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APPLICATIONS OF DETERMINISTIC OPTIMIZATION TECHNIQUES TO SOME PROBABILISTIC CHOICE MODELS FOR PRODUCT PRICING USING RESERVATION PRICES*

R. Shioda, L. Tunçel† and B. Hui

Abstract: We consider revenue management models for pricing a product line with several customer segments, working under the assumption that every customer segment's product choice is determined entirely by their reservation price. We model the customer choice behavior by several probabilistic choice models and formulate the resulting problems as mixed-integer programming problems. We study special properties of these formulations and compare the resulting optimal prices of the different probabilistic choice models. We also explore some heuristics and valid inequalities to improve the running time of the mixed-integer programming problems. We illustrate the computational results of our models on real and generated customer data taken from a company in the tourism sector.

Key words: *mixed integer programming, optimal product pricing, global optimization, probabilistic choice models, second order cone*

Mathematics Subject Classification: *90C11, 90C26, 90C50*

1 Introduction

One of the key revenue management challenges for a company is to determine the "right" price for each of their product line. Generally speaking, a company wants to set the prices to maximize their total profit. The challenge arises from the complex relationship between the product prices and the total profit. For example, how do the prices affect the demand for each product? In cases where multiple products are offered by the company, the price and demand for a product cannot be considered in isolation from the other products. That is, the company must take into account the fact that in addition to competitors' products and prices, a customer's decision to purchase a product can be swayed by the relative prices of similar products offered by the same company. Thus, prices need to be set not only to "beat" the competitors products but also to avoid "cannibalizing" the company's own product line. For example, if there are high margin and low margin products, setting the price of the latter too low may decrease demand for the high margin product, thus resulting in lower profit.

In this paper, we study several different models for product pricing from a mathematical optimization perspective. The models differ from one another according to different assumptions on customer purchasing behavior. Before we discuss the details of our approach, let us first give a brief overview of the relevant work in this area.

†Corresponding Author:

1.1 Background

There are many variations on product pricing models depending on the setting. For example, there is the single-product, multi-customer setting, which is primarily concerned with what price to offer to different customer segments. Airline revenue management is one of the most popular examples in this context, where business travelers, leisure travelers and budget travelers are offered different prices for the same flight, depending on the lead time of purchase and additional options (e.g., partially refundable tickets). An alternative framework is the multi-product, multi-customer setting where every customer is offered the same price for a given product, but different customer segments have varying preferences. This is more of a combinatorial problem where given the customer preference information, the prices need to be set to maximize total revenue. We will focus on the second type of problem in this paper.

In general, suppose a company has m different products and market analysis tells them that there are n distinct customer segments, where customers of the same segment behave the "same". A key revenue management problem is to determine optimal prices for each product to maximize total revenue, given the customer choice behavior. There are multitudes of models for customer choice behavior [27], but this paper focuses solely on those based on reservation prices.

Let R_{ij} denote the *reservation price* of segment i for product j, $i = 1, \ldots, n$, $j = 1, \ldots, m$, which reflects how much customers of segment i are *willing* and *able* to spend on product j. R_{ij} is not only the dollar amount that product j is worth to customers in segment i, but it also reflects how much they are able to pay for it. For example, if a customer segment believes that a 7 day vacation to St. Lucia is worth \$2,000, but they can only afford \$1,000 for a vacation, then their reservation price for St. Lucia is \$1,000. We assume that reservation prices are the same for every customer in a given segment and each segment pays the same price for each product. (Note however that our definition of "product" in this paper is abstract enough that our mathematical models are ale to handle any differential pricing strategy by introducing new "products," see [26].) Customer choice models based on reservation prices assume that customer purchasing behavior can be fully determined by their reservation price and the price of products. Without loss of generality, we make the following assumption:

Assumption 1.1. R_{ij} is a nonnegative integer for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

If the price of product j is set to $\$\pi_j$, $\pi_j \geq 0$, then the *surplus* of segment i for product j is the difference between the reservation price and the price, i.e., $R_{ij} - \pi_j$. It is often assumed that a segment will only consider purchasing a product with nonnegative utility, i.e.,

Assumption 1.2. If segment i buys product j, then $R_{ij} - \pi_j \geq 0$, $i = 1, \ldots, m$, $j = 1, \ldots, m$.

Even in a reservation price framework, there are several different models for customer choice behavior in the literature (see [20] and the references therein). In [6, 7], the authors proposed a pricing model that maximizes profits with the assumption that each customer segment only buys the product with the maximum surplus if the surplus is nonnegative. This model is often referred to as the *maximum utility* or *envy-free pricing* model. In this model, each segment buys at most one product. The authors modeled the problem as a non-convex, nonlinear mixed-integer programming problem and solved the problem using a variety of heuristic approaches.

In [15], the authors examined a Share-of-Surplus Choice Model in which the probability that a segment will choose a product is the ratio of its surplus versus the total surplus for the segment across all products with nonnegative surplus. They proposed a heuristic which involves decomposing the problem into hypercubes and used a simulated annealing algorithm to find the best hypercube. Solutions found by the heuristic for problems with sizes up to 5 products and 10 segments were shown to be near-optimal.

Another approach of pricing multiple products is to consider the problem of bundle pricing [11]. It is the problem of determining whether it is more profitable to offer some of the products together as a package or individually, and what prices should be assigned to the bundles or individual products to maximize profit. The bundle pricing problem in [11] was formulated as a mixed integer linear programming problem using a disjunctive programming technique [3].

Some research has been done on partitioning customers into segments by the probability that they would buy each product. In [12], the authors proposed a segmentation approach that groups the customers according to their reservation prices and price sensitivity. The probability of a segment choosing a product j is modeled as a multinomial logit model with the segment's reservation price, price sensitivity, and the price of the product j as parameters. Unlike their model, we do not consider variances in price sensitivity in this paper as a criterion when we partition customers into segments and we assume that all segments react to price changes in the same way.

In our approach, the company sets the prices but it does not manipulate the offers each customer receives. Therefore, the prices will be the main decision variables in our mathematical models. A parallel stream of research uses choice models to decide which subset of customers should be offered which subset of products as well as the duration of the offer. Let us call the first approach *universal prices approach* and the second one *distinguishing prices approach*.

Distinguishing prices approach has been successfully applied in many settings, in particular, in airline revenue management, see [25, 16]. In the framework of the distinguishing prices approach, another related situation is that of *flexible products*. The customers are offered *flexible products* each of which is a subset of similar products (e.g., the customers are offered the flexible products {product 1, product 2}, {product 3, product 4, product 5} and {product 1, product 6}). Then, when a customer buys a flexible product (say {product 1, product 6}), the customer knows that s/he will end up with one of the products in the subset for her/his choice (either product 1 or product 6); however, the company is the one who decides which specific product from the subset chosen by the customer will be assigned to that customer (either product 1 or product 6). See, [10, 9].

An important avenue of research in revenue management is concerned with dynamic pricing situations (see, [4, 1, 2]). In this paper, we focus on the static version of the problem. In practice, static models may be used in dynamic situations, provided reoptimization of the prices can be done quickly (this was shown to be so for the maximum utility models using reservation prices in [23]).

There are many other aspects of customer choices that can influence the prices. For instance, a number of firms competing based on their local inventory levels may create an environment where the demand shifts dramatically from one company to the other almost purely based on the local availability of the products and hence the local inventory levels (see [17]).

Many optimization models and techniques have been effectively utilized in the area of revenue management. Classical approaches have used dynamic programming models and their reformulations by very large scale linear programming problems. Also used are Markov decision process formulations and their affine approximations (see [28]). Mixed-integer bilevel programs are utilized in [5] to attack the joint network design and optimal pricing problems.

In this paper, we assume that the reservation prices for each customer segment and product are given. Given different models of customer purchasing behavior, we aim to formulate and solve the corresponding revenue maximization problem as a mixed-integer programming problem. Note that we take the customer choice (behavior) models (e.g., Share-of-Surplus, Price Sensitive) as *axioms* in the corresponding sections of this paper. The axioms (customer choice models) are probabilistic in nature (hence, the phrase *Probabilistic Choice Models* in the title) as they define the probability that a customer segment will buy a particular product (or what fraction of a group of customers in a customer segment will buy a particular product). However, our main focus is in solving the underlying mathematical problem (i.e., computing optimal or near-optimal prices maximizing the expected revenue or profit). Our mathematical optimization formulations of the underlying problem are deterministic and exact (not an approximation to a stochastic problem). For a discussion of various axioms related to the axioms of this paper, see [26]. In the Appendix, we discuss how we performed the customer segmentation and estimated the mean values for the reservation prices from customer purchase orders of a Canadian company in the tourism sector.

1.2 Probabilistic Choice Models

In this section, we introduce the general framework of probabilistic customer choice models that determines the probability that customer segment i will purchase product j, $i = 1, \ldots, n$, $j = 1, \ldots, m$. Let β_{ij} be binary decision variables where

$$\beta_{ij} := \begin{cases} 1, & \text{if the surplus of product } j \text{ is nonnegative for segment } i; \\ 0, & \text{otherwise.} \end{cases}$$

That is, $\beta_{ij} = 1$ if and only if $R_{ij} - \pi_j \geq 0$ and $\beta_{ij} = 0$ if and only if $R_{ij} - \pi_j < 0$, where, again, π_j is the decision variable for the price of product j. This relationship can be modeled by:

$$\begin{aligned} (R_{ij} - \pi_j)\beta_{ij} &\geq 0, \\ (R_{ij} - \pi_j)(1 - \beta_{ij}) &\leq 0, \\ (R_{ij} - \pi_j + 1) &\leq (R_{ij} - \min_i \{R_{ij}\} + 1)\, \beta_{ij}, \end{aligned}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ (the third inequality is valid under Assumption 1.1). To linearize the above inequalities, we can use a disjunctive programming technique. Let p_{ij} be an auxiliary variable where $p_{ij} = \pi_j \beta_{ij}$, i.e,

$$p_{ij} := \begin{cases} \pi_j, & \text{if } \beta_{ij} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

This relationship can be modeled by the following set of linear inequalities (together with the restriction that $\beta_{ij} \in \{0, 1\}$, for every i, j):

$$\begin{aligned} p_{ij} &\geq 0, \\ p_{ij} &\leq \pi_j, \\ p_{ij} &\leq R_{ij}\beta_{ij}, \\ p_{ij} &\geq \pi_j - (\max_{i=1,\ldots,n} R_{ij} + 1)(1 - \beta_{ij}), \end{aligned}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. The first and the third inequalities set $p_{ij} = 0$ when $\beta_{ij} = 0$; the second and the fourth inequalities set $p_{ij} = \pi_j$ when $\beta_{ij} = 1$. R_{ij} is a valid upper-bound for p_{ij} since if $p_{ij} > R_{ij}$, then $\beta_{ij} = 0$ and thus $p_{ij} = 0$. Also, $\max_{i=1,\ldots,n} \{R_{ij}\}+1$ is a valid upper-bound for π_j since no segment will buy product j if $\pi_j > R_{ij}$ for all $i = 1, \ldots, n$.

Here $\boldsymbol{\pi}, \boldsymbol{\beta}$, and $\boldsymbol{p}$ are vectors of π_j, β_{ij} and p_{ij}, respectively; let P be the following polyhedron:

$$
\begin{aligned}
P := \{(\boldsymbol{\pi}, \boldsymbol{\beta}, \boldsymbol{p}) : \quad & R_{ij}\beta_{ij} - p_{ij} \geq 0, && i = 1, \ldots, n, j = 1, \ldots, m, \\
& R_{ij}(1 - \beta_{ij}) - \pi_j \leq 0, && i = 1, \ldots, n, j = 1, \ldots, m, \\
& p_{ij} \leq \pi_j, && i = 1, \ldots, n, j = 1, \ldots, m, \\
& p_{ij} \geq \pi_j - \left(\max_{i=1,\ldots,n} \{R_{ij}\} + 1 \right)(1 - \beta_{ij}), && i = 1, \ldots, n, j = 1, \ldots, m, \\
& R_{ij} - \pi_j + 1 \leq \left(R_{ij} - \min_{i=1,\ldots,n} \{R_{ij}\} + 1 \right) \beta_{ij}, && i = 1, \ldots, n, j = 1, \ldots, m, \\
& p_{ij} \geq 0, \pi_j \geq 0, && i = 1, \ldots, n, j = 1, \ldots, m\}.
\end{aligned} \tag{1.1}
$$

Thus, to model the condition in Assumption 1.2, we need to set prices π_j and β_{ij} such that $\beta \in \{0, 1\}^{n \times m}$ and $(\boldsymbol{\pi}, \boldsymbol{\beta}, \boldsymbol{p}) \in P$.

There are ambiguities regarding the choices between multiple products with nonnegative utility. Given all the products with nonnegative surplus, which products would the customer buy? Are there some products they are more likely to buy than others? In a probabilistic choice framework, we need to determine the probability Pr_{ij} that segment i buys product j. Let N_i be the number of customers in segment i. Then the expected revenue for the company is

$$\sum_{i=1}^{n} N_i E[\text{revenue earned from segment } i] = \sum_{i=1}^{n} N_i \sum_{j=1}^{m} \pi_j Pr_{ij}.$$

In our revenue management problem, we can interpret Pr_{ij} as the fraction of customers of segment i that buys product j, i.e., the expected revenue is

$$\sum_{j=1}^{m} \pi_j E[\text{number of customers in segment } i \text{ that buys product } j] = \sum_{j=1}^{m} \pi_j \sum_{i=1}^{n} N_i Pr_{ij}.$$

Furthermore, let Pr_{ij} be positive if and only if the surplus of product j is nonnegative for segment i.

Thus, the expected revenue maximization problem is:

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} \sum_{j=1}^{m} N_i \pi_j Pr_{ij}, \\
\text{s.t.} \quad & Pr_{ij} > 0 \Leftrightarrow \beta_{ij} = 1, \quad i = 1, \ldots, n; j = 1, \ldots, m, \\
& Pr_{ij} = 0 \Leftrightarrow \beta_{ij} = 0, \quad i = 1, \ldots, n; j = 1, \ldots, m, \\
& (\boldsymbol{\pi}, \boldsymbol{\beta}, \boldsymbol{p}) \in P, \\
& \beta_{ij} \in \{0, 1\}, \quad i = 1, \ldots, n; j = 1, \ldots, m.
\end{aligned} \tag{1.2}
$$

All the probabilistic choice models explored in this paper are based on the optimization problem (1.2). What differentiates the models is how Pr_{ij} is defined.

One of the most popular probabilistic choice models in the marketing literature may be the *multinomial logit* (MNL) model,

$$Pr_{ij} := \frac{e^{v_{ij}}}{\sum_{k=1}^{m} e^{v_{ik}}},$$

where v_{ij} represent the utility or desirability of the product j to segment i. Clearly, there are wide variations in how this v_{ij} is modeled as well. Note that in the MNL model, due to the use of the exponential function, v_{ij} may take on any real value. For example, an alternative model is to have

$$Pr_{ij} := \frac{v_{ij}}{\sum_k v_{ik}},$$

but we would then require $v_{ij} \geq 0$ and $\sum_k v_{ik} > 0$, which could be easily addressed in many cases. One of the major differences in our work is that we model optimal pricing problem for big-ticket items (just like the dichotomy between convex optimization and integer programming) we are not able to assume that the buy/do-not-buy decisions are treatable by continuous variables. Indeed, in the markets or decision making situations when the volume of sales are so high one can define the decision variables using the inverse market share variable as compared to the price variables, and generate tractable, convex optimization problems; see for example [24] and [8] and the references therein. Using the reservation price approach, Schön [21] extended such models to Bradley-Terry-Luce model; however, these models still do not resolve the discreteness of the 0,1 decision making situations.

Main contributions of this paper are: new mathematical models of customer behavior for big-ticket items (e.g., buying cars, trucks, airplanes or vacation packages), new MIP formulations, new MIP-SOCP (Second Order Cone Programming) formulations, some of their mathematical properties, heuristics and valid inequalities, algorithm to extract reservation prices from raw, purchase order data. In this paper, we examine several probabilistic choice models from a mathematical programming perspective. Depending on how Pr_{ij} is modeled, we can formulate the optimization problem (1.2) as a convex mixed-integer programming problem (MIP). We present one of the simplest models of our paper, the *Weighted Uniform Model* in Section 2. In this model, customers are more likely to purchase products with higher reservation prices. We call a special case of it *Uniform Model* (purely of academic interest). In Section 3, we explore mathematical optimization formulations of the Share-of-Surplus Model proposed in [15], including an MIP formulation for the case with restricted prices. Section 4 explores the *Price Sensitive Model* where Pr_{ij} decreases as the price of product j increases. We then discuss properties of the optimal solutions on particular data sets (Section 5) and compare the optimal prices π_j and variables β_{ij} of the different models (Section 5.1). We also consider enhancements to the models, including heuristics to determine good feasible solutions quickly (Section 6) and valid inequalities to speed up the solution time of the MIPs (Section 7). In Section 8, we show how we can incorporate product capacity limits and product costs into the models. We illustrate some computational results of our models in Section 9 and conclude and discuss future work in Section 10. Our main focus is on the *Share-of-Surplus Model* and the *Price Sensitive Model*. For the former model, we propose a modification and provide a new formulation. To the best of our knowledge, the *Price Sensitive Model* although very intuitive is new, the novel part being the mathematical formulation. Among other new ideas, we also utilize second order cone constraints (and related solvers) in our convex relaxations and branch-and-bound algorithms. We introduce these mathematical models slowly, building up towards more sophisticated (and more realistic) models from more elementary (but much less applicable) *Weighted Uniform Model*. We consider these simpler models for theoretical and computational reasons only. These models by no means correspond to estimating the consumer behavior. We utilize them for approximating optimal prices (for the more realistic *Share-of-Surplus* and the *Price Sensitive Models*) and for improving our understanding of the underlying mathematical structures in the more realistic models.

Note that proofs for all theorems and lemmas are in Appendix A.

1.3 Notation

Following are common parameters and notation used throughout the paper:

n	number of segments,
m	number of products,
N_i	size of segment i,
R_{ij}	reservation price of segment i for product j,
$\overline{R}_j$	$:= \max_i\{R_{ij}\}$,
$\underline{R}_j$	$:= \min_i\{R_{ij}\}$,
$\widetilde{R}_i$	$:= \max_j\{R_{ij}\}$,
π_j	price of product j (decision variable),
β_{ij}	equals 1 iff $R_{ij} - \pi_j \geq 0$, equals 0 otherwise (decision variable),
P	polyhedron (1.1).

2 Uniform and Weighted Uniform Models

In this section, we introduce the *Weighted Uniform Model* which is inspired by the multinomial-logit (MNL) model discussed in Section 1.2. We let the utilities v_{ij} be represented by the reservations prices R_{ij}, but only consider products with nonnegative surplus. Let

$$Pr_{ij} := \begin{cases} 0, & \text{if } \sum_{j=1}^{m} R_{ij}\beta_{ij} = 0, \\ \dfrac{u(R_{ij})\beta_{ij}}{\sum_{k=1}^{m} u(R_{ik})\beta_{ik}}, & \text{otherwise,} \end{cases}$$

where $u(\cdot)$ is a monotone nondecreasing function of R_{ij}. Thus, with this definition of Pr_{ij}, out of all products with nonnegative surplus, a customer is more likely to buy a product with higher reservation price. In the marketing literature, $u(x) = \exp(x)$ is a common choice for the MNL model since $u(x) > 0$ for all $x \in \mathbb{R}$. However, since from Assumption 1.1 $R_{ij} \geq 0, \forall i, j$, we may define $u(x) := x$, i.e.,

$$Pr_{ij} := \frac{R_{ij}\beta_{ij}}{\sum_{k=1}^{m} R_{ik}\beta_{ik}}, \quad \text{if } \sum_{j=1}^{m} R_{ij}\beta_{ij} \geq 1.$$

This leads to our *Weighted Uniform Model*, we name the model arising from the trivial constant function $u(x) := 1$, the *Uniform Model*.

2.1 The Formulation

The corresponding expected revenue maximizing problem is

$$
\begin{array}{lll}
\max & \displaystyle\sum_{i=1}^{n} N_i t_i, & \\
\text{s.t.} & \sum_{j=1}^{m} R_{ij} a_{ij} \le \sum_{j=1}^{m} R_{ij} p_{ij}, & \forall i, \\
& t_i \le \widetilde{R}_i \sum_{j=1}^{m} R_{ij} \beta_{ij}, & \forall i, \\
& a_{ij} \le \widetilde{R}_i \beta_{ij}, & \forall i, \forall j, \\
& a_{ij} \le t_i, & \forall i, \forall j, \\
& a_{ij} \ge t_i - \widetilde{R}_i (1 - \beta_{ij}), & \forall i, \forall j, \\
& (\boldsymbol{p}, \boldsymbol{\pi}, \boldsymbol{\beta}) \in P, & \\
& \beta_{ij} \in \{0, 1\}, & \forall i, j.
\end{array}
\tag{2.1}
$$

2.2 Alternative Formulation

The Weighted Uniform Model has an alternative formulation using the binary variables x_{ij}:

$$
\begin{array}{lll}
\max & \displaystyle\sum_{i=1}^{n} N_i t_i, & \\
\text{s.t.} & \sum_{i=0}^{n} x_{ij} = 1, & \forall j, \\
& \sum_{j=1}^{m} R_{ij} a_{ij} \le \sum_{j=1}^{m} \sum_{l: R_{lj} \le R_{ij}} R_{ij} R_{lj} x_{lj}, & \forall i, \\
& a_{ij} \le t_i, & \forall i, j, \\
& a_{ij} \le \widetilde{R}_i \sum_{l: R_{lj} \le R_{ij}} x_{lj}, & \forall i, j, \\
& a_{ij} \ge t_i - \widetilde{R}_i \sum_{l: R_{lj} > R_{ij}} x_{lj}, & \forall i, j, \\
& t_i \le \widetilde{R}_i \sum_{j=1}^{m} \sum_{l: R_{lj} \le R_{ij}} x_{lj}, & \forall i, \\
& t_i \ge 0, & \forall l, i, \\
& a_{ij} \ge 0, & \forall i, j, \\
& x_{ij} \in \{0, 1\}, & \forall i, j.
\end{array}
\tag{2.2}
$$

This alternative formulation results in a stronger integer programming formulation (even in the special case of the Uniform Model). Section 9 illustrates the running time of the Weighted Uniform Model (2.2) on problem instances of various sizes.

Theorem 2.1. *Let π_j^*, $j = 1, \ldots, m$, be the optimal prices of the Weigthed Uniform Model above. Then, for every product k that is bought, π_k^* equals R_{ik} for some $i = 1, \ldots, n$. In particular, let the vectors $\boldsymbol{\pi}^*$ and $\boldsymbol{\beta}^*$ be optimal for Problem (2.2). If $\sum_{i=1}^{n} \beta_{ik} \ge 1$, then*

$$\pi_k^* = \min_{i: \beta_{ik}^* = 1} \{R_{ik}\}.$$

If $\sum_{i=1}^{n} \beta_{ik} = 0$, then $\pi_k^* = \overline{R}_k + 1$.

3 Share-of-Surplus Model

It seems realistic to assume that the probability of a customer buying a product is related to the surplus. A similar scenario is when a customer prefers buying the product that has the most discount at the moment, rather than picking a product randomly or preferring the product with the highest reservation price. We want a model where higher the surplus, higher the fraction of the segment that buys that product. That is, the probability that a customer buys a product depends on the customer's reservation price as well as the price of the product. A monotone nondecreasing function is needed to describe the relationship between the probability and the surplus. The *Share-of-Surplus Choice Model* [15] is a form of a probabilistic choice model where the probability that a segment will choose a product is the ratio of its surplus versus the total surplus for the segment across all products with nonnegative surplus.

3.1 The Formulation

In this model, the probability that segment i buys product j is given by:

$$Pr_{ij} := \frac{(R_{ij} - \pi_j)\beta_{ij}}{\sum_k (R_{ik} - \pi_k)\beta_{ik}}.$$

For the moment, let us assume that $\sum_k (R_{ik} - \pi_k)\beta_{ik} > 0$ for all $i = 1, \ldots, n$ for notational simplicity. We will relax this assumption in Section 3.2. With the above definition, $Pr_{ij} = 0$ if $R_{ij} = \pi_j$, which may not be desirable. To ensure that the probability Pr_{ij} is strictly positive when $R_{ij} = \pi_j$, we may define the probability as follows:

$$Pr^*_{ij} := \frac{(R_{ij} - \pi_j + \eta)\beta_{ij}}{\sum_k (R_{ik} - \pi_k + \eta)\beta_{ik}}, \tag{3.1}$$

where η is a small positive constant. For the sake of simplicity of presentation, we will use the first definition of the probability throughout the rest of this section. Note that this differs from the standard MNL model since we do not consider negative surplus products. The expected revenue given by this model is

$$\sum_{i=1}^{n} \sum_{j=1}^{m} N_i \pi_j \left(\frac{(R_{ij} - \pi_j)\beta_{ij}}{\sum_k (R_{ik} - \pi_k)\beta_{ik}} \right).$$

We can model this Share-of-Surplus Choice Model as the following nonlinear mixed-integer programming model:

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} \sum_{j=1}^{m} N_i \pi_j \left(\frac{(R_{ij} - \pi_j)\beta_{ij}}{\sum_k (R_{ik} - \pi_k)\beta_{ik}} \right) \\ \text{s.t.} \quad & (\boldsymbol{\pi}, \boldsymbol{\beta}, \boldsymbol{p}) \in P, \\ & \beta_{ij} \in \{0, 1\}, \qquad i = 1, \ldots, n; j = 1, \ldots, m, \end{aligned} \tag{3.2}$$

where P is the polyhedron defined in Section 1.2.

The objective function can further be reformulated to a sum of ratios, where the numerator is a concave quadratic and the denominator is linear:

$$\max \sum_{i=1}^{n} \sum_{j=1}^{m} N_i \pi_j \left(\frac{(R_{ij} - \pi_j)\beta_{ij}}{\sum_k (R_{ik} - \pi_k)\beta_{ik}} \right) \quad \Leftrightarrow \quad \max \sum_{i=1}^{n} \sum_{j=1}^{m} N_i \left(\frac{R_{ij} p_{ij} - p_{ij}^2}{\sum_k R_{ik}\beta_{ik} - p_{ik}} \right).$$

Thus, Model (3.2) can be formulated as the following mixed-integer fractional programming problem with linear constraints:

$$\begin{aligned} \max \quad & \sum_{i=1}^{n}\sum_{j=1}^{m} N_i \left(\frac{R_{ij}p_{ij} - p_{ij}^2}{\sum_k R_{ik}\beta_{ik} - p_{ik}} \right), \\ \text{s.t.} \quad & (\boldsymbol{p}, \boldsymbol{\pi}, \boldsymbol{\beta}) \in P, \\ & \beta_{ij} \in \{0,1\}, \qquad \forall i, j. \end{aligned} \tag{3.3}$$

The formulation (3.3) is a non-convex optimization problem. Unfortunately, there is no apparent convex relaxation of this formulation that yields a tight relaxation. In the next section, we find a mixed-integer programming formulation that approximates the Share-of-Surplus model by restricting the prices.

3.2 Restricted Prices

Unlike the Uniform and Weighted Uniform Models, the computation of optimal prices, given β_{ij}'s, is not immediate for the Share-of-Surplus model. Define $B_i = \{j : \beta_{ij} = 1\}$. Then the optimal prices is the solution to

$$\begin{aligned} \max \quad & \sum_{i=1}^{n}\sum_{j \in B_i} N_i \pi_j \left(\frac{(R_{ij} - \pi_j)}{\sum_{k \in B_i}(R_{ik} - \pi_k)} \right) \\ \text{s.t.} \quad & R_{ij} - \pi_j \geq 0, \qquad \forall i, j \in B_i, \\ & R_{ij} - \pi_j < 0, \qquad \forall i, j \notin B_i, \\ & \pi_j \geq 0, \qquad \forall j. \end{aligned} \tag{3.4}$$

If β_{ij} equals one for at least one segment, then we know that

$$\pi_j \in \left(\max_{i:\beta_{ij}=0} \{R_{ij}\}, \min_{i:\beta_{ij}=1} \{R_{ij}\} \right].$$

Suppose product l is bought by at least one segment and its price is increased by $\epsilon > 0$ such that β_{ij}'s do not change. Define $S_j = \{i : \beta_{ij} = 1\}$. Then the change in the objective value is:

$$\sum_{i \in S_l} \epsilon N_i \left(\frac{(R_{il} - (\pi_l + \epsilon)) \sum_{j \in B_i}(R_{ij} - \pi_j) + \sum_{j \in B_i}(\pi_j - \pi_l)(R_{ij} - \pi_j)}{(\sum_{k \in B_i}(R_{ik} - \pi_k))(\sum_{k \in B_i}(R_{ik} - \pi_k) - \epsilon)} \right). \tag{3.5}$$

Increasing the price of product l by ϵ would result in an increased objective value if (3.5) is positive. The β_{ij}'s do not change after the price increase, which implies that $R_{il} \geq \pi_l + \epsilon$. Therefore, all the terms in (3.5) are nonnegative except perhaps $(\pi_j - \pi_l)$. Thus, we can expect (3.5) to be positive if π_l is relatively low compared to other prices. Intuitively, this means that if π_l is low enough relative to other prices, then we want to raise π_l so that the surplus of product j decreases, hence decreasing the probability that the customers will buy this low-priced product. On the other hand, if π_l is high enough relative to other prices, we want to decrease π_l so that the probability that the customers will buy this expensive product increases, thus generating more revenue. Suppose we restrict π_j to be equal to $\min_{i:\beta_{ij}=1} \{R_{ij}\}$ (this was a property of the Uniform and Weighted Uniform Models, see Theorem 2.1). Then the Share-of-Surplus Model can be modeled as a mixed-integer linear programming model. Again, let x_{ij} equal 1 if segment i has the smallest reservation price out of all segments with nonnegative surplus for product j; 0 otherwise. Again, we introduce a dummy segment 0 with $R_{0j} := \overline{R}_j, \forall j$, $N_0 := 0$ and add the constraint $\sum_{i=0}^{n} x_{ij} = 1$. As before, $\beta_{ij} = \sum_{l:R_{lj} \le R_{ij}} x_{ij}$ and let us restrict π_j to equal $\sum_i R_{ij} x_{ij}$. Then the objective function of the Share-of-Surplus Model is:

$$\sum_{i=0}^{n}\sum_{j=1}^{m} N_i \pi_j \left(\frac{(R_{ij}-\pi_j)\beta_{ij}}{\sum_k (R_{ik}-\pi_k)\beta_{ik}} \right) = \sum_i N_i \left(\frac{\sum_j \sum_{l:R_{lj}\le R_{ij}} R_{lj}(R_{ij}-R_{lj})x_{lj}}{\sum_k \sum_{l:R_{lk}\le R_{ik}} (R_{ik}-R_{lk})x_{lk}} \right).$$

Let us now relax the assumption that the denominator $\sum_{k=1}^{m}\sum_{l:R_{lk}\le R_{ik}}(R_{ik}-R_{lk})x_{lk} > 0$ for all i. Define:

$$t_i := \begin{cases} \frac{\sum_j \sum_{l:R_{lj}\le R_{ij}} R_{lj}(R_{ij}-R_{lj})x_{lj}}{\sum_k \sum_{l:R_{lk}\le R_{ik}} (R_{ik}-R_{lk})x_{lk}}, & \text{if } \sum_k \sum_{l:R_{lk}\le R_{ik}} (R_{ik}-R_{lk})x_{lk} \ne 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let us introduce an auxiliary continuous variable u_{lij} where $u_{lij} := t_i x_{lj}$ for all segments l, i and products j where $R_{lj} \le R_{ij}$. Then we can formulate the problem as a linear mixed-integer programming problem:

$$\begin{aligned}
\max \quad & \sum_{i=1}^{n} N_i t_i, && (3.6)\\
\text{s.t.} \quad & \textstyle\sum_{i=0}^{n} x_{ij} = 1, && \forall j,\\
& \textstyle\sum_{j=1}^{m}\sum_{l:R_{lj}\le R_{ij}}(R_{ij}-R_{lj})u_{lij} \le \sum_{j=1}^{m}\sum_{l:R_{lj}\le R_{ij}} R_{lj}(R_{ij}-R_{lj})x_{lj}, && \forall i,\\
& u_{lij} \le t_i, && \forall l,i,j, R_{lj}\le R_{ij},\\
& u_{lij} \le \widetilde{R}_i x_{lj}, && \forall l,i,j, R_{lj}\le R_{ij},\\
& u_{lij} \ge t_i - \widetilde{R}_i(1-x_{lj}), && \forall l,i,j, R_{lj}\le R_{ij},\\
& t_i \le \widetilde{R}_i \textstyle\sum_{j=1}^{m}\sum_{l:R_{lj}\le R_{ij}}(R_{ij}-R_{lj})x_{lj}, && \forall i,\\
& t_i \ge 0, && \forall l,i,\\
& u_{lij} \ge 0, && \forall l,i,j, R_{lj}\le R_{ij},\\
& x_{ij} \in \{0,1\}, && \forall i,j.
\end{aligned}$$

If we use the probability Pr^*_{ij} (3.1) instead, then the objective function is:

$$\sum_i N_i \left(\frac{\sum_j \sum_{l:R_{lj}\le R_{ij}} R_{lj}(R_{ij}-R_{lj}+\eta)x_{lj}}{\sum_k \sum_{l:R_{lk}\le R_{ik}} (R_{ik}-R_{lk}+\eta)x_{lk}} \right).$$

Then the problem can be formulated along the same lines, with the exception that if $\eta < 1$, then we need to replace the constraint

$$t_i \le \widetilde{R}_i \sum_{j=1}^{m} \sum_{l:R_{lj}\le R_{ij}} (R_{ij}-R_{lj}+\eta)x_{lj}, \forall i$$

by

$$t_i \le \frac{1}{\eta}\widetilde{R}_i \sum_{j=1}^{m} \sum_{l:R_{lj}\le R_{ij}} (R_{ij}-R_{lj}+\eta)x_{lj}, \forall i$$

so that the right-hand-side is at least $\widetilde{R}_i$ whenever the summation is non-zero.

The constant η used in the formulation is assumed to be small enough such that the difference between Pr^*_{ij} and Pr_{ij} is almost negligible but that the probability is positive when the surplus is nonnegative. The examination of the effect of the value of η on the problem and the determination of the ideal value for the constant are left for future work. In our computational experiments, we found that the total computation time of the Share-of-Surplus Model with restricted prices (3.6) is significantly longer than that of the Uniform and the Weighted Uniform Models. We would like to explore other ways to formulate it or perhaps find cuts in order to decrease the solution time. We may also want to investigate other monotonically increasing functions to describe the probability which would perhaps lead to formulations that are easier to solve. The experimental results are discussed further in Section 9.

4 Price Sensitive Model

A common economic assumption is that as the price of a product decreases, the demand increases. In this section, we discuss a probabilistic choice model where the probability of a customer buying a particular product with nonnegative surplus is inversely proportional to the price of the product.

4.1 The Formulation

Again, let p_{ij} be the auxiliary variable where $p_{ij} := \pi_j \beta_{ij}$. Consider the probability of customer segment i buying product j as defined below:

$$Pr_{ij} := \begin{cases} 0, & \text{if } \beta_{ij} = 0 \text{ (Case 0)}, \\ 1, & \text{if } \beta_{ij} = 1, \sum_k \beta_{ik} = 1 \text{ (Case 1)}, \\ \frac{1}{\sum_k \beta_{ik} - 1}\left(\beta_{ij} - \frac{p_{ij}}{\sum_k p_{ik}}\right), & \text{otherwise (Case 2)}. \end{cases}$$

In this model, $Pr_{ij} = 0$ if product j has a negative surplus for segment i (Case 0), $Pr_{ij} = 1$ if product j is the only product with nonnegative surplus (Case 1), and if there are multiple products with nonnegative surplus (Case 2), Pr_{ij} is inversely proportional to the price of those products. Thus, we call this model the *Price Sensitive Model*. With some reformulation, the expected revenue maximization problem corresponding to this model can be formulated as a second-order cone programming problem with integer variables. In this model, the expected revenue from segment i, Rev_i, is

$$Rev_i := \begin{cases} 0, & \text{if } \sum_j p_{ij} = 0, \\ \left(\frac{\sum_j p_{ij}}{\sum_j \beta_{ij} - 1 + z_i} - \frac{\sum_j p_{ij}^2}{(\sum_j \beta_{ij} - 1 + z_i)(\sum_k p_{ik})}\right) + (\sum_j p_{ij}) z_i, & \text{otherwise.} \end{cases}$$

In the above $z_i \in \{0, 1\}$ an indicator variable that will be zero if and only if there are at least two products for segment i with nonnegative surplus.

Let s_i be an auxiliary variable where $s_i := (\sum_j p_{ij}) z_i$, which we know is a relationship that can be modeled by linear constraints. Also let

$$t_i := \begin{cases} 0, & \text{if } \sum_j p_{ij} = 0, \\ \frac{\sum_j p_{ij}}{\sum_j \beta_{ij} - 1 + z_i} - \frac{\sum_j p_{ij}^2}{(\sum_j \beta_{ij} - 1 + z_i)(\sum_k p_{ik})}, & \text{otherwise.} \end{cases}$$

Then, the expected revenue maximization problem corresponding to the Price Sensitive Model is:

$$\begin{array}{lll} \max & \sum_{i=1}^{n} N_i t_i + \sum_{i=1}^{n} N_i s_i, & \\ \text{s.t.} & \sum_j p_{ij}^2 \leq (\sum_j p_{ij})^2 - t_i(\sum_j \beta_{ij} - 1 + z_i)(\sum_j p_{ij}), & \forall i, \\ & t_i \leq \sum_j p_{ij}, & \forall i, \\ & s_i \leq \sum_j p_{ij}, & \forall i, \\ & s_i \leq \sum_j R_{ij} z_i, & \forall i, \\ & \sum_j \beta_{ij} \leq z_i + m(1 - z_i), & \forall i, \\ & z_i \geq \beta_{ij} - \sum_{k \neq j} \beta_{ik}, & \forall i, \forall j, \\ & (\boldsymbol{p}, \boldsymbol{\pi}, \boldsymbol{\beta}) \in P, & \\ & \beta_{ij} \in \{0, 1\}, & \forall i, j, \\ & z_i \in \{0, 1\}, s_i \geq 0, & \forall i, j, \end{array} \tag{4.1}$$

where P is the polyhedron (1.1) defined in Section 1.2.

We need to reformulate the first set of constraints to make the continuous relaxation of (4.1) a convex programming problem. Let us look at the first set of constraints:

$$\sum_j p_{ij}^2 \leq (\sum_j p_{ij})^2 - t_i(\sum_j \beta_{ij} - 1 + z_i)(\sum_j p_{ij}), \quad \forall i. \tag{4.2}$$

If $t_i > 0$ then $z_i = 0$ and if $z_i = 1$ then $t_i = 0$. Thus, we can eliminate the z_i term from the above inequality if we include the constraint

$$t_i \leq \widetilde{R}_i(1 - z_i).$$

Also, let b_{ij} be auxiliary variables where $b_{ij} := t_i \beta_{ij}$. Again, such relations can be modeled by linear constraints. Then, (4.2) becomes

$$\sum_j p_{ij}^2 \leq (\sum_j p_{ij})(\sum_j p_{ij} - \sum_j b_{ij} + t_i), \quad \forall i.$$

Let us further introduce auxiliary variables x_i and y_i such that:

$$x_i + y_i = \sum_j p_{ij} - \sum_j b_{ij} + t_i, \qquad \forall i,$$

$$x_i - y_i = \sum_j p_{ij}, \qquad \forall i.$$

Thus, the constraint becomes $\sum_j p_{ij}^2 \leq (x_i + y_i)(x_i - y_i) = x_i^2 - y_i^2$. Then, (4.2) can be represented by the second-order cone constraints and linear inequalities, and the Price Sensitive

Model becomes:

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} N_i t_i + \sum_{i=1}^{n} N_i s_i, && && (4.3)\\
\text{s.t.} \quad & \sqrt{\textstyle\sum_j p_{ij}^2 + y_i^2} \le x_i, && \forall i,\\
& x_i + y_i = \textstyle\sum_j p_{ij} - \sum_j b_{ij} + t_i, && \forall i,\\
& x_i - y_i = \textstyle\sum_j p_{ij}, && \forall i,\\
& t_i \le \widetilde{R}_i(1 - z_i), && \forall i, \forall j,\\
& b_{ij} \le \widetilde{R}_i \beta_{ij}, && \forall i,\\
& b_{ij} \le t_i, && \forall i, \forall j,\\
& b_{ij} \ge t_i - \widetilde{R}_i(1 - \beta_{ij}), && \forall i, \forall j,\\
& t_i \le \textstyle\sum_j p_{ij}, && \forall i,\\
& s_i \le \textstyle\sum_j p_{ij}, && \forall i,\\
& s_i \le \textstyle\sum_j R_{ij} z_i, && \forall i,\\
& \textstyle\sum_j \beta_{ij} \le z_i + m(1 - z_i), && \forall i,\\
& z_i \ge \beta_{ij} - \textstyle\sum_{k \ne j} \beta_{ik}, && \forall i, \forall j,\\
& (\boldsymbol{p}, \boldsymbol{\pi}, \boldsymbol{\beta}) \in P,\\
& \beta_{ij} \in \{0, 1\}, && \forall i, j,\\
& z_i \in \{0, 1\}, s_i \ge 0, b_{ij} \ge 0, && \forall i, j.
\end{aligned}
$$

We can easily eliminate the variables x_i's or y_i's from the above formulation, but we kept them in the above formulation to illustrate the second order cone constraint in a canonical form. Some preliminary computational results for the Price Sensitive Model are illustrated in Section 9.

5 Special Properties

In this section, we discuss properties of the optimal solutions of our models for data sets with special characteristics.

Lemma 5.1. *Suppose $n \le m$ and for every segment i, we can find a unique product $p(i)$ such that $R_{ip(i)} = \max_j \{R_{ij}\}$. Further suppose that for each of such product $p(i)$, segment i is the unique segment such that $R_{ip(i)} = \max_k \{R_{kp(i)}\}$. Let $J := \{j : j = p(i)$ for some segment $i \ne 0\}$. Then in an optimal solution,*

$$\beta_{ij} := \begin{cases} 1, & \text{if } j = p(i),\\ 0, & \text{otherwise.} \end{cases}$$

In the alternative formulation, an optimal solution is

$$x_{ij} := \begin{cases} 1, & \text{if } j = p(i),\\ 1, & \text{if } i = 0 \text{ and } j \notin J,\\ 0, & \text{otherwise,} \end{cases}$$

where segment 0 is the dummy segment.

The following lemmas apply to the Uniform Model, the Weighted Uniform Model, and the Share-of-Surplus Model with restricted prices.

Lemma 5.2. *If the optimal values for the x (or β) variables are known, then the optimal prices can be determined. Furthermore, if the optimal prices are known, then the optimal values for the x (or β) variables can be determined.*

Lemma 5.3. *Suppose R_{st} is the maximum reservation price over all segments and products and only one pair of segment and product has that reservation price. Then in any optimal solution, segment s buys product t.*

5.1 Comparisons

In this subsection, we compare the optimal solution, in terms of the prices π_j's and β_{ij}'s, of the different models. We notice in most examples, the four models have the same optimal solutions [22]. Of the ones where they have different optimal solutions, usually the Uniform Model, the Weighted Uniform model and the Price Sensitive model have the same optimal solution, while the Share-of-Surplus Model has a different optimal solution. We also compared the models' optimal solutions on random data in which the reservation prices are uniformly generated from a specified range. The differences in the optimal values of the Uniform, the Weighted Uniform, and the Price Sensitive Models were quite small in many problem instances, but the Share-of-Surplus Model gave smaller optimal values than the other three models in most cases. It is most likely because the probability for a segment to buy a lower-priced product is usually higher in the Share-of-Surplus Model than in the other three models.

6 Heuristics

As we will see in Section 9, CPLEX takes significant amount of time just to find a feasible solution for larger problems. Fortunately, we can easily find a feasible mixed-integer solution for the formulations of all our models. Thus, we can provide the solver with a "good" starting feasible solution in hopes of decreasing the solution times.

6.1 Heuristic 0

One possible strategy, which we call *Heuristic 0*, is to set $\beta_{ij*} = 1$ for each segment i where $R_{ij*} = \max_j \{R_{ij}\}$. The other β variables are set accordingly to ensure feasibility. The details can be found in [22]. For some very special data sets, this heuristic is guaranteed to deliver the optimal solution.

Lemma 6.1. *Suppose the conditions are the same as those stated in Lemma 5.1. That is, for every segment i, we can find a unique product $p(i)$ such that $R_{ip(i)} = \max_j \{R_{ij}\}$, and for each of such product $p(i)$, segment i is the unique segment such that $R_{ip(i)} = \max_k \{R_{kp(i)}\}$. Then Heuristic 0 gives an optimal solution.*

However, Heuristic 0 may not yield a strong solution in general. For the rest of this section, we discuss a few simple techniques for improving on the feasible solution found by Heuristic 0.

6.2 Heuristic 1

After running Heuristic 0, we select a product k that is bought by at least one customer segment, and let l be the segment with the lowest reservation price that buys product k. We consider the change in the objective value if the segment does not buy product k anymore and perhaps buys another product q that it does not currently buy (i.e., β_{lq} currently equals to 0). This can be thought of as swapping β_{lk} with β_{lq}. We select the option that increases the objective value the most and modify the β variables accordingly. That is, segment l either does not buy product k anymore, or it buys another product instead of product k. If none of the options increases the objective value, we make no changes. We repeat until no swaps can be made to increase the objective value. This algorithm terminates because there are finitely many possible values for the β's and the objective value strictly increases after each swap. The details can be found in [22].

The order in which we select the products to be examined affects the final solution that will be given by the heuristic. The goal is to use an order that maximizes the total increase in the objective value. In this heuristic, we sort the products by the price and examine the products in the order of the lowest price to the highest price. If we make a change in any iteration, we sort the products again since the prices may change, and start with the lowest-priced product again. The heuristic stops when no changes can be made after examining all the products consecutively from the lowest price to the highest price.

This simple heuristic can be used to find a feasible integral solution for any of the models. The only part that needs to be changed is how the objective value is calculated. The version shown here makes use of β, but it can be easily modified to use the x variables as in the alternative formulation.

6.3 Heuristic 2

Heuristic 1 can be modified to have a polynomial runtime if the price of the product that we examine is non-decreasing in each iteration. From experiments of Heuristic 1, we noticed that if a swap can be made when product k at price π_k is selected, it is very unlikely that a swap can be made for a product at a price lower than π_k in subsequent iterations. Therefore, we would expect the results to be similar if we do not examine products with lower prices again.

Heuristic 2 is the same as Heuristic 1 but the products are selected in a different order. After a customer is swapped out of product k with price π_k before the swap, only products with prices at least π_k are examined. The price of product k increases after a swap, so it will be examined again if there are still customers buying product k. If a new product s is bought and if its new price π_s^{new} is less than π_k, then product s will never be examined. If a product cannot be swapped to increase the objective value, then it will not be examined again. The details can be found in [22].

Let $O(f(n,m))$ be the runtime to calculate the increase in objective value if segment l does not buy product k anymore or if segment l buys product s instead of product k, where n is the number of customer segments and m is the number of products. Clearly, $f(n,m)$ is polynomial in n and m, since the runtime to calculate the objective value is polynomial.

Lemma 6.2. *The runtime of Heuristic 2 is polynomial.*

6.4 Heuristic 3

Heuristic 3 is a hybrid between Heuristic 1 and Heuristic 2. It examines the products in the same way, but after a swap in which segment l buys product s instead of product k and

$\pi_s^{new} < \pi_k$ (equivalently, $R_{ls} < R_{lk}$), it would examine all the products with prices $\geq \pi_s^{new}$. That is, the price of the products that it examines decreases only if a product has a lower price after a swap. The details can be found in [22]. It is not yet clear if this heuristic has an exponential worst-case runtime. However, experimental results shows that it has a similar runtime to Heuristic 2 and the resulting objective value is usually better (see [22]).

6.5 Comparison of the Heuristics

All of the heuristics terminate in a very short time. The time required for Heuristic 1 to terminate increases significantly as the problem size increases. The objective values found are better than or at least as good as the ones found by the other two heuristics, except in one problem instance (when n = 60, m = 20) where Heuristic 2 has a better solution. Experimental results show that Heuristic 3 has a similar runtime as Heuristic 2 and the resulting objective value is usually better. The effect of using a starting solution found by the heuristics for the Uniform Model is explored in Section 9.

7 Valid Inequalities

To further improve the solution time for the mixed-integer programming models, we considered several mixed-integer cuts for the various choice models.

7.1 Convex Quadratic Valid Inequalities

In the original Uniform Model, the variable a_{ij} were introduced to convexify the bilinear inequalities:

$$\sum_{j=1}^{m} t_i \beta_{ij} \leq \sum_{j=1}^{m} p_{ij}, \quad \forall i. \tag{7.1}$$

We wish to include a convex constraint in the mixed-integer programming formulation that is implied by the above inequalities and some valid convex inequalities.

Let M_i be a positive number (as small as possible) such that $t_i^2 \leq M_i$, for every feasible solution $(t_1, \dots, t_n, \beta_{11}, \dots, \beta_{nm}, p_{11}, \dots, p_{nm})$ of the mixed integer programming problem. Also, note that $\beta_{ij}^2 \leq \beta_{ij}$. Combining these relations together yields the following set of valid inequalities:

$$a_i t_i^2 + b_i \sum_{j=1}^{m} (\beta_{ij}^2 - \beta_{ij}) + \sum_{j=1}^{m} t_i \beta_{ij} - \sum_{j=1}^{m} p_{ij} \leq a_i M_i, \quad i = 1, \dots, n, \tag{7.2}$$

where a_i and b_i are nonnegative constants. With appropriate values of a_i and b_i, the above set of quadratic inequalities represents a convex set.

Lemma 7.1. *The function* $f(t, \beta_1, \dots, \beta_m, p_1, \dots, p_m) := at^2 + b\sum_{j=1}^{m}(\beta_j^2 - \beta_j) + \sum_{j=1}^{m} t\beta_j - \sum_{j=1}^{m} p_j$ *is a convex function iff* $a > 0$, $b > 0$ *and* $ab \geq \frac{m}{4}$.

Next, we generalize the above construction to allow different coefficients b_j for the inequalities $\beta_{ij}^2 \leq \beta_{ij}$. Let $\boldsymbol{b}$ denote the vector $(b_1, b_2, \dots, b_m)^T$ and let $\boldsymbol{B}$ denote the $m \times m$ diagonal matrix with entries $b_1, b_2, \dots, b_m$ on the diagonal.

Lemma 7.2. *The function* $F(t, \beta_1, \dots, \beta_m, p_1, \dots, p_m) := at^2 + \sum_{j=1}^{m} b_j(\beta_j^2 - \beta_j) + \sum_{j=1}^{m} t\beta_j - \sum_{j=1}^{m} p_j$ *is a convex function iff* $\boldsymbol{b} > 0$ *and* $a \geq \sum_{j=1}^{m} \frac{1}{4b_j}$.

Corollary 7.3. *Let M_i be as above, and $b_1 > 0, b_2 > 0, \ldots, b_m > 0$, and $a \geq \sum_{j=1}^m \frac{1}{4b_j}$ be given. Then the inequality*

$$at_i^2 + \sum_{j=1}^m [b_j \beta_{ij}^2 + (t_i - b_j)\beta_{ij} - p_{ij}] \leq aM_i$$

is a valid convex quadratic inequality for the feasible region of the mixed integer programming problem.

In the alternate formulation of the Uniform Model (2.2) with x_{ij} variables instead of β_{ij}'s, the bilinear constraint corresponding to (7.1) is

$$\sum_{j=1}^m t_i \sum_{l:R_{lj} \leq R_{ij}} x_{ij} \leq \sum_{j=1}^m \sum_{l:R_{lj} \leq R_{ij}} R_{lj} x_{ij}, \quad \forall i.$$

As before, we add a times $t_i^2 \leq M_i$ and b_{lj} times $x_{lj}^2 \leq x_{lj}$ for all j and l such that $R_{lj} \leq R_{ij}$ to get a valid convex quadratic inequality for (2.2).

Corollary 7.4. *Let M_i be as above, and $b_{l1} > 0, b_{l2} > 0, \ldots, b_{lm} > 0$ for l such that $R_{lj} \leq R_{ij}$, and $a \geq \sum_{j=1}^m \sum_{l:R_{lj} \leq R_{ij}} \frac{1}{4b_{lj}}$ be given. Then the inequality*

$$at_i^2 + \sum_{j=1}^m \sum_{l:R_{lj} \leq R_{ij}} [b_{lj} x_{lj}^2 + (t_i - b_{lj} - R_{lj}) x_{lj}] \leq aM_i$$

is a valid convex quadratic inequality for the feasible region of the mixed integer programming problem. If $b = b_{l1} = b_{l2} = \cdots = b_{lm}$, then we need $ab \geq \sum_{j=1}^m \sum_{l:R_{lj} \leq R_{ij}} \frac{1}{4}$.

Clearly, the tighter the upperbound M_i for t_i^2, the stronger the valid convex quadratic inequality. One approach to generate such M_i would be to optimize t_i over the current convex relaxation and square the result. However, such upperbounds for t_i^2 may not be effective. Instead of going after a constant M_i let us consider another upperbound for t_i^2, allowing M_i to be a linear function of the existing variables. For the alternative formulation of the Uniform Model (2.2), we know that if $\sum_{j=1}^m \sum_{l:R_{lj} \leq R_{ij}} x_{lj} \geq 1$ then $t_i = \frac{\sum_{j=1}^m \sum_{l:R_{lj} \leq R_{ij}} R_{lj} x_{lj}}{\sum_{j=1}^m \sum_{l:R_{lj} \leq R_{ij}} x_{lj}}$ and

$$t_i^2 = \frac{(\sum_{j=1}^m \sum_{l:R_{lj} \leq R_{ij}} R_{lj} x_{lj})^2}{(\sum_{j=1}^m \sum_{l:R_{lj} \leq R_{ij}} x_{lj})^2} \leq \sum_{j=1}^m \sum_{l:R_{lj} \leq R_{ij}} R_{lj}^2 x_{lj},$$

where we used the fact that t_i will be the square of the average of certain R_{ij} values (depending on x_{ij}); clearly, such a value is at most the square of the maximum, which is at most the sum of squares of all such R_{ij} involved in the average. Thus,

Corollary 7.5. *Let $b_{l1} > 0, \ldots, b_{lm} > 0$ for l such that $R_{lj} \leq R_{ij}$, and $a \geq \sum_{j=1}^m \sum_{l:R_{lj} \leq R_{ij}} \frac{1}{4b_{lj}}$ be given. Then the inequality*

$$at_i^2 + \sum_{j=1}^m \sum_{l:R_{lj} \leq R_{ij}} [b_{lj} x_{lj}^2 + (t_i - b_{lj} - R_{lj} - aR_{lj}^2) x_{lj}] \leq 0 \tag{7.3}$$

is a valid convex quadratic inequality for the feasible region of the mixed integer programming problem. If $b = b_{l1} = b_{l2} = \cdots = b_{lm}$, then we need $ab \geq \sum_{j=1}^m \sum_{l:R_{lj} \leq R_{ij}} \frac{1}{4}$.

We tested the efficacy of the convex quadratic inequality (7.3) on the Uniform Model, where the reservations prices were randomly generated from a uniform distribution. Table 1 illustrates the optimal value of the mixed-integer programming problem (MIP), the optimal value of the corresponding LP relaxation (LP), and the optimal value of the quadratically constrained problem (QCP) resulting from adding the inequalities (7.3) to the LP relaxation. We see that these inequalities are indeed cuts since the optimal solution of the LP violates them in most instances.

n	m	v	MIP	LP	With Cut
10	10	1	8980.67	9020.83	9007.12
		2	7576.50	8136.93	8076.50
		3	8656.75	8814.38	8799.53
		4	8767.67	8956.24	8950.98
		5	7369.68	7977.77	7860.84
10	20	1	9658.00	9691.25	9690.69
		2	9373.00	9433.28	9430.34
		3	9276.83	9424.39	9423.05
		4	8603.58	8970.78	8939.18
		5	9473.00	9534.28	9532.12
10	40	1	9777.50	9782.75	9782.05
		2	9798.67	9839.42	9837.89
		3	9788.50	9806.82	9805.72
		4	9592.50	9654.70	9653.06
		5	9771.00	9771.00	9770.49
10	60	1	9836.00	9836.00	9833.83
		2	9836.00	9868.33	9866.21
		3	9860.50	9868.95	9868.08
		4	9865.00	9865.00	9863.50
		5	9854.00	9854.00	9852.26

Table 1: Uniform Model with the convex quadratic valid inequalities (7.3). n is the number of customer segments, m is the number of products, and v is a label of the problem instance. The column "MIP" is the optimal objective value (2.2), the column "LP" is the optimal objective value of the LP relaxation of (2.2), and the column "With Cut" is the optimal objective value of the continuous relaxation of (2.2) with the convex quadratic inequality (7.3).

There are four anomalies in Table 1, namely, the instances (n, m, v) with (10, 40, 5), (10, 60, 1), (10, 60, 4) and (10, 60, 5). For each of these instances, the objective value of the QCP relaxation is *strictly less* than the optimal objective value of the MIP. The convex quadratic inequalities were indeed valid for the MIP optimal solution. However, we determined that CPLEX's barrier method returned a suboptimal solution for these QCPs. When observing the details of the CPLEX run, we saw that in each of these four instances the initial solution of the QCP relaxation had primal infeasibility in the order of 10^{10} and dual infeasibility in order of 10^3. CPLEX stopped after 30 to 40 barrier iterations, declaring the current primal solution, with complementarity gap around 10^{-8}, dual infeasibility about 10^{-5} and primal infeasibility around 10^2, as "primal optimal" with "no dual solution available". While these preliminary experiments show the potential usefulness of our convex cuts, it is also clear that to use them in a robust and effective manner, one needs to work with QCP or second-order cone (SOCP) algorithms that generate dual feasible solutions and use the dual objective function value in the related SOCP-IP computations. We leave the design of such specialized interior-point algorithms to future work.

7.2 Knapsack Covers

We can formulate the Uniform Model as a pure 0,1 optimization problem. For $k = 0, \ldots, m$, let

$$y_{ik} := \begin{cases} 1, & \text{if segment } i \text{ has exactly } k \text{ products with nonnegative surplus,} \\ 0, & \text{otherwise.} \end{cases}$$

Then, the probability that segment i buys product j is $\sum_{k=1}^{m} \frac{1}{k}\beta_{ij} y_{ik}$ and the Uniform Model can be formulated as the following 0,1 programming problem:

$$\begin{array}{lll} \max & \displaystyle\sum_{i=1}^{n} N_i \sum_{j=1}^{m} \sum_{l:R_{lj} \leq R_{ij}} R_{lj} \sum_{k=1}^{m} \frac{1}{k} z_{l,j,i,k}, & (7.4) \\ \text{s.t.} & \displaystyle\sum_{i=1}^{n} x_{ij} = 1, & j = 1, \ldots, m, \\ & \displaystyle\sum_{k=0}^{m} y_{ik} = 1, & i = 1, \ldots, n, \\ & \displaystyle\sum_{k=1}^{m} \sum_{j=1}^{m} \sum_{l:R_{l,j} \leq R_{i,j}} \frac{1}{k} z_{l,j,i,k} = 1 - y_{i,0}, & i = 1, \ldots, n, \\ & \displaystyle\sum_{j=1}^{m} \sum_{l:R_{l,j} \leq R_{ij}} x_{lj} = \sum_{k=0}^{m} k y_{ik}, & i, \ldots, m, \\ & z_{l,j,i,k} \leq x_{l,j}, & \forall i, \forall j, k = 1, \ldots, m; l : R_{l,j} \leq R_{i,j}, \\ & z_{l,j,i,k} \leq y_{i,k}, & \forall i, \forall j, k = 1, \ldots, m; l : R_{l,j} \leq R_{i,j}, \\ & z_{l,j,i,k} \geq x_{l,j} + y_{i,k} - 1, & \forall i, \forall j, k = 1, \ldots, m; l : R_{l,j} \leq R_{i,j}, \\ & x_{i,j} \in \{0, 1\}, & \forall i, \forall j, \\ & y_{i,k} \in \{0, 1\}, & \forall i, k = 0, \ldots, m, \\ & 0 \leq z_{l,j,i,k} \leq 1, & \forall i, \forall j, k = 1, \ldots, m; l : R_{l,j} \leq R_{i,j}. \end{array}$$

In the above, $z_{l,j,i,k}$ is a continuous variable defined for the segment pairs i, l and the product j (defined for $R_{lj} \leq R_{ij}$); the final index k is a counter (recall the definition of y_{ik}). The constraints imply

$$z_{l,j,i,k} \leq \min \{x_{lj}, y_{ik}\}.$$

So, $z_{l,j,i,k}$ is zero unless segment i has exactly k products with nonzero surplus and segment l has the smallest reservation price out of all segments with a nonnegative surplus for product j. In the latter case $z_{l,j,i,k}$ will be one.

The pure 0,1 formulation (7.4) may not be as strong as the mixed-integer formulation (2.2). However, we may be able to exploit the vast amount of work done in developing strong valid inequalities for pure 0-1 programming problems for formulation (7.4). One obvious family of valid inequalities are the knapsack covers [19]. From (7.4), we have the constraints

$$\sum_{j=1}^{m} \beta_{ij} = \sum_{k=0}^{m} k y_{ik}, \quad i = 1, \ldots, n,$$

(where we substituted $\beta_{ij} := \sum_{l:R_{l,j} \le R_{ij}} x_{lj}$ purely for notational ease) and

$$\sum_{k=0}^{m} y_{ik} = 1, \quad i = 1, \ldots, n.$$

From these, for a given i and k, we get:

$$\sum_{j=1}^{m} \beta_{ij} \le \sum_{l=0}^{k} k y_{il} + \sum_{l=k+1}^{m} m y_{il} = \sum_{l=0}^{k} k y_{il} + \sum_{l=k+1}^{m} m y_{il} + m - m \sum_{k=0}^{m} y_{ik}$$

$$\Rightarrow \sum_{j=1}^{m} \beta_{ij} + (m-k) \sum_{l=0}^{k} y_{il} \le m,$$

where the last inequality is a knapsack constraint (note that $\sum_{l=0}^{k} y_{il} \in \{0,1\}$ in the integer solution so we can treat the term as a 0-1 variable). For a given i and k, let P_{ik} be a subset of $k+1$ products , i.e, $P_{ik} \subseteq \{1, \ldots, m\}$, $|P_{ik}| = k+1$. Thus, the corresponding knapsack cover inequality is

$$\sum_{j \in P_{ik}} \beta_{ij} + \sum_{l=0}^{k} y_{il} \le k+1. \tag{7.5}$$

Given a fractional solution to (7.4), separating (7.5) can be done in polynomial time. Given x_{ij}'s , and thus β_{ij}'s, we rank β_{ij} for each i, $i = 1, \ldots, n$. For each k, let

$$P^*_{ik} = \{j : \beta_{ij} \text{ is one of the } k^{th} \text{ largest } \beta_{ij}\text{'s}, j = 1, \ldots, m\}.$$

Thus, for each i and k, the corresponding cover inequality is violated by the current solution if and only if $\sum_{j \in P^*_{ik}} \beta_{ij} + \sum_{l=0}^{k} y_{il} > k+1$. We can also incorporate all of the inequalities (7.5) into (7.4) with only polynomial numbers of additional constraints and variables.

Lemma 7.6. *Given i and k, there exists β_{ij}, $j = 1, \ldots, m$ and y_{il}, $l = 0, \ldots, k$ satisfying (7.5) for all $P_{ik} \subseteq \{1, \ldots, m\}$, $|P_{ik}| = k+1$ if and only if there exists q and p_j, $j = 1, \ldots, m$ such that*

$$\begin{aligned} (k+1)q + \sum_{j=1}^{m} p_j + \sum_{l=0}^{k} y_{il} &\le k+1, \\ q + p_j &\ge \beta_{ij}, \quad j = 1, \ldots, m, \\ p_j &\ge 0, \quad j = 1, \ldots, m. \end{aligned}$$

Thus, we can either iteratively separate the knapsack cover inequalities, or from Lemma 7.6, add the following constraints to (7.4):

$$\begin{aligned} (k+1)q_{ik} + \sum_{j=1}^{m} p_{i,j,k} + \sum_{l=0}^{k} y_{il} &\le k+1, i = 1, \ldots, n; k = 0, \ldots, m, \\ q_{ik} + p_{ijk} &\ge \beta_{ij}, j = 1, \ldots, m; i = 1, \ldots, n; k = 0, \ldots, m, \\ p_{ijk} &\ge 0, j = 1, \ldots, m; i = 1, \ldots, n; k = 0, \ldots, m. \end{aligned} \tag{7.6}$$

Table 2 illustrates that these knapsack covers (7.5) are indeed cuts. It compares formulation (7.4) with and without the cover inequalities (7.6) in terms of the objective value

n	m	v	MIP (2.2)	Pure 0-1 (7.4) without (7.6)	Pure 0-1 (7.4) with (7.6)
4	4	1	2304.79	2564.71	2399.63
		2	3447.79	**3404.00**	**3404.00**
		3	333.60	**333.00**	**333.00**
		4	**3005.67**	3060.92	**3005.67**
		5	3294.81	3360.95	3271.48
4	10	1	382.54	406.42	390.50
		2	381.85	398.19	391.36
		3	358.60	397.36	373.89
		4	355.97	389.98	365.59
		5	394.23	402.74	384.18
10	4	1	744.71	802.93	799.31
		2	845.80	856.12	853.60
		3	799.50	850.95	848.58
		4	809.58	856.85	842.16
		5	883.05	925.44	911.67
10	10	1	985.58	997.40	990.70
		2	991.44	1008.53	1003.15
		3	1016.35	1021.94	1016.75
		4	825.48	872.92	864.30
		5	1014.14	1021.50	1013.01

Table 2: Objective values of the LP relaxation for the mixed-integer programming formulation (2.2) and Pure 0-1 formulation with and without the Knapsack Cover inequalities (7.5), where n is the number of customer segments, m is the number of products, and v is a label of the problem instance. LP objective values in bold corresponds to the IP optimal value.

of their linear programming relaxation on randomly generated instances. However, even with all the knapsack cover inequalities, the Pure 0,1 formulation is still weaker than the mixed-integer formulation (2.2) in most cases.

These knapsack cover inequalities (7.5) can also be used to generate valid inequalities for the mixed-integer programming formulation (2.2).

Lemma 7.7. *Suppose $\bar{x}_{ij}$ is a fractional solution of (2.2) and let $\bar{\beta}_{ij} = \sum_{l:R_{lj}\le R_{ij}} \bar{x}_{lj}$. For a given i, $i = 1, \ldots, n$, if there are no y_{ik}'s that satisfies*

$$\begin{aligned} \textstyle\sum_{k=0}^m y_{ik} &= 1, \\ \textstyle\sum_{k=0}^m k y_{ik} &= \textstyle\sum_{j=1}^m \bar{\beta}_{ij}, \\ \textstyle\sum_{l=0}^k y_{il} &\le k+1-\textstyle\sum_{j\in P^*_{ik}} \bar{\beta}_{ij}, \quad k=0,\ldots,m \end{aligned} \tag{7.7}$$

where $P^*_{ik} := \{j : \bar{\beta}_{ij}$ is one of the k largest $\bar{\beta}_{ij}, j = 1, \ldots, m\}$, then

$$\sum_{j=1}^m v\beta_{ij} + \sum_{j\in P^*_{ik}} w_k \beta_{ij} \le \sum_{k=0}^m (k+1) w_k \tag{7.8}$$

is a valid inequality for (2.2) that cuts off $\bar{x}_{ij}$*, where*

$$\begin{aligned} u + kv + \textstyle\sum_{l=0}^{k} w_k &\geq 0, && k = 0, \ldots, m, \\ u + \textstyle\sum_{j=1}^{m} \bar{\beta}_{ij} v + \left(k + 1 - \textstyle\sum_{j \in P_{ik}^*} \bar{\beta}_{ij}\right) w_k &< 0, \\ w_k &\geq 0, && k = 0, \ldots, m, \end{aligned}$$

for some u*.*

8 Product Capacity and Cost

In all of our discussions thus far, we have assumed that there are no capacity limits nor costs for our products. Clearly, this is not a realistic assumption in many applications. In this section, we discuss how we can incorporate capacity limits and product costs into some of our customer choice models.

8.1 Product Capacity

Product capacity limits are crucial constraints for products such as airline seats and hotel rooms. Certain consumer choice models handle capacity constraints easily, whereas it poses a challenge to others. We present this extension for the Uniform Model, the Weighted Uniform Model, and the Share-of-Surplus Model with restricted prices. We were not able to incorporate the capacity constraint in the Price Sensitive Model while maintaining the convexity of the continuous relaxation. In all of the following subsections, we assume that the company can sell up to Cap_j units of product j, $Cap_j \geq 0$, $j = 1, \ldots, m$.

Uniform and Weighted Uniform Model

Capacity constraints can be incorporated to the mixed-integer formulations of the Uniform Model and the Weighted Uniform Model with some additional variables. We discuss the formulation for the Uniform Model only, since it extends easily to the Weighted Uniform Model. In the Uniform Model, the expected number of customers that buy product j is $\sum_i N_i \frac{\beta_{ij}}{\sum_k \beta_{ik}}$ if $\sum_k \beta_{ik} \geq 1$ and is 0 if $\sum_k \beta_{ik} = 0$. Let B_{ij} be an auxiliary variable such that $B_{ij} := \frac{\beta_{ij}}{\sum_k \beta_{ik}}$ if $\sum_k \beta_{ik} \geq 1$ and is 0 if $\sum_k \beta_{ik} = 0$, i.e., the fraction of customers from segment i buying product j, Pr_{ij}. Thus, $\beta_{ij} = B_{ij} \sum_k \beta_{ik}$. Let $b_{ijk} := B_{ij}\beta_{ik}$. The capacity constraint can be represented by the following set of linear constraints:

$$\begin{aligned} \textstyle\sum_i N_i B_{ij} &\leq Cap_j, && \forall j, \\ \beta_{ij} &= \textstyle\sum_k b_{ijk}, && \forall i, \forall j, \\ b_{ijk} &\leq \beta_{ik}, && \forall i, \forall j, \forall k, \\ b_{ijk} &\geq B_{ij} - (1 - \beta_{ik}), && \forall i, \forall j, \forall k, \\ b_{ijk} &\leq B_{ij}, && \forall i, \forall j, \forall k, \\ b_{ijk} &\geq 0, && \forall i, \forall j, \forall k. \end{aligned} \tag{8.1}$$

The above constraints can also be represented by x_{ij} variables of earlier sections, instead of the β_{ij} variables.

Share-of-Surplus Model

For the Share-of-Surplus Model with restricted prices (3.6), the expected number of customers that buy product j is $\sum_i N_i \left(\frac{\sum_{l:R_{lj} \leq R_{ij}} (R_{ij} - R_{lj}) x_{lj}}{\sum_k (\sum_{l:R_{lk} \leq R_{ik}} (R_{ik} - R_{lk}) x_{lk})} \right)$ if $\sum_k [\sum_{l:R_{lk} \leq R_{ik}} (R_{ik} - R_{lk}) x_{lk}] \neq 0$. Let B_{ij} be an auxiliary variable such that

$$B_{ij} := \begin{cases} \left(\frac{\sum_{l:R_{lj} \leq R_{ij}} (R_{ij} - R_{lj}) x_{lj}}{\sum_k (\sum_{l:R_{lk} \leq R_{ik}} (R_{ik} - R_{lk}) x_{lk})} \right), & \text{if } \sum_k [\sum_{l:R_{lk} \leq R_{ik}} (R_{ik} - R_{lk}) x_{lk}] \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Again, B_{ij} is the fraction of customers from segment i buying product j, or Pr_{ij}. Thus,

$$\sum_{l:R_{lj} \leq R_{ij}} (R_{ij} - R_{lj}) x_{lj} = B_{ij} \sum_k \left[\sum_{l:R_{lk} \leq R_{ik}} (R_{ik} - R_{lk}) x_{lk} \right].$$

Let $b_{ijlk} := B_{ij} x_{lk}$. Just as before, the capacity constraint can be represented by the following set of linear constraints:

$$\begin{aligned} \textstyle\sum_i N_i B_{ij} &\leq Cap_j, && \forall j, && (8.2) \\ \textstyle\sum_{l:R_{lj} \leq R_{ij}} (R_{ij} - R_{lj}) x_{lj} &= \textstyle\sum_k \left[\sum_{l:R_{lk} \leq R_{ik}} (R_{ik} - R_{lk}) b_{ijlk} \right] && \forall i, \forall j, \\ b_{ijlk} &\leq x_{lk}, && \forall i, \forall j, \forall l, \forall k, \\ b_{ijlk} &\geq B_{ij} - (1 - x_{lk}), && \forall i, \forall j, \forall l, \forall k, \\ b_{ijlk} &\leq B_{ij}, && \forall i, \forall j, \forall l, \forall k, \\ b_{ijlk} &\geq 0, && \forall i, \forall j, \forall l, \forall k. \end{aligned}$$

Risk Products

In some cases, companies may want to penalize against under-shooting a capacity. For example, if there is a large fixed cost or initial investment for product j, the company may sacrifice revenue and decrease its price to ensure that all of the product is sold. We call such products *risk* products. For these products, we may add a penalty for under-shooting in the objective, i.e., given a user-defined penalty coefficient $w_j > 0$ for under-selling product j, we modify the objective to

$$\sum_{i=1}^{n} N_i \sum_{j=1}^{m} \pi_j Pr_{ij} - \sum_{j=1}^{m} w_j \left(Cap_j - \sum_{i=1}^{n} N_i B_{ij} \right)$$

or

$$\sum_{i=1}^{n} \sum_{j=1}^{m} N_i (\pi_j Pr_{ij} + w_j B_{ij})$$

where B_{ij} is as before.

From a profit optimization point of view, it is sub-optimal to forcibly sell unprofitable products. Such a policy implies that the company is overstocked with these risk products, i.e., Cap_j is too large. In some cases, we may want to treat Cap_j as a variable. For example, in the travel industry, the product procurement division will seek out contracts with hotels to secure certain numbers of rooms for a given time period. However, if that travel destination is not profitable for the company, they may be better off securing very few rooms or not securing any rooms at all. In all of our models, making Cap_j a variable will not affect the linearity of the constraints. Also, there will most likely be an upperbound for Cap_j for all $j = 1, \ldots, m$. If procuring a unit of product j costs v_j, then the objective function can be modified to:

$$\sum_{i=1}^{n} N_i \sum_{j=1}^{m} \pi_j Pr_{ij} - \sum_{j=1}^{m} v_j Cap_j.$$

By determining the optimal value for Cap_j, it should no longer be necessary for the company to penalize under-selling of products[‡].

8.2 Product Cost

Suppose each product j has a variable cost of c_j per unit. In the objective function, we want to subtract c_j multiplied by the expected number of customers that buy product j. For all the probabilistic choice models discussed in this paper, the objective function becomes

$$\sum_i N_i \sum_j (\pi_j - c_j) Pr_{ij},$$

where Pr_{ij} is the probability that the customer segment i buys product j. This is equivalent to lowering all the reservation prices of product j by c_j in all of the models except the Price Sensitive Model. We can also easily incorporate fixed costs for products and capacities (resources) with additional constraints and 0,1 variables.

9 Computational Results

This section illustrates the empirical performances of the Uniform, Weighted Uniform, Share-of-Surplus and Price Sensitive Models on randomly generated and real data sets. The randomly generated reservation prices are generated from a uniform distribution. For each n (number of segments) and m (number of products) pair, five instances were generated. The real data are subsets of reservation prices estimated from actual booking orders of a travel company (our procedure in estimating reservation prices are discussed in Appendix B). There is one instance for each n and m pair. The same data set were used for the Uniform, Weighted Uniform and the Share-of-Surplus Model, however the Price Sensitive Model was tested on smaller data sets due to its significantly longer computation time.

The models were run with default parameter settings of CPLEX 9.1 and a time limit of two hours (7200 CPU seconds) unless indicated otherwise. They were run on a machine with four processors and 8 gigabyte of RAM, with at most one process running at a time on each processor.

The Uniform, Weighted Uniform and the Share-of-Surplus Model all began with the solution of Heuristic 1 of Section 6. For every problem instance, the heuristic took at most one CPU second to solve. Although Heuristic 1 found strong feasible integer solutions, it did not significantly improve the total solution time and the total branch-and-bound nodes explored by CPLEX. Thus, even when starting out with a good integer solution, proving optimality was a difficult task for many of these problems.

The tables show the number of segments n, the number of products m, total number of dual simplex iterations ("# of SimplxItns"), total number of branch-and-bound nodes visited

[‡]It is possible that a company may procure large quantities of a currently non-profitable product to increase their long-term market share. We will not consider such long-term marketing strategy in this paper.

("# of Nodes"), the relative optimality gap when CPLEX was terminated ("FinalGap"), total CPU seconds ("Time"), and the relative gap between the objective value of the best integer solution found by CPLEX versus that of Heuristic 1 ("Heuristic Gap"). For the randomly generated data, the geometric mean over the five instances is illustrated. The column "Provably Optimal" indicates the percentage of the five instances solved to provable optimality within the two hour time limit. For the real data, where there is just one instance for each (n,m) pair, the column "CPLEX Status" indicates whether the problem was solved to provable optimality ("Optimal") or not ("Feasible") within the two hour time limit.

9.1 Uniform Model

We use the alternative formulation (2.2) of the Uniform Model for all experiments. In this formulation, there are $2nm + n$ variables (mn of them are binary), $2n + m + 3nm$ rows and $m(n + 1)(2n + 1) + 6nm + n$ non-zeros.

Table 3 shows the results of the Uniform Models on the randomly generated data sets. CPLEX solves problems with $n \leq 10$ within seconds to provable optimality, but requires significantly longer time for larger problems. However, even for those problems that CPLEX could not solve to provable optimality within two hours, the optimality gap at termination is very low (largest at 5.58%). Thus, proving optimality seems to be the main difficulty though it succeeds in finding good integer solutions. The heuristic also yields good initial solutions, especially for smaller problems (i.e., $n \leq 20$). For $n = 60$ and $n = 100$, it appears as though the heuristic gives better solution for problems with larger m, but this may be because CPLEX is unable to find better integer solutions for these larger problems.

It is interesting to note that the value of n drives the solution time of the problem. For example, problems with $(n, m) = (10, 100)$ was solved to optimality in 3.98 CPU seconds on average, whereas none of the problems with $(n, m) = (100, 10)$ was solved to provable optimality within two hours. Clearly, the number of variables, number of rows and especially the number of non-zeros differ between the two problem types. For $(n, m) = (10, 100)$, there 2010 variables, 3120 rows, and 29110 non-zeros, whereas $(n, m) = (100, 10)$ has 2100 variables, 3210 rows, and 209110 non-zeros.

The heuristic also appears to perform better, in terms of the heuristic gap, in problems where $n \leq m$. For example, the heuristic gaps for $(n, m) = (60, 2)$ and $(n, m) = (100, 2)$ are 20.34% and 50.27%, respectively. However, the heuristic finds the optimal solution for all instances with $n = 2$. This is probably due to properties described in Lemmas 5.1 and 6.1. With randomly generated data and $n \ll m$, the special properties described in the lemmas are more likely to occur.

Table 4 illustrates the results of the Uniform Model on the real data set. It appears that these problems are harder to solve than those with random data. For example, three of the problems with $n = 10$ for the real data did not solve to provable optimality, whereas all the problems with $n = 10$ solved within 4 CPU seconds for the random data. The final optimality gaps are also significantly worse than that of the random data of similar sizes. This is also true for the Heuristic Gap. Thus, it is clear that the relative values of R_{ij}'s are critical in the solution time of these formulations, not just n and m.

9.2 Weighted Uniform Model

We use the alternative formulation (2.2) of the Weighted Uniform Model for all experiments. The number of variables, number of rows and number of non-zeros are the same as that for the Uniform Model with alternate formulation, namely, there are $2nm + n$ variables (mn of them are binary), $2n + m + 3nm$ rows and $m(n + 1)(2n + 1) + 6nm + n$ non-zeros.

Given that the formulation of the Weighted Uniform Model is very similar to that of the Uniform Model, it is no surprise that the computational results are also very similar. Table 5 shows the results of the Weighted Uniform Model on the randomly generated data set. Again, for most instances, the performance metrics are very similar to that of the Uniform Model. Table 6 illustrates the results on the real data. Just as in the Uniform Model, the optimality gap, running time and the heuristic gap are not as good as with the randomly generated data.

9.3 Share-of-Surplus Model

We use the restricted price formulation of the Share-of-Surplus Model, where $\eta := 1$. In this formulation, there are $\frac{1}{2}mn(n+1) + nm + n$ variables (nm of them being binary), $\frac{3}{2}mn(n+1) + 2n + m$ rows, and $m(n+1)(5n+1) + n$ non-zeros.

Table 7 illustrates the results of the Share-of-Surplus model on the randomly generated data. The running time for this model is significantly longer than that of the Uniform and Weighted Uniform models, especially for large n. This is not completely surprising since the Share-of-Surplus formulation has more variables than the other models (e.g., for $(n, m) = (20, 10)$ the Uniform model has 420 variables, 650 rows and 9830 non-zeros, whereas the Share-of-Surplus model has 2310 variables, 6350 rows and 21230 non-zeros). However, it was still surprising to find that for several problems, the LP relaxation could not be solved within the two hour time limit. For $(n, m) = (20, 100)$ and $(n, m) = (100, 5)$, the LP relaxation of two out of the five instances could not be solved. For $(n, m) = (60, 10)$, the LP relaxation of three out of the five instances could not be solved. The "Final Gap" value presented for these (n, m) pairs is the geometric mean of the optimality gap over the instances whose LP relaxation was solved. For $(n, m) = (60, 20)$, $(60, 60)$, $(60, 100)$, $(100, 10)$, $(100, 20)$, $(100, 60)$, $(100, 100)$, the LP relaxation of none of the five instances could be solved. These are not terribly large problems – $(n, m) = (20, 100)$ has 23100 variables, 63140 rows, and 212120 non-zeros. Thus, there must be some structural properties that make these LPs difficult to solve.

By observing the CPLEX output when solving the LP relaxation, we noticed frequent occurrences of unscaled infeasibility. CPLEX's preprocessor scales the rows of the mixed-integer programming formulation before solving it, and unscaled infeasibility occurs if the optimal solution found for the scaled problem is not feasible for the original problem. This seems to imply that our problem is ill-conditioned. The reservation prices in the problem instances are generally in the range of 500 to 1500. That is, the coefficients of some of the variables are more than 1500 times the coefficients of other variables, making the problem quite ill-conditioned. We can attempt to solve this problem by scaling the reservation prices before using them in the model since the optimal solution is the same regardless of the unit of the reservation prices. In a future work, the impact of this type of specialized scaling on the total computation time should be examined.

Table 8 shows the results with the real data. Similar to the Uniform and Weighted Uniform Model, the real data is more difficult to solve, in general, than the randomly generated data. This is most evident for $n = 10$ and 20, where the optimality gap and the heuristic gap is significantly lower with the real data than with the randomized data.

9.4 Price Sensitive Model

Table 9 shows some preliminary computational results of running small problem instances with the Price Sensitive Model formulation (4.3). The first ten cases (t*) each has 3 products and 3 segments. The next six cases (rand*) each has 5 products and 5 segments and the reservation prices are random numbers that range from 500 to 1200. The rest of the cases are subsets of real data and the file name ($n \times m$) indicates the number of segments and the number of products, respectively, in the inputs. The model was run with default parameter settings of CPLEX 9.1 and a time limit of two hours (7200 CPU seconds). Since the formulation has a second-order cone constraint, only small problems could be solved quickly. The smaller cases can be solved to optimality fairly quickly, but the solutions for the last two cases ("10×10" and "10×20") found by CPLEX after 2 hours have large optimality gaps.

10 Conclusion

We presented various ways to formulate and solve product pricing models using mathematical programming approaches. We have discussed four different probabilistic choice models, all of which are based on reservation prices and are formulated as convex mixed-integer programming problems. The Uniform Distribution Model assumes that Pr_{ij}, the probability that segment i buys product j, is uniform among all products with nonnegative surplus. The Weighted Uniform Model assumes that Pr_{ij} is proportional to the reservation price R_{ij}. In the Share-of-Surplus Model, the probability Pr_{ij} depends on the surplus of the products. Using the assumption that demand increases as price decreases, the Price Sensitive Model uses Pr_{ij} that is inversely proportional to the price of the products with nonnegative surplus. A few special properties of the models have been shown and comparisons of the models' optimal solutions provide some indication of how the models behave. We have proposed and tested a few simple heuristics for finding feasible solutions, which results in strong, often optimal, integer solutions from empirical experiments. Computational results of the various models are also presented and they show that the proposed models are difficult to solve for larger problem instances.

Further research is required to explore ways to improve the solution time of all the models. For example, more investigations on different cuts may be useful, especially on the valid inequalities discussed in Section 7. We also need to examine the structural properties of the problem instances that are especially difficult to solve. For the Share-of-Surplus Model, we may want to investigate other monotone nondecreasing functions to describe the probability which would perhaps lead to formulations that are easier to solve. We may examine the effect of the value of the constant η on the problem and determine the ideal value for the constant. In addition, we currently do not fully understand the effect of scaling the reservation prices and this area should be explored further.

All the models discussed in this paper assume that the company has no competitors. We should explore ways to consider competitor products in our models in order to correctly model the loss of revenue when the customers buy from other companies. We can easily incorporate competitor products in our formulations by considering the surplus of every segment for every competitor product. However, this may unrealistically increase the denominator of Pr_{ij} and collecting such detailed competitor information is very difficult. The challenge is to determine how to include competitor information without explicitly considering each competitor product individually.

The motive of this paper is to show how some marketing models of customer choice behavior can be modelled exactly using mixed-integer programming. This work illustrates the modeling power of integer and convex nonlinear programming techniques and we hope that our work and approach will be extended to other product pricing and customer choice models in the future.

n	m	Provably Optimal(%)	# of SimplxItns	# of Nodes	Final Gap(%)	Time (CPUsec)	Heuristic Gap(%)
2	2	100	9	0	0.00	0.01	0.00
2	5	100	19	0	0.00	0.01	0.00
2	10	100	31	0	0.00	0.01	0.00
2	20	100	58	0	0.00	0.01	0.00
2	60	100	125	0	0.00	0.03	0.00
2	100	100	154	0	0.00	0.04	0.00
5	2	100	38	0	0.00	0.01	0.00
5	5	100	106	0	0.00	0.03	0.00
5	10	100	142	0	0.00	0.04	0.00
5	20	100	217	0	0.00	0.06	0.00
5	60	100	261	0	0.00	0.08	0.00
5	100	100	573	0	0.00	0.26	0.00
10	2	100	109	0	0.00	0.02	0.00
10	5	100	307	11	0.00	0.16	0.00
10	10	100	2252	0	0.00	0.90	0.00
10	20	100	5134	488	0.00	3.83	0.00
10	60	100	1885	0	0.00	1.26	0.00
10	100	100	3259	0	0.00	3.98	0.00
20	2	100	214	0	0.00	0.04	0.00
20	5	100	4401	454	0.00	1.12	0.00
20	10	100	356098	24550	0.01	64.34	0.56
20	20	60	6150940	232358	0.05	1659.42	0.43
20	60	60	1193300	38903	0.04	1102.32	0.06
20	100	80	73764	0	0.00	141.27	0.00
60	2	100	702	7	0.00	0.41	20.34
60	5	100	265222	11806	0.01	81.94	15.02
60	10	0	12324100	444651	3.01	7200.00	2.32
60	20	0	5982610	184930	2.81	7200.00	0.51
60	60	0	977550	27395	0.98	7200.00	0.00
60	100	0	333850	9169	0.35	7200.00	0.00
100	2	100	1025	9	0.00	0.95	50.37
100	5	80	8067590	273307	0.03	4100.81	19.49
100	10	0	7034250	189925	5.58	7200.00	4.80
100	20	0	1562800	50215	4.75	7200.00	0.00
100	60	0	241156	6956	1.20	7200.00	0.00
100	100	0	86500	836	0.65	7200.00	0.00

Table 3: Uniform Model on Randomly Generated Data.

n	m	CPLEX Status	# of SimplxItns	# of Nodes	Final Gap(%)	Time (CPUsec)	Heuristic Gap(%)
2	2	Optimal	5	0	0.00	0.06	0.00
2	5	Optimal	37	0	0.00	0.06	0.00
2	10	Optimal	28	0	0.00	0.06	0.00
2	20	Optimal	28	0	0.00	0.06	0.00
2	60	Optimal	96	0	0.00	0.08	0.00
2	100	Optimal	125	0	0.00	0.10	0.00
5	2	Optimal	40	1	0.00	0.07	0.00
5	5	Optimal	501	99	0.00	0.15	0.18
5	10	Optimal	14520	3066	0.01	1.61	0.00
5	20	Optimal	124	0	0.00	0.10	0.00
5	60	Optimal	146	0	0.00	0.17	0.00
5	100	Optimal	537	27	0.01	1.17	0.12
10	2	Optimal	123	7	0.00	0.08	2.94
10	5	Optimal	14702	4473	0.01	1.80	0.46
10	10	Optimal	3099905	976193	0.01	478.43	1.47
10	20	Feasible	36883946	4521011	7.01	7200.00	1.19
10	60	Feasible	16801245	3206741	2.27	7200.00	1.24
10	100	Feasible	14817894	2201874	0.69	7200.00	0.00
20	2	Optimal	152	0	0.00	0.13	0.34
20	5	Optimal	148607	33304	0.01	22.45	6.08
20	10	Feasible	33346360	5977052	1.18	7200.00	9.66
20	20	Feasible	20586030	1288903	19.08	7200.00	5.53
20	60	Feasible	9038508	435610	11.20	7200.00	2.93
20	100	Feasible	6982060	250213	5.95	7200.00	13.42
60	2	Optimal	5757	420	0.00	2.10	9.57
60	5	Feasible	21893848	3472747	1.77	7200.00	20.11
60	10	Feasible	15824025	1460216	18.64	7200.00	8.83
60	20	Feasible	8570162	290388	21.44	7200.00	13.74
60	60	Feasible	2180690	43495	25.11	7200.00	14.93
60	100	Feasible	996247	6782	21.66	7200.00	8.28
100	2	Optimal	20970	1360	0.00	6.78	9.71
100	5	Feasible	14311720	2001758	7.43	7200.00	15.85
100	10	Feasible	9416746	644540	18.70	7200.00	16.89
100	20	Feasible	3513959	142581	30.80	7200.00	19.42
100	60	Feasible	785803	5144	26.30	7200.00	15.80
100	100	Feasible	375860	659	30.80	7200.00	0.00

Table 4: Uniform Model on Real Data with Heuristic

n	m	Provably Optimal(%)	# of SimplxItns	# of Nodes	Final Gap(%)	Time (CPUsec)	Heuristic Gap(%)
2	2	100	9	0	0.00	0.01	0.00
2	5	100	8	0	0.00	0.01	0.00
2	10	100	11	0	0.00	0.01	0.00
2	20	100	13	0	0.00	0.01	0.00
2	60	100	24	0	0.00	0.02	0.00
2	100	100	126	0	0.00	0.04	0.00
5	2	100	43	0	0.00	0.01	0.00
5	5	100	99	0	0.00	0.02	0.00
5	10	100	123	0	0.00	0.03	0.00
5	20	100	121	0	0.00	0.05	0.00
5	60	100	199	0	0.00	0.07	0.00
5	100	100	466	0	0.00	0.22	0.00
10	2	100	91	0	0.00	0.03	0.00
10	5	100	272	17	0.00	0.11	0.00
10	10	100	1937	0	0.00	0.60	0.00
10	20	100	3054	306	0.00	2.30	0.00
10	60	100	1647	0	0.00	1.15	0.00
10	100	100	2565	0	0.00	3.46	0.00
20	2	100	241	0	0.00	0.06	0.00
20	5	100	3114	390	0.00	0.84	0.00
20	10	100	218660	18240	0.01	39.31	0.55
20	20	80	2910690	138381	0.02	726.41	0.42
20	60	60	994118	29001	0.03	861.67	0.06
20	100	80	45913	0	0.00	98.81	0.00
60	2	100	1938	106	0.00	0.89	19.75
60	5	100	804911	37932	0.01	202.56	14.83
60	10	0	14506800	420194	2.68	7200.00	2.22
60	20	0	6361410	236692	2.42	7200.00	0.49
60	60	0	1229010	31682	0.92	7200.00	0.00
60	100	0	428255	13577	0.30	7200.00	0.00
100	2	100	2720	122	0.00	2.29	56.39
100	5	20	16598900	594915	1.10	7177.44	19.09
100	10	0	7778290	202018	6.45	7200.00	3.88
100	20	0	1845340	56619	4.39	7200.00	0.00
100	60	0	309144	8180	1.16	7200.00	0.00
100	100	0	93974	1500	0.64	7200.00	0.00

Table 5: Weighted Uniform Model on Randomly Generated Data.

n	m	CPLEX Status	# of SimplxItns	# of Nodes	Final Gap(%)	Time (CPUsec)	Heuristic Gap(%)
2	2	Optimal	3	0	0.00	0.01	0.00
2	5	Optimal	45	0	0.00	0.01	0.00
2	10	Optimal	20	0	0.00	0.01	0.00
2	20	Optimal	18	0	0.00	0.01	0.00
2	60	Optimal	4	0	0.00	0.02	0.00
2	100	Optimal	3	0	0.00	0.03	0.00
5	2	Optimal	42	0	0.00	0.01	0.00
5	5	Optimal	391	96	0.00	0.06	0.18
5	10	Optimal	23329	4752	0.01	2.03	1.09
5	20	Optimal	50	0	0.00	0.02	0.00
5	60	Optimal	90	0	0.00	0.03	0.00
5	100	Optimal	421	70	0.01	0.42	0.10
10	2	Optimal	101	9	0.00	0.03	3.08
10	5	Optimal	12343	3907	0.01	1.41	0.42
10	10	Optimal	3348430	1045730	0.01	507.41	1.04
10	20	Feasible	39369300	6194330	6.74	7200.00	0.99
10	60	Feasible	18731400	3423080	1.45	7200.00	1.30
10	100	Feasible	15682000	2204020	0.58	7200.00	0.00
20	2	Optimal	147	2	0.00	0.06	0.36
20	5	Optimal	123083	28175	0.01	18.33	6.00
20	10	Feasible	36021100	5402250	3.93	7200.00	9.05
20	20	Feasible	24832400	1800900	18.30	7200.00	6.04
20	60	Feasible	10603800	862087	8.79	7200.00	3.17
20	100	Feasible	8833870	694551	4.20	7200.00	11.46
60	2	Optimal	4894	684	0.00	1.46	9.66
60	5	Feasible	22890000	3716240	1.99	7200.00	20.18
60	10	Feasible	15735000	1173870	15.68	7200.00	10.66
60	20	Feasible	8696510	373766	20.35	7200.00	14.73
60	60	Feasible	2530950	116001	18.26	7200.00	16.78
60	100	Feasible	1312000	42641	14.40	7200.00	9.19
100	2	Optimal	17588	2030	0.00	5.44	9.68
100	5	Feasible	14179400	2392980	7.23	7200.00	16.28
100	10	Feasible	9981990	790005	18.38	7200.00	16.71
100	20	Feasible	5245070	175370	26.13	7200.00	20.11
100	60	Feasible	1121010	33001	17.46	7200.00	16.05
100	100	Feasible	625020	21315	16.72	7200.00	7.80

Table 6: Weighted Uniform Model on Real Data.

n	m	Provably Optimal(%)	# of SimplxItns	# of Nodes	Final Gap(%)	Time (CPUsec)	Heuristic Gap(%)
2	2	100	14	0	0.00	0.01	0.00
2	5	100	15	0	0.00	0.01	0.00
2	10	100	33	0	0.00	0.01	0.00
2	20	100	48	0	0.00	0.01	0.00
2	60	100	134	0	0.00	0.03	0.00
2	100	100	203	0	0.00	0.06	0.00
5	2	100	171	0	0.00	0.04	0.00
5	5	100	261	0	0.00	0.10	0.00
5	10	100	320	0	0.00	0.18	0.00
5	20	100	1056	0	0.00	0.48	0.00
5	60	100	600	0	0.00	0.28	0.00
5	100	100	1134	0	0.00	0.75	0.00
10	2	100	1392	41	0.00	0.33	0.00
10	5	100	6969	247	0.00	2.55	1.73
10	10	100	91468	3385	0.00	40.18	0.00
10	20	80	307804	8433	0.03	275.30	0.00
10	60	80	8988	0	0.00	10.54	0.00
10	100	100	6067	0	0.00	19.02	0.00
20	2	100	10312	117	0.00	4.59	0.00
20	5	100	406673	7447	0.00	279.50	1.42
20	10	0	5066530	62721	6.82	7200.00	1.42
20	20	0	1937260	13536	3.41	7200.00	0.00
20	60	0	191894	15	0.40	7200.00	0.00
20	100	20	71510	0	0.00*	1528.05	0.00
60	2	100	519426	2345	0.00	1095.90	21.39
60	5	0	466665	161	29.10	7200.00	0.00
60	10	0	206470	0	14.86**	7200.00	0.00
60	20	0	159213	0	noLP	7200.00	0.00
60	60	0	128080	0	noLP	7200.00	0.00
60	100	0	159868	0	noLP	7200.00	0.00
100	2	0	636346	403	35.18	7200.00	51.89
100	5	0	153009	0	43.76*	7200.00	0.00
100	10	0	139430	0	noLP	7200.00	0.00
100	20	0	135284	0	noLP	7200.00	0.00
100	60	0	173941	0	noLP	7200.00	0.00
100	100	0	171525	0	noLP	7200.00	0.00

Table 7: Share-of-Surplus Model with Restricted Prices on Randomly Generated Data. In column "OptGap", ' *', ' **' , and 'noLP' indicate that the LP relaxation of two out of the five, three out of the five, and all five of the instances, respectively, could not be solved within the two hour time limit.

n	m	CPLEX Status	# of SimplxItns	# of Nodes	Final Gap(%)	Time (CPUsec)	Heuristic Gap(%)
2	2	Optimal	9	0	0.00	0.01	0.00
2	5	Optimal	45	0	0.00	0.01	0.00
2	10	Optimal	29	0	0.00	0.01	0.00
2	20	Optimal	52	0	0.00	0.01	0.00
2	60	Optimal	177	0	0.00	0.03	0.00
2	100	Optimal	216	0	0.00	0.04	0.00
5	2	Optimal	174	14	0.00	0.03	7.40
5	5	Optimal	1366	172	0.00	0.23	1.90
5	10	Optimal	88171	13920	0.01	11.90	3.39
5	20	Optimal	184	0	0.00	0.05	0.00
5	60	Optimal	369	0	0.00	0.21	0.00
5	100	Optimal	30968	17285	0.01	112.99	0.02
10	2	Optimal	1097	44	0.00	0.24	0.98
10	5	Optimal	41285	3391	0.01	7.67	10.39
10	10	Optimal	20140700	1218800	0.01	5328.99	18.76
10	20	Feasible	11489200	678326	13.59	7200.00	1.70
10	60	Feasible	2528930	27068	13.72	7200.00	4.97
10	100	Feasible	1398250	7331	12.28	7200.00	0.00
20	2	Optimal	7378	113	0.00	2.75	0.52
20	5	Optimal	3348750	137019	0.01	1680.05	10.60
20	10	Feasible	5765580	67567	24.20	7200.00	20.22
20	20	Feasible	1944430	16779	34.80	7200.00	11.01
20	60	Feasible	159487	250	40.17	7200.00	0.00
20	100	Feasible	122578	0	42.21	7200.00	0.00
60	2	Optimal	619097	2249	0.00	1089.83	13.97
60	5	Feasible	514955	95	37.08	7200.00	13.39
60	10	Feasible	184065	0	43.32	7200.00	0.00
60	20	Feasible	120800	0	noLP	7200.00	0.00
60	60	Feasible	60800	0	noLP	7200.00	0.00
60	100	Feasible	39400	0	noLP	7200.00	0.00
100	2	Feasible	684915	681	15.93	7200.00	14.29
100	5	Feasible	139800	0	34.75	7200.00	17.14
100	10	Feasible	134000	0	noLP	7200.00	0.00
100	20	Feasible	103800	0	noLP	7200.00	0.00
100	60	Feasible	46900	0	noLP	7200.00	0.00
100	100	Feasible	32700	0	noLP	7200.00	0.00

Table 8: Share-of-Surplus Model with Restricted Prices on Real Data.

Problem Name	CPLEX Status	# of SimplxItns	# of Nodes	Final Gap(%)	Time (CPUsec)
t1	Optimal	861	106	0.00	0.61
t2	Optimal	616	54	0.00	0.74
t3	Optimal	453	48	0.00	0.53
t4	Optimal	980	86	0.00	1.00
t5	Optimal	716	84	0.00	0.69
t6	Optimal	1023	98	0.00	0.76
t7	Optimal	579	46	0.00	0.67
t8	Optimal	980	86	0.00	1.01
t9	Optimal	940	86	0.00	0.78
t10	Optimal	507	52	0.00	0.61
rand1	Optimal	20625	1682	0.00	21.30
rand2	Optimal	12025	735	0.00	14.47
rand3	Optimal	7254	498	0.00	9.67
rand4	Optimal	4899	316	0.00	6.34
rand5	Optimal	7443	474	0.00	10.08
rand6	Optimal	11705	873	0.00	12.85
2x2	Optimal	8	0	0.00	0.01
2x5	Optimal	768	56	0.00	0.59
5x2	Optimal	785	81	0.00	0.54
5x5	Optimal	25662	2151	0.00	23.62
5x10	Optimal	2685098	182506	0.01	4232.98
10x5	Optimal	970831	79289	0.01	1331.47
10x10	Feasible	1655571	105230	71.96	7200.00
10x20	Feasible	168276	7984	85.77	7200.00

Table 9: Price Sensitive Model on Small-Scale Randomly Generated Data.

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A Supplemental Proofs

Proof of Theorem 2.1. Suppose we are given the optimal β_{ij}'s for Problem (2.1). Then, the problem simplifies to:

$$\begin{aligned}
\max \quad & \sum_{i=1}^{n} N_i t_i, & \\
\text{s.t.} \quad t_i &\leq \frac{\sum_{j:\beta_{ij}=1} R_{ij}\pi_j}{\sum_{j:\beta_{ij}=1} R_{ij}\beta_{ij}}, & \forall i : \sum_j \beta_{ij} \geq 1, \\
t_i &= 0, & \forall i : \sum_j \beta_{ij} = 0, \\
\pi_j &\leq R_{ij}, & \forall j : \sum_j \beta_{ij} \geq 1, \forall i : \beta_{ij} = 1, \\
\pi_j &\geq \overline{R}_j + 1, & \forall j : \sum_j \beta_{ij} = 0, \\
\pi_j &\leq \overline{R}_j + 1, & \forall j : \sum_j \beta_{ij} = 0.
\end{aligned}$$

The first constraint is from the first constraint in (2.1) using the implications $a_{ij} = \beta_{ij} t_i$ and $p_{ij} = \beta_{ij}\pi_j$. The last three sets of constraints are from the inequalities defining P (1.1). Thus, in the optimal solution, $t_i = \frac{\sum_{j:\beta_{ij}=1} R_{ij}\pi_j}{\sum_j R_{ij}\beta_{ij}}$ if $\sum_j \beta_{ij} \geq 1$ and $t_i = 0$ otherwise. Then, $\pi_j = \min_{i:\beta_{ij}=1} \{R_{ij}\}$ if $\sum_i \beta_{ij} \geq 1$ and $\pi_j = \overline{R}_j + 1$ otherwise. □

Proof of Lemma 5.1. The maximum revenue we can get from segment i is $N_i (\max_j \{R_{ij}\}) = N_i R_{ip(i)}$. This happens when segment i only buys product $p(i)$. Hence, the objective value of any feasible solution is at most $\sum_{i=1}^n N_i R_{ip(i)}$. Consider the solution with the x variables assigned as in the lemma and

$$\pi_j := \begin{cases} \max_{k=1,\ldots,n} \{R_{kj}\}, & \text{if } j \in J, \\ R_{0j}, & \text{otherwise.} \end{cases}$$

Because of the assumptions in the lemma, the solution is feasible with exactly one segment with nonnegative surplus for each product $j \in J$ and no segment buying any products $j \notin J$. Thus, every segment only buys the product with the maximum reservation price. The corresponding objective value is $\sum_{i=1}^n N_i R_{ip(i)}$. Therefore, the solution is optimal. □

Proof of Lemma 5.2. If the x variables are known, then $\pi_j = R_{ij}$ where i is the segment such that $x_{ij} = 1$. If the β variables are known, then $\pi_j = \min_{i:\beta_{ij}=1} R_{ij}$. If the optimal prices are known, we know that each π_j equals the reservation price of some segment. Then in the optimal solution,

$$x_{ij} := \begin{cases} 1, & \text{if } R_{ij} = \pi_j, \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad \beta_{ij} := \begin{cases} 1, & \text{if } R_{ij} \geq \pi_j, \\ 0, & \text{otherwise.} \end{cases}$$

These are the only values that would make the solution feasible. □

Proof of Lemma 5.3. Suppose in an optimal solution, $\beta_{st} = 0$. We know that $\beta_{it} = 0$ for all segments i in all three models. Let v be the optimal value. Consider the objective value v' if β_{st} is set to 1. We will have $\pi_t = R_{st}$.

In the Uniform Model, if $\sum_j \beta_{sj} = 0$, then clearly the objective value increases by $N_s R_{st}$. If $\sum_j \beta_{sj} \geq 1$, then

$$v' - v = N_s \left(\frac{R_{st} + \sum_j p_{sj}}{1 + \sum_j \beta_{sj}} - \frac{\sum_j p_{sj}}{\sum_j \beta_{sj}} \right) = N_s \left(\frac{\sum_j (R_{st} - \pi_j)\beta_{sj}}{(\sum_j \beta_{sj})(1 + \sum_j \beta_{sj})} \right). \tag{A.1}$$

R_{st} is the maximum reservation price and each of the π_j's equals to a reservation price, so $R_{st} \geq \pi_j \quad \forall j$. The condition $\sum_j \beta_{sj} \geq 1$ implies that $\beta_{sk} = 1$ for some product $k \neq t$, and we know that $R_{st} > \pi_k$. So the expression (A.1) is strictly positive. This contradicts the fact that it is an optimal solution. Therefore, $\beta_{st} \geq 1$ in every optimal solution.

Similarly, in the Weighted Uniform Model, if $\sum_j \beta_{sj} = 0$, then clearly the objective value increases by $N_s R_{st}$. If $\sum_j \beta_{sj} = 1$, then

$$v' - v = N_s R_{st} \left(\frac{\sum_j (\pi_t - \pi_j) R_{sj} \beta_{sj}}{(\sum_j R_{sj}\beta_{sj})(R_{st} + \sum_j R_{sj}\beta_{sj})} \right) > 0,$$

where $\pi_t = R_{st} > \pi_j, \quad \forall j$.

In the Share-of-Surplus Model with restricted prices, if $\sum_j \beta_{sj} = 0$, then the objective value increases by $N_s R_{st}$. Otherwise,

$$\begin{aligned} rclv' - v = N_s & \left(\frac{\pi_t (R_{st} - \pi_t + \eta) + \sum_j \pi_j (R_{sj} - \pi_j + \eta)\beta_{sj}}{(R_{st} - \pi_t + \eta) + \sum_j (R_{sj} - \pi_j + \eta)\beta_{sj}} - \frac{\sum_j \pi_j (R_{sj} - \pi_j + \eta)\beta_{sj}}{\sum_j (R_{sj} - \pi_j + \eta)\beta_{sj}} \right) \\ =& N_s (R_{st} - \pi_t + \eta) \left(\frac{\sum_j (\pi_t - \pi_j)(R_{sj} - \pi_j + \eta)\beta_{sj}}{(\sum_j (R_{sj} - \pi_j + \eta)\beta_{sj})((R_{st} - \pi_t + \eta) + \sum_j (R_{sj} - \pi_j + \eta)\beta_{sj})} \right) \\ >& 0, \end{aligned}$$

where we let $\eta > 0$ to avoid singularity.

In all three models, we showed that the solution is not optimal if $\beta_{st} = 0$. So in every optimal solution, segment s buys product t. □

Proof of Lemma 6.1. The proof directly follows from Lemma 5.1. □

Proof of Lemma 6.2. The time it takes to examine a product k is $O(mf(n,m))$ since we consider up to m products that product k can swap with. A product is examined multiple times only if its price increases after a swap. Since a product's price always equals to a segment's reservation price, it can only increase at most n times. So there are at most $O(nm)$ iterations to examine a product, and each iteration has a runtime of $O(mf(n,m))$. Therefore, the runtime of Heuristic 2 is $O(nm^2 f(n,m))$. □

Proof of Lemma 7.1. The Hessian of f is

$$\nabla^2 f = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0}^T & 0 \end{bmatrix},$$

where $\boldsymbol{A} := \begin{bmatrix} 2a & \boldsymbol{e}^T \\ \boldsymbol{e} & 2\boldsymbol{B} \end{bmatrix}$, $\boldsymbol{B}$ is an $m \times m$ diagonal matrix with $b, b, \ldots, b$ on the diagonal, and $\boldsymbol{e}$ is a vector of all ones. Since f is twice continuously differentiable, f is convex iff $\boldsymbol{A}$ is a positive semi-definite matrix. If $b \leq 0$, then $\boldsymbol{A}$ is not positive semi-definite and f is not convex. So we may assume $b > 0$.

The Schur-complement of $\boldsymbol{B}$ in $\boldsymbol{A}$ is $2a - \frac{1}{2b}(\boldsymbol{e}^T\boldsymbol{e})$. Thus,

$$\boldsymbol{A} \succeq 0 \quad \Leftrightarrow \quad a - \frac{m}{4b} \geq 0 \quad \Leftrightarrow \quad ab \geq \frac{m}{4}.$$

□

Proof of Lemma 7.2. As in the proof of the previous lemma, F is twice continuously differentiable. The Hessian of F is

$$\nabla^2 F = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0}^T & \boldsymbol{0} \end{bmatrix},$$

where $\boldsymbol{A} := \begin{bmatrix} 2a & \boldsymbol{e}^T \\ \boldsymbol{e} & 2\boldsymbol{B} \end{bmatrix}$. Therefore, F is convex iff $\boldsymbol{A}$ is positive semidefinite. If for some j, $b_j \leq 0$, then $\boldsymbol{A}$ is not positive semidefinite. Therefore, $\boldsymbol{b} > \boldsymbol{0}$. Let $\bar{\boldsymbol{b}} := (\frac{1}{\sqrt{b_1}}, \frac{1}{\sqrt{b_2}}, \ldots, \frac{1}{\sqrt{b_m}})^T$. Also, if $a \leq 0$ then $\boldsymbol{A}$ is not positive semidefinite; thus, $a > 0$. The Schur complement of a in $\boldsymbol{A}$ is $2\boldsymbol{B} - \frac{1}{2a}\boldsymbol{e}\boldsymbol{e}^T$. Thus,

$$\begin{aligned} \boldsymbol{A} \succeq 0 \quad &\Leftrightarrow \quad 2\boldsymbol{B} - \frac{1}{2a}\boldsymbol{e}\boldsymbol{e}^T \succeq 0 \\ &\Leftrightarrow \quad 4I - \frac{1}{a}\bar{\boldsymbol{b}}\bar{\boldsymbol{b}}^T \succeq 0 \\ &\Leftrightarrow \quad 4\bar{\boldsymbol{b}}^T\bar{\boldsymbol{b}} - \frac{1}{a}(\bar{\boldsymbol{b}}^T\bar{\boldsymbol{b}})^2 \geq 0 \\ &\Leftrightarrow \quad a \geq \frac{\bar{\boldsymbol{b}}^T\bar{\boldsymbol{b}}}{4} = \sum_{j=1}^{m} \frac{1}{4b_j}. \end{aligned}$$

□

Proof of Lemma 7.6. For given β_{ij}'s, finding the most violated subset P_{ik}^* for (7.5) is equivalent to solving

$$\begin{aligned} \max \quad & \textstyle\sum_{j=1}^m \beta_{ij} z_j, \\ \text{s.t.} \quad & \textstyle\sum_{j=1}^m z_j = k+1, \\ & 0 \leq z_j \leq 1, \qquad j = 1, \ldots, m. \end{aligned}$$

Since the feasible region of the above LP is an integral polyhedron, and since the LP is clearly feasible and bounded, it has an optimal 0,1 solution corresponding to the characteristic vector of P_{ik}^*. The Dual of this LP is:

$$\begin{aligned} \min \quad & (k+1)q + \textstyle\sum_{j=1}^m p_j, \\ \text{s.t.} \quad & q + p_j \geq \beta_{ij}, \qquad j = 1, \ldots, m, \\ & p_j, \qquad j = 1, \ldots, m. \end{aligned}$$

If there exist β_{ij}'s and y_{il} which satisfy (7.5) for all covers P_{ik}, then it must satisfy (7.5) for P_{ik}^*. Thus, from strong duality, there exist q and p_j satisfying the constraints for the Dual LP and $\sum_{j \in P_{ik}^*} \beta_{ij} = (k+1)q + \sum_{j=1}^m p_j$.

Conversely, if there exist q and p_j which satisfy the constraints of the Dual LP and there is a y_{il} such that $(k+1)q + \sum_{j=1}^m p_j + \sum_{l=0}^k y_{il} \leq k+1$, then from weak duality, $\sum_{j \in P_{ik}} \beta_{ij} \leq (k+1)q + \sum_{j=1}^m p_j$ for all P_{ik}'s and thus, $\sum_{j \in P_{ik}} \beta_{ij} + \sum_{l=0}^k y_{il} \leq k+1$ for all P_{ik}'s. □

Proof of Lemma 7.7. The system (7.7) are valid inequalities for the pure 0,1 formulation (7.4). Thus, $\hat{x}_{ij}$ is a feasible integer solution to (2.2) if and only if $\hat{x}_{ij}$ and $\hat{y}_{ik} = 1$, where $k = \sum_{l:R_{lj} \le R_{ij}} x_{lj}$ is a feasible integer solution to (7.4).

Using Farkas' Lemma, (7.7) is infeasible if and only if there exist u, v, and w_k, $k = 0, \ldots, m$, where

$$\begin{aligned} u + kv + \textstyle\sum_{l=0}^{k} w_k \ge 0, &\qquad k = 0, \ldots, m, \\ u + \textstyle\sum_{j=1}^{m} \bar{\beta}_{ij} v + \left(k + 1 - \sum_{j \in P_{ik}^*} \bar{\beta}_{ij}\right) w_k < 0, & \\ w_k \ge 0, &\qquad k = 0, \ldots, m. \end{aligned}$$

Therefore, $\sum_{j=1}^{m} v\beta_{ij} + \sum_{j \in P_{ik}^*} w_k \beta_{ij} \le \sum_{k=0}^{m} (k+1) w_k$ is a valid inequality for (2.2) and it is violated by $\bar{\beta}_{ij}$. □

B Estimating the Reservation Price

The reservation price data used in the computational experiments of Section 9 are estimated from actual purchase orders of a Canadian travel company. The customers are partitioned into segments according to their demographic information, purchase lead time and other characteristics. Suppose after the segmentation, there are n customers, with N_i customers in segment i, $i = 1, \ldots, n$. The company offers m products.

From the historical data, we know what fraction of customers of each segment purchased each product and how much they paid for it. Let

$$\begin{aligned} fr_{ij} &:= \text{the fraction of segment } i \text{ customers who purchased product } j, \\ B_i &:= \{j : fr_{ij} > 0\}, \text{ i.e., set of products purchased by segment } i, \\ p_{ij} &:= \text{the price that customers of segment } i \text{ paid for product } j. \end{aligned}$$

The price paid for a particular product may be slightly different from customer to customer depending on the time of sales and other anomalies. Thus, the above p_{ij} value is the average price paid by segment i for product j.

To estimate the reservation price R_{ij} of segment i for product j, we assumed that customers behaved according to the share-of-surplus model of Section 3. Thus, fr_{ij} should be approximately equal to

$$\frac{R_{ij} - p_{ij}}{\sum_{k \in B_i} (R_{ik} - p_{ik})},$$

where R_{ij}'s are now variables and p_{ij}'s are data.

We fit R_{ij}'s and the share-of-surplus model to the data using least squares regression, i.e., for each segment i, we solved for R_{ij}'s, $j = 1, \ldots, m$, that minimizes

$$\sum_{j \in B_i} \left(f_{ij} - \frac{R_{ij} - p_{ij}}{\sum_{k \in B_i} R_{ik} - p_{ik}} \right)^2 \text{ or } \sum_{j \in B_i} \left(f_{ij} \Big(\sum_{k \in B_i} R_{ik} - p_{ik} \Big) - R_{ij} - p_{ij} \right)^2$$

$$\text{subject to:} \quad \begin{aligned} R_{ij} - p_{ij} &\ge 0, \quad j \in B_i, \\ \textstyle\sum_{k \in B_i} R_{ik} - p_{ik} &\ge \delta \end{aligned}$$

where $\delta > 0$.

There are some further details that need to be addressed. One of the key issues is estimating R_{ij} for $j \notin B_i$. Currently, we have these R_{ij}'s set to 0, which is clearly an underestimate. Although we do not have any direct information about segment i's preference level of product j, we may be able to infer this from other segments that do purchase product j. As a future work, we can consider using data mining techniques such as clustering and collaborative filtering to determine these R_{ij}'s (see for instance [13, 14]).

Also note that in the above we only described a way to roughly estimate the *expected values* of the reservation prices R_{ij}. Indeed, most companies have much more detailed information on their customers (especially those customers with frequent flier accounts) and the related market pricing and volume data. Using these, one usually observes the empirical distribution of the sales, and also one accounts for the potential customers' activities on the firm's website (those who search but perhaps do not buy) and fits a distribution function. Finally, the actual R_{ij}s are generated from these distributions using certain sampling techniques. Much of the details of these procedures are sector dependent and more importantly company dependent and are usually kept confidential.

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R. Shioda
Bank of America Merrill Lynch, London
E-mail address: shioda@baml.com

L. Tunçel
Department of Combinatorics and Optimization
Faculty of Mathematics, University of Waterloo
Waterloo, Ontario N2L 3G1, Canada
E-mail address: ltuncel@math.uwaterloo.ca

B. Hui
E-mail address: Betsy.Hui@gmail.com