x}) : F_i(\bar{x}, y_i) \not\subset -intC_i(\bar{x}), y_i \in A_i(\bar{x}).$$

It follows that $\bar{x} \in \bar{X}$, and for each $i \in I$ and for any $y_i \in A_i(\bar{x})$, $v_i \notin -intC_i(\bar{x})$ for some $v_i \in F_i(\bar{x}, y_i)$. Again by Proposition 2.3 in [8], we have

$$\inf_{z_i \in F_i(\bar{x}, y_i)} \{-\xi_{e_i}(\bar{x}, z_i)\} \leq -\xi_{e_i}(\bar{x}, v_i) \leq 0,$$

and hence for each $i \in I$, $\alpha(\bar{x}, i) = \sup_{y_i \in A_i(\bar{x})} \inf_{z_i \in F_i(\bar{x}, y_i)} \{-\xi_{e_i}(\bar{x}, z_i)\} \leq 0$. It follows that

$$\phi_1(\bar{x}) = \sup_{i \in I} \alpha(\bar{x}, i) \leq 0. \tag{3.3}$$

Now (3.2) and (3.3) imply that $\phi(\bar{x}) = 0$. □

Now we present an important property of the gap function for (SGVQEP) as follows.

Proposition 3.3. *Assume that for each $i \in I$,*

(i) *the set-valued map $F_i : X \times X_i \to 2^{X_i}$ is upper semi-continuous and compact-valued;*

(ii) *the set-valued map $W_i : X \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus intC_i(x)$ is upper semi-continuous;*

(iii) *the set-valued map A_i is lower semi-continuous on X.*

Then the function $\phi : X \to \mathbb{R} \cup \{+\infty\}$ defined by (3.1) is lower semi-continuous. Further assume that the solution set Ω of (SGVQEP) is nonempty, then $\mathrm{Dom}(\phi) \neq \emptyset$.

Proof. First, it is obvious that $\phi(x) > -\infty, \forall x \in X$. Otherwise, suppose that there exists $x_0 \in X$ such that

$$\phi(x_0) = \sup_{i \in I} \sup_{y_i \in A_i(x_0)} \inf_{z_i \in F_i(x_0, y_i)} \{-\xi_{e_i}(x_0, z_i)\} = -\infty.$$

Then for each $i \in I$,

$$\inf_{z_i \in F_i(x_0, y_i)} \{-\xi_{e_i}(x_0, z_i)\} = -\infty., \forall y_i \in A_i(x_0)$$

It follows from (ii) and Theorem 2.1(i) in [8] that $\xi_{e_i}(\cdot, \cdot)$ is upper semi-continuous. Since F_i is compact-valued on $X \times X_i$, for all $y_i \in A_i(x_0)$, there exists $v_i \in F(x_0, y_i)$ such that $\xi_{e_i}(x_0, v_i)) = +\infty$, which is impossible since $\xi_{e_i}(x_0, \cdot)$ is a finite function on Y_i.

From (i), the upper semi-continuity of ξ_{e_i} and Proposition 21 (pp. 119) in [5], we know that for each $i \in I$, and for each $x \in X$, $\inf_{z_i \in F_i(x, y_i)} \{-\xi_{e_i}(x, z_i)\} = -\sup_{z_i \in F_i(x, y_i)} \{\xi_{e_i}(x, z_i)\}$ is lower semi-continuous on X_i. It follows from (iii) and Proposition 19 (pp. 118) in [5] that for each $i \in I$, $\alpha(x, i) = \sup_{y_i \in A_i(x)} \inf_{z_i \in F_i(x, y_i)} \{-\xi_{e_i}(x, z_i)\}$ is lower semi-continuous on X.

Thus,

$$\phi(x) = \sup_{i \in I} \alpha(x, i) = \sup_{i \in I} \sup_{y_i \in A_i(x)} \inf_{z_i \in F_i(x, y_i)} \{-\xi_{e_i}(x, z_i)\}$$

is lower semi-continuous on X.

Furthermore, if $\Omega \neq \emptyset$, by Proposition 3.2, we see that $\mathrm{Dom}(\phi) \neq \emptyset$. This completes the proof. □

In order to relate the well-posedness of (SGVQEP) to that of constrained minimization problems, we consider the well-posedness of the following general constrained mathematical program:

$$(P) \qquad \begin{cases} \min \phi(x) \\ s.t. \ \ x_i \in A_i(x), \forall i \in I, \end{cases}$$

where $\phi : X \to \mathbf{R} \cup \{\infty\}$ is proper and lower semicontinuous. The optimal set and optimal value of (P) are denoted by $\bar{\Omega}$ and $\bar{v}$, respectively.

Definition 3.4 ([27])**.** A sequence $\{x^n\} \subset X$ is called a minimizing sequence for (P) if

$$\limsup_{n\to+\infty} \phi(x^n) \le \bar{v}, \tag{3.4}$$

and for each $i \in I$,

$$d_i(x_i^n, A_i(x^n)) \to 0. \tag{3.5}$$

Definition 3.5 ([27])**.** (P) is said to be generalized Tykhonov well-posed if $\bar{\Omega} \neq \emptyset$, and for any minimizing sequence $\{x^n\}$ for (P), there exist a subsequence $\{x^{n_j}\}$ of $\{x^n\}$ and $\bar{x} \in \bar{\Omega}$ such that $x^{n_j} \to \bar{x}$.

The following result reveals the relationship between generalized Tykhonov well-posedness of (SGVQEP) and that of (P).

Theorem 3.6. *Let all Assumptions in Proposition 3.2 hold. Then (SGVQEP) is generalized Tykhonov well-posed if and only if (P) is generalized Tykhonov well-posed with $\phi(x)$ defined by (3.1).*

Proof. Let $\phi(x)$ be defined by (3.1). Since $\Omega \neq \emptyset$, it follows from Proposition 3.2 that $\bar{x} \in \Omega$ is a solution of (SGVQEP) if and only if $\bar{x}$ is an optimal solution of (P) with $\bar{v} = \phi(\bar{x}) = 0$.

We first prove the necessity. Assume that $\{x^n\}$ is an approximating solution sequence of (SGVQEP). Then there exists $\{\epsilon^n\} \subseteq \mathbf{R}_+$ with $\epsilon^n \to 0$ such that (2.1) and (2.2) hold for each $i \in I$. It follows from (2.1) that (3.5) holds. By (2.2) and Proposition 2.3 in [8], we know that for each $i \in I$, for all $y_i \in A_i(x^n)$, there exists $v_i^n \in F_i(x^n, y_i)$ such that $\xi_{e_i}(x^n, v_i^n) \ge -\epsilon^n$. It follows that for each $i \in I$, for all $y_i \in A_i(x^n)$, we have $\inf_{v_i\in F_i(x^n,y_i)} \{-\xi_{e_i}(x^n, v_i)\} \le \epsilon^n$. Thus, $\phi(x^n) = \sup_{i\in I} \sup_{y_i\in A_i(x^n)} \inf_{v_i\in F_i(x^n,y_i)} \{-\xi_{e_i}(x^n, v_i)\} \le \epsilon^n$, which implies that (3.4) holds with $\bar{v} = 0$. Hence, $\{x^n\}$ is an approximating solution sequence of (P). It follows from the generalized Tykhonov well-posedness of (SGVQEP) that (P) is generalized Tykhonov well-posed.

Now we show the Sufficiency. Assume that $\{x^n\}$ is an approximating solution sequence of (P). Then there exists $\{\epsilon^n\} \subseteq \mathbf{R}_+$ with $\epsilon^n \to 0$ such that (3.4) and (3.5) hold for each $i \in I$. It follows from (3.5) that (2.1) holds.

Furthermore, it follows from (3.4) that there exists $\{\epsilon^n\} \subseteq R_+$ with $\epsilon^n \to 0$ such that $\phi(x^n) = \phi(x^n) \le \epsilon^n$. Then, for each $i \in I$, for all $y_i \in A_i(x^n)$, $\inf_{v_i\in F_i(x^n,y_i)} \{-\xi_{e_i}(x^n, v_i)\} \le \epsilon^n$. Since $-\xi_{e_i}(x^n, \cdot)$ is lower semi-continuous and $F_i(x^n, .)$ is compact-valued on X_i, for all $y_i \in A_i(x^n)$, there exists $v_i^n \in F(x^n, y_i)$ such that $-\xi_{e_i}(x^n, v_i^n) = \inf_{v_i\in F_i(x^n,y_i)} \{-\xi_{e_i}(x^n, v_i)\} \le \epsilon^n$. So, for each $i \in I$, for all $y_i \in A_i(x^n)$, there exists $v_i^n \in F(x^n, y_i)$ such that $\xi_{e_i}(x^n, v_i^n) \ge -\epsilon^n$. It follows from Proposition 2.3 in [8] that for each $i \in I$, (2.2) holds. Thus, $\{x_n\}$ is an approximating solution sequence of (SGVQEP). It follows from the generalized Tykhonov well-posedness of (P) that (SGVQEP) is generalized Tykhonov well-posed. This completes the proof. □

Remark 3.7. (i) If for each $i \in I$, F_i is replaced by a single-valued map f_i, then by Propositions 3.2, 3.3 and Theorem 3.6 reduce to Lemmas 3.1, 3.2, and Theorem 3.3 in [39], respectively.

(ii) If I is a singleton, then by Theorem 3.6, we can obtain Theorem 3.9(i) in [27].

(iii) Proposition 3.2 improves Theorem 3.1 in [15] since the continuity of ξ_{e_i} has been replaced by the upper-semicontinuity of ξ_{e_i}.

4 Metric Characterizations for Generalized Tykhonov Well-Posedness of (SGVQEP)

In this section, we give some metric characterizations for the generalized Tykhonov well-posedness of (SGVQEP).

Now we consider the Kuratowski measure of noncompactness for a nonempty subset A of X (see [25]) defined by

$$\alpha(A) = \inf\{\epsilon > 0 : A \subset \cup_{i=1}^{n} A_i, \text{ for every } A_i, diam A_i < \epsilon\},$$

where $\mathrm{diam}A_i$ is the diameter of A_i defined by

$$\mathrm{diam}A_i = \sup\{d(x_1, x_2) : x_1, x_2 \in A_i\}.$$

Given two nonempty subsets A and B of X, the excess of set A and B is defined by

$$e(A, B) = \sup\{d(a, B) : a \in A\},$$

and the Hausdorff distance between A and B is defined by

$$H(A, B) = \max\{e(A, B), e(B, A)\}.$$

For any $\epsilon > 0$, the approximating solution set for (SGVQEP), is defined by

$$\Theta(\epsilon) := \{x \in X : \forall i \in I, d_i(x_i, A_i(x)) \leq \epsilon \ \text{ and } \ F_i(x, y_i) + \epsilon e_i(x) \not\subset -intC_i(x), \forall y_i \in A_i(x)\}.$$

Theorem 4.1. *(SGVQEP) is generalized Tykhonov well-posed if and only if the solution set Ω is nonempty, compact and*

$$e(\Theta(\epsilon), \Omega) \to 0 \ as \ \epsilon \to 0. \tag{4.1}$$

Proof. Let (SGVQEP) be generalized Tykhonov well-posed. Then Ω is nonempty and compact. Now we show that (4.1) holds. Suppose to the contrary that there exist $l > 0$, $\epsilon^n > 0$ with $\epsilon^n \to 0$ and $z^n \in \Theta_1(\epsilon^n)$ such that

$$d(z^n, \Omega) \geq l. \tag{4.2}$$

Since $\{z^n\} \subset \Theta_1(\epsilon^n)$, we know that $\{z^n\}$ is an approximating solution sequence for (SGVQEP). By the generalized Tykhonov well-posedness of (SGVQEP), there exists a subsequence $\{z^{n_j}\}$ of $\{z^n\}$ converging to some element of Ω. This contradicts (4.2). Hence, (4.1) holds.

Conversely, suppose that Ω is nonempty, compact and (4.1) hold. Let $\{x^n\}$ be an approximating solution sequence for (SGVQEP). Then there exist a sequence $\{\epsilon^n\}$ with $\{\epsilon^n\} \subseteq \mathbf{R}_+$ and $\epsilon^n \to 0$ such that (2.1) and (2.2) hold for each $i \in I$. Thus, $\{x^n\} \subset \Theta(\epsilon^n)$. It follows from (4.1) that there exists a sequence $\{z^n\} \subseteq \Omega$ such that

$$d(x^n, z^n) = d(x^n, \Omega) \leq e(\Theta_1(\epsilon^n), \Omega) \to 0.$$

Since Ω is compact, there exists a subsequence $\{z^{n_k}\}$ of $\{z^n\}$ converging to $x_0 \in \Omega$. So the corresponding subsequence $\{x^{n_k}\}$ of $\{x^n\}$ converging to x_0. Therefore, (SGVQEP) is generalized Tykhonov well-posed. This completes the proof. □

Theorem 4.2. *Assume that for each $i \in I$,*

(i) *the set-valued map F_i is upper semi-continuous with compact-valued on $X \times X_i$;*

(ii) *the set-valued map $W_i : X \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus -intC_i(x)$ is closed;*

(iii) *the set-valued map A_i is lower semi-continuous and closed on X.*

Then (SGVQEP) is generalized Tykhonov well-posed if and only if

$$\Theta(\epsilon) \neq \emptyset, \forall \epsilon > 0 \text{ and } \lim_{\epsilon\to 0} \alpha(\Theta(\epsilon)) = 0. \tag{4.3}$$

Proof. Suppose that (4.3) holds. Then, for any $\epsilon > 0$, $\mathrm{Cl}(\Theta(\epsilon))$ is nonempty closed and increasing with $\epsilon > 0$, where $\mathrm{Cl}(\Theta(\epsilon))$ is the closure of $\Theta(\epsilon)$. By (4.3), we obtain that $\lim_{\epsilon\to 0} \alpha(\mathrm{Cl}(\Theta(\epsilon))) = \lim_{\epsilon\to 0} \alpha(\Theta(\epsilon)) = 0$. By the generalized Cantor theorem (P. 412 in [25]), we know that

$$H(\mathrm{Cl}(\Theta(\epsilon)), \Delta) \to 0, \text{as } \ \epsilon \to 0, \tag{4.4}$$

where $\Delta = \cap_{\epsilon>0}\mathrm{Cl}(\Theta(\epsilon))$ is nonempty compact.

Now we show that

$$\Delta = \Omega. \tag{4.5}$$

Let $\bar{x} \in \Delta$. Then $d(\bar{x}, \Theta(\epsilon)) = 0$, for every $\epsilon > 0$. Given $\epsilon^n > 0$, $\epsilon^n \to 0$, for every n there exists $u^n \in \Theta(\epsilon^n)$ such that $d(\bar{x}, u^n) < \epsilon^n$. Hence, $u^n \to \bar{x}$ and for each $i \in I$,

$$d_i(u_i^n, A_i(u^n)) \leq \epsilon^n, \tag{4.6}$$

and

$$F_i(u^n, y_i) + \epsilon^n e_i(u^n) \not\subset -intC_i(x^n), \forall y_i \in A_i(u^n). \tag{4.7}$$

By (4.6) and $u^n \to \bar{x}$, for each $i \in I$, there exists $w_i^n \in A_i(u^n)$ such that $w_i^n \to \bar{x}_i$. It follows from the closedness of A_i that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$. For any $y_i \in A_i(\bar{x})$, by the lower semicontinuity of A_i and (4.7), we have a sequence $\{y_i^n\}$ with $y_i^n \in A_i(x^n)$ convergeing to y_i such that for each $i \in I$,

$$F_i(u^n, y_i^n) + \epsilon^n e_i(u^n) \notin -intC_i(x^n). \tag{4.8}$$

From (i) and Proposition 2.59 in [14] (page 59), we know that for each $i \in I$, $G_i(.,.,.)$ is upper semi-continuous with compact-valued on $X \times Y_i \times \mathbf{R}^+$, where $G_i(x, y_i, \epsilon) := F_i(x, y_i) + \epsilon e_i(x)$, $\forall (x, y_i, \epsilon) \in X \times Y_i \times \mathbf{R}^+$. It follows from (ii) and Theorem 8 in [5] (page 110) that for each $i \in I$ and for any $(x, y_i, \epsilon) \in X \times Y_i \times \mathbf{R}$, $G_i(x, y_i, \epsilon) \cap W_i(x) = (F_i(x, y_i) + \epsilon e_i(x)) \cap W_i(x)$ is upper semi-continuous. Hence, by (4.8), we get for each $i \in I$, there exists $u_i^n \in (F_i(x^n, y_i^n) + \epsilon^n e_i(x^n)) \cap W_i(x^n)$. It follows from the upper semi-continuity and compact-valuedness of $G_i(.,.,.) \cap W_i(.)$ that for each $i \in I$, $\{u_i^n\}$ has a subsequence converging to a point $\bar{u}_i$ with $\bar{u}_i \in F_i(\bar{x}, y_i) \cap W_i(\bar{x})$. That is, for each $i \in I$, $F_i(\bar{x}, y_i) \not\subset -intC_i(\bar{x}), \forall y_i \in A_i(\bar{x})$. Thus, $\bar{x} \in \Omega$, which implies that $\Delta \subset \Omega$. The opposite inclusion $\Omega \subset \Delta$ is obvious. So, (4.5) holds. It follows from (4.5) and (4.4) that $e(\Theta_1(\epsilon), \Omega) \to 0$ as $\epsilon \to 0$. Thus, (SGVQEP) is generalized Tykhonov well-posed by Theorem 4.1.

Conversely, let (SGVQEP) be generalized Tykhonov well-posed. Observe that for every $\epsilon > 0$,

$$H(\Theta(\epsilon), \Omega) = max\{e(\Theta(\epsilon), \Omega), e(\Omega, \Theta(\epsilon))\} = e(\Theta(\epsilon), \Omega).$$

Hence,

$$\begin{aligned}\alpha(\Theta(\epsilon)) &\leq 2H(\Theta(\epsilon), \Omega) + \alpha(\Omega) \\ &= 2e(\Theta(\epsilon), \Omega), \end{aligned} \tag{4.9}$$

where $\alpha(\Omega) = 0$ since Ω is compact. By Theorem 4.1, we get that $e(\Theta(\epsilon), \Omega) \to 0$ as $\epsilon \to 0$. It follows from (4.9) that (4.3) holds. This completes the proof. □

Remark 4.3. (i) If I is a singleton, then Theorem 4.2 reduces to Theorem 3.2 [27] with the continuity of A replaced by the lower semi-continuity of A.

(ii) If F_i is replaced by a single-valued function $f_i : X \times X_i \to Y_i$, then Theorems 4.1 and 4.2 reduce to Theorems 4.1 and 4.2 in [39], respectively.

(iii) Theorems 4.1 and 4.2 extend and improve the corresponding results in [16,28,29,31] and the references therein.

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