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A REGULARIZED OUTER APPROXIMATION METHOD FOR MONOTONE SEMI-INFINITE VARIATIONAL INEQUALITY PROBLEMS*

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Abstract: The semi-infinite variational inequality problem (SIVIP) is a variational inequality problem (VIP) whose feasible set is given by infinitely many convex inequalities. To solve SIVIP, we propose an outer approximation method with a regularization technique, which approximately solves a VIP with a finite number of inequality constraints at each iteration. In the algorithm, the regularized gap function is used to specify a criterion for approximate solutions of such VIPs. We establish global convergence of the algorithm by assuming the monotonicity of the problem, Slater's condition, and the existence of a solution. We also report some numerical results to examine the effectiveness of the algorithm.

Key words: *semi-infinite variational inequality problem, outer approximation method, regularized gap function*

Mathematics Subject Classification: *65K15, 90C33, 90C34*

1 Introduction

The variational inequality problem (VIP), denoted VI(S, F), is to find a point $x^* \in \mathbb{R}^n$ such that

$$x^* \in S, \quad \langle F(x^*), x - x^* \rangle \geq 0 \quad \text{for all} \quad x \in S,$$

where S is a nonempty closed convex subset of $\mathbb{R}^n$, F is a mapping from $\mathbb{R}^n$ into itself, and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$. Throughout the paper, we assume that the mapping F is continuously differentiable. The VIP has been used to model various equilibrium problems in economics, operations research, transportation and regional science. A survey of theory, algorithms and applications of the VIP can be found in [8, 13]. The VIP has been well studied and a lot of algorithms are available for solving it. However, many of the existing algorithms work practically only when S exhibits a certain tractable structure such as the nonnegative orthant of $\mathbb{R}^n$ or a polyhedral set.

In this paper, we consider VI(S, F) with S defined by

$$S = \{\, x \in \mathbb{R}^n \mid g(x, t) \leq 0 \quad \text{for all } t \in T \,\}, \tag{1.1}$$

where $T \subset \mathbb{R}^m$ is a nonempty compact set with infinitely many elements and $g : \mathbb{R}^n \times T \to \mathbb{R}$ is a continuous function such that $g(\cdot, t)$ is a convex function for each fixed $t \in T$. We call

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the VIP with S defined by (1.1) the semi-infinite variational inequality problem (SIVIP), and refer to as SIVI(S, F). Note that S can be expressed as

$$S = \{ x \in \mathbb{R}^n \mid h(x) \le 0 \}, \tag{1.2}$$

where the function $h : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$h(x) = \max_{t \in T} g(x, t). \tag{1.3}$$

Despite numerous works on the VIP and the semi-infinite programming problem (SIP) [19, 22], which is the mathematical programming problem with the feasible set defined by (1.1), the study on theory and algorithms for solving the SIVIP is relatively recent.

Since S is defined by infinitely many inequalities, it is hard to deal with the problem directly in practice. Then, for solving VI(S, F) with such S, several algorithms have been proposed by utilizing an outer approximation technique. The fundamental idea underlying the outer approximation method is to generate a sequence $\{ x^k \}$ by solving subproblems whose feasible sets S^k contain S and are relatively easy to deal with. Fukushima [11] introduced a relaxed projection method for solving VI(S, F) where S is defined by (1.2) with a general convex function h. This method computes at each iteration the projection onto a half space S^k defined by using the subdifferential of h at the current iterate x^k. It has been showed that this method is globally convergent under the strong monotonicity assumption on F. A method with S^k replaced by a general half space separating x^k from S has been proposed by Censor and Gibali [7]. Bello Cruz and Iusem [2] provided an algorithm based on the method in [11] and showed its global convergence under the weaker condition that F is paramonotone [15]. Moreover, the same authors [3] presented a relaxed projection-type method which is globally convergent under the mere monotonicity assumption on F.

For SIVI(S, F) with S particularly defined by (1.1), there have been proposed several outer approximation methods. These methods solve VI(S^k, F) at each iteration k, where S^k is an outer approximation of S defined by finitely many inequalities. Under the assumption that F is paramonotone and S is compact, Burachik *et al.* [4] proposed an outer approximation scheme for solving the SIVIP. Assuming S is compact and F is Lipschitz continuous and pseudomonotone-plus [17], implementable algorithms have been presented in [9, 21]. Under the same assumptions, Fang *et al.* [10] provided an inexact method which uses an approximate solution of VI(S^k, F) at each iteration. They use the gap function [1] to check a criterion for approximate solutions. Without the compactness of S, Burachik *et al.* [6] introduced a scheme combining the outer approximation method with a regularization method, which solves VI(S^k, F^k) at each iteration, where F^k is an approximate mapping of F. They proved that when F is paramonotone, a sequence of exact solutions of VI(S^k, F^k) is bounded and any of its accumulation points solves SIVI(S, F).

We propose an algorithm for solving the SIVIP, which is an implementable version of the regularized outer approximation scheme [6]. This method needs only an approximate solution of VI(S^k, F^k), while the scheme in [6] assumes the exact solution of VI(S^k, F^k). We use the regularized gap function [12] to specify a criterion for approximate solution of VI(S^k, F^k). Moreover, we establish global convergence of the proposed algorithm by assuming the existence of a solution of SIVI(S, F), Slater's condition, and the monotonicity of F, which is a weaker assumption on F than the paramonotonicity assumed in [6].

This paper is organized as follows. In Section 2, we review some preliminary results concerning monotone mappings and the regularized gap function. In Section 3, we present an outer approximation method and show that it is globally convergent under the strong monotonicity assumption on the mapping F. In Section 4, we propose the main algorithm by combining the method proposed in Section 3 with a regularization method, and establish its global convergence under the mere monotonicity assumption on the mapping F. In Section 5, we give some numerical results to examine the effectiveness of the proposed algorithm. In Section 6, we conclude the paper with some remarks.

2 Preliminaries

In this section, we simply assume that S is a general nonempty closed convex set. For $\mathrm{VI}(S,F)$, Auslender [1] showed that the gap function $f_0 : \mathbb{R}^n \to \mathbb{R} \cup \{ +\infty \}$ defined by

$$f_0(x) = \sup_{y \in S} \langle F(x), x - y \rangle$$

attains its minimum on S at a solution of $\mathrm{VI}(S,F)$. However, the function f_0 is in general nondifferentiable and may even fail to be finite-valued. To overcome such drawbacks of the gap function, Fukushima [12] proposed the regularized gap function $f_\alpha : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f_\alpha(x) = \sup_{y \in S} \left\{ \langle F(x), x - y \rangle - \frac{1}{2} \alpha \, \|y - x\|^2 \right\},$$

where $\alpha > 0$ is a parameter. This function enjoys a similar property to that of the gap function, as shown in the following proposition.

Proposition 2.1 ([12, Theorem 3.1])**.** *The function f_α satisfies $f_\alpha(x) \geq 0$ for all $x \in S$. Moreover, $f_\alpha(x) = 0$ and $x \in S$ if and only if x solves* $\mathrm{VI}(S,F)$*. Hence x solves* $\mathrm{VI}(S,F)$ *if and only if it solves the following optimization problem and $f_\alpha(x) = 0$:*

$$\begin{array}{ll} \text{minimize} & f_\alpha(x) \\ \text{subject to} & x \in S. \end{array}$$

Unlike the gap function f_0, the regularized gap function f_α is always finite-valued. When the mapping F is continuously differentiable, so is f_α [12, Theorem 3.2]. In addition, the function f_α enjoys some useful properties. Several methods utilizing this function have been proposed for solving $\mathrm{VI}(S,F)$ [24–26].

Recall that the mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *monotone* if

$$\langle F(x) - F(y), x - y \rangle \geq 0 \quad \text{for any} \quad x, y \in \mathbb{R}^n, \tag{2.1}$$

strictly monotone if strict inequality holds in (2.1) whenever $x \neq y$, and *strongly monotone* with modulus $\mu > 0$ if

$$\langle F(x) - F(y), x - y \rangle \geq \mu \, \|x - y\|^2 \quad \text{for any} \quad x, y \in \mathbb{R}^n.$$

Clearly any strongly monotone mapping is strictly monotone, and any strictly monotone mapping is monotone.

3 Algorithm for the Strongly Monotone SIVIP

In this section, we make the following assumption, which ensures that $\mathrm{SIVI}(S,F)$ has a unique solution.

Assumption 3.1. The mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is strongly monotone with modulus $\mu > 0$.

We propose a method for solving SIVI(S, F) with S given by (1.1) and show its global convergence under Assumption 3.1. The proposed algorithm consists of major iterations and inner iterations within each major iteration. On the rth inner iteration of the kth major iteration, a nonempty *finite* set $T^{k,r} \subset T$ is given. We define the convex set $S^{k,r} \subseteq \mathbb{R}^n$ by

$$S^{k,r} = \{\, x \in \mathbb{R}^n \mid g(x,t) \le 0 \quad \text{for all } t \in T^{k,r} \,\}, \tag{3.1}$$

and consider VI$(S^{k,r}, F)$, which is easier than SIVI(S, F) to deal with, since $S^{k,r}$ is given by finitely many inequalities. Moreover, define the function $f_\alpha^{k,r} : \mathbb{R}^n \to \mathbb{R}$ by

$$f_\alpha^{k,r}(x) = \sup_{y \in S^{k,r}} \left\{ \langle F(x), x - y \rangle - \frac{1}{2}\alpha \, \|y - x\|^2 \right\}, \tag{3.2}$$

where $\alpha > 0$. The function $f_\alpha^{k,r}$ is the regularized gap function for VI$(S^{k,r}, F)$, which is continuously differentiable. By Proposition 2.1, VI$(S^{k,r}, F)$ is equivalent to the optimization problem

$$\begin{array}{ll} \text{minimize} & f_\alpha^{k,r}(x) \\ \text{subject to} & x \in S^{k,r}. \end{array} \tag{3.3}$$

Note that Assumption 3.1 ensures that, for every k and r, VI$(S^{k,r}, F)$ has the unique solution $\bar{x}^{k,r}$ which satisfies $f_\alpha^{k,r}(\bar{x}^{k,r}) = 0$ and $\bar{x}^{k,r} \in S^{k,r}$.

The next proposition shows that $f_\alpha^{k,r}$ can be used as an error bound for VI$(S^{k,r}, F)$, provided the parameter α is chosen sufficiently small.

Proposition 3.2 ([26, Proposition 3.4]). *The function $f_\alpha^{k,r}$ satisfies the inequality*

$$f_\alpha^{k,r}(x) \ge \left(\mu - \frac{1}{2}\alpha \right) \|x - \bar{x}^{k,r}\|^2 \quad \textit{for all } x \in S^{k,r}.$$

In the remainder of this section, we assume the following.

Assumption 3.3. The parameter α satisfies $0 < \alpha < 2\mu$.

We propose the following algorithm which only requires an approximate solution $x^{k,r}$ of VI$(S^{k,r}, F)$ for each k and r. In the stopping criterion of the algorithm, we use the functions $\theta_\alpha^k : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\theta_\alpha^k(x) = \max \left(f_\alpha^{k,r(k)}(x), \max_{t \in T} g(x,t) \right), \tag{3.4}$$

where $r(k)$ denotes the number of inner iterations within the kth major iteration.

Algorithm 1.

Step 0. Choose $x^0 \in \mathbb{R}^n$, $\alpha \in (0, 2\mu)$, TOL ≥ 0 and sequences $\{\delta_k\}, \{\sigma_k\} \subset \mathbb{R}_{++}$ such that $\lim_{k\to\infty} \delta_k = \lim_{k\to\infty} \sigma_k = 0$. Set $k := 1$.

Step 1. Obtain x^k by the following procedure.

Step 1-0. Choose a nonempty finite set $T^{k,1} \subset T$. Set $r := 1$ and $x^{k,0} := x^{k-1}$.

Step 1-1. Find an approximate solution $x^{k,r}$ of VI$(S^{k,r}, F)$ such that $f_\alpha^{k,r}(x^{k,r}) \le \delta_k$ and $x^{k,r} \in S^{k,r}$.

Step 1-2. Find $t^{k,r} \in T$ such that $g(x^{k,r}, t^{k,r}) > \sigma_k$.

(i) If such $t^{k,r}$ does not exist, then set $r(k) := r$, $x^k := x^{k,r}$. Go to Step 2.

(ii) Otherwise, set $T^{k,r+1} := T^{k,r} \cup \{ t^{k,r} \}$, $r := r+1$ and return to Step 1-1.

Step 2. If $\theta_\alpha^k(x^k) \leq \mathrm{TOL}$, then output x^k and stop. Otherwise, set $k := k+1$ and return to Step 1,

Some remarks about Algorithm 1 are in order. Similar remarks also apply to Algorithm 2 to be presented in the next section. First, to find $x^{k,r}$ in Step 1-1, we may use an iterative method such as [12, 26] for solving (3.3). Second, to find $t^{k,r}$ in Step 1-2, we need to carry out global maximization of the function $g(t, x^{k,r})$ over $t \in T$. This is by no means an easy task unless the function $g(t, x^{k,r})$ is concave with respect to t. However, some kind of global optimization is generally unavoidable in semi-infinite problems. Here we simply presume that such a $t^{k,r}$ can be found whenever one exists. In Section 5, we suggest a practical procedure to find a $t^{k,r}$ for the case where T is an interval in $\mathbb{R}$.

We first show that the inner iterations within Step 1 do not repeat infinitely, which ensures that Algorithm 1 is well-defined.

Proposition 3.4. *The inner iterations in Step 1 terminate finitely by producing x^k for every k.*

Proof. Assume that, on some kth iteration, Step 1 does not terminate finitely, that is, an infinite sequence $\{ x^{k,r} \}$ is generated for some k. Choose $\hat{x} \in S$ arbitrarily and define

$$M := \max_{y \in \mathbb{R}^n} \left\{ \langle F(\hat{x}), \hat{x} - y \rangle - \frac{1}{2}\alpha \|y - \hat{x}\|^2 \right\},$$

which is finite since the maximand is a strongly concave quadratic function of y. For any k and r, we have $M \geq f_\alpha^{k,r}(\hat{x})$ by the definition of $f_\alpha^{k,r}$. It follows from $\hat{x} \in S \subseteq S^{k,r}$ and Proposition 3.2 that

$$f_\alpha^{k,r}(\hat{x}) \geq \left(\mu - \frac{1}{2}\alpha \right) \|\hat{x} - \bar{x}^{k,r}\|^2,$$

where $\bar{x}^{k,r}$ is the unique solution of $\mathrm{VI}(S^k, F)$. Hence we have

$$\frac{M}{\mu - \frac{1}{2}\alpha} \geq \|\hat{x} - \bar{x}^{k,r}\|^2.$$

Moreover, we have

$$\delta_k \geq f_\alpha^{k,r}(x^{k,r}) \geq \left(\mu - \frac{1}{2}\alpha \right) \|x^{k,r} - \bar{x}^{k,r}\|^2,$$

and hence the following inequalities hold:

$$\begin{aligned} \|x^{k,r} - \hat{x}\| &\leq \|x^{k,r} - \bar{x}^{k,r}\| + \|\bar{x}^{k,r} - \hat{x}\| \\ &\leq \sqrt{\frac{\delta_k}{\mu - \frac{1}{2}\alpha}} + \sqrt{\frac{M}{\mu - \frac{1}{2}\alpha}}, \end{aligned} \tag{3.5}$$

which implies that $\{ x^{k,r} \}$ is bounded. Moreover, since T is compact, $\{ (x^{k,r}, t^{k,r}) \}$ has at least one accumulation point. Let $(x^{k,\infty}, t^{k,\infty})$ be an arbitrary accumulation point of $\{ (x^{k,r}, t^{k,r}) \}$, and $\{ (x^{k,r}, t^{k,r}) \}_{r \in \mathbb{K}}$ be a subsequence of $\{ (x^{k,r}, t^{k,r}) \}$ which converges to $(x^{k,\infty}, t^{k,\infty})$. We claim that $g(x^{k,\infty}, t^{k,r}) \leq 0$ for all $r \in \mathbb{K}$. Assume to the contrary that there exists $\bar{r} \in \mathbb{K}$ such that $g(x^{k,\infty}, t^{k,\bar{r}}) > 0$. Then, there exists a sufficiently large $\tilde{r}$ such that $\tilde{r} > \bar{r}$, $\tilde{r} \in \mathbb{K}$, and $g(x^{k,\tilde{r}}, t^{k,\bar{r}}) > 0$ by the continuity of g. On the other hand, since $t^{k,\bar{r}} \in T^{k,\tilde{r}}$, we have $g(x^{k,\tilde{r}}, t^{k,\bar{r}}) \leq 0$. This is a contradiction. Hence we have $g(x^{k,\infty}, t^{k,r}) \leq 0$ for all $r \in \mathbb{K}$, which implies $g(x^{k,\infty}, t^{k,\infty}) \leq 0$. However, it follows from $g(x^{k,r}, t^{k,r}) > \delta_k$ and the continuity of g that $g(x^{k,\infty}, t^{k,\infty}) \geq \delta_k > 0$, which again yields a contradiction. Then the result follows. ☐

The following proposition validates the use of the function θ_α^k, which is defined by (3.4), in the stopping criterion of the algorithm.

Proposition 3.5. *For any k, the function θ_α^k satisfies $\theta_\alpha^k(x) \geq 0$ for all $x \in \mathbb{R}^n$. Moreover, $\theta_\alpha^k(x) = 0$ if and only if x solves* SIVI(S, F).

Proof. Let x be an arbitrary point in $\mathbb{R}^n$. If $x \notin S$, then we have $\max_{t \in T} g(x,t) > 0$. If $x \in S$, we have $f_\alpha^{k,r(k)}(x) \geq f_\alpha(x) \geq 0$ from Proposition 2.1 and the definitions of $f_\alpha^{k,r(k)}$ and f_α. This proves the first part of the proposition. Since $\theta_\alpha^k(x) = 0$ implies that $f_\alpha(x) \leq f_\alpha^{k,r(k)}(x) \leq 0$ and $\max_{t \in T} g(x,t) \leq 0$, the last part of the proposition also follows from Proposition 2.1. ☐

By the above proposition, if the algorithm with TOL $= 0$ terminates at some kth major iteration, then x^k solves SIVI(S, F). Otherwise, the algorithm generates an infinite sequence $\{ x^k \}$. We next show that the sequence $\{ x^k \}$ is bounded.

Lemma 3.6. *The sequence $\{ x^k \}$ is bounded.*

Proof. The result follows from (3.5) in the proof of Proposition 3.4, since $x^k = x^{k,r(k)}$ for each k and $\lim_{k \to \infty} \delta_k = 0$. ☐

We are ready to show global convergence of Algorithm 1 with TOL $= 0$, which implies that Algorithm 1 with TOL > 0 terminates finitely.

Theorem 3.7. *Let* TOL $= 0$. *Then the sequence $\{ x^k \}$ converges to the unique solution of* SIVI(S, F).

Proof. We assume that the sequence is infinite, since otherwise the algorithm terminates at a solution by Proposition 3.5. It then follows from Lemma 3.6 that $\{ x^k \}$ is bounded and has at least one accumulation point. Let x^∞ be an arbitrary accumulation point of $\{ x^k \}$, and $\{ x^k \}_{k \in \mathbb{K}}$ be a subsequence of $\{ x^k \}$ which converges to x^∞. By the definitions of f_α and $f_\alpha^{k,r(k)}$, we have

$$f_\alpha(x^k) \leq f_\alpha^{k,r(k)}(x^k) \leq \delta_k$$

for all k. Thus we have $f_\alpha(x^\infty) \leq 0$, since f_α is continuous and $\{ \delta_k \}$ converges to 0. Moreover, by the construction of Algorithm 1, we have

$$h(x^k) \leq \sigma_k \quad \text{for all } k \in \mathbb{N},$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is defined by (1.3). This yields $h(x^\infty) \leq 0$ by the continuity of h and $\sigma_k \to 0$, which implies $x^\infty \in S$. Therefore by Proposition 2.1, x^∞ solves SIVI(S, F). Since the solution of SIVI(S, F) is unique by the strong monotonicity of F, we conclude that the entire sequence $\{ x^k \}$ converges to the solution of SIVI(S, F). ☐

$\boxed{4}$ Algorithm for the Monotone SIVIP

In the previous section, under the strong monotonicity assumption on F, we have proved that a sequence of approximate solutions of $\mathrm{VI}(S^k, F)$ converges to the solution of $\mathrm{SIVI}(S, F)$. However, the strong monotonicity of F is very restrictive in practice. In this section, we propose a method that incorporates a regularization technique with the previous algorithm, and establish its global convergence without assuming the strong monotonicity of F. Throughout this section, we make the following assumption.

Assumption 4.1.

(i) $\mathrm{SIVI}(S, F)$ has at least one solution.

(ii) The mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is monotone on $\mathbb{R}^n$.

(iii) There exists $w \in \mathbb{R}^n$ such that $h(w) = \max_{t\in T} g(w, t) < 0$ (Slater's condition).

Let $\{ \varepsilon_k \}$ be a positive sequence converging to 0 and w be a Slater point as given in Assumption 4.1 (iii). Such a point w can be obtained by applying a descent method for minimizing the nonsmooth convex function h, whenever the Slater condition holds. For each k, define the mapping $F^k : \mathbb{R}^n \to \mathbb{R}^n$ by

$$F^k(x) = F(x) + \varepsilon_k(x - w). \tag{4.1}$$

Then, F^k is strongly monotone under Assumption 4.1 (ii). Let $\{ \sigma_k \}$ be a positive sequence converging to 0. In addition, for each k, let S^k be a convex set such that $S \subseteq S^k$ and assume the unique solution $\bar{x}^k$ of $\mathrm{VI}(S^k, F^k)$ satisfies $h(\bar{x}^k) \le \sigma_k$. Burachik *et al.* [6] showed that, when F is paramonotone, $\{ \bar{x}^k \}$ is bounded and its accumulation point is a solution of $\mathrm{SIVI}(S, F)$ if $\{ \varepsilon_k \}$ and $\{ \sigma_k \}$ are chosen properly. Notice that the method in [6] needs to solve $\mathrm{VI}(S^k, F^k)$ exactly at each iteration, which is hardly implementable in practice. Now we propose a method which solves $\mathrm{VI}(S^k, F^k)$ only approximately at each iteration k. Moreover, we show that the poposed algorithm is globally convergent under the mere monotonicity assumption on F, which is a weaker assumption on F than the paramonotonicity.

The algorithm also consists of major iterations and inner iterations within each major iteration. On the rth inner iteration of the kth major iteration, a nonempty finite set $T^{k,r} \subset T$ is given. Let the convex set $S^{k,r} \subseteq \mathbb{R}^n$ be given by (3.1) and the function $f^{k,r} : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f^{k,r}(x) = \sup_{y\in S^{k,r}} \left\{ \langle F^k(x), x - y\rangle - \frac{1}{2}\varepsilon_k \|y - x\|^2 \right\}.$$

Note that $f^{k,r}$ is the regularized gap function for $\mathrm{VI}(S^{k,r}, F^k)$. In particular, notice that the parameter ε_k is *common* to the regularization parameter used to define F^k in (4.1). This particular choice of the parameters enables us to ensure that the inner iterations terminate finitely at each major iteration, see Proposition 4.2.

The detailed steps of the regularized outer approximation method are described as follows.

Algorithm 2 (Regularized outer approximation method).

Step 0. Choose $x^0 \in \mathbb{R}^n$, a Slater point $w \in \mathbb{R}^n$ such that $h(w) = \max_{t\in T} g(w,t) < 0$, $\alpha > 0$, TOL ≥ 0 and sequences $\{\delta_k\}, \{\sigma_k\}, \{\varepsilon_k\} \subset \mathbb{R}_{++}$ such that $\lim_{k\to\infty} \delta_k = \lim_{k\to\infty} \sigma_k = \lim_{k\to\infty} \varepsilon_k = 0$. Set $k := 1$.

Step 1. Obtain x^k by the following procedure.

Step 1-0. Choose a nonempty finite set $T^{k,1} \subset T$. Set $r := 1$ and $x^{k,0} := x^{k-1}$.

Step 1-1. Find an approximate solution $x^{k,r}$ of $\mathrm{VI}(S^{k,r}, F^k)$ such that $f^{k,r}(x^{k,r}) \leq \delta_k$ and $x^{k,r} \in S^{k,r}$.

Step 1-2. Find $t^{k,r}$ such that

$$g(x^{k,r}, t^{k,r}) > \sigma_k. \tag{4.2}$$

(i) If such $t^{k,r}$ does not exist, then set $r(k) := r$, $x^k := x^{k,r}$. Go to Step 2.

(ii) Otherwise, set $T^{k,r+1} := T^{k,r} \cup \{t^{k,r}\}$, $r := r+1$ and return to Step 1-1.

Step 2. If $\theta^k_\alpha(x^k) \leq$ TOL, then output x^k and stop. Otherwise, set $k := k+1$ and return to Step 1.

Recall that θ^k_α in Step 2 is defined by (3.4). We next show that the inner iterations in Step 1 terminate finitely.

Proposition 4.2. *The inner iterations in Step 1 terminate finitely by producing x^k for every k.*

Proof. For each k, we can regard Step 1 of Algorithm 2 as Step 1 of Algorithm 1 with $\mu = \alpha = \varepsilon_k$. Then the result follows from Proposition 3.4. □

Now we are ready to state the main theorem that establishes global convergence of Algorithm 2.

Theorem 4.3. *Let* TOL $= 0$ *and parameters $\{\delta_k\}$, $\{\sigma_k\}$ and $\{\varepsilon_k\}$ be chosen to satisfy $\delta_k = O(\varepsilon_k)$ and $\sigma_k = O(\varepsilon_k)$. Then, the following statements hold:*

(i) *The sequence $\{x^k\}$ is bounded.*

(ii) *Any accumulation point of $\{x^k\}$ solves* $\mathrm{SIVI}(S, F)$.

Proof. Proposition 4.2 ensures that $r(k) < \infty$ for each k. Define $f^k : \mathbb{R}^n \to \mathbb{R}$ as $f^k := f^{k,r(k)}$. If the algorithm terminates at some kth major iteration, then x^k solves $\mathrm{SIVI}(S,F)$ by Proposition 3.5. Below we assume that the algorithm generates an infinite sequence $\{x^k\}$.

(i) There exist $\lambda > 0$ and $\xi > 0$ such that $\sup_k \sigma_k/\varepsilon_k < \lambda < \infty$ and $\sup_k \delta_k/\varepsilon_k < \xi < \infty$. By Assumption 4.1 (iii), we have $h(w) \leq -\phi < 0$ for some ϕ. Since $\varepsilon_k \to 0$, we have $0 < \lambda\varepsilon_k/\phi < 1$ for all k sufficiently large. Without loss of generality, we assume that these inequalities hold for all $k \in \mathbb{N}$. Define $\tilde{x}^k$ as

$$\tilde{x}^k = x^k + \frac{\lambda\varepsilon_k}{\phi}(w - x^k). \tag{4.3}$$

Then, we have $\tilde{x}^k \in S$. In fact, by the convexity of h, we have

$$\begin{aligned} h(\tilde{x}^k) &\leq \frac{\lambda\varepsilon_k}{\phi} h(w) + \left(1 - \frac{\lambda\varepsilon_k}{\phi}\right) h(x^k) \\ &\leq \frac{\lambda\varepsilon_k}{\phi}(-\phi) + \left(1 - \frac{\lambda\varepsilon_k}{\phi}\right)\lambda\varepsilon_k \\ &= -\frac{\lambda^2\varepsilon_k^2}{\phi} \leq 0, \end{aligned}$$

where the second inequality follows from $h(w) \leq -\phi$ and $h(x^k) \leq \sigma_k \leq \lambda\varepsilon_k$. Therefore we have

$$\langle F(x^*), \tilde{x}^k - x^* \rangle \geq 0, \tag{4.4}$$

where x^* is a solution of SIVI(S, F). It follows from the definition of f^k that

$$\xi\varepsilon_k \geq \delta_k \geq f^k(x^k) \geq \langle F(x^k) + \varepsilon_k(x^k - w), x^k - x^* \rangle - \frac{1}{2}\varepsilon_k \|x^* - x^k\|^2,$$

which implies

$$\begin{aligned}
&\langle F(x^k), x^* - x^k \rangle \\
&\quad \geq \quad \varepsilon_k \left\{ \|x^k\|^2 - \langle w, x^k - x^* \rangle - \langle x^k, x^* \rangle - \frac{1}{2}(\|x^k\|^2 - 2\langle x^*, x^k \rangle + \|x^*\|^2) - \xi \right\} \\
&\quad = \quad \varepsilon_k \left\{ \frac{1}{2}\|x^k\|^2 - \langle w, x^k - x^* \rangle - \frac{1}{2}\|x^*\|^2 - \xi \right\}.
\end{aligned} \tag{4.5}$$

Moreover, by the definition (4.3) of $\tilde{x}^k$, we have

$$\begin{aligned}
\frac{\lambda\varepsilon_k}{\phi}\langle F(x^*), w - x^k \rangle &= \langle F(x^*), \tilde{x}^k - x^k \rangle \\
&\geq \langle F(x^*), x^* - x^k \rangle \\
&\geq \langle F(x^k), x^* - x^k \rangle,
\end{aligned} \tag{4.6}$$

where the first inequality follows from (4.4), and the second inequality follows from the monotonicity of F. Combining (4.5) with (4.6) and rearranging terms yield

$$\left\| x^k + \frac{\lambda}{\phi}F(x^*) - w \right\|^2 \leq \frac{\lambda^2}{\phi^2}\|F(x^*)\|^2 + \|x^* - w\|^2 + 2\xi.$$

Since this inequality holds for all k, we conclude that $\{ x^k \}$ is bounded.

(ii) Let x^∞ be an arbitrary accumulation point of $\{ x^k \}$, and $\{ x^k \}_{k \in \mathbb{K}}$ be a subsequence of $\{ x^k \}$ which converges to x^∞. Similarly to the proof of Theorem 3.7, we have $x^\infty \in S$. For an arbitrary positive scalar $\bar{\alpha}$, define $y^{\bar{\alpha}} \in S$ as

$$y^{\bar{\alpha}} = \operatorname*{argmax}_{y \in S} \left\{ \langle F(x^\infty), x^\infty - y \rangle - \frac{1}{2}\bar{\alpha}\|y - x^\infty\|^2 \right\}.$$

Note that $f_{\bar{\alpha}}(x^\infty) = \langle F(x^\infty), x^\infty - y^{\bar{\alpha}} \rangle - \frac{1}{2}\bar{\alpha}\|y^{\bar{\alpha}} - x^\infty\|^2$. By the construction of Algorithm 2 and the definition of f^k, we have

$$\begin{aligned}
\delta_k &\geq f^k(x^k) \\
&\geq \langle F(x^k) + \varepsilon_k(x^k - w), x^k - y^{\bar{\alpha}} \rangle - \frac{1}{2}\varepsilon_k\|y^{\bar{\alpha}} - x^k\|^2 \\
&\geq \langle F(x^k) + \varepsilon_k(x^k - w), x^k - y^{\bar{\alpha}} \rangle - \frac{1}{2}(\varepsilon_k + \bar{\alpha})\|y^{\bar{\alpha}} - x^k\|^2.
\end{aligned}$$

It then follows from $x^k \to x^\infty$, $\delta_k \to 0$, $\varepsilon_k \to 0$ and the continuity of F that

$$0 \geq \left\langle F(x^\infty), x^\infty - y^{\bar{\alpha}} \right\rangle - \frac{1}{2}\bar{\alpha}\|y^{\bar{\alpha}} - x^\infty\|^2 = f_{\bar{\alpha}}(x^\infty),$$

which implies that $f_{\bar{\alpha}}(x^\infty) = 0$ and hence x^∞ solves SIVI(S, F) from Proposition 2.1. $\square$

5 Numerical Experiments

In this section, we report some numerical results. The program was coded in MATLAB 2012b and run on a machine with Intel Core i7 3.00 GHz CPU and 4GB RAM. We observe the behavior of Algorithm 2 applied to several examples of SIVI(S, F) that have the following common features: The mapping F is merely monotone, and it is neither paramonotone nor strongly monotone. The origin $w = (0, 0, \dots, 0)^\top$ is a Slater point. The set T is the unit interval $[0, 1]$ in $\mathbb{R}$, and the function $g : \mathbb{R}^n \times T \to \mathbb{R}$ is given by

$$g(x, t) := \langle a(t), x \rangle - b(t),$$

where $a : T \to \mathbb{R}^n$ and $b : T \to \mathbb{R}$ are continuous functions.

Actual implementation of Algorithm 2 is carried out as follows. In Step 0, we set $x^0 = (-5, -5, \dots, -5)^\top$, $\alpha = 0.1$ and TOL $= 10^{-5}$, and choose parameters $\{ \delta_k \}$, $\{ \sigma_k \}$ and $\{ \varepsilon_k \}$ as $\delta_k = \sigma_k = 0.5^k$ and $\varepsilon_k = 30 \cdot 0.5^k$. In Step 1-0, $T^{k,1}$ is chosen as $T^{1,1} = \{ 0, 1 \}$ and $T^{k,1} = T^{k-1,r(k-1)}$ for each $k \geq 2$. In Step 1-1, $x^{k,r}$ is obtained by applying the descent method of [12] to (3.3). In Step 1-2, to find $t^{k,r}$ that satisfies (4.2), we first choose N grid points $\bar{t}_1, \bar{t}_2, \dots, \bar{t}_N$ from the interval T and compute $g(x^{k,r}, t)$ for $t = \bar{t}_1, \bar{t}_2, \dots, \bar{t}_N \in T$. If we find a $\bar{t} \in \{ \bar{t}_1, \bar{t}_2, \dots, \bar{t}_N \}$ that satisfies (4.2), then we set $t^{k,r} := \bar{t}$. Otherwise, we solve

$$\begin{array}{ll} \text{maximize} & g(x^{k,r}, t) \\ \text{subject to} & t \in T, \end{array} \tag{5.1}$$

and check whether the computed solution t^* of (5.1) satisfies (4.2). To solve (5.1), we apply Newton's method with the starting point $\bar{t}_0 := \operatorname{argmax} \{ g(x^{k,r}, t) \mid t = \bar{t}_1, \bar{t}_2, \dots, \bar{t}_N \}$. Although there is no theoretical guarantee, in practice we may expect to find a global maximizer of (5.1) by taking a sufficiently large N. In these experiments, we set $N = 101$ and $\bar{t}_i = (i-1)/(N-1)$ for $i = 1, \dots, N$. Moreover, we regard $g(x^k, t^*)$ as $\max_{t \in T} g(x^k, t)$ when computing $\theta_\alpha^k(x^k)$ in Step 2.

Example 1.

$$F(x) = \begin{pmatrix} x_2 - 1 \\ -x_1 - 1 \end{pmatrix}, \; a(t) = \begin{pmatrix} \cos \pi t \\ \sin \pi t \end{pmatrix}, \; b(t) = 1.$$

Note that

$$S = \{ x \in \mathbb{R}^2 \mid \|x\| \leq 1 \} \cup \{ x \in \mathbb{R}^2 \mid -1 \leq x_1 \leq 1, x_2 \leq 0 \}.$$

Upon termination, the algorithm found the point $x = (0.0003, 1.0000)^\top$ after 15 major iterations, while the exact solution of this problem is $x^* = (0, 1)^\top$.

Example 2.

$$F(x) = \begin{pmatrix} x_2 - 23/5 \\ -x_1 + 15/2 \\ x_3^3 + x_4 - 37/5 \\ -x_3 + 27/10 \end{pmatrix}, \; a(t) = \begin{pmatrix} 4t \\ -13t^2 \\ 18t^3 \\ -9t^4 \end{pmatrix}, \; b(t) = \frac{4}{9}.$$

The algorithm found the point $x = (1.0003, 1.0002, 0.9998, 0.9990)^\top$ after 17 major iterations. The exact solution is $x^* = (1, 1, 1, 1)^\top$.

Example 3.

$$F(x) = \begin{pmatrix} e^{x_1-1} + x_2 - 6 \\ e^{x_2-1} - x_1 - 5/3 \\ x_4 + 41/9 \\ -x_3 - 10/3 \\ x_5^3 + 8/9 \end{pmatrix}, \; a(t) = \begin{pmatrix} 4t \\ 5t^3 \\ -10t^2 \\ 13t^3 \\ -9t^4 \end{pmatrix}, \; b(t) = 3t^2 + \frac{4}{9}.$$

The algorithm found the point $x = (1.0008, 1.0001, 1.0014, 1.0011, 1.0000)^\top$ after 17 major iterations. The exact solution is $x^* = (1,1,1,1,1)^\top$.

Example 4.

$$F(x) = \begin{pmatrix} x_2 + 395/2 \\ -x_1 - 43061/64 \\ x_4 + 6117/8 \\ -x_3 - 3371/4 \\ x_5^3 + x_6 + 586 \\ x_6^3 - x_5 + 32077/64 \\ x_7^3 - 2605/4 \end{pmatrix}, \; a(t) = \begin{pmatrix} -256t^6 \\ 625t^5 \\ -500t^4 \\ 375t^3 \\ -168t^2 \\ 143t^5 - 428t^4 \\ 201t^3 + 33t \end{pmatrix}, \; b(t) = 25t^2 + \frac{9}{4}.$$

The algorithm found the point $x = (0.9949, 0.9954, 0.9955, 0.9976, 0.9991, 0.9994, 0.9998)^\top$ after 17 major iterations. The exact solution is $x^* = (1,1,1,1,1,1,1)^\top$.

More detailed computational results are shown in Table 1 and Figure 1, where

ite : the number of major iterations,

$r_{\rm sum}$: the sum of $r(k)$'s, $k = 1, 2, \ldots, \text{ite}$, where $r(k)$ denotes the number of inner iterations within the kth major iteration,

$\{ r(k) \}$: the values of $r(k)$ for $k = 1, 2, \ldots, \text{ite}$,

$|T^{\text{ite}, r(\text{ite})}|$: the number of elements of $T^{\text{ite}, r(\text{ite})}$,

time(sec) : the CPU time in seconds.

In the column of $\{ r(k) \}$, p^q means that, for some k, we have $r(k) = p$ in q consecutive iterations, and $\{ p_1, \ldots, p_{\bar{N}} \}^q$ means that, for some k, $r(k + i\bar{N} + j - 1) = p_j$ for $i = 0, 1, \ldots, q-1$ and $j = 1, \ldots, \bar{N}$. For example, $1^{10}, 3, \{ 1, 2 \}^2$ means that $r(1) = r(2) = \cdots = r(10) = 1$, $r(11) = 3$, $r(12) = 1$, $r(13) = 2$, $r(14) = 1$ and $r(15) = 2$. Figure 1 depicts the values of $\log_{10} \theta_\alpha^k(x^k)$ for $k = 1, 2, \ldots, \text{ite}$.

Table 1: Computational results of Algorithm 2

Example	ite	$r_{\rm sum}$	$\{ r(k) \}$	$\lvert T^{\text{ite}, r(\text{ite})} \rvert$	time(sec)
1	15	22	$1^4, 2, 1^4, 4, \{ 2, 1 \}^2, 2$	9	1.41
2	17	26	$1, 2^2, 1^2, 2, 1^2, 2^2, 1, 2^4, 1^2$	11	7.05
3	17	26	$1, 2^2, 1^2, \{ 1, 2 \}^4, 2, 1, 2^2$	11	108.25
4	17	36	$4, 1, 2, 1^2, 2, 3, 1, 3, 2^3, 3, 2, 1, 4, 2$	21	194.07

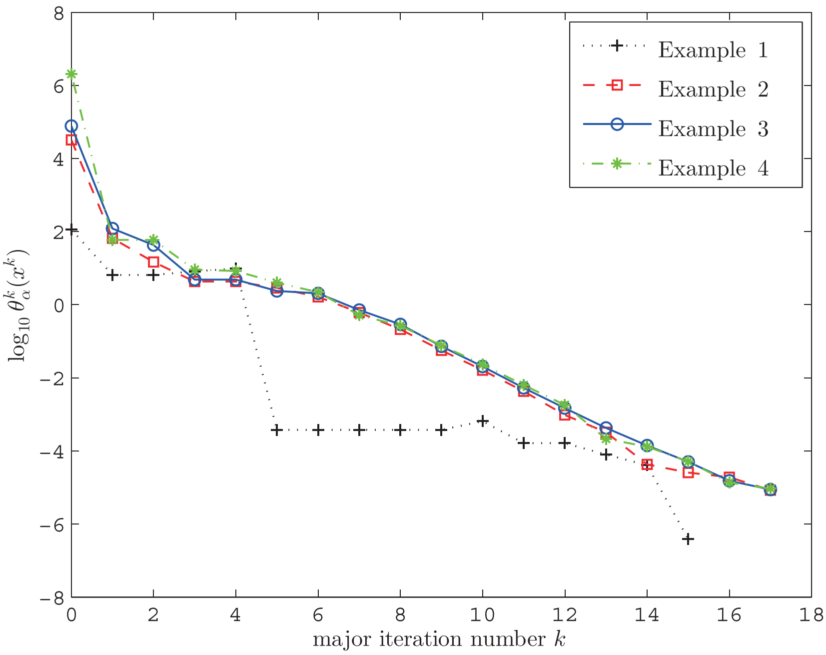

Figure 1: Behavior of Algorithm 2

6 Conclusion

We have proposed a regularized outer approximation method for solving the semi-infinite variational inequality problem (SIVIP). Utilizing the properties of the regularized gap function, we have established global convergence of the algorithm by assuming the monotonicity of the problem, Slater's condition and the existence of a solution. Moreover, we have shown the effectiveness of the proposed algorithm through numerical experiments. However, we have to admit that the complexity of the subproblem VI($S^{k,r}, F^k$) grows

(i) as the inner iteration proceeds, since the size of the index set $T^{k,r}$ increases monotonically, and

(ii) as the major iteration proceeds, since the numerical instability may occur due to the diminishing effect of the regularization parameter ε_k.

We may mitigate (i) by introducing a constraint dropping scheme. For the semi-infinite programming problem (SIP), the explicit exchange algorithms [14,18,27] have been proposed, and in particular, Okuno *et al.* [20] proposed an explicit exchange method combined with a regularization method. It is an interesting subject of research to extend their scheme to the SIVIP. To avoid (ii), proximal-type methods have been proposed for the variational inequality problem [5, 23] and the SIP [16]. It is also an interesting future work to explore the possibility of applying the proximal point method to the SIVIP.

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