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# APPROXIMATE PROPER EFFICIENCY ON REAL LINEAR VECTOR SPACES*

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**Abstract:** The aim of the paper is presenting necessary and sufficient conditions to characterize the approximate (weak/proper) efficient solutions of vector optimization problems under real linear vector spaces without any particular topology. To this end, we use different scalarization approaches based upon the dual cone, Gerstewitz's scalarization function, and the Lagrangian mapping notions. Since there is not any topology here, we utilize some algebraic concepts instead of topological (relative) interior, closure, and dual cone. Also, some separation and alternative theorems play a fundamental role in establishing the main results.

**Key words:** *vector optimization, approximate (weak/proper) efficiency, algebraic interior, vectorial closure, separation theorems*

**Mathematics Subject Classification:** *49M29, 90C48, 90C26, 90C29*

## 1 Introduction

The importance of the vector optimization is well known, acknowledging its numerous applications; see, e.g., [8, 9, 13, 26, 31, 32]. In recent decades, these problems have been studied from different points of view. One of the important subjects in vector optimization is characterization of the efficient set utilizing the scalarization approaches. In fact, scalarization techniques play a vital role in sketching the numerical algorithms and duality results; see, e.g., [8–10, 13, 17, 26, 30, 33]. In this paper, we are going to present some necessary and sufficient conditions, via scalarization tools, to characterize the approximate (weak/proper) efficient solutions of vector optimization problems under real linear vector spaces without any particular topology.

Approximate solutions of vector optimization problems, which were first introduced by Kutateladze [18], have been investigated in several frameworks with different goals. As can be seen in the numerical optimization literature, most of the practical algorithms in complex real world problems are convergent to a stationary point in a limiting sense. So, we must resort some practical rules for terminating the iterative procedures. Some of these terminating criteria can be found in Chapter 7 of [4]. These rules produce the points which may not be optimal or even critical (stationary) but may still serve the decision maker's purposes from practical points of view (see an example given in [7]). In fact, the iterative algorithms give the solutions which satisfy the decision maker(s) approximately with

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a sufficient degree accuracy. On the other hand, one of the basic tools for obtaining the (weak/proper) efficient solutions of the vector optimization problems is converting the problem to a scalar optimization problem, and then solving the obtained problem by iterative numerical algorithms. So, some important questions arise naturally. How is the efficiency situation of the approximate solutions generated from the scalar problems? Are these solutions approximately (weak/proper) efficient? What is the degree of accuracy if we consider these solutions as approximately efficient? Answering these questions is one of the aims of studying the approximate efficient solutions. Another reason is related to the existence. Existing the efficient solutions is guaranteed under some conditions, say compactness, which are restrictive in some situations, while approximate efficient solutions exist under weaker assumptions. Furthermore, from a computational point of view, obtaining approximate efficient solutions is economical in compared to obtaining the exact ones, especially in large scale problems. Some applications of approximate solutions in large scale practical problems can be seen in [28]. Also, some large scale multiobjective optimization problems arising in Biology have been reported in [31, 32].

After introducing the approximate solutions of vector optimization models by Kutateladze, some scholars [11,15,24,35,39] dealt with the properties of these solutions. White [37] investigated different kinds of these solutions. Further extensions of approximate efficiency can be seen in [11, 15, 36]. Theoretical results, including necessary and sufficient conditions and duality results, for characterizing different kinds of approximate efficient solutions can be found in [5, 7, 10–12, 14, 21–24, 27, 33]. Shao and Ehrgott [28] addressed an application of approximate efficiency in radiotherapy treatment planning.

The vector optimization problems under real linear vector spaces, without any topology, have been studied by some authors [1–3] in recent years. Studying these problems opens new connections between Optimization, Functional Analysis, and Convex analysis. Since there is no topology here, we have to use some algebraic concepts to define and characterize approximate efficiency. To this end, we use the concepts of algebraic (relative) interior, vectorial closure, and algebraic dual cone, addressed in [1–3]. Three scalarization tools, including dual cone, Gerstewitz's function [25, 30, 34], and Lagrangian mapping, are utilized and new necessary and sufficient conditions are provided. The results of the present paper extend those given in the literature and provide some new applications for the notions introduced in [1–3]. Using the presented results, one can answer the above-mentioned questions about approximate proper efficiency and the degree of accuracy of the approximate solutions generated from the scalar problems by iterative numerical procedures. The established results fill the gap between two fields, optimization in linear vector spaces and approximate solutions of scalar optimization problems. Using the results of the present paper, any development in numerical/theoreteical scalar optimization under linear vector spaces can be helpful in developing the vector optimization on linear vector spaces; especially the results related to convergent analysis and existence.

The rest of the paper unfolds as follows: Section 2 contains some preliminaries and three separation and alternative results. In Section 3, the concept of approximate (weak/proper) efficiency is defined and some propositions, highlighting the relationships between different kinds of approximate efficiency, are provided. Section 4 contains the main results, presenting necessary and sufficient conditions utilizing above-mentioned scalarization tools.

## 2 Preliminaries

Throughout this paper, X is a real linear vector space, $A$ is a subset of $X$, and $K \subseteq X$ is a nontrivial nonempty ordering convex cone. $K$ is called pointed if $K \cap (-K) = \{0\}$. $cone(A)$,

$conv(A)$, and $aff(A)$ denote the cone generated by $A$, the convex hull of $A$, and the affine hull of $A$, respectively.

Considering $\lambda \in (0,1)$, $A$ is called $\lambda-$convex if for each $a_1, a_2 \in A$, $\lambda a_1 + (1-\lambda)a_2 \in A$. $A$ is nearly convex if there exists a $\lambda \in (0,1)$ such that $A$ is $\lambda-$convex.

For two sets $A, B \subseteq X$ ant a vector $\bar{a} \in X$, we use the following notations:

$$A + B = \{a + b : \ a \in A, \ b \in B\},$$

$$A - B = \{a - b : \ a \in A, \ b \in B\},$$

$$\bar{a} + A = \{\bar{a} + a : \ a \in A\},$$

$$A \backslash B = \{a \in A : \ a \notin B\}.$$

The algebraic interior of $A \subseteq X$, denoted by cor(A), and the relative algebraic interior of $A$, denoted by icr(A), are defined as follows [2]:

$$cor(A) = \{x \in A : \ \forall x' \in X \ , \ \exists \lambda' > 0; \ \ \forall \lambda \in [0, \lambda'], \ \ x + \lambda x' \in A\},$$

$$icr(A) = \{x \in A : \ \forall x' \in L(A) \ , \ \exists \lambda' > 0; \ \ \forall \lambda \in [0, \lambda'], \ \ x + \lambda x' \in A\},$$

where $L(A) = span(A - A)$ is the linear hull of $A - A$. When $cor(A) \neq \emptyset$ we say that $A$ is solid; and we say that $A$ is relatively solid if $icr(A) \neq \emptyset$. It is known that, if $cor(K) \neq \emptyset$, then $cor(K) \cup \{0\}$ is a convex cone, $cor(K) + K = cor(K)$ and $cor(cor(K)) = cor(K)$; see [3].

**Note:** In [3] the authors showed that $icr(K) + K = icr(K)$, while the relation $cor(K) + K = cor(K)$ can be proved similar to the proof given in [3]. So, we directly use $cor(K) + K = cor(K)$ as a result in this paper.

Although the (relative) algebraic interior set is usually defined in linear vector spaces without topology, in some cases it might be useful under topological vector spaces too. It is because the algebraic (relative) interior can be nonempty while (relative) interior set is empty. The algebraic (relative) interior preserves most of the properties of (relative) interior.

The algebraic dual of $X$ is denoted by $X'$, and $\langle ., . \rangle$ exhibits the duality pairing, i.e., for $l \in X'$ and $x \in X$ we have $\langle l, x \rangle = l(x)$. The positive dual and the strict positive dual of $K$ are, respectively, defined by

$$K^+ = \{l \in X' : \ \langle l, a \rangle \geq 0, \ \ \forall a \in K\},$$

$$K^{+s} = \{l \in X' : \ \langle l, a \rangle > 0, \ \ \forall a \in K \backslash \{0\}\}.$$

The vectorial closure of $A$, which is considered instead of closure in the absence of topology, is defined by [2]

$$vcl(A) = \{b \in X : \exists x \in X \ ; \ \forall \lambda' > 0 \ , \ \exists \lambda \in [0, \lambda'] \ ; \ b + \lambda x \in A\}.$$

$A$ is called vectorially closed if $A = vcl(A)$. It is known that, if $K$ is a relatively solid convex cone, then $vcl(K) = K^{++}$, where $K^{++}$ is the positive dual of $K^+$; see [3].

In the rest of this section, two separation results and an alternative theorem under real linear vector spaces are addressed. These theorems will be used in sequel.

The proofs of Theorems 2.2 and 2.3 can be found in [1, 3]. Proposition 2.1 results from Theorem 3.14 in [13], though we have provided a short proof here.

**Proposition 2.1.** *Let $M, K$ be solid nontrivial convex cones in $X$. If $M \cap cor(K) = \emptyset$, then there exists a functional $l \in X^{'} \backslash \{0\}$ such that,*

$$\langle l, m \rangle \leq 0 \leq \langle l, k \rangle \qquad \forall (k \in K, m \in M),$$

*and furthermore, $\langle l, k \rangle > 0$ for all $k \in cor(K)$, and $\langle l, m \rangle < 0$ for all $m \in cor(M)$.*

*Proof.* Since $M \cap cor(K) = \emptyset$ and $cor(M) \subseteq M$, we have $0 \notin cor(M) - cor(K) = cor(M - K)$. Hence, by Theorem 3.14 in [13], there exists a functional $l \in X' \backslash \{0\}$ such that

$$\langle l, m - k \rangle \leq 0 \qquad \forall k \in K \ , \ \forall m \in M.$$

Since $0 \in M \cap K$, we obtain

$$\langle l, m \rangle \leq 0 \ \leq \langle l, k \rangle \qquad \forall k \in K \ , \ \forall m \in M.$$

If there exists $k \in cor(K)$ such that $\langle l, k \rangle = 0$, then

$$\forall x \in X, \ \ \exists \lambda^{'} > 0; \quad \forall \lambda \in [0, \lambda^{'}] \quad k + \lambda x \in K.$$

Therefore $\langle l, k \rangle + \lambda \langle l, x \rangle \geq 0$ for each $x \in X$. Hence $\langle l, x \rangle \geq 0$ for each $x \in X$. On the other hand, $-x \in X$. Thus $\langle l, x \rangle = 0$ for each $x \in X$, which makes a contradiction, because $l \neq 0$. Therefore, $\langle l, k \rangle > 0$ for each $k \in cor(K)$.
Similarly it can be shown that $\langle l, m \rangle < 0$ for each $m \in cor(M) \backslash \{0\}$. Note that, $0 \notin cor(M)$ because $M \neq X$. □

**Theorem 2.2.** *Let $M, K$ be two convex, nontrivial, and vectorially closed cones in $X$ such that $M, K$ are relatively solid and $K^+$ is solid. If $M \cap K = \{0\}$, then there exists a functional $l \in X' \backslash \{0\}$ such that,*

$$\langle l, k \rangle \geq 0 \geq \langle l, m \rangle \qquad \forall (k \in K, m \in M),$$

*and furthermore,*

$$\langle l, k \rangle > 0 \qquad \forall k \in K \backslash \{0\}.$$

**Theorem 2.3** (Alternative theorem)**.** *Let $K$ be a nontrivial solid pointed convex cone and let $A$ be a nonempty subset of $X$. If $vcl(cone(A) + K)$ is convex, then one and only one of the following alternatives is valid:*
(i) $A \cap (-cor(K)) \neq \emptyset$
(ii) $A^+ \cap K^+ \neq \{0\}$.

## 3 Approximate Proper Efficiency

Let $X, Y, Z$ be three real linear vector spaces such that $Y, Z$ are partially ordered by nontrivial ordering convex cones $K, M$, respectively. Consider the following unconstrained and constrained vector optimization problems:

$$\begin{array}{ll} \text{(UVOP)} & K - Min \, \{f(x) : x \in E\}, \\ \text{(CVOP)} & K - Min \, \{f(x) : x \in E, \ \ g(x) \in -M\}, \end{array}$$

where $E \subseteq X$ is a nonempty set, $g : E \to Z$ and $f : E \to Y$.

The set of feasible solutions of (UVOP) and (CVOP) are, respectively,

$$\Omega = E \quad \text{and} \quad \Omega = \{x \in E : g(x) \in -M\}.$$

So, in general we consider the following vector optimization problem:

$$\text{(VOP)} \qquad K - Min\,\{f(x) : x \in \Omega\}.$$

Hereafter, $\epsilon \in K\backslash\{0\}$ is a fixed vector, which is used as a tolerance in approximate efficiency. In fact, $\epsilon$ exhibits the degree of accuracy.

The following definition is a generalization of the corresponding definitions given in [3, 7, 18, 24, 29]. Comparing the approximate weak efficiency defined here with the standard one, shows that here $int(K)$ has been replaced by $cor(K)$, because there is not any topology here. Also, note that even in topological vector spaces the set of (weakly/properly) efficient solutions might be empty in a noncompact instance, while the set of approximate solutions can be nonempty under very weaker hypotheses. Furthermore, in applied optimization, the mathematical programming models characterize simplified versions of the real problems, and moreover, numerical algorithms may produce approximate solutions. These are some of the reasons which make approximate solutions worth studying.

**Definition 3.1.** $x_0 \in \Omega$ is called an $\epsilon-$efficient solution of (VOP) if

$$\Big(f(\Omega) - f(x_0) + \epsilon\Big) \cap \Big(-K\backslash\{0\}\Big) = \emptyset.$$

Furthermore, when $K$ is solid, $x_0 \in \Omega$ is called an $\epsilon-$weak efficient solution of (VOP) if

$$\Big(f(\Omega) - f(x_0) + \epsilon\Big) \cap \Big(-cor(K)\Big) = \emptyset.$$

**Note:** Hereafter, whenever we talk about the (approximate) weak efficiency, it is assumed that $cor(K) \neq \emptyset$, i.e., $K$ is solid. Also, notice that $0 \notin cor(K)$ because $K \neq Y$. In fact, it can be shown that $K = Y$ if $0 \in cor(K)$.

One of the most important solution notions in vector optimization theory is proper efficiency. This concept has been studied in many publications to eliminate the situations in which the trade-off between criteria is unbounded. The following definition generalizes the proper efficiency notions defined in [3, 29].

**Definition 3.2.** $x_0 \in \Omega$ is called an $\epsilon-$Hurwicz vectorial proper efficient solution ($\epsilon-$HuV) of (VOP) if

$$vcl\Bigg(conv\Bigg(cone\Bigg((f(\Omega) - f(x_0)) \cup (K + \epsilon)\Bigg)\Bigg)\Bigg) \cap (-K) = \{0\};$$

and $x_0 \in \Omega$ is called an $\epsilon-$Benson vectorial proper efficient solution ($\epsilon-$BeV) of (VOP) if

$$vcl\Bigg(cone\Bigg(f(\Omega) - f(x_0) + K + \epsilon\Bigg)\Bigg) \cap (-K) = \{0\}.$$

The following proposition shows that the set of $\epsilon-$BeV solutions is a subset of that of $\epsilon-$efficient solutions.

**Proposition 3.3.** *If $x_0$ is $\epsilon-BeV$ for (VOP), then $x_0$ is an $\epsilon-$efficient solution of (VOP).*

*Proof.* Suppose, on the contrary, $x_0$ is not $\epsilon-$efficient. Then there exists

$$y \in \Big(f(\Omega) - f(x_0)\Big) \cap \Big(-K\backslash\{0\} - \epsilon\Big).$$

Thus,

$$y+\epsilon \in f(\Omega)-f(x_0)+(0+\epsilon) \subseteq f(\Omega)-f(x_0)+(K+\epsilon),$$

and

$$y+\epsilon \in (-K\backslash\{0\}).$$

Therefore, we have

$$y+\epsilon \in vcl\Bigg(cone\Bigg(f(\Omega)-f(x_0)+K+\epsilon\Bigg)\Bigg) \cap (-K\backslash\{0\}),$$

which contradicts the assumption and completes the proof. $\square$

From the above proposition and due to $cor(K) \subseteq K$, it can be seen that, the set of $\epsilon-$BeV solutions is a subset of that of $\epsilon-$weak efficient points. The following proposition provides a relationship between $\epsilon-$HuV and $\epsilon-$BeV notions. Part (i) of this proposition generalizes part (i) of Proposition 3.1 in [3]. From the following result, it can be seen that the set of $\epsilon-$HuV solutions is a subset of that of $\epsilon-$ (weak) efficient points, as well.

**Proposition 3.4.** (i) *If $x_0$ is $\epsilon-$HuV for (VOP), then $x_0$ is $\epsilon-$BeV for (VOP).*
(ii) *Suppose that $-\epsilon \in K$. If $x_0$ is $\epsilon-$BeV for (VOP) and $cone\Bigg(f(\Omega)-f(x_0)+K+\epsilon\Bigg)$ is convex, then $x_0$ is $\epsilon-$HuV for (VOP).*

*Proof.* (i) The proof of this part is simalr to that of Proposition 3.1 in [3], and is hence omitted.
(ii) Since $-\epsilon \in K$, we get

$$f(\Omega)-f(x_0) \subseteq f(\Omega)-f(x_0)+K+\epsilon,$$

On the other hand

$$K+\epsilon \subseteq f(\Omega)-f(x_0)+K+\epsilon.$$

Therefore,

$$\Bigg(f(\Omega)-f(x_0)\Bigg) \cup \Bigg(K+\epsilon\Bigg) \subseteq f(\Omega)-f(x_0)+K+\epsilon.$$

Hence,

$$vcl\Bigg(conv\Bigg(cone\Bigg(\big(f(\Omega)-f(x_0)\big)\cup(K+\epsilon)\Bigg)\Bigg)\Bigg) \subseteq vcl\Bigg(conv\Bigg(cone\Bigg(f(\Omega)-f(x_0)+K+\epsilon\Bigg)\Bigg)\Bigg).$$

Since $cone\Bigg(f(\Omega)-f(x_0)+K+\epsilon\Bigg)$ is convex, we have

$$vcl\Bigg(conv\Bigg(cone\Bigg(\big(f(\Omega)-f(x_0)\big)\cup(K+\epsilon)\Bigg)\Bigg)\Bigg) \subseteq vcl\Bigg(cone\Bigg(f(\Omega)-f(x_0)+K+\epsilon\Bigg)\Bigg).$$

Thus,

$$vcl\Bigg(cone\Bigg(f(\Omega)-f(x_0)+K+\epsilon\Bigg)\Bigg)\cap(-K)=\{0\}$$

implies

$$vcl\Bigg(conv\Bigg(cone\Bigg(\big(f(\Omega)-f(x_0)\big)\cup(K+\epsilon)\Bigg)\Bigg)\Bigg)\cap(-K)=\{0\}.$$

This completes the proof. $\square$

**Remark 3.5.** Some definitions of approximate efficiency are given by means of an $\epsilon$-translation of the image set, where $\epsilon$ is a member of the ordering cone. If one follows the same scheme in defining the $\epsilon-$Hurwicz proper efficiency, then this notion will be defined by the following relation:

$$vcl\Bigg(conv\Bigg(cone\Bigg((f(\Omega) - f(x_0) + \epsilon) \cup (K + \epsilon)\Bigg)\Bigg)\Bigg) \cap (-K) = \{0\}. \tag{3.1}$$

If we define the $\epsilon-$Hurwicz proper efficiency based upon relation (3.1), then it can be shown that part (ii) of the above proposition holds without the assumption $-\epsilon \in K$. Notice that this assumption fails for a pointed ordering cone.

Also, if we consider the $\epsilon-$Hurwicz proper efficiency using (3.1), then similar to the proof of Proposition 3.1 in [3], it can be shown that: If $x_0$ is $\epsilon-$HuV for (VOP), then $x_0$ is $2\epsilon-$BeV for (VOP).

## 4 Scalarization

This section contains the main results of the paper. In the first part of this section, a scalar optimization problem, based upon the algebraic dual cone, is dealt with and some connections between approximate minimality for scalar problem and approximate efficiency for (VOP) are presented. The following definition is used in sequel.

**Definition 4.1.** Considering the scalar function $h : X \to \mathbb{R}$ and the given positive scalar $\varepsilon > 0$, the vector $x_0 \in \Omega$ is called an $\varepsilon-$minimum of the scalar optimization problem

$$Min\ \{h(x) : x \in \Omega\}$$

if

$$h(x_0) \leq h(x) + \varepsilon \qquad \forall x \in \Omega.$$

The vector $x_0 \in \Omega$ is called an $\varepsilon-$ strict minimum of the above scalar optimization problem if

$$h(x_0) < h(x) + \varepsilon \qquad \forall x \in \Omega.$$

Theorems 4.2 and 4.3 provide necessary and sufficient conditions for approximate efficiency and approximate weak efficiency for (VOP).

**Theorem 4.2.** *Let $\varepsilon > 0$. If there exists $l \in K^+\backslash\{0\}$ such that $x_0 \in \Omega$ is an $\varepsilon-$minimum of scalar program,*

$$Min\ \{l \circ f(x) : x \in \Omega\},$$

*then $x_0$ is an $\epsilon-$weak efficient solution to (VOP) for each $\epsilon \in K\backslash\{0\}$ which satisfies $\varepsilon \leq \langle l, \epsilon\rangle$. Furthermore, if $x_0 \in \Omega$ is a strict $\varepsilon-$minimum of the above scalar program, then $x_0$ is an $\epsilon-$efficient solution to (VOP) for each $\epsilon \in K\backslash\{0\}$ which satisfies $\varepsilon \leq \langle l, \epsilon\rangle$.*

*Proof.* Assume that $x_0$ is not an $\epsilon-$weak efficient solution to (VOP) for some $\epsilon \in K\backslash\{0\}$ which satisfies $\varepsilon \leq \langle l, \epsilon\rangle$. Then

$$\exists y \in \Big(f(\Omega) - f(x_0) + \epsilon\Big) \cap \Big(-cor(K)\Big).$$

Since $l \in K^+\backslash\{0\}$ and $y \in -cor(K)$, we have $-y \in K$ which implies $\langle l, y\rangle \leq 0$. Since $-y \in cor(K)$,

$$\forall x^{'} \in X\ ,\ \ \exists \lambda^{'} > 0\ ,\ \ \forall \lambda \in [0, \lambda^{'}]\ ,\ \ -y + \lambda x^{'} \in K.$$

If $\langle l, y\rangle = 0$, then $0 \leq \langle l, -y + \lambda x^{'}\rangle = \lambda \langle l, x^{'}\rangle$. Hence, $\langle l, x^{'}\rangle \geq 0$ for each $x^{'} \in X$. This implies $l = 0$, which contradicts the assumption of the theorem. Hence $\langle l, y\rangle < 0$.
Furthermore, there exists $x \in \Omega$ such that

$$y = f(x) - f(x_0) + \epsilon.$$

Thus,

$$\langle l, f(x) - f(x_0) + \epsilon\rangle < 0.$$

Hence,

$$\langle l, f(x)\rangle + \varepsilon \leq \langle l, f(x)\rangle + \langle l, \epsilon\rangle < \langle l, f(x_0)\rangle.$$

This contradicts the $\epsilon-$minimality of $x_0$ and completes the proof of the first part. The second part of the theorem can be proved similarly. □

**Theorem 4.3.** *Assume that* $cone\Big(f(\Omega) - f(x_0) + K + \epsilon\Big)$ *is convex and solid. If* $x_0$ *is an* $\epsilon-$*weak efficient solution to (VOP), then there exists* $l \in K^{+}\backslash\{0\}$ *such that* $x_0$ *is an* $\varepsilon-$*minimum of scalar optimization problem*

$$Min\ \{l \circ f(x) : x \in \Omega\},$$

*for each* $\varepsilon > 0$ *which satisfies* $\varepsilon \geq \langle l, \epsilon\rangle$.

*Proof.* Since $x_0$ is $\epsilon-$weak efficient,

$$\Big(f(\Omega) - f(x_0) + \epsilon\Big) \cap \Big(-cor(K)\Big) = \emptyset.$$

It can be shown that $cor(K) + K = cor(K)$. Also, $0 \notin cor(K)$ because $K \neq Y$. Hence

$$cone\Big(f(\Omega) - f(x_0) + K + \epsilon\Big) \cap \Big(-cor(K)\Big) = \emptyset.$$

By Theorem 2.2, there exists $l \in K^{+}\backslash\{0\}$ such that

$$\langle l, f(x) - f(x_0) + k + \epsilon\rangle \geq 0 \qquad \forall (x \in \Omega\ ,\ k \in K).$$

Setting $k = 0$, we have

$$\langle l, f(x) - f(x_0) + \epsilon\rangle \geq 0 \qquad \forall x \in \Omega.$$

Hence,

$$\langle l, f(x)\rangle + \varepsilon \geq \langle l, f(x)\rangle + \langle l, \epsilon\rangle \geq \langle l, f(x_0)\rangle \qquad \forall x \in \Omega,$$

and the proof is completed. □

Regarding Propositions 3.3 and 3.4, if $x_0$ is an $\epsilon-$HuV (or $\epsilon-$BeV or $\epsilon-$efficient), then the above theorem is still valid.

In the above two theorems, the functional $l$ belongs to $K^{+}$, that is, this functional is nonnegative on $K$. In the following two theorems, we consider positive functionals belonging to $K^{+s}$, to obtain necessary and sufficient conditions to characterize the $\epsilon-$BeV solutions of (VOP).

**Theorem 4.4.** *Let $\varepsilon > 0$. If there exists $l \in K^{+s}$ such that $x_0 \in \Omega$ is an $\varepsilon-$minimum of the scalar program*

$$Min\,\{l \circ f(x) : x \in \Omega\},$$

*then $x_0$ is an $\epsilon-BeV$ solution to (VOP) for each $\epsilon \in K\backslash\{0\}$ which satisfies $\varepsilon \leq \langle l, \epsilon\rangle$.*

*Proof.* Assume that $x_0$ is not an $\epsilon-$BeV solution to (VOP) for some $\epsilon \in K\backslash\{0\}$ with $\varepsilon \leq \langle l, \epsilon\rangle$. Then there exists $y \neq 0$ such that

$$y \in vcl\Big(cone\Big(f(\Omega) - f(x_0) + K + \epsilon\Big)\Big)\cap(-K).$$

Then $y \in (-K)$ and, since $l \in K^{+s}$, we have

$$\langle l, y\rangle < 0. \tag{4.1}$$

On the other hand, since $y \in vcl\Big(cone\Big(f(\Omega) - f(x_0) + K + \epsilon\Big)\Big)$, there exist $x^{'} \in X$ and a sequence $\lambda_n \to 0^+$ such that

$$y + \lambda_n x^{'} \in cone\Big(f(\Omega) - f(x_0) + K + \epsilon\Big) \qquad \forall n.$$

So, there exist sequences $\{\alpha_n\} \subseteq [0, +\infty)$, $\{y_n\} \subseteq \Omega$ and $\{k_n\} \subseteq K$ such that

$$y + \lambda_n x^{'} = \alpha_n\Big(f(y_n) - f(x_0) + k_n + \epsilon\Big).$$

Since $l$ is linear, we deduce

$$\langle l, y + \lambda_n x^{'}\rangle = \alpha_n\Big(\langle l, f(y_n)\rangle - \langle l, f(x_0)\rangle + \langle l, k_n\rangle + \langle l, \epsilon\rangle\Big).$$

Since $x_0$ is an $\varepsilon-$minimum of the scalar program

$$Min\,\{l \circ f(x) : x \in \Omega\},$$

we have

$$\langle l, f(x_0)\rangle \leq \langle l, f(x)\rangle + \varepsilon \qquad \forall x \in \Omega.$$

So, we obtain

$$\langle l, f(y_n)\rangle - \langle l, f(x_0)\rangle + \langle l, \epsilon\rangle \geq \langle l, f(y_n)\rangle - \langle l, f(x_0)\rangle + \varepsilon \geq 0.$$

Furthermore,

$$\langle l, k_n\rangle \geq 0 \qquad \forall n,$$

because $l \in K^{+s}$. From these, it follows that for each $n$

$$\langle l, y + \lambda_n x^{'}\rangle \geq 0 \qquad \forall n.$$

As $\lambda_n \to 0^+$, we have $\langle l, y\rangle \geq 0$, which contradicts (4.1). Therefore, $x_0$ is an $\epsilon - BeV$ point. $\square$

In the above theorem, if one further assumes that $f$ is $K-$convexlike, then $x_0$ is an $\epsilon-$HuV solution to (VOP) for each $\epsilon \in K\backslash\{0\}$ which satisfies $\varepsilon \leq \langle l, \epsilon\rangle$ and $-\epsilon \in K$.

The following definition is used in Theorem 4.6. In this definition a generalized convexity notion, introduced by Adan and Novo [3], is presented. See [1, 2, 38] for more details about generalized cone convexity.

**Definition 4.5** ( [3]). The mapping $f : E \subseteq X \to Y$ is called a generalized vector convexlike ($GVCL$) function in $E$ with respect to $K$ if

$$vcl\Big(cone(f(E)) + K\Big)$$

is convex.

If one assumes that $cone(f(E)) + K$ is nearly convex and relatively solid, then, by Proposition 3 in [1], $f$ is generalized vector convexlike.

Theorem 4.6 provides a necessary condition for $\epsilon-$Benson proper efficiency utilizing the dual cone.

**Theorem 4.6.** *Suppose that $K^+$ and $K$ are solid and $K$ is vectorially closed. Let $x_0 \in \Omega$ and let $f(.) - f(x_0) + \epsilon$ be a GVCL mapping in $\Omega$ with respect to $K$. If $x_0$ is an $\epsilon - BeV$ solution to (VOP), then there exists a functional $l \in K^{+s}$ such that $x_0$ is an $\varepsilon-$minimum of the scalar program $Min\,\{l \circ f(x) : x \in \Omega\}$ for $\varepsilon \geq \langle l, \epsilon\rangle$.*

*Proof.* Since $x_0$ is $\epsilon - BeV$,

$$vcl\Big(cone\Big(f(\Omega) - f(x_0) + K + \epsilon\Big)\Big)\cap(-K) = \{0\}.$$

Hence,

$$vcl\Big(cone\Big(f(\Omega) - f(x_0) + \epsilon\Big)+K\Big)\cap(-K) = \{0\}.$$

Since $f(.) - f(x_0) + \epsilon$ is GVCL in $\Omega$ with respect to the K,

$$vcl\Big(cone\Big(f(\Omega) - f(x_0) + \epsilon\Big)+K\Big)$$

is convex. Also, by Proposition 6 in [1],

$$cor\Big(vcl\Big(cone\Big(f(\Omega) - f(x_0) + \epsilon\Big)+K\Big)\Big)= cone\Big(f(\Omega) - f(x_0) + \epsilon\Big)+cor(K).$$

On the other hand, $cor(K) \neq \emptyset$, which implies that

$$cor\Big(vcl\Big(cone\Big(f(\Omega) - f(x_0) + \epsilon\Big)+K\Big)\Big)\neq \emptyset.$$

Therefore, $vcl\Big(cone\Big(f(\Omega) - f(x_0) + \epsilon\Big)+K\Big)$ is solid. Thus, by separation Theorem 2.3, there exists $l \in K^{+s}$ such that,

$$\langle l, f(x) - f(x_0) + k + \epsilon\rangle \geq 0 \quad \forall(x \in \Omega\,,\, k \in K)$$

which implies that

$$\langle l, f(x_0)\rangle \leq \langle l, f(x)\rangle + \langle l, k+\epsilon\rangle \quad \forall (x \in \Omega\,,\, k \in K).$$

Setting $k = 0$, we get

$$\langle l, f(x_0)\rangle \leq \langle l, f(x)\rangle + \langle l, \epsilon\rangle \leq \langle l, f(x)\rangle + \varepsilon \qquad \forall x \in \Omega,$$

which completes the proof. □

Since each $\epsilon-$HuV solution is an $\epsilon-$BeV solution, the above theorem is also valid for $\epsilon - HuV$. In the above theorem if one sets $\epsilon = 0$, then Theorem 4.2 in [3] is obtained.

From Theorems 4.4 and 4.6 we have the following corollary.

**Corollary 4.7.** *Suppose that $K$ and $K^+$ are solid and $K$ is vectorially closed. Let $x_0 \in \Omega$, and $f(.) - f(x_0) + \epsilon$ be a GVCL mapping in $\Omega$ with respect to $K$. Then, $x_0$ is an $\epsilon-BeV$ solution of (VOP) if and only if there exists a functional $l \in K^{+s}$ such that $x_0$ is an $\varepsilon-$minimum of the scalar program,*

$$Min\,\{l \circ f(x) : x \in \Omega\},$$

*for $\varepsilon = \langle l, \epsilon\rangle$.*

The second part of this section contains a characterization of the approximate (weak) efficient solutions of (VOP) utilizing the notion of Gerstewitz's scalarization function.

Considering $e \in cor(K)$ and $v \in Y$, suppose that $K$ is vectorially closed. The Gerstewitz's function $h_{e,v} : Y \longrightarrow \mathbb{R}$, is defined by

$$h_{e,v}(y) := min\{t \in \mathbb{R} : y \in v + te - K\}.$$

Notice that the set $\{t \in \mathbb{R} : y \in v + te - K\}$ is nonempty, closed and bounded from below (Lemma 2.2 in [19]). The Gerstewitz's function is well-known and widely used in optimization, see e.g., [17, 25, 34]

In the following, an equivalent form of the Gerstewitz's function, utilizing the concept of the base of cone $K$, is provided.

**Definition 4.8** ([25])**.** A convex subset $\Theta$ of a convex cone $K$ is called a base of $K$ if $0 \notin vcl(\Theta)$ and $K = cone(\Theta)$. We denote the set of all bases of $K$ by $B(K)$.

Now, we define the set $B$ as follows:

$$B = \{l \in K^+ \;:\; l(e) = 1\} \tag{4.2}$$

where $e \in cor(K)$. It is not difficult to show that $B$ is a base of $K^+$.

**Proposition 4.9.** *Considering $(e, v) \in cor(K) \times Y$, if $K$ is vectorially closed, then*

$$h_{e,v}(y) = Sup\,\{\langle l, y - v\rangle \;:\; l \in B\}.$$

*Proof.* By proposition 2.2 in [3], we have $K = K^{++}$. Hence

$$\begin{aligned} h_{e,v}(y) &= Min\,\{t \in \mathbb{R} \;: y \in v + te - K\} \\ &= Min\,\{t \in \mathbb{R} \;: v + te - y \in K = K^{++}\} \\ &= Min\,\{t \in \mathbb{R} \;: \langle l, v + te - y\rangle \geq 0 \qquad \forall l \in K^+\} \\ &= Min\,\{t \in \mathbb{R} \;: \langle l, y - v\rangle \leq t \qquad \forall l \in B\} \\ &= Sup\,\{\langle l, y - v\rangle \;: l \in B\}. \end{aligned}$$

□

Theorem 4.10 introduces a sufficient condition for approximate (weak) efficiency using the Gerstewitz's function.

**Theorem 4.10.** *Let $\varepsilon > 0$ and $K$ be pointed and vectorially closed. If $e \in cor(K)$ and $\alpha \in \mathbb{R}$ such that $x_0$ is a strict $\varepsilon-$minimum solution to*

$$Min\,\{h_{e,f(x_0)-\alpha e} \circ f(x) : x \in \Omega\}, \tag{4.3}$$

*then $x_0$ is $\bar{\varepsilon}e-$efficient for $(VOP)$ for each $\bar{\varepsilon} \geq \varepsilon$.*

*Proof.* Since $x_0$ is a strict $\varepsilon-$minimum solution of (4.3), we get,

$$h_{e,f(x_0)-\alpha e} \circ f(x) + \varepsilon > h_{e,f(x_0)-\alpha e} \circ f(x_0) \qquad \forall x \in \Omega.$$

Considering

$$t_1 = h_{e,f(x_0)-\alpha e}(f(x_0)) = Min\,\{t \in \mathbb{R}\,;\ f(x_0) \in f(x_0) + te - e\alpha - K\},$$

we have $t_1 \leq \alpha$ and $(t_1 - \alpha)e \in K$. Since $K \neq Y$, $0 \notin cor(K)$. Hence $e \neq 0$. Also, $(\alpha - t_1)e \in K$, because $\alpha - t_1 \geq 0$. Hence $\pm(t_1 - \alpha)e \in K$, which implies that $t_1 - \alpha = 0$ because $K$ is pointed and $e \neq 0$. Therefore,

$$h_{e,f(x_0)-\alpha e} \circ f(x_0) = \alpha.$$

So, we have

$$h_{e,f(x_0)-\alpha e} \circ f(x) + \varepsilon - \alpha > 0 \qquad \forall x \in \Omega.$$

Therefore, by Proposition 4.9, we get

$$Sup\,\{\langle l, (f(x) - f(x_0) + \alpha e)\rangle \ : \ l \in B\} + \varepsilon - \alpha > 0 \qquad \forall x \in \Omega.$$

So, there exists $l \in B$ such that $\langle l, (f(x) - f(x_0) + \alpha e)\rangle + \varepsilon - \alpha > 0$. Hence, there exists $l \in K^+\backslash\{0\}$ such that

$$\langle l, (f(x) - f(x_0))\rangle + \varepsilon > 0 \qquad \forall x \in \Omega.$$

Therefore, $x_0$ is a strict $\varepsilon-$minimum solution of the scalar program

$$Min\,\{l \circ f(x) : x \in \Omega\}.$$

So, by Theorem 4.2, $x_0$ is an $\epsilon-$efficient solution of (VOP) for each $\epsilon \in K\backslash\{0\}$ which satisfies $\varepsilon \leq \langle l, \epsilon\rangle$. Hence, $x_0$ is $\bar{\varepsilon}e-$efficient solution of $(VOP)$ for each $\bar{\varepsilon} \geq \varepsilon$. □

The following theorem provides a stronger result in being compared to the above theorem. In fact, in Theorem 4.11 the vector $v$ of the considered Gerstewitz's function is an arbitrary vector belonging to $Y$. In this theorem, if one replaces the "$\varepsilon-$minimality" assumption with "$\varepsilon-$strict minimality", then the theorem is still valid even for $\bar{\varepsilon} = \varepsilon$.

**Theorem 4.11.** *Let $\varepsilon > 0$ and $K$ be vectorially closed. If $x_0$ is an $\varepsilon-$minimum solution to*

$$Min\,\{h_{e,v} \circ f(x) : x \in \Omega\},$$

*for some $e \in cor(K)$ and $v \in X$, then $x_0$ is $\bar{\varepsilon}e-$efficient solution for $(VOP)$ for each $\bar{\varepsilon} > \varepsilon$.*

*Proof.* Considering $\bar{\varepsilon} > \varepsilon$, by contradiction, assume that there exists $\bar{x} \in \Omega$ such that

$$f(\bar{x}) - f(x_0) + \bar{\varepsilon}e \in -K.$$

Thus

$$\begin{aligned} f(\bar{x}) \in f(x_0) - \bar{\varepsilon}e - K &\subseteq v + h_{e,v}(f(x_0))e - K - \bar{\varepsilon}e - K \\ &\subseteq v + \Big(h_{e,v}(f(x_0)) - \bar{\varepsilon}\Big)e - K. \end{aligned}$$

Thus

$$h_{e,v}(f(\bar{x})) \leq h_{e,v}(f(x_0)) - \bar{\varepsilon} < h_{e,v}(f(x_0)) - \varepsilon.$$

This contradicts the assumption and completes the proof. □

The following result provides a necessary condition for approximate weak efficient solutions of (VOP). It can be seen that this theorem is valid for $\varepsilon e-$BeV, $\varepsilon e-$HuV, and $\varepsilon e-$efficient points too.

**Theorem 4.12.** *Let $K$ be pointed and vectorially closed, and $\varepsilon > 0$. Suppose that $cone(f(\Omega) - f(x_0) + K + \varepsilon e)$ is convex and solid for some $e \in cor(K)$. If $x_0$ is $\varepsilon e-$weak efficient for (VOP), then $x_0$ is $\varepsilon-$minimum of the scalar program*

$$Min\,\{h_{e,f(x_0)-\alpha e} \circ f(x) : x \in \Omega\}$$

*for each $\alpha \in \mathbb{R}$.*

*Proof.* Since $x_0$ is $\varepsilon e-$weak efficient, then

$$(f(\Omega) - f(x_0) + \varepsilon e) \cap (-cor(K)) = \emptyset$$

Suppose that $x_0$ is not an $\varepsilon-$minimum solution to the scalar program

$$Min\,\{h_{e,f(x_0)-\alpha e} \circ f(x) : x \in \Omega\},$$

for some $\alpha \in \mathbb{R}$. Then there exists some $\bar{x} \in \Omega$ such that

$$h_{e,f(x_0)-\alpha e}(f(x_0)) > h_{e,f(x_0)-\alpha e}(f(\bar{x})) + \varepsilon.$$

Thus,

$$\begin{aligned} \alpha - \varepsilon > h_{e,f(x_0)-\alpha e}(f(\bar{x})) &= Sup\,\{\langle l, (f(\bar{x}) - f(x_0) + \alpha e)\rangle \ : \ l \in B\} \\ &\geq \langle l, (f(\bar{x}) - f(x_0) + \alpha e)\rangle \qquad \forall l \in B. \end{aligned}$$

Therefore,

$$\langle l, (f(\bar{x}) - f(x_0) + \alpha e)\rangle < \alpha - \varepsilon \qquad \forall l \in B$$

Hence

$$\langle l, f(\bar{x})\rangle + \varepsilon < \langle l, f(x_0)\rangle \qquad \forall l \in K^+\backslash\{0\}. \tag{4.4}$$

On the other hand, due to Theorem 4.3, there exists $l \in K^+\backslash\{0\}$ such that $x_0$ is $\varepsilon-$minimum of scalar program

$$Min\,\{l \circ f(x) : x \in \Omega\},$$

which leads to

$$\langle l, f(x)\rangle + \varepsilon \geq \langle l, f(x_0)\rangle \qquad \forall x \in \Omega.$$

This contradicts (4.4), and completes the proof. □

Now, we start the last part of this section. In this part, we consider the constrained problem (CVOP). This constrained vector optimization problem is converted to an unconstrained one using a Lagrangian mapping. Then some connections between two problems, constrained and unconstrained, are given, utilizing some of the results established so far. To this end, a constraint qualification condition is required. In the following definition, we adapt the Slater constraint qualification to our context.

**Definition 4.13** ( [3]). We say that the Slater constraint qualification for (CVOP) holds if there exists $x \in \Omega$ such that $g(x) \in -cor(M)$.

Hereafter, the set of all linear operators from $Z$ into $Y$ is denoted by $O(Z, Y)$, and $\Gamma$ is the set of positive linear operators, i.e.,

$$\Gamma = \{T \in O(Z, Y) : T(M) \subseteq K\}.$$

The Lagrangian mapping $L : E \times \Gamma \to Y$ corresponding to the program (CVOP) is defined by $L(x, T) = f(x) + T(g(x))$. The above-defined map helps us to convert (CVOP) to an unconstrained vector optimization problem, under appropriate assumptions. Theorem 4.14 is a generalization of Theorem 4.3 in [3].

**Theorem 4.14.** *In (CVOP), let $x_0 \in \Omega$ and $T \in \Gamma$. Consider the following unconstrained problem*

$$Min\,\{f(x) + (T \circ g)(x) \,:\, x \in E\} \tag{4.5}$$

(i) *If $T(g(x_0)) = 0$ and $x_0$ is $\epsilon-BeV$ for* (4.5)*, then $x_0$ is $\epsilon-BeV$ for (CVOP).*
(ii) *Assume that $M$ is pointed and solid, $K^+$ and $K$ are solid, $K$ is vectorially closed, $f(.) - f(x_0) + \epsilon$ is a GVCL mapping in $\Omega$ with respect to $K$ and $\Big(f(.) - f(x_0) + \epsilon, g(.)\Big)$ is a GVCL mapping in $E$ with respect to $(K, M)$. If the Slater constraint qualification holds and $x_0$ is $\epsilon-BeV$ for (CVOP), then $x_0$ is an $\epsilon-BeV$ solution to unconstrained optimization problem* (4.5) *for some $T \in \Gamma$.*

*Proof.* (i) For each $x \in \Omega$, we have $T(g(x)) \in -K$. Hence, $0 \in T(g(\Omega)) + K$. Therefore,

$$\begin{aligned} f(\Omega) - f(x_0) + \epsilon + K &\subseteq f(\Omega) - f(x_0) + \epsilon + K + T(g(\Omega)) + K \\ &= f(\Omega) - f(x_0) + \epsilon + K + T(g(\Omega)) + K - T(g(x_0)) \\ &= \Big(f + T \circ g\Big)(\Omega) - \Big(f + T \circ g\Big)(x_0) + \epsilon + K \\ &\subseteq \Big(f + T \circ g\Big)(E) - \Big(f + T \circ g\Big)(x_0) + \epsilon + K. \end{aligned}$$

Hence,

$$vcl\bigg(cone\bigg(\Big(f + T \circ g\Big)(E) - \Big(f + T \circ g\Big)(x_0) + \epsilon + K\bigg)\bigg) \cap (-K) = \{0\}$$

implies

$$vcl\bigg(cone\Big(f(\Omega) - f(x_0) + \epsilon + K\Big)\bigg) \cap (-K) = \{0\}.$$

This completes the proof of part (i).
(ii) Since $f(.) - f(x_0) + \epsilon$ is GVCL in $\Omega$, because of Corollary 4.7, there exists a functional $l \in K^{+s}$ such that $x_0$ is an $\varepsilon-$minimum of the scalar program,

$$Min\,\{(l\circ f)(x) : x\in\Omega\}$$

for each $\varepsilon$ which satisfies $\varepsilon=\langle l,\epsilon\rangle$. This implies

$$\langle l,f(x_0)\rangle \le \langle l,f(x)\rangle+\varepsilon \qquad \forall x\in\Omega.$$

Equivalently, for each $x\in E$ which satisfies $g(x)\in -M$, we have

$$\langle l,f(x)-f(x_0)\rangle+\varepsilon\ge 0.$$

Thus, there is no $x\in E$ such that

$$\Big(\langle l,f(x)-f(x_0)\rangle+\varepsilon\ ,\ g(x)\Big)\in cor(-\mathbb{R}_+)\times cor(-M)=-cor(\mathbb{R}_+\times M).$$

Thus

$$\Big(\langle l,f(E)-f(x_0)\rangle+\varepsilon\ ,\ g(E)\Big)\cap\Big(-cor(\mathbb{R}_+\times M)\Big)=\emptyset.$$

Due to Proposition 4.1 in [3], $vcl\Big(cone\Big(\langle l,f(E)-f(x_0)\rangle+\varepsilon\ ,\ g(E)\Big)+\Big(\mathbb{R}_+\times M\Big)\Big)$ is convex. Therefore, by Theorem 2.4,

$$\Big(\langle l,f(E)-f(x_0)\rangle+\varepsilon\ ,\ g(E)\Big)^+\cap\Big(\mathbb{R}_+\times M\Big)^+\ne\{0\}.$$

So, there exists $(r,\mu)\in(\mathbb{R}_+\times M)^+\backslash\{0\}$ such that

$$\Big\langle (r,\mu)\ ,\ \Big(\langle l,f(x)-f(x_0)\rangle+\varepsilon\ ,\ g(x)\Big)\Big\rangle\ge 0, \qquad \forall x\in E.$$

This implies that

$$r\Big(\langle l,f(x)-f(x_0)\rangle+\varepsilon\Big)+\langle\mu,g(x)\rangle\ge 0, \qquad \forall x\in E.$$

Furthermore $r\ne 0$, otherwise

$$\langle\mu,g(x)\rangle\ge 0 \qquad \forall x\in E$$

which contradicts the Slater constraint qualification. Hence $r>0$.
Also, there exists $k_1\in K$ such that $\langle l,k_1\rangle>0$, because $l\in K^{+s}$. Considering

$$y_1=\frac{k_1}{r\langle l,k_1\rangle},$$

we have $y_1\in K\backslash\{0\}$ and $\langle l,y_1\rangle=\dfrac{1}{r}$.
Now, we define a linear mapping $T:Z\to Y$ by

$$T(z)=\langle\mu,z\rangle y_1.$$

Since for $m\in M$, $T(m)=\langle\mu,m\rangle y_1\in K$, thus $T\in\Gamma$. Also, we have

$$r\Big(\langle l,f(x)-f(x_0)\rangle+\varepsilon\Big)+\langle\mu,g(x)\rangle-\langle\mu,g(x_0)\rangle\ge 0, \qquad \forall x\in E.$$

Thus,

$$\langle l, f(x) - f(x_0)\rangle + \varepsilon + \langle l, y_1\rangle \langle \mu, g(x) - g(x_0)\rangle \geq 0, \qquad \forall x \in E,$$

which leads to

$$\langle l, f(x) - f(x_0)\rangle + \varepsilon + \langle l, T(g(x)) - T(g(x_0))\rangle \geq 0, \qquad \forall x \in E.$$

Therefore, we have

$$\langle l, f(x) + T(g(x))\rangle - \langle l, f(x_0) + T(g(x_0))\rangle + \varepsilon \geq 0, \qquad \forall x \in E.$$

So, $x_0$ is an $\varepsilon-$minimum to the scalar program

$$Min\,\{\langle l, (f + T \circ g)(x)\rangle \,:\, x \in E\}.$$

By Theorem 4.4, $x_0$ is $\epsilon-$BeV for

$$Min\,\{(f + T \circ g)(x) \,:\, x \in E\},$$

and the proof is completed. □

**Theorem 4.15.** (i) *Let $T \in \Gamma$ and $x_0 \in \Omega$ such that $T(g(x_0)) = 0$. If $x_0$ is an $\epsilon-$weak efficient solution of* (4.5), *then $x_0$ is an $\epsilon-$weak efficient solution of (CVOP).*
(ii) *Suppose that $K, M$ are pointed and $M$ is solid. Also, assume that $x_0 \in \Omega$ and $(f(.) - f(x_0) + \epsilon\ ,\ g(.))$ is GVCL in $E$ with respect to $(K, M)$. If the Slater constraint qualification holds, and $x_0$ is $\epsilon-$weak efficient for (CVOP), then there exists a $T \in \Gamma$ such that $x_0$ is $\epsilon-$weak efficient for unconstrained program* (4.5).

*Proof.* (i) If $x_0$ is not $\epsilon-$weak efficient for (CVOP), then there exists $\bar{x} \in E$ such that $g(\bar{x}) \in -M$ and

$$f(x_0) \in f(\bar{x}) + cor(K) + \epsilon.$$

Therefore

$$f(x_0) - T(g(\bar{x})) \in f(\bar{x}) + cor(K) + \epsilon - T(g(\bar{x})) \subseteq f(\bar{x}) + cor(K) + \epsilon + K = f(\bar{x}) + cor(K) + \epsilon.$$

Since $T(g(x_0)) = 0$, we have

$$f(x_0) + T(g(x_0)) \in f(\bar{x}) + cor(K) + \epsilon + T(g(\bar{x})),$$

leading to

$$\Big((f + T \circ g)(E) - (f + T \circ g)(x_0) + \epsilon\Big) \cap \Big(-cor(K)\Big) \neq \emptyset.$$

It makes a contradiction and completes the proof of part (i).
(ii) Since $x_0 \in \Omega$ is an $\epsilon-$weak efficient solution of (CVOP), then

$$\Big(f(E) - f(x_0) + \epsilon\ ,\ g(E)\Big) \cap -\,cor\Big(K \times M\Big) = \emptyset.$$

Since $\Big(f(.) - f(x_0) + \epsilon\ ,\ g(.)\Big)$ is GVCL in $E$ with respect to $(K, M)$,

$$vcl\bigg(cone\Big(f(E) - f(x_0) + \epsilon\ ,\ g(E)\Big) + (K, M)\bigg)$$

is convex. Hence, applying Theorem 2.4, there exists $(\mu_K, \mu_M) \in (K \times M)^+$ such that

$$\langle \mu_K, f(x) - f(x_0) + \epsilon \rangle + \langle \mu_M, g(x) \rangle \geq 0 \qquad \forall x \in E.$$

Furthermore $\mu_K \neq 0$, otherwise

$$\langle \mu_M, g(x) \rangle \geq 0 \qquad \forall x \in E$$

which contradicts the Slater constraint qualification. On the other hand,

$$\langle \mu_M, g(x_0) \rangle \leq 0,$$

because $x_0 \in \Omega$. Since $K$ is solid, there exists $k_1 \in K$, such that $\langle \mu_K, k_1 \rangle > 0$; otherwise $l(K) = \{0\}$ and $l(-K) = \{0\}$. So $l(K-K) = \{0\}$, but since K is solid, we get $span(K-K) = Y$; and thus $l = 0$, which makes a contradiction.

Setting $y_1 = \dfrac{k_1}{\langle \mu_K, k_1 \rangle}$, we have $y_1 \in K \backslash \{0\}$. Now, we define a linear mapping $T : Z \to Y$ by $T(z) = \langle \mu_M, z \rangle y_1$. Since for $m \in M$ we have $T(m) = \langle \mu_M, m \rangle y_1 \in K \backslash \{0\}$, thus $T \in \Gamma$. On the other hand,

$$(\mu_K \circ T)(z) = \mu_K \Big( \langle \mu_M, z \rangle y_1 \Big) = \langle \mu_M, z \rangle \langle \mu_K, y_1 \rangle = \langle \mu_M, z \rangle.$$

Therefore,

$$\begin{aligned} &\langle \mu_K, f(x) - f(x_0) + \epsilon \rangle + \langle \mu_M, g(x) \rangle - \langle \mu_M, g(x_0) \rangle \geq 0 \quad \forall x \in E \\ \Rightarrow &\langle \mu_K, f(x) - f(x_0) + \epsilon \rangle + \langle \mu_K \circ T, g(x) \rangle - \langle \mu_K \circ T, g(x_0) \rangle \geq 0 \quad \forall x \in E \\ \Rightarrow &\langle \mu_K, (f + T \circ g)(x) \rangle + \langle \mu_K, \epsilon \rangle - \langle \mu_K, (f + T \circ g)(x_0) \rangle \geq 0 \quad \forall x \in E \end{aligned} \tag{4.6}$$

Therefore, $x_0$ is $\epsilon-$weak efficient for Problem (4.5), otherwise there exists $\bar{x} \in E$ such that

$$(f + T \circ g)(x_0) \in (f + T \circ g)(\bar{x}) + \epsilon + cor(K).$$

Hence

$$\langle \mu_K, (f + T \circ g)(x_0) - (f + T \circ g)(\bar{x}) - \epsilon \rangle > 0.$$

So, we have

$$\langle \mu_K, (f + T \circ g)(x_0) \rangle > \langle \mu_K, (f + T \circ g)(\bar{x}) \rangle + \langle \mu_K, \epsilon \rangle,$$

which contradicts (4.6). Hence $x_0$ is $\epsilon-$weak efficient for (4.5), and the proof is completed. □

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