Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 11, No. 1, pp. 177–193, January 2015

SUBDERIVATIVE AND SUBDIFFERENTIAL OF DUAL GAP FUNCTIONS IN VARIATIONAL INEQUALITIES*

Jinchuan Zhou, Naihua Xiu, Soonyi Wu and Ziyan Luo

Abstract: By employing the tools of variational analysis, we provide new characterizations for subderivatives and subdifferentials of the dual gap functions for pseudo-monotone variational inequality problems (VIP) in $\mathbb{R}^n$. Based on these, we give an exact expression of the largest global error bound constant, and establish the equivalences among (i) global error bound, (ii) weak sharpness, (iii) minimum principle sufficiency (MPS) property, and (iv) linear regularity for the pseudo-monotone VIP over polyhedral sets. Among the established equivalences as mentioned above, the relationship of (iv) linear regularity to (i), (ii), and (iii) is the most important new result.

Key words: *dual gap function, subdifferential, global error bound, weak sharpness, linear regularity*

Mathematics Subject Classification: *49J52, 65K10*

1 Introduction

This paper is concerned with the following *variational inequality problem* (VIP): find $x^* \in X$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in X, \tag{1.1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$, and $X \subseteq \mathbb{R}^n$ is a nonempty closed convex set. Denote by X^* the solution set of problem (1.1). Throughout the paper, we assume that F is continuous and pseudo-monotone on X, and the solution set X^* is nonempty.

VIPs have been extensively studied with a number of applications in many fields, such as the economic equilibrium, control theory, game theory, transportation science, and operations research. For more details, see [9] and references therein. How to solve this kind of problems? One approach is to transform VIP into an equivalent optimization problem by utilizing various gap functions, such as the primal gap function [2], the regularized gap function [1, 10, 22], D-gap function [18, 21, 26] and the dual gap function [3, 12, 14, 24, 25, 27]. Much attention has been focused on the last kind of gap function, defined as

$$G(x) = \sup\{\langle F(y), x - y\rangle \mid y \in X\}.$$

This dual gap function has been investigated on the relationships among the global error bound, weak sharpness of the solution set, minimum principle sufficiency property, and finite

*This research was supported by National Natural Science Foundation of China (11101248, 71271021, 11301022), the Fundamental Research Funds for the Central Universities of China (2013JBM123), and Shandong Province Natural Science Foundation (ZR2010AQ026, ZR2012AM016).

termination of iterative algorithms for solving the VIP. For example, Marcotte and Zhu [15] established some equivalences among the above properties of the VIP under the condition that F is pseudo-monotone$^+$ and X is compact. Some generalizations of these results were developed by many authors. One extension is to relax the compactness assumption on X and the pseudo-monotonicity$^+$ of F [27]; another extension is to treat the problem in an infinite-dimensional setting [23, 24]. It should be noted, however, that the characterization of subdifferential of the dual gap function established in [27] was given only at the points restricted in the relative interior of effective domain of G, and explicit formulae for global error bound were not provided in [15, 23, 24, 27].

We shall show in this paper that these problems can be overcome by using the variational analysis technique. First, the characterization of the subderivative of the dual gap function is developed and the expression of the subdifferential is established in the whole domain of the gap function without the restriction in the relative interior. The explicit expression of subdifferential plays an essential role in the following two aspect: one is to provide an estimate of the largest global error bound constant; another is to establish the equivalence among weak sharpness of X^*, minimum principle sufficiency property, global error bound, and linear regularity.

The paper is organized as follows. Section 2 contains some preliminary results. In Section 3 we present characterizations of subderivatives and subdifferentials of the dual gap function. Based on these, we study the relationship among the concepts of global error bound, weak sharpness of the solution set, MPS property, and linear regularity for pseudo-monotone VIP in Section 4. Conclusion is drawn in Section 5.

We shall make use of the following notations throughout the paper. Let A be a nonempty set in $\mathbb{R}^n$. The closure, convex hull, interior, and relative interior of A are denoted by cl$A$, co$A$, int$A$, and ri$A$, respectively. We denote the cone of feasible directions by $F_A(x)$, and the normal cone by $N_A(x)$. The tangent cone is defined dually by the relation $T_A(x) = N_A(x)^\circ$, where A° denotes the polar set of $A \subseteq \mathbb{R}^n$, i.e., $A^\circ = \{v \in \mathbb{R}^n | \ \langle v, x\rangle \leq 0, \forall x \in A\}$. Let $\delta(x|S)$ and $\sigma^*(x|S)$ stand for the indicator function and the support function of some set S, respectively; i.e., $\delta(x|S) = 0$ if $x \in S$ and $\delta(x|S) = \infty$ if $x \notin S$, and $\sigma^*(x|S) := \sup\{\langle x, y\rangle | \ y \in S\}$. We denote by $\|\cdot\|$ the usual Euclidean norm, and by $\mathbb{B}$ the closed unit ball in $\mathbb{R}^n$.

2 Preliminaries

A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *monotone* on $X \subset \mathbb{R}^n$ if for any $x, y \in X$,

$$\langle F(y) - F(x), y - x\rangle \geq 0.$$

The mapping F is said to be *pseudo-monotone* on X if for any $x, y \in X$,

$$\langle F(x), y - x\rangle \geq 0 \Longrightarrow \langle F(y), y - x\rangle \geq 0.$$

The VIP is said to have the *minimum principle sufficiency* property (MPS property) if

$$\Gamma(x^*) = X^*, \quad \forall x^* \in X^*,$$

where $\Gamma(x) := \arg\max_{y\in X} \langle F(x), x - y\rangle$. Similarly, the VIP has the *maximum principle sufficiency* property if

$$\Lambda(x^*) = X^*, \quad \forall x^* \in X^*,$$

where $\Lambda(x) := \arg\max_{y\in X}\langle F(y), x-y\rangle$.

The following geometric characterization was adopted in [15,17] as the definition of weak sharpness, i.e., the solution set X^* of VIP is said to be *weakly sharp* if

$$-F(x^*) \in \operatorname{int} \bigcap_{x\in X^*} \Big[T_X(x) \bigcap N_{X^*}(x)\Big]^\circ, \quad \forall x^* \in X^*. \tag{2.1}$$

We say that G has a *global error bound* on X, if there exists some positive scalar α such that

$$G(x) \geq \alpha \operatorname{dist}(x, X^*), \quad \forall x \in X, \tag{2.2}$$

where $\operatorname{dist}(x, X^*) = \inf\{\|x - x^*\| | \ x^* \in X^*\}$. Let $\{C_i | i = 1, 2, \dots, m\}$ be an arbitrary collection of nonempty sets in $\mathbb{R}^n$ with a nonempty intersection $C = \bigcap_{i=1}^m C_i$. The collection $\{C_i | i = 1, 2, \dots, m\}$ is said to be *linearly regular* over some set $S \subseteq \mathbb{R}^n$ with modulus β (see [4,7,13,28] for more information) if

$$\beta \operatorname{dist}(x, C) \leq \max_{i=1,\dots,m} \operatorname{dist}(x, C_i), \quad \forall x \in S.$$

Let $f : \mathbb{R}^n \to \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be an extended-real-valued function. Denote the effective domain by $\operatorname{dom} f := \{x \in \mathbb{R}^n | \ f(x) < +\infty\}$ and the epigraph by $\operatorname{epi} f := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} | \ f(x) \leq \alpha\}$. The function f is said to be Gâteaux differentiable at $x \in \operatorname{dom} f$ provided that there exists $\xi \in \mathbb{R}^n$ such that

$$f'(x; w) := \lim_{\lambda \downarrow 0} \frac{f(x + \lambda w) - f(x)}{\lambda} = \langle \xi, w\rangle, \quad \forall w \in \mathbb{R}^n.$$

As shown in [23, 24], the Gâteaux differentiability of G is essentially important for studying the weak sharpness of the solution set and for deriving finite termination of iterative algorithms for solving VIP. The function f is said to be (Fréchet) differentiable at x if $f(x + h) = f(x) + \langle \nabla f(x), h\rangle + o(\|h\|)$.

In this paper, we mainly employ the following two concepts, the subderivative and the subdifferential. The subderivative of f at $\bar{x}$, denoted by $df(\bar{x})$, is defined as

$$df(\bar{x})(\bar{w}) = \liminf_{\substack{w \to \bar{w} \\ \lambda \downarrow 0}} \frac{f(\bar{x} + \lambda w) - f(\bar{x})}{\lambda},$$

and the subdifferential of G at $\bar{x}$, denoted by $\partial f(\bar{x})$, is specified as

$$\partial f(\bar{x}) = \{v \in \mathbb{R}^n | \langle v, w\rangle \leq df(\bar{x})(w), \quad \forall w \in \mathbb{R}^n\}.$$

3 Subdifferentiability

In this section, we study the subderivative and the subdifferential of dual gap function, respectively.

3.1 Subderivative

In order to state the structural expression of the subderivative of the dual gap function, we first review the following two key lemmas. Letting $\delta \in \mathbb{R}$, for the notation convenience, write the set $\{y \in X | \ \langle F(y), x - y\rangle \geq \delta\}$ simply by $\theta(x, \delta)$.

Lemma 3.1 ([27, Theorem 3.3]). *If* $\bar{x} \in \mathrm{dom}G$, *then*

$$G'(\bar{x}; w) = \begin{cases} \lim\limits_{\delta \nearrow G(\bar{x})} \sup\{\langle F(y), w\rangle |\ y \in \theta(\bar{x}, \delta)\}, & w \in F_{\mathrm{dom}G}(\bar{x}), \\ +\infty, & w \notin F_{\mathrm{dom}G}(\bar{x}). \end{cases}$$

Lemma 3.2 ([20, Proposition 8.21] and [19, Theorem 23.2]). *For a convex function* $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ *and any point* $\bar{x}$ *where* f *is finite, the directional derivative* $f'(\bar{x}; w)$ *exists for all* $w \in \mathbb{R}^n$, *and*

$$df(\bar{x})(w) = \liminf_{w' \to w} f'(\bar{x}; w'), \quad \forall w \in \mathbb{R}^n.$$

With the preparations, the characterization of $dG(\bar{x})$ can be stated as follows.

Theorem 3.3. *If* $\bar{x} \in domG$ *and*

$$\limsup_{\substack{w' \to w \\ w' \in T_{\mathrm{dom}G}(\bar{x})}} \lim_{\delta \nearrow G(\bar{x})} \sup \{\langle F(y), w - w'\rangle |\ y \in \theta(\bar{x}, \delta)\} \le 0, \quad \forall w \in T_{\mathrm{dom}G}(\bar{x}) \tag{3.1}$$

then

$$dG(\bar{x})(w) = \begin{cases} \lim\limits_{\delta \nearrow G(\bar{x})} \sup \{\langle F(y), w\rangle |\ y \in \theta(\bar{x}, \delta)\}, & w \in T_{\mathrm{dom}G}(\bar{x}), \\ +\infty, & w \notin T_{\mathrm{dom}G}(\bar{x}). \end{cases}$$

Proof. We first show that $dG(\bar{x})(w) = +\infty$ for any $w \notin T_{\mathrm{dom}G}(\bar{x})$. In fact, if $w \notin T_{\mathrm{dom}G}(\bar{x})$, then for any positive sequences $\{\lambda_i\}$ converging to 0 and $\{w_i\}$ converging to w, we have $\bar{x} + \lambda_i w_i \notin \mathrm{dom}G$ for all i sufficiently large. Hence, by the definition of subderivative of G at $\bar{x}$, we get $dG(\bar{x})(w) = +\infty$.

Now let $w \in T_{\mathrm{dom}G}(\bar{x})$ be given. Since $T_{\mathrm{dom}G}(\bar{x}) = \mathrm{cl}F_{\mathrm{dom}G}(\bar{x})$ due to the convexity of $\mathrm{dom}G$ [5, Proposition 2.55], we get

$$\begin{aligned} dG(\bar{x})(w) &= \liminf_{w' \to w} G'(\bar{x}; w') \\ &= \liminf_{\substack{w' \to w \\ w' \in F_{\mathrm{dom}G}(\bar{x})}} G'(\bar{x}; w') \\ &= \liminf_{\substack{w' \to w \\ w' \in F_{\mathrm{dom}G}(\bar{x})}} \lim_{\delta \nearrow G(\bar{x})} \sup \{\langle F(y), w'\rangle |\ y \in \theta(\bar{x}, \delta)\}, \end{aligned} \tag{3.2}$$

where the first equation is due to Lemma 3.2 and the second equality comes from Lemma 3.1, since $G'(\bar{x}; w) = +\infty$ for all $w \notin F_{\mathrm{dom}G}(\bar{x})$.

Define

$$\pi(w) := \lim_{\delta \nearrow G(\bar{x})} \sup \{\langle F(y), w\rangle |\ y \in \theta(\bar{x}, \delta)\}. \tag{3.3}$$

Note that π is convex. In fact, for any $w_1, w_2 \in \mathbb{R}^n$ and $\lambda \in [0, 1]$, we have

$$\begin{aligned} \sup_{y \in \theta(\bar{x}, \delta)} \langle F(y), \lambda w_1 + (1 - \lambda) w_2\rangle &= \sup_{y \in \theta(\bar{x}, \delta)} \Big[\lambda\langle F(y), w_1\rangle + (1 - \lambda)\langle F(y), w_2\rangle\Big] \\ &\le \lambda \sup_{y \in \theta(\bar{x}, \delta)} \langle F(y), w_1\rangle + (1 - \lambda) \sup_{y \in \theta(\bar{x}, \delta)} \langle F(y), w_2\rangle. \end{aligned}$$

Taking the limits on both sides yields

$$\pi(\lambda w_1 + (1 - \lambda) w_2) \le \lambda \pi(w_1) + (1 - \lambda)\pi(w_2).$$

We now show that π is lower semi-continuous over $T_{\text{dom}G}(\bar{x})$. This is equivalent to showing that the level set $\text{lev}_{\leq\alpha}\pi := \{w \in T_{\text{dom}G}(\bar{x}) |\ \pi(w) \leq \alpha\}$ is closed for every $\alpha \in \mathbb{R}$ [19]; i.e., for any given sequence $\{w_n\}$ in $\text{lev}_{\leq\alpha}\pi$ with limit $\bar{w}$, we need to show that $\bar{w} \in \text{lev}_{\leq\alpha}\pi$. Note that $\bar{w} \in T_{\text{dom}G}(\bar{x})$, since $w_n \in T_{\text{dom}G}(\bar{x})$ and $T_{\text{dom}G}(\bar{x})$ is closed. From $w_n \in \text{lev}_{\leq\alpha}\pi$ for all n, we have

$$\lim_{\delta\nearrow G(\bar{x})} \sup\{\langle F(y), w_n\rangle |\ y \in \theta(\bar{x}, \delta)\} \leq \alpha. \tag{3.4}$$

Note that

$$\begin{aligned} \sup\{\langle F(y), \bar{w}\rangle |\ y \in \theta(\bar{x}, \delta)\} &= \sup\{\langle F(y), \bar{w} - w_n + w_n\rangle |\ y \in \theta(\bar{x}, \delta)\} \\ &\leq \sup\{\langle F(y), \bar{w} - w_n\rangle |\ y \in \theta(\bar{x}, \delta)\} \\ &\quad + \sup\{\langle F(y), w_n\rangle |\ y \in \theta(\bar{x}, \delta)\}. \end{aligned}$$

Hence

$$\begin{aligned} & \lim_{\delta\nearrow G(\bar{x})} \sup\{\langle F(y), \bar{w}\rangle |\ y \in \theta(\bar{x}, \delta)\} \\ \leq\ & \lim_{\delta\nearrow G(\bar{x})} \sup\{\langle F(y), \bar{w} - w_n\rangle |\ y \in \theta(\bar{x}, \delta)\} + \lim_{\delta\nearrow G(\bar{x})} \sup\{\langle F(y), w_n\rangle |\ y \in \theta(\bar{x}, \delta)\} \\ \leq\ & \limsup_{\substack{w'\to\bar{w} \\ w'\in T_{\text{dom}G}(\bar{x})}} \lim_{\delta\nearrow G(\bar{x})} \sup\{\langle F(y), \bar{w} - w'\rangle |\ y \in \theta(\bar{x}, \delta)\} + \alpha \\ \leq\ & \alpha, \end{aligned}$$

where the second inequality is due to the fact that $w_n \in T_{\text{dom}G}(\bar{x})$ and (3.4) and the last inequality comes from (3.1). Hence $\pi(\bar{w}) \leq \alpha$, i.e., $\bar{w} \in \text{lev}_{\leq\alpha}\pi$.

It follows from (3.2) that

$$\begin{aligned} dG(\bar{x})(w) &= \liminf_{\substack{w'\to w \\ w'\in F_{\text{dom}G}(\bar{x})}} \pi(w') \\ &\geq \liminf_{\substack{w'\to w \\ w'\in T_{\text{dom}G}(\bar{x})}} \pi(w') \\ &= \pi(w), \end{aligned} \tag{3.5}$$

where the last step is due to the lower semicontinuity of π as just shown above.

To obtain the reverse inequality, choose some $u \in \text{ri}F_{\text{dom}G}(\bar{x}) = \text{ri}T_{\text{dom}G}(\bar{x})$, where the equality is due to [19, Theorem 6.3]. Consider the following two cases. Case 1. If $\pi(w) = +\infty$, then $dG(\bar{x})(w) = \pi(w) = +\infty$ by (3.5). Case 2. If $\pi(w) < +\infty$, using the convexity of π and the fact that $\lambda u + (1 - \lambda)w \in \text{ri}F_{\text{dom}G}(\bar{x})$ for $\lambda \in (0, 1)$ (since $u \in \text{ri}F_{\text{dom}G}(\bar{x})$ and $w \in T_{\text{dom}G}(\bar{x})$, see [19, Theorem 6.1]), we have

$$\begin{aligned} dG(\bar{x})(w) &= \liminf_{\substack{w'\to w \\ w'\in F_{\text{dom}G}(\bar{x})}} \pi(w') \\ &\leq \liminf_{\lambda\searrow 0} \pi\big(\lambda u + (1 - \lambda)w\big) \\ &\leq \liminf_{\lambda\searrow 0} \big[\lambda\pi(u) + (1 - \lambda)\pi(w)\big] \\ &= \pi(w). \end{aligned}$$

The above inequality, together with (3.5), implies that $dG(\bar{x})(w) = \pi(w)$. This completes the proof. ∎

Remark 3.4. The condition (3.1) is satisfied if

i) $\sup\{\|F(y)\|\mid y\in\theta(\bar{x},\delta)\}<+\infty$ for some $\delta<G(\bar{x})$. In fact, this condition guarantees the existence of $\delta_0\in\mathbb{R}$ and $M>0$ such that

$$\sup\{\|F(y)\|\mid y\in\theta(\bar{x},\delta)\}<M,\quad \forall\delta\in(\delta_0,G(\bar{x})],$$

since the set $\theta(\bar{x},\delta)$ is monotonically contractive as δ increases to $G(\bar{x})$. In this case, because

$$\sup\{\langle F(y),w-w'\rangle\mid y\in\theta(\bar{x},\delta)\}\le\sup\{\|F(y)\|\mid y\in\theta(\bar{x},\delta)\}\cdot\|w-w'\|\le M\|w-w'\|,$$

taking the limits on both sides of the above inequality yields

$$\limsup_{\substack{w'\to w\\ w'\in T_{\mathrm{dom}G}(\bar{x})}}\ \lim_{\delta\nearrow G(\bar{x})}\sup\{\langle F(y),w-w'\rangle\mid y\in\theta(\bar{x},\delta)\}\le\limsup_{\substack{w'\to w\\ w'\in T_{\mathrm{dom}G}(\bar{x})}} M\|w-w'\|=0.$$

Hence our result extends [27, Theorem 3.3], since as shown above our assumption is weaker.

ii) X is compact. In this case, it naturally has

$$\sup\{\|F(y)\|\mid y\in\theta(\bar{x},\delta)\}\le\sup\{\|F(y)\|\mid y\in X\}<+\infty,$$

where the first inequality follows from the fact that $\theta(x,\delta)\subset X$ by definition and the last step is due to the continuity of F and the compactness of X.

Corollary 3.5. *If $\bar{x}\in\mathrm{dom}G$ and $\theta(\bar{x},\delta)$ is bounded for some $\delta<G(\bar{x})$, then*

$$dG(\bar{x})(w)=\begin{cases}\max\{\langle F(y),w\rangle\mid y\in\Lambda(\bar{x})\}, & w\in T_{\mathrm{dom}G}(\bar{x}),\\ +\infty, & w\notin T_{\mathrm{dom}G}(\bar{x}).\end{cases}$$

Proof. According to [27, Theorem 3.5], we have

$$G'(\bar{x};w)=\max\{\langle F(y),w\rangle\mid y\in\Lambda(\bar{x})\},\quad \forall w\in F_{\mathrm{dom}G}(\bar{x}).$$

The boundedness of $\theta(\bar{x},\delta)$ and the continuity of F imply that $\Lambda(\bar{x})$ is compact, which in turn implies that the convex function π defined in (3.3) is finite everywhere on $\mathbb{R}^n$. Thus, π is continuous on $T_{\mathrm{dom}G}(\bar{x})$ by [19, Corollary 10.1.1]. In view of Lemma 3.2, for each $w\in T_{\mathrm{dom}G}(\bar{x})$, we have

$$\begin{aligned}dG(\bar{x})(w)&=\liminf_{w'\to w}G'(\bar{x};w')\\&=\liminf_{\substack{w'\to w\\ w'\in F_{\mathrm{dom}G}(\bar{x})}}G'(\bar{x};w')\\&=\liminf_{\substack{w'\to w\\ w'\in F_{\mathrm{dom}G}(\bar{x})}}\pi(w')\\&=\pi(w).\end{aligned}$$

This completes the proof. □

The crucial distinction between the characterizations of directional derivative and subderivative lies at the relative boundary points of the cone of feasible directions, i.e., $G'(\bar{x};\cdot)$ agrees with $dG(\bar{x})(\cdot)$ everywhere except at the points in $T_{\mathrm{domG}}(\bar{x})\backslash F_{\mathrm{domG}}(\bar{x})$. Since $F_{\mathrm{domG}}(\bar{x})$ may be open, it prevents us from deriving the structural expression of subdifferentials directly. This is the reason why we make use of the subderivative, which is always a lower semi-continuous function; see [20] for more properties of subderivative.

3.2 Subdifferential

Now let us establish the characterization of the subdifferential of G at $x \in$ domG. For a closed convex set A, let A_∞ represent its recession cone

$$A_\infty := \{y \in \mathbb{R}^n |\ x + \lambda y \in A \text{ for some } x \in A \text{ and all } \lambda \geq 0\}.$$

Theorem 3.6. *If $\bar{x} \in \mathrm{dom}G$ then*

$$\partial G(\bar{x}) = \bigcap_{\delta < G(\bar{x})} \mathrm{cl}\big[\mathrm{co}F\big(\theta(\bar{x}, \delta)\big) + N_{\mathrm{dom}G}(\bar{x})\big].$$

Proof. According to [11, Proposition 3], we know

$$\partial G(\bar{x}) = \bigcap_{\delta < G(\bar{x})} \mathrm{cl}\left[\mathrm{co}F\big(\theta(\bar{x}, \delta)\big) + \left\{v | (v, v^T\bar{x}) \in \left(\mathrm{cl}\left[\mathrm{co}\big(\cup_{y \in X} \{(F(y), F(y)^T y)\}\big)\right]\right)_\infty\right\}\right]. \tag{3.6}$$

Note that

$$\mathrm{gph} f_y^* = (F(y), F(y)^T y),$$

where $f_y(x) := \langle F(y), x - y\rangle$. In fact, by the definition of conjugate function, we obtain

$$f_y^*(x^*) = \sup_{x \in \mathbb{R}^n} \{\langle x^*, x\rangle - f_y(x)\} = \sup_{x \in \mathbb{R}^n} \{\langle x^* - F(y), x\rangle + \langle F(y), y\rangle\} = \begin{cases} \langle F(y), y\rangle, & x^* = F(y) \\ +\infty, & \text{otherwise.} \end{cases}$$

We know from [11, Proposition 4] that

$$N_{\mathrm{dom}G}(\bar{x}) = \left\{v \in \mathbb{R}^n |\ (v, v^T\bar{x}) \in \left(\mathrm{cl}\left[\mathrm{co}\big(\cup_{y \in X} \mathrm{gph} f_y^*\big)\right]\right)_\infty\right\}. \tag{3.7}$$

Putting (3.6)-(3.7) yields

$$\partial G(\bar{x}) = \bigcap_{\delta < G(\bar{x})} \mathrm{cl}\big[\mathrm{co}F(\theta(\bar{x}, \delta)) + N_{\mathrm{dom}G}(\bar{x})\big].$$

□

The above theorem can be regarded as an extension of Theorem 3.8 in [27] since the point $\bar{x}$ is not restricted to the relative interior of domG. The relation between subderivative and subdifferential is given below, which will be used in our subsequent analysis.

Theorem 3.7. *Let $x \in \mathrm{dom}G$. If $\Lambda(\bar{x})$ is nonempty, then*

$$dG(\bar{x})(w) = \sigma^*\big(w | \partial G(\bar{x})\big). \tag{3.8}$$

Proof. According to Theorem 3.6, if $\Lambda(\bar{x})$ is nonempty, then $\partial G(\bar{x})$ is also nonempty, since $\Lambda(\bar{x}) \subset \theta(\bar{x}, \delta)$ for all $\delta < G(\bar{x})$. Hence the desired result follows from [20, Theorem 8.30]. □

In particular, when $\theta(\bar{x}, \delta)$ is bounded for some $\delta < G(\bar{x})$, then $\partial G(\bar{x})$ can be simply described in terms of $\Lambda(\bar{x})$.

Corollary 3.8. *If $\bar{x} \in \mathrm{dom}G$ and $\theta(\bar{x}, \delta)$ is bounded for some $\delta < G(\bar{x})$, then*

$$\partial G(\bar{x}) = \mathrm{co}F\big(\Lambda(\bar{x})\big) + N_{\mathrm{dom}G}(\bar{x}),$$

where $F(\Lambda(\bar{x})) := \{F(y) |\ y \in \Lambda(\bar{x})\}$.

Proof. It follows from Corollary 3.5 that

$$\begin{aligned} dG(\bar{x})(w) &= \sigma^*\big(w|\ F(\Lambda(\bar{x}))\big) + \delta\big(w|\ T_{\mathrm{dom}G}(\bar{x})\big) \\ &= \sigma^*\big(w|\ F(\Lambda(\bar{x}))\big) + \sigma^*(w|\ N_{\mathrm{dom}G}(\bar{x})) \\ &= \sigma^*\big(w|\ F(\Lambda(\bar{x})) + N_{\mathrm{dom}G}(\bar{x})\big), \end{aligned} \tag{3.9}$$

where in the second equality we use the fact that $\sigma^*(w|K) = \delta(w|K^\circ)$ for a closed convex cone K, see [19] for the detailed discussion on the conjugate duality between support function and indicator function.

Note that the boundedness of $\theta(\bar{x}, \delta)$ implies that $\Lambda(\bar{x})$ is nonempty. Hence Theorem 3.7 is applicable. This together with (3.9) yields

$$\sigma^*\big(w|\partial G(\bar{x})\big) = \sigma^*\big(w|\ F(\Lambda(\bar{x})) + N_{\mathrm{dom}G}(\bar{x})\big).$$

Hence according to [19, Corollary 13.1.1] we have

$$\partial G(\bar{x}) = \mathrm{cl}[\mathrm{co}F(\Lambda(\bar{x})) + N_{\mathrm{dom}G}(\bar{x})].$$

The boundedness of $\theta(\bar{x}, \delta)$ implies that $\Lambda(\bar{x})$ is compact, which further implies that the compactness of $F(\Lambda(\bar{x}))$ due to the continuity of F. Hence, according to [20, Exercise 3.12], the closure operation can be dropped. □

4 Solution Properties

Utilizing the characterizations of subderivatives and subdifferentials of the dual gap function G given in the previous section, we provide an estimate of the largest global error bound constant and study the relationship among global error bound, weak sharpness, MPS property, and linear regularity for the pseudo-monotone VIP. First, we review the concept of the basic constraint qualification due to Rockafellar [19]. First, recall from [20] that $\partial^\infty f(x) := \limsup\limits_{\substack{x' \xrightarrow{f} x \\ \lambda \searrow 0}} \lambda \widehat{\partial} f(x')$.

Definition 4.1. [8, Definition 2.5] Consider the problem of minimizing G over X. We say that the basic constraint qualification (BCQ) is satisfied at $x \in X$ if for every $u \in \partial^\infty G(x)$ and $v \in N_X(x)$ satisfying $u + v = 0$, it must be the case that $u = v = 0$. The BCQ is said to be satisfied on X^* if it is satisfied at every point of X^*.

Remark 4.2. According to [20, Proposition 8.12], we know $\partial^\infty G(x) = N_{\mathrm{dom}G}(x)$. Hence if the BCQ holds at x, then it follows from [20, Theorem 6.42] that $N_\Omega(x) = N_{\mathrm{dom}G}(x) + N_X(x)$ and $T_\Omega(x^*) = T_X(x^*) \cap T_{\mathrm{dom}G}(x^*)$, where $\Omega := X \cap \mathrm{dom}G$.

Lemma 4.3. [8, Theorem 2.6] *Consider the optimization problem of minimizing G over X with zero optimal value. If the BCQ holds on X^*, then G has an global error bound on X with modulus $\alpha > 0$ if and only if the following inclusion holds with the same modulus,*

$$\alpha\mathbb{B} \subseteq \partial G(x^*) + \big[T_X(x^*) \cap N_{X^*}(x^*)\big]^\circ, \quad \forall x^* \in X^*. \tag{4.1}$$

Based on Lemma 4.3 and Theorem 3.3, an alternative equivalent description for the concept of global error bound is given below, where $T_X(x^*)$ can be replaced by $T_\Omega(x^*)$.

Theorem 4.4. *If the BCQ holds on X^*, then G has an global error bound on X with modulus $\alpha > 0$ if and only if the following inclusion holds*

$$\alpha'\mathbb{B} \subseteq \partial G(x^*) + \big[T_\Omega(x^*) \cap N_{X^*}(x^*)\big]^\circ, \quad \forall\, x^* \in X^* \text{ and } \forall\, \alpha' \in (0,\alpha). \tag{4.2}$$

Proof. We first note that the formula (3.8) holds at every $x^* \in X^*$, since $X^* \subset \Lambda(x^*)$ for all $x^* \in X^*$ [27, Proposition 3.12] and X^* is nonempty by hypothesis. According to Lemma 4.3, we only need to show the equivalence between (4.1) and (4.2). In view of [19, Corollary 13.1.1], (4.1) holds for some $\alpha > 0$ if and only if for any $x^* \in X^*$ and $w \in \mathbb{R}^n$, we have

$$\begin{aligned}\sigma^*(w|\alpha\mathbb{B}) &\le \sigma^*\big(w|\partial G(x^*) + [T_X(x^*) \cap N_{X^*}(x^*)]^\circ\big)\\ &= \sigma^*\big(w|\partial G(x^*)\big) + \sigma^*\big(w|[T_X(x^*) \cap N_{X^*}(x^*)]^\circ\big)\\ &= dG(x^*)(w) + \delta\big(w|\ T_X(x^*) \cap N_{X^*}(x^*)\big).\end{aligned} \tag{4.3}$$

Note that $\sigma^*(w|\alpha\mathbb{B}) = \alpha\,\sigma^*(w|\mathbb{B}) = \alpha\|w\|$. It follows from Theorem 3.3 that (4.3) is equivalent to

$$\alpha\|w\| \le dG(x^*)(w), \quad \forall\ w \in T_X(x^*) \cap N_{X^*}(x^*) \cap T_{\mathrm{dom}G}(x^*). \tag{4.4}$$

Taking into account of Remark 4.2, (4.4) can be equivalently written as follows:

$$\begin{aligned}&dG(x^*)(w) \ge \alpha\|w\|, \quad \forall w \in T_\Omega(x^*) \cap N_{X^*}(x^*), && (4.5)\\ \Leftrightarrow\ &\sigma^*\big(w|\partial G(x^*)\big) + \delta\big(w|\ T_\Omega(x^*) \cap N_{X^*}(x^*)\big) \ge \sigma^*(w|\alpha\mathbb{B}), \quad \forall w \in \mathbb{R}^n, &&\\ \Leftrightarrow\ &\sigma^*\big(w|\partial G(x^*)\big) + \sigma^*\big(w|[T_\Omega(x^*) \cap N_{X^*}(w)]^\circ\big) \ge \sigma^*(w|\alpha\mathbb{B}), \quad \forall w \in \mathbb{R}^n, &&\\ \Leftrightarrow\ &\sigma^*\big(w|\partial G(x^*) + \big[T_\Omega(x^*) \cap N_{X^*}(w)\big]^\circ\big) \ge \sigma^*(w|\alpha\mathbb{B}), \quad \forall w \in \mathbb{R}^n, &&\\ \Leftrightarrow\ &\alpha\mathbb{B} \subseteq \mathrm{cl}\{\partial G(x^*) + \big[T_\Omega(x^*) \cap N_{X^*}(x^*)\big]^\circ\}, && (4.6)\end{aligned}$$

the last equivalence following from [19, Corollary 13.1.1]. Noticing that the sets involved in (4.6) are convex, then according to the basic topological properties of convex sets, (4.6) is equivalent to

$$\mathrm{int}(\alpha\mathbb{B}) \subseteq \mathrm{int}\Big(\mathrm{cl}\{\partial G(x^*) + \big[T_\Omega(x^*) \cap N_{X^*}(x^*)\big]^\circ\}\Big) = \mathrm{int}\big(\partial G(x^*) + \big[T_\Omega(x^*) \cap N_{X^*}(x^*)\big]^\circ\big),$$

which shows the validity of (4.2). Conversely, we readily get (4.6) by (4.2), since $\alpha\mathbb{B} = \mathrm{cl}\{\cup_{\alpha'<\alpha}\alpha'\mathbb{B}\}$. This completes the proof. □

Corollary 4.5. *If $X^* \subseteq \mathrm{int}(\mathrm{dom}G)$ and $\theta(x^*,\delta)$ is bounded for some $\delta < G(x^*)$, then G has an global error bound on X with modulus $\alpha > 0$ if and only if the following inclusion holds*

$$\alpha\mathbb{B} \subseteq \mathrm{co}F\big(\Lambda(x^*)\big) + \big[T_\Omega(x^*) \cap N_{X^*}(x^*)\big]^\circ, \quad \forall x^* \in X^*.$$

Proof. Since $X^* \subseteq \mathrm{int}(\mathrm{dom}G)$, then $N_{\mathrm{dom}G}(x^*) = \{0\}$ for all $x^* \in X^*$. Hence it follows from Corollary 3.8 that $\partial G(x^*) = \mathrm{co}F(\Lambda(x^*))$. This, together with Lemma 4.3, yields the desired result. □

Define the largest global error bound constant as

$$\begin{aligned}\alpha_{\max} &:= \sup\{\alpha \ge 0|\ G(x) \ge \alpha\,\mathrm{dist}(x,X^*),\ \forall x \in X\}\\ &= \inf\left\{\frac{G(x)}{\mathrm{dist}(x,X^*)}\Big|x \in X/X^*\right\}.\end{aligned}$$

This is slightly different from that given in [16], where α is restricted to be positive. Clearly, the case of $\alpha_{\max} > 0$ (or $\alpha_{\max} = 0$) corresponds to the existence (or the failure) of global error bound. Using the characterizations of subderivatives and subdifferentials of G developed in Section 3, we provide an estimate of the largest global error bound constant, which answers the question mentioned in [27, Page 148] in the affirmative.

Theorem 4.6. *If the BCQ holds on X^*, then*

$$\alpha_{\max} = \inf\{dG(x^*)(w)|w \in S(x^*), x^* \in X^*\}, \tag{4.7}$$

where $S(x^*) := \{w \in \mathbb{R}^n|\ w \in T_\Omega(x^*) \cap N_{X^*}(x^*), \|w\| = 1\}$.

Proof. We first consider the case when $\alpha_{\max} > 0$; i.e., G has an error bound on X. As evident from the proof of Lemma 4.4, (2.2) is equivalent to the validity of (4.5), from which and the positive homogeneity of $dG(x^*)$ [20], the formulas claimed for $\alpha_{\max}$ follows. We now show that $\alpha_{\max} = 0$ is equivalent to

$$\inf\{dG(x^*)(w)|\ w \in S(x^*), x^* \in X^*\} = 0.$$

First, we have

$$\inf\{dG(x^*)(w)|w \in S(x^*), x^* \in X^*\} \le 0. \tag{4.8}$$

Indeed, if

$$\inf\{dG(x^*)(w)|w \in S(x^*), x^* \in X^*\} > 0,$$

then (4.5) holds, which ensures the validity of (4.6) (and hence (4.2)) by the argument given in Theorem 4.4. Hence G has a global error bound, i.e., $\alpha_{\max} > 0$.

To obtain the reverse inequality, consider an arbitrary $x^* \in X^*$. Note that problem of minimizing $G(x)$ over X is equivalent to the problem of minimizing $G(x) + \delta(x|X)$ over $\mathbb{R}^n$. The optimality of x^* implies $0 \in \partial\big(G + \delta(\cdot|X)\big)(x^*)$. Since the BCQ holds at x^*, we get $0 \in \partial G(x^*) + N_X(x^*)$ by [19, Theorem 23.8]. This further implies that

$$0 \le \sigma^*(w|\partial G(x^*) + N_X(x^*)) = \sigma^*(w|\partial G(x^*)) + \delta(w|T_X(x^*)) = dG(x^*)(w), \quad \forall w \in T_X(x^*).$$

Since $T_\Omega(x^*) \bigcap N_{X^*}(x^*) \subseteq T_\Omega(x^*) \subseteq T_X(x^*)$, the above inequality implies that

$$\inf\{dG(x^*)(w)|w \in S(x^*), x^* \in X^*\} \ge 0.$$

This, together with (4.8), establishes the equation as claimed. □

Corollary 4.7. *Let x^* be the unique solution of VIP, i.e., $X^* = \{x^*\}$. If BCQ holds at x^* and $\theta(x^*, \delta)$ is bounded for some $\delta < G(x^*)$, then*

$$\alpha_{\max} = \min\{\sigma^*\big(w|F(\Lambda(x^*))\big)|\ w \in T_\Omega(x^*),\ \|w\| = 1\},$$

where $F(\Lambda(x^*)) := \{F(x)|\ x \in \Lambda(x^*)\}$.

Proof. Since $X^* = \{x^*\}$, then $N_{X^*}(x^*) = \mathbb{R}^n$. Hence $S(x^*)$ given in (4.7) coincides with $\{w \in T_\Omega(x^*)|\|w\| = 1\}$. Since $\Omega = X \cap \mathrm{dom}G$, then $T_\Omega(x^*) \subset T_{\mathrm{dom}G}(x^*)$. Hence $dG(x^*)(w) = \sigma^*\big(w|F(\Lambda(x^*))\big)$ for all $w \in T_\Omega(x^*)$ by Corollary 3.5. Plugging these facts in (4.7) yields

$$\alpha_{\max} = \inf\{\sigma^*\big(w|F(\Lambda(x^*))\big)|\ w \in T_\Omega(x^*),\ \|w\| = 1\}.$$

Finally, note that the "infimum" can be replaced by "min", due to the compactness of $\{w \in T_\Omega(x^*)|\|w\| = 1\}$ and the lower semicontinuity of the support function [19]. This completes the proof. □

Lemma 4.8. *Let A, B be two subsets in $\mathbb{R}^n$. If $A \subseteq B \subseteq \{\lambda A | \lambda > 0\}$, then $A^\circ = B^\circ$.*

Proof. It is clear that

$$A^\circ \supseteq B^\circ \supseteq \{\lambda A | \lambda > 0\}^\circ. \tag{4.9}$$

If $v \in A^\circ$, then $\langle v, a\rangle \leq 0$ for all $a \in A$. For any $z \in \{\lambda A | \lambda > 0\}$, i.e., $z = \lambda_0 a_0$ for some $\lambda_0 > 0$ and $a_0 \in A$, we have $\langle v, z\rangle = \langle v, \lambda_0 a_0\rangle = \lambda_0 \langle v, a_0\rangle \leq 0$, i.e., $v \in \{\lambda A | \lambda > 0\}^\circ$. Hence $A^\circ \subseteq \{\lambda A | \lambda > 0\}^\circ$. Combining this with (4.9) yields the desired result. □

We now show the equivalence among global error bound, weak sharpness, MPS property regularity, and linear regularity, provided that G is differentiable. The relationship among global error bound, weak sharpness, and MPS property regularity has been studied by many authors under various assumptions; for example, in [15, Theorems 4.1 and 4.2], F is pseudo-monotone$^+$ and X is compact polyhedral; in [23, Theorem 5.3] F is pseudo-monotone$^+$ and X is polyhedral. These results were extended in [27, Theorems 4.4 and 4.7], where the pseudo-monotonicity$^+$ was replaced by pseudo-monotonicity. Here we further show that, under the Slater condition, these properties are equivalent to linear regularity as well. First, we note that because G is convex, then the concept of Gâteaux differentiability coincides with Fréchet differentiability in finite dimensional spaces. Indeed, by definition the Gâteaux differentiability means that the directional derivative $G'(x, \cdot)$ is linear. This implies that G is Fréchet differentiable at x by [19, Theorem 25.2].

Lemma 4.9. *Let F be continuous and pseudo-monotone on X. If G is differentiable on X^*, then F is constant on X^*, i.e., there exists $F^* \in \mathbb{R}^n$ such that $F(x) = F^*$ for all $x \in X^*$, and $\nabla G(x) = F^*$ for all $x \in X^*$.*

Proof. See [24, Theorem 2.3] and [27, Proposition 3.13]. □

Theorem 4.10. *Let F be pseudo-monotone and continuous on the polyhedral set X. Suppose that G is differentiable on X^*. Then the following statements are equivalent:*

(a) *X^* is weakly sharp;*

(b) VIP *possesses the* MPS *property;*

(c) *G has a global error bound on X.*

Moreover, if the Slater condition holds, i.e., there exists $x_0 \in \mathbb{R}^n$ such that $G(x_0) < 0$, the above statements are further equivalent to

(d) *the pair $\{X, \mathrm{lev}_{\leq 0} G\}$ is linearly regular over X.*

Proof. The equivalence among (a), (b), and (c) can be found in [27, Theorems 4.4 and 4.7]. Now let us show that (c) is equivalent to (d) under the Slater condition. Since G is (Fréchet) differentiable at $x^* \in X^*$, then $x^* \in \mathrm{int}(\mathrm{dom} G)$. So the BCQ holds at x^*, due to $\partial^\infty G(x^*) = N_{\mathrm{dom} G}(x^*) = \{0\}$.

Note that for a closed set A we always have $\partial \mathrm{dist}(x, A) = N_A(x) \cap \mathbb{B}$ by [20, Example 8.53]. The global error bound of G over X implies the existence of a positive scalar α_1 such that

$$\begin{aligned}
& \alpha_1 \mathrm{dist}(x, X^*) \leq G(x), && \forall x \in X,\\
\Leftrightarrow\ & \alpha_1 \mathrm{dist}(x, X^*) \leq G(x) + \delta_X(x), && \forall x \in \mathbb{R}^n,\\
\Leftrightarrow\ & \alpha_1 \mathbb{B} \cap N_{X^*}(x) \subseteq \partial(G(x) + \delta_X(x)), && \forall x \in X^*,\\
\Leftrightarrow\ & \alpha_1 \mathbb{B} \cap N_{X^*}(x) \subseteq \partial G(x) + N_X(x), && \forall x \in X^*,\\
\Leftrightarrow\ & \alpha_1 \mathbb{B} \cap N_{X^*}(x) \subseteq F(x) + N_X(x), && \forall x \in X^*\\
\Leftrightarrow\ & \alpha_1 \mathbb{B} \cap N_{X^*}(x) \subseteq F^* + N_X(x), && \forall x \in X^*,
\end{aligned} \tag{4.10}$$

where the second equivalence follows from [6, Theorem 2.3], the third equivalence from the subdifferential calculus rules given in [20, Corollary 10.9] (since BCQ holds on X^* as shown above), the forth equivalence from the fact $\partial G(x) = \{F(x)\}$ for all $x \in X^*$ (since G is differential on X^*), and the last equivalence from the fact that F is constant over X^* by Lemma 4.9.

The linear regularity of the pair $\{X, \mathrm{lev}_{\le 0}G\}$ over X implies the existence of a positive scalar α_2 such that

$$\begin{aligned} & \alpha_2 \mathrm{dist}(x, X^*) \le \mathrm{dist}(x, \mathrm{lev}_{\le 0}G), && \forall x \in X, \\ \Leftrightarrow\quad & \alpha_2 \mathrm{dist}(x, X^*) \le \mathrm{dist}(x, \mathrm{lev}_{\le 0}G) + \delta_X(x), && \forall x \in \mathbb{R}^n, \\ \Leftrightarrow\quad & \alpha_2 \mathbb{B} \cap N_{X^*}(x) \subseteq \partial\big(\mathrm{dist}(x, \mathrm{lev}_{\le 0}G) + \delta_X(x)\big), && \forall x \in X^*, \\ \Leftrightarrow\quad & \alpha_2 \mathbb{B} \cap N_{X^*}(x) \subseteq \mathbb{B} \cap N_{\mathrm{lev}_{\le 0}\mathrm{G}}(x) + N_X(x), && \forall x \in X^*, \end{aligned} \tag{4.11}$$

where the second equivalence comes from [6, Theorem 2.3], the last equivalence follows from [20, Corollary 10.9] (since $\mathrm{dom}\big(\mathrm{dist}(\cdot, \mathrm{lev}_{\le 0}\mathrm{G})\big) = \mathbb{R}^n$).

Under the Slater condition, we now claim that

$$N_{\mathrm{lev}_{\le 0}\mathrm{G}}(x) = \mathrm{cone} F^* := \{\lambda F^* | \lambda \ge 0\},\ \forall x \in X^*.$$

Let

$$\Gamma := \{v | \langle v, z - x\rangle \le 0,\ \forall z \text{ with } G(z) < 0\}.$$

First, note that

$$N_{\mathrm{lev}_{\le 0}\mathrm{G}}(x) = \{v | \langle v, z - x\rangle \le 0,\ \forall z \text{ with } G(z) \le 0\} = \Gamma,$$

where the first equality is due to the definition of normal cone. It is clear that $N_{\mathrm{lev}_{\le 0}\mathrm{G}}(x) \subseteq \Gamma$. Conversely, choose $v \in \Gamma$. If z satisfies $G(z) \le 0$, then $G(\lambda z + (1-\lambda)x_0) \le \lambda G(z) + (1-\lambda)G(x_0) < 0$ for $\lambda \in (0, 1)$, since $G(x_0) < 0$ by hypothesis. Hence $\langle v, \lambda z + (1-\lambda)x_0 - x\rangle \le 0$. Taking the limits as $\lambda \to 1^-$ yields $\langle v, z - x\rangle \le 0$. Hence $v \in N_{\mathrm{lev}_{\le 0}\mathrm{G}}(x)$.

Second, note that

$$\{z - x |\ z \text{ with } G(z) < 0\} \subseteq \{w |\ G'(x; w) < 0\} \subseteq \bigcup_{\lambda > 0} \lambda \{z - x |\ z \text{ with } G(z) < 0\}.$$

In fact, if z satisfies $G(z) < 0$, then

$$G'(x; z - x) = \inf_{t > 0} \frac{G(x + t(z - x)) - G(x)}{t} \le G(x + z - x) - G(x) = G(z) < 0,$$

where the first equality is due to [19, Theorem 23.1], the first inequality holds by taking $t = 1$, and the second equality follows from the fact $G(x) = 0$ since $x \in X^*$ as required above. Similarly, if w satisfies $G'(x; w) < 0$, i.e.,

$$0 > G'(x; w) = \inf_{t > 0} \frac{G(x + tw) - G(x)}{t} = \inf_{t > 0} \frac{G(x + tw)}{t},$$

then there exists $t_0 > 0$ such that $0 > G(x + t_0 w)/t_0$, which further implies $0 > G(x + t_0 w)$. Let $z = x + t_0 w$. Then $w = (z - x)/t_0 \in \cup_{\lambda > 0} \lambda \{z - x |\ G(z) < 0\}$.

Hence according to Lemma 4.8 that

$$\{z - x |\ z \text{ with } G(z) < 0\}^\circ = \{w |\ G'(x; w) < 0\}^\circ.$$

Therefore, similar to [7] we have

$$\begin{aligned}
N_{\mathrm{lev}_{\leq 0}\mathrm{G}}(x) &= \{v|\langle v, z-x\rangle \leq 0, \quad \forall z \text{ with } G(z)<0\} \\
&= \{z-x|\ z \text{ with } G(z)<0\}^\circ \\
&= \{w|\ G'(x;w)<0\}^\circ \\
&= (\mathrm{cl}\{w|G'(x;w)<0\})^\circ \\
&= \{w|dG(x)w \leq 0\}^\circ \\
&= \{w|\langle F^*, w\rangle \leq 0\}^\circ \\
&= ((F^*)^\circ)^\circ \\
&= \mathrm{cone}(F^*),
\end{aligned}$$

where the fourth equality is due to the fact $(\mathrm{cl}K)^\circ = K^\circ$ for any convex set K (see [19, Page 121], the fifth equality comes from Lemma 3.2 and the fact $\mathrm{cl}\{x|f(x) \leq \alpha\} = \{x|(\mathrm{cl}f)(x) \leq \alpha\}$ when $\alpha > \inf f$ (see [19, Theorem 7.6]), the sixth equality follows from the fact $dG(x)(w) = \nabla G(x)^T w = F(x)^T w = (F^*)^T w$ by Lemma 4.9, and the last step from [20, Corollary 6.21].

Combining this and (4.11) yields

$$\alpha_2 \mathbb{B} \cap N_{X^*}(x) \subseteq \mathbb{B} \cap \mathrm{cone}(F^*) + N_X(x), \quad \forall x \in X^*. \tag{4.12}$$

To complete the proof, we need to show the equivalence between (4.10) and (4.12). Note that the Slater condition means that for the unconstrained problem of minimizing G over $\mathbb{R}^n$, the optimal value is negative and is not achieved at any point in X^*. Thus, $0 \notin \partial G(x)$ for each $x \in X^*$, which in turn implies that $F^* \neq 0$. Therefore, all that we need to show is the implication (4.12) $\Rightarrow$ (4.10), since the converse part holds trivially by taking $\alpha_2 = \frac{\alpha_1}{\|F^*\|}$. Now choosing $z \in \alpha_2\mathbb{B} \cap N_{X^*}(x)$, (4.12) implies the existence of $z_1 = \lambda F^*$ for some $\lambda \in [0, 1/\|F^*\|]$ and $z_2 \in N_X(x)$ such that $z = z_1 + z_2$. According to the optimality of $x \in X^*$ for the convex programming problem of minimizing $G(x)$ over X implies that $0 \in \partial G(x) + N_X(x) = \nabla G(x) + N_X(x) = F^* + N_X(x)$, i.e., $-F^* \in N_X(x)$. Therefore, $z = \lambda F^* + z_2 = F^* + (1-\lambda)(-F^*) + z_2$, which further implies that $z \in F^* + N_X(x)$, since $N_X(x) + N_X(x) = N_X(x)$ by [19, Theorem 2.6]. This completes the proof. □

Refined results can be obtained for the affine variational inequalities (AVI), which corresponds to the VIP with F being affine and X being polyhedral. More precisely, the affine variational inequalities is to find $x^* \in X$ such that

$$\langle Mx^* + q, x - x^*\rangle \geq 0, \quad \forall x \in X,$$

where $q \in \mathbb{R}^n$, $M \in \mathbb{R}^{n\times n}$, and X is a polyhedral set.

Theorem 4.11. *If M is a positive definite matrix, VIP has a unique solution, say x^*, then the following statements are equivalent:*

(a) *X^* is weakly sharp;*

(b) VIP *possesses the* MPS *property;*

(c) *G has a global error bound on X;*

(d) *x^* is a nondegenerate solution, i.e., $-(Mx^* + q) \in \mathrm{int}N_X(x^*)$;*

(e) *there is a positive scalar* γ *such that*

$$g(x) \geq \gamma \operatorname{dist}(x, X^*) = \gamma \|x - x^*\|, \quad x \in X,$$

where $g(x) := \langle Mx + q, x\rangle - \inf\{\langle Mx + q, y\rangle |\ y \in X\}$ *is the primal gap function.*

Moreover, if there exists some $x_0 \in \mathbb{R}^n$ *such that*

$$M^\top x_0 - q \in X^\circ \quad \text{and} \quad \langle x_0, q\rangle < 0, \tag{4.13}$$

then the above statements are further equivalent to

(f) *the pair* $\{X, \operatorname{lev}_{\leq 0} G\}$ *is linearly regular over* X.

Proof. We first show that G is (Fréchet) differentiable and X^* is a single-point set. For any given $x \in \mathbb{R}^n$, let us consider the following problem:

$$\min_{y \in X} \ \langle My + q, y - x\rangle. \tag{4.14}$$

Since M is positive definite, this is a convex quadratic programming whose objective function is strongly convex in y. Thus, the problem (4.14) has a unique solution, i.e., the set $\Lambda(x)$ is a single-point set for each $x \in \mathbb{R}^n$. The nonemptiness of $\Lambda(x)$ for each $x \in \mathbb{R}^n$ implies that $\operatorname{dom} G = \mathbb{R}^n$, whereas the uniqueness of $\Lambda(x)$ implies the boundedness of the set $\theta(x, \delta)$ for any $x \in \mathbb{R}^n$ and $\delta \in \mathbb{R}$ by [19, Corollary 8.7.1]. According to Corollary 3.8, we get that ∂G is single-point set, which in turn implies that G is differentiable (see [19, Theorem 25.1]). The uniqueness of X^* follows from the fact $X^* \subseteq \Lambda(x^*)$ for each $x^* \in X^*$ by [23, Proposition 2.3], since $\Lambda(x^*)$ is a single-point set as shown above. Because M is positive definite, then $F(x) = Mx + q$ is pseudo-monotone (in fact is monotone). This together with the differentiability of G, allows us to establish the equivalence between the statements (a), (b), and (c) by applying Theorem 4.10. Due to X^* consisting of a single point, we have $N_{X^*}(x^*) = \mathbb{R}^n$ by definition. Thus, the formula (2.1) of weak sharpness takes the form $-Mx^* - q \in \operatorname{int}\big(T_X(x^*)\big)^\circ = \operatorname{int} N_X(x^*)$. This establishes the equivalence between (a) and (d). The equivalent between (d) and (e) can be found in [9, Theorem 6.4.6].

To complete the proof, according to Theorem 4.10, it suffices to show the validity of the Slater condition. Since $M^\top x_0 - q \in X^\circ$, we have $\langle M^\top x_0 - q, y\rangle \leq 0$ for all $y \in X$. This, together with the positive definiteness of M and the fact $\langle x_0, q\rangle < 0$ in (4.13), implies that for any $y \in X$ we have

$$\langle My + q, x_0 - y\rangle = \langle y, M^T x_0 - q\rangle - \langle My, y\rangle + \langle q, x_0\rangle < 0. \tag{4.15}$$

Hence $G(x_0) = \sup\{\langle My + q, x_0 - y\rangle | y \in X\} = \langle My_0 + q, x_0 - y_0\rangle < 0$, where the maximum can be attained at some y_0 due to positive definiteness of M (see the argument following (4.14)), and the inequality follows from (4.15). This means that the Slater condition holds. Applying Theorem 4.10 yields the desired result. □

At the end of this paper, we need to point out that Theorem 4.10 can be regarded as a complement to [15, 23, 24, 27], where they do not reveal the close relationship between weak sharpness and linear regularity. For AVI, a special case of VIP, we get a series of equivalences under the condition of positive definiteness in Theorem 4.11. In [27, Theorem 4.8], the authors establish the relation $(b) \Longrightarrow (a)$, (c), (d), and (e), when M is positive semi-definite. It would be interesting to know whether these equivalent relations remain true when the positive definiteness is replaced by positive semidefiniteness.

5 Conclusion

The novelty of this paper is twofold. First, we develop some new characterizations of subderivatives and subdifferentials of the dual gap function. An outstanding advantage of our result over the ones proposed in [27] is that the characterization of subdifferentials is presented at any point x in domG, rather than at the point x restricted to ri(dom$G$). Second, we further establish the equivalences of weak sharpness of X^*, minimum principle sufficiency property, global error bound, to linear regularity in the presence of the Slater condition. In addition, an estimate of the largest global error bound constant is provided.

Acknowledgements

We are gratefully indebted to anonymous referees for their valuable suggestions that helped us to essentially improve the original presentation of the paper.

References

[1] G. Auchmuty, Variational principles for variational inequalities, *Numer. Funct. Anal. Optim.* 10 (1989) 863–874.

[2] A. Auslender, *Optimisation, Méthodes numériques*, Masson, Paris, 1976.

[3] D. Aussel, R. Correa and M. Marechal, Gap functions for quasivariational inequalities and generalized Nash equilibrium problems, *J. Optim. Theory and Appl.* 151 (2011) 474–488.

[4] A. Bakan, F. Deutsch and W. Li, Strong CHIP, normality, and linear regularity of convex sets, *Trans. Amer. Math. Soc.* 357 (2005) 3831–3863.

[5] J.F. Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer-Verlag, New York, 2000.

[6] J.V. Burke and S. Deng, Weak sharp minima revisited, part I: basic theory, *Control and Cybernetics* 31 (2002) 439–469.

[7] J.V. Burke and S. Deng, Weak sharp minima revisited, part II: application to linear regularity and error bounds, *Math. Programming* 104 (2005) 235–261.

[8] J.V. Burke and M.C. Ferris, Weak sharp minima in mathematical programming, *SIAM J. Control Optim.* 31 (1993) 1340–1359.

[9] F. Facchinei and J.S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Vol: I, II, Springer Verlag, New York, 2003.

[10] M. Fukushima, Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems, *Math. Programming* 53 (1992) 99–110.

[11] A. Hantoute and M.A. López, A complete characterization of the subdifferential set of the supremum of an arbitrary family of convex functions , *J. Convex Anal.* 15 (2008) 831–858.

[12] N.J. Huang, J. Li and X.Q. Yang, Weak sharpness for gap functions in vector variational inequalities, *J Math. Anal. Appl.* 394 (2012) 449–457.

[13] C. Li, K.F. Ng and T.K. Pong, The SECQ, linear regularity, and the strong CHIP for an infinite system of closed convex sets in normed linear spaces, *SIAM J. Optim.* 18 (2007) 643–665.

[14] S.J. Li and X.W. Xue, Sensitivity analysis of gap functions for vector variational inequality via coderivatives, *Optimization* 63 (2014) 1075–1098.

[15] P. Marcotte and D. Zhu, Weak sharp solutions of variational inequalities, *SIAM J. Optim.* 9 (1998) 179–189.

[16] K.F. Ng and X.Y. Zheng, Global weak sharp minima on Banach spaces, *SIAM J. Control Optim.* 41 (2003) 1868–1885.

[17] M. Patriksson, A unified framework of descent algorithms for nonlinear programs and variational inequalities, Ph.D. thesis, Department of Mathematics, Linköping Institute of Technology, Linköping, Sweden, 1993.

[18] J.M. Peng, Equivalence of variational inequality problems to unconstrained optimization, *Math. Programming* 78 (1997) 347–355.

[19] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.

[20] R.T. Rockafellar and R.J.Wets, *Variational Analysis*, Springer, New York, 1998.

[21] M.V. Solodov and P. Tseng, Some methods based on the D-gap function for solving monotone variational inequalities, *Comput. Optim. Appl.* 17 (2000) 255–277.

[22] J.H. Wu, M. Florian and P. Marcotte, A general descent framework for the monotone variational inequality problem, *Math. Programming* 61 (1993) 281–300.

[23] Z.L. Wu and S.Y. Wu, Weak sharp solutions of variational inequalities in Hilbert spaces, *SIAM J. Optim.* 14 (2004) 1011–1027.

[24] Z.L. Wu and S.Y. Wu, Gâteaux differentiability of the dual gap function of a variational inequality, *Eur. J. Oper. Res.* 190 (2008) 328–344.

[25] Y.D. Xu and S.J. Li, Gap functions and error bounds for weak vector variational inequalities, *Optimization* 63 (2014) 1339–1352.

[26] N. Yamashita, K. Taji and M.Fukushima, Unconstrained optimization reformulations of variational inequality problems, *J. Optim. Theory Appl.* 92 (1997) 439–456.

[27] J.Z. Zhang, C.Y. Wang and N.H. Xiu, The dual gap function for variational inequalities, *Appl. Math. Optim.* 48 (2003) 129–148.

[28] X.Y. Zheng and K.F. Ng, Linear regularity for a collection of subsmooth sets in Banach spaces, *SIAM J. Optim.* 19 (2008) 62–76.

Manuscript received 18 March 2014
revised 24 May 2014
accepted for publication 20 August 2014

Jinchuan Zhou
Department of Mathematics, School of Science
Shandong University of Technology
Zibo 255049, P.R.China
E-mail address: jinchuanzhou@163.com

Naihua Xiu
Department of Mathematics, School of Science
Beijing Jiaotong University, Beijing 100044, P.R.China
E-mail address: nhxiu@bjtu.edu.cn

Soonyi Wu
Department of Mathematics, National Cheng Kung University
Tainan 700, Taiwan
E-mail address: soonyi@mail.ncku.edu.tw

Ziyan Luo
State Key Laboratory of Rail Traffic Control and Safety
Beijing Jiaotong University, Beijing 100044, P.R.China
E-mail address: zyluo@bjtu.edu.cn