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DISTRIBUTED ITERATIVE METHODS FOR SOLVING NONMONOTONE VARIATIONAL INEQUALITY OVER THE INTERSECTION OF FIXED POINT SETS OF NONEXPANSIVE MAPPINGS

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Abstract: We consider a networked system which consists of a finite number of users and discuss a nonmonotone variational inequality with the gradient of the sum of all users' nonconvex objective functions over the intersection of all users' constraint sets. Centralized iterative methods, which require us to use the explicit forms of all users' objective functions and constraint sets, have been proposed to solve the variational inequality. However, it would be difficult to apply them to the variational inequality in this system because none of the users in the system can know the explicit forms of all users' objective functions and constraint sets. In this paper, we translate the variational inequality into a nonmonotone variational inequality over the intersection of the fixed point sets of certain nonexpansive mappings and devise distributed iterative methods, which enable each user to solve the variational inequality without using the explicit forms of other users' objective functions and constraint sets, based on fixed point theory for nonexpansive mappings. We also present convergence analyses for them and provide numerical examples for the bandwidth allocation. The analyses and numerical examples suggest that distributed iterative methods with slowly diminishing step-size sequences converge to a solution to the variational inequality.

Key words: *nonmonotone variational inequality, fixed point set, nonexpansive mapping, incremental subgradient method, broadcast iterative method, conjugate gradient method, fixed point optimization algorithm*

Mathematics Subject Classification: *49M37, 65K05, 90C26, 90C90*

1 Introduction

1.1 Background

Nonmonotone variational inequality problems [7, Chapter I], [24, Subchapter 6.D], related to important nonlinear problems such as nonconvex optimization problems and Nash equilibrium problems, are the central topic of optimization theory. Iterative methods [8] for solving nonmonotone variational inequalities have been widely studied.

In this paper, we consider a networked system in which each user has its own private nonconvex objective function and constraint set, and deal with a nonmonotone variational inequality for the gradient of the sum of all users' objective functions over the intersection of all users' constraint sets. Moreover, we assume that each user's constraint set does not

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always have a simple form (for examples of such a constraint set, see Examples (II) and (III) in Subsection 1.2). This implies that the metric projection onto each user's constraint set cannot be easily computed.† The nonmonotone variational inequality problem in this case includes, for instance, power control problems in direct-sequence code-division multiple-access (CDMA) data networks [12] and in wireless networks [25], and bandwidth allocation problems [13].

References [12,13] showed that practical problems, such as power control and bandwidth allocation problems, can be translated into a nonmonotone variational inequality problem over the *fixed point set* of a certain *nonexpansive* mapping [1], [9, Chapter 3], [10, Chapter 1], and presented iterative methods for solving the nonmonotone variational inequality. These iterative methods must use the explicit forms of the nonexpansive mapping and the nonmonotone operator, and hence, they are referred to as *centralized iterative methods*.

In the case of the power control for the uplink or downlink in CDMA data network, the base station plays the role of the centralized operator, and hence, it can get user information such as the explicit forms of the objective functions and constraint sets from the start. To control the power allocation in the network, the base station executes a centralized iterative method and transmits the powers computed by the method to all users in the network. However, there is no centralized operator in peer-to-peer (P2P) networks for data storage allocation [20], wireless networks for power allocation [25], and wired networks for bandwidth allocation [14, 19]. Therefore, in large-scale and complex networked systems, there is an inconvenient possibility that none of the users can get the explicit forms of their own objective functions and constraint sets, and hence, centralized iterative methods cannot be applied to such systems. Moreover, since such networks can grow in size, *distributed mechanisms* should be used for network resource allocation instead of centralized ones that involve extra infrastructure. Distributed mechanisms enable each user to adjust its own resource allocation without using the private information of other users such as their objective functions and constraint sets.

Many *distributed iterative methods*, which can be applied to such systems, have been proposed. The conventional distributed iterative methods enable each user in the networked system to solve a monotone variational inequality without using other users' convex, nondifferentiable objective functions and simple constraint sets. The well-known distributed iterative method is the *incremental subgradient method* (see [3, Subchapter 8.2], [4,17,18,21] and references therein). The incremental subgradient methods can be implemented through cooperation, whereby each user communicates with its neighbor user. Broadcast types of distributed iterative methods [5, 6, 25] were proposed for solving the monotone variational inequality. The broadcast iterative methods can be implemented through cooperation, whereby all users communicate with each other. Reference [14] proposed an incremental gradient method and a broadcast iterative method for solving a monotone variational inequality over the intersection of the fixed point sets of nonexpansive mappings. These methods [14] with slowly diminishing step-size sequences converge to a unique solution to the monotone variational inequality.

From the above viewpoint, we can conclude that distributed iterative methods should be devised to solve a nonmonotone variational inequality over the intersection of the fixed point sets of nonexpansive mappings, which is defined properly in the next subsection.

†The projection onto a simple set (e.g., a half-space) can be computed within a finite number of arithmetic operations.

1.2 Main problem

This paper considers a networked system, which consists of K users, under the following assumptions:

Assumption 1.1. Suppose that user i ($i \in I := \{1, 2, \dots, K\}$) has its own private constraint set $C^{(i)}$ and objective function $f^{(i)}$ satisfying the following:

(A1) $X^{(i)}$ ($i \in I$) is a nonempty, bounded, closed, convex set of $\mathbb{R}^N$ onto which the projection, denoted by $P_{X^{(i)}}$, can be easily computed.

(A2) User i ($i \in I$) can use a firmly nonexpansive mapping[‡] $T^{(i)} \colon \mathbb{R}^N \to \mathbb{R}^N$ with ($X^{(i)} \supset$) $\mathrm{Fix}(T^{(i)}) := \{x \in \mathbb{R}^N : T^{(i)}(x) = x\} = C^{(i)}$ and $\bigcap_{i \in I} \mathrm{Fix}(T^{(i)}) \neq \emptyset$.

(A3) $f^{(i)} \colon \mathbb{R}^N \to \mathbb{R}$ ($i \in I$) is continuously differentiable.

User i in actual networked systems [14, 19, 20, 25] has a bounded constraint set $C^{(i)}$. Then, user i can set a bounded $X^{(i)}$ ($\supset C^{(i)}$) ($i \in I$) in advance (e.g., $X^{(i)}$ is a closed ball with a large enough radius). Section 4 discusses the network bandwidth allocation problem when user i (source i) has a bounded set $C^{(i)}$ and a box constant set $X^{(i)}$. See [14,19,20,25] for other examples of a bounded $C^{(i)}$.

Let us provide three examples (I)–(III) [14] of $T^{(i)}$ ($i \in I$) satisfying Assumption (A2). (I) Suppose that user i has a simple, closed convex constraint set $C^{(i)}$ (e.g., $C^{(i)}$ is a closed ball or a half-space). The typical example of $T^{(i)}$ is the metric projection, denoted by $P_{C^{(i)}}$, onto $C^{(i)}$ because $P_{C^{(i)}}$ can be easily computed and satisfies the firm nonexpansivity condition and $\mathrm{Fix}(P_{C^{(i)}}) = C^{(i)}$. (II) Let us consider the case where $C^{(i)}$ is the intersection of simple, closed convex sets $D_j^{(i)}$s ($j \in J(i) := \{1, 2, \dots, m(i)\}$), i.e.,

$$C^{(i)} := \bigcap_{j \in J(i)} D_j^{(i)}. \tag{1.1}$$

Since $P_{D_j^{(i)}}$ can be easily computed, user i can use $T^{(i)}$ defined by

$$T^{(i)} := \frac{1}{2} \left(\mathrm{Id} + \prod_{j \in J(i)} P_{D_j^{(i)}} \right), \tag{1.2}$$

where Id stands for the identity mapping on $\mathbb{R}^N$. $T^{(i)}$ satisfies the firm nonexpansivity condition [2, Definition 4.1, Proposition 4.2] and $\mathrm{Fix}(T^{(i)}) = \mathrm{Fix}(\prod_{j \in J(i)} P_{D_j^{(i)}}) = \bigcap_{j \in J(i)} D_j^{(i)} = C^{(i)}$. Section 4 describes that source i has $T^{(i)}$ defined by Equation (1.2) with $\mathrm{Fix}(T^{(i)}) = C^{(i)} \subset X^{(i)}$. (III) Let us consider the case where $C^{(i)}$ is the set of all minimizers of a differentiable, convex functional $g^{(i)}$ with the Lipschitz continuous gradient $\nabla g^{(i)}$ over a simple, closed convex set $D^{(i)}$, i.e.,

$$C^{(i)} := \left\{ x \in D^{(i)} : g^{(i)}(x) = \min_{y \in D^{(i)}} g^{(i)}(y) \right\}. \tag{1.3}$$

‡ $T \colon \mathbb{R}^N \to \mathbb{R}^N$ is said to be *firmly nonexpansive* [1], [9, Chapter 12], [10, Subchapter 1.11] if $\|T(x) - T(y)\|^2 \leq \langle x - y, T(x) - T(y) \rangle$ ($x, y \in \mathbb{R}^N$), where $\langle \cdot, \cdot \rangle$ stands for the inner product of $\mathbb{R}^N$ and $\| \cdot \|$ is the norm of $\mathbb{R}^N$.

$P_{D^{(i)}}(\mathrm{Id} - \lambda \nabla g^{(i)})$ is nonexpansive with an adequate λ (> 0) [11, Proposition 2.3]. Accordingly, user i can use a firmly nonexpansive $T^{(i)}$ defined by

$$T^{(i)} := \frac{1}{2}\left(\mathrm{Id} + P_{D^{(i)}}\left(\mathrm{Id} - \lambda \nabla g^{(i)}\right)\right) \tag{1.4}$$

with $\mathrm{Fix}(T^{(i)}) = \mathrm{Fix}(P_{D^{(i)}}(\mathrm{Id} - \lambda \nabla g^{(i)})) = C^{(i)}$ [27, Theorem 46.C (1) and (2)]. It would be difficult to compute $P_{C^{(i)}}$ onto $C^{(i)}$ in Equation (1.1) and $P_{C^{(i)}}$ onto $C^{(i)}$ in Equation (1.3). Meanwhile, we can see that $T^{(i)}$ in Equation (1.2) and $T^{(i)}$ in Equation (1.4) are firmly nonexpansive with $\mathrm{Fix}(T^{(i)}) = C^{(i)}$ and can be easily computed.

The main problem in this paper is the following nonmonotone variational inequality problem with information on the whole system:

Problem 1.1. Under Assumption 1.1, find a point in

$$\begin{aligned} &\mathrm{VI}\left(\bigcap_{i \in I} \mathrm{Fix}\left(T^{(i)}\right), \nabla\left(\sum_{i \in I} f^{(i)}\right)\right) \\ &:= \left\{ x^\star \in X := \bigcap_{i \in I} \mathrm{Fix}\left(T^{(i)}\right) : \left\langle x - x^\star, \nabla\left(\sum_{i \in I} f^{(i)}\right)(x^\star)\right\rangle \geq 0 \ (x \in X)\right\}. \end{aligned}$$

1.3 Main objective and contributions of the paper

The main objective of the paper is to devise distributed iterative methods for solving Problem 1.1, which the existing methods in Subsection 1.1 cannot solve, and to prove that the proposed methods converge to a solution to Problem 1.1 under certain assumptions. The contribution of this paper is that it is the first study to tackle nonmonotone variational inequalities over the intersection of the fixed point sets of nonexpansive mappings and it proposes two distributed iterative methods for them. Section 2 presents an *incremental fixed point optimization algorithm*, based on the conventional incremental subgradient methods, for solving Problem 1.1 and its convergence analysis. Section 3 presents a *broadcast fixed point optimization algorithm*, based on the conventional broadcast iterative methods, for solving Problem 1.1 and its convergence analysis. Section 4 applies the algorithms to a network bandwidth allocation problem and provides numerical examples for network bandwidth allocation. The convergence analyses and numerical examples describe that the algorithms with slowly diminishing step-size sequences converge to a solution to Problem 1.1. Section 5 concludes the paper.

2 Incremental Gradient Method for Nonmonotone Variational Inequality and Its Convergence Analysis

We first present the following algorithm for solving Problem 1.1:

Algorithm 2.1 (Incremental Fixed Point Optimization Algorithm)**.**

Step 0. User i $(i \in I)$ sets $(\alpha_n)_{n\in\mathbb{N}}, (\lambda_n)_{n\in\mathbb{N}}$, and $(\beta_n)_{n\in\mathbb{N}}$, chooses $x_{-1}^{(i)} \in \mathbb{R}^N$ arbitrarily, and computes $d_{-1}^{(i)} := -\nabla f^{(i)}(x_{-1}^{(i)})$. User K sets $x_0 \in \mathbb{R}^N$ and transmits $x_0^{(0)} := x_0$ to user 1.

Step 1. Given $x_n = x_n^{(0)} \in \mathbb{R}^N$ and $d_{n-1}^{(i)} \in \mathbb{R}^N$ $(i \in I)$, user i computes $x_n^{(i)} \in \mathbb{R}^N$ cyclically by

$$\begin{cases} d_n^{(i)} := -\nabla f^{(i)}\left(x_n^{(i-1)}\right) + \beta_n d_{n-1}^{(i)}, \\ y_n^{(i)} := T^{(i)}\left(x_n^{(i-1)} + \lambda_n d_n^{(i)}\right), \\ x_n^{(i)} := P_{X^{(i)}}\left(\alpha_n x_n^{(i-1)} + (1-\alpha_n) y_n^{(i)}\right) \ (i = 1, 2, \ldots, K). \end{cases}$$

Step 2. User K defines $x_{n+1} \in \mathbb{R}^N$ by

$$x_{n+1} := x_n^{(K)}$$

and transmits $x_{n+1}^{(0)} := x_{n+1}$ to user 1. Put $n := n+1$, and go to Step 1.

The conventional incremental subgradient method (see [3, Subchapter 8.2], [4, 17, 18, 21] and references therein) is defined as follows:

$$\begin{cases} x_n^{(i)} := P_C\left(x_n^{(i-1)} - \lambda_n g_n^{(i)}\right), \ g_n^{(i)} \in \partial f^{(i)}\left(x_n^{(i-1)}\right) \ (i = 1, 2, \ldots, K), \\ x_{n+1} := x_n^{(K)}, \end{cases} \tag{2.1}$$

where all users have a simple, closed convex set C and $\partial f^{(i)}(x)$ stands for the subdifferential of a nonsmooth, convex functional $f^{(i)}$ at $x \in H$. References [3, Proposition 8.2.6] and [21, Proposition 2.4] describe that, when $(\lambda_n)_{n\in\mathbb{N}}$ satisfies $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $\sum_{n=0}^{\infty} \lambda_n^2 < \infty$, $(x_n)_{n\in\mathbb{N}}$ in Algorithm (2.1) converges to a minimizer of $\sum_{i\in I} f^{(i)}$ over C.

The incremental fixed point optimization algorithm [14] was presented as a way to solve Problem 1.1 when $f^{(i)}$ is differentiable and convex and $T^{(i)}$ is firmly nonexpansive. This algorithm cannot be applied to the case where $f^{(i)}$ is nonsmooth and convex because the proof of its convergence analysis essentially uses the Lipschitz continuity of the gradient of $f^{(i)}$ (see [14, Proposition 2.1]).[§]

Meanwhile, Algorithm 2.1 can use a firmly nonexpansive $T^{(i)}$ satisfying $\mathrm{Fix}(T^{(i)}) = C^{(i)}$ and can be applied even when $f^{(i)}$ $(i \in I)$ is not always convex. Algorithm 2.1 uses the *conjugate gradient direction* [22, Chapter 5], [15, 16] generated by $d_n^{(i)} := -\nabla f^{(i)}(x_n^{(i-1)}) + \beta_n d_{n-1}^{(i)}$ $(i \in I, n \in \mathbb{N})$ to accelerate algorithms with the steepest descent direction $d_n^{(i)} := -\nabla f^{(i)}(x_n^{(i-1)})$ $(i \in I, n \in \mathbb{N})$.

We need the following assumptions to guarantee that Algorithm 2.1 converges to a solution to Problem 1.1:

Assumption 2.1. User i $(i \in I)$ has $(\alpha_n)_{n\in\mathbb{N}} \subset [0,1]$, $(\beta_n)_{n\in\mathbb{N}} \subset [0,1]$, and $(\lambda_n)_{n\in\mathbb{N}} \subset (0,1]$ satisfying[¶]

$$\text{(C1)}\ 0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1, \ \text{(C2)} \sum_{n=0}^{\infty} \lambda_n < \infty, \ \text{(C3)} \lim_{n\to\infty} \beta_n = 0.$$

Assumption 2.2. $(\nabla f^{(i)}(x_n^{(i-1)}))_{n\in\mathbb{N}}$ $(i \in I)$ is bounded.

[§]In the future, we should consider developing distributed iterative methods for solving minimization problems over the fixed point sets of nonexpansive mappings in which all users' objective functions are nonsmooth and convex.

[¶]Examples of $(\alpha_n)_{n\in\mathbb{N}}, (\lambda_n)_{n\in\mathbb{N}}$, and $(\beta_n)_{n\in\mathbb{N}}$ are $\alpha_n := a$ $(\in (0,1))$, $\lambda_n := 1/(n+1)^b$ $(b > 1)$, and $\beta_n := 1/(n+1)^c$ $(c > 0)$.

Let us provide a condition to satisfy Assumption 2.2. The boundedness of $X^{(i)}$ guarantees that $(x_n^{(i)})_{n\in\mathbb{N}}$ $(i \in I)$ is bounded. Now suppose that $\nabla f^{(i)}$ $(i \in I)$ is Lipschitz continuous with $L^{(i)} > 0$ ($L^{(i)}$-Lipschitz continuous), i.e., $\|\nabla f^{(i)}(x) - \nabla f^{(i)}(y)\| \leq L^{(i)}\|x - y\|$ $(x, y \in \mathbb{R}^N)$. Accordingly, $\|\nabla f^{(i)}(x_n^{(i-1)}) - \nabla f^{(i)}(y)\| \leq L^{(i)}\|x_n^{(i-1)} - y\|$ $(i \in I, y \in \mathbb{R}^N)$. This inequality and the boundedness of $(x_n^{(i)})_{n\in\mathbb{N}}$ $(i \in I)$ ensure that Assumption 2.2 is satisfied under the Lipschitz continuity of $\nabla f^{(i)}$ $(i \in I)$.

Now let us do a convergence analysis on Algorithm 2.1.

Theorem 2.3. *Suppose that Assumptions 1.1, 2.1, and 2.2 are satisfied. Then,* $(x_n^{(i)})_{n\in\mathbb{N}}$ $(i \in I)$ *in Algorithm 2.1 has the following properties:*

(a) $(x_n^{(i)})_{n\in\mathbb{N}}$ $(i \in I)$ *is bounded and* $\lim_{n\to\infty} \|x_n - y\|$ *exists for all* $y \in \bigcap_{i\in I} \mathrm{Fix}(T^{(i)})$.

(b) $\lim_{n\to\infty} \|x_n - x_n^{(i-1)}\| = 0$ $(i \in I)$, $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, $\lim_{n\to\infty} \|x_n - T^{(i)}(x_n)\| = 0$ $(i \in I)$.

(c) $(x_n^{(i)})_{n\in\mathbb{N}}$ $(i \in I)$ *converges to a common fixed point of* $T^{(i)}$*s.*

(d) *If* $\|x_{n+1} - x_n\| = o(\lambda_n)$, $(x_n^{(i)})_{n\in\mathbb{N}}$ $(i \in I)$ *converges to a solution to Problem 1.1.*

Item (c) in Theorem 2.3 guarantees that all $(x_n^{(i)})_{n\in\mathbb{N}}$s converge to the same point $x^\star$ in $\bigcap_{i\in I} \mathrm{Fix}(T^{(i)})$. This means that all users which use Algorithm 2.1 under Assumptions 2.1 and 2.2 can find the feasible point in Problem 1.1.

From Item (b) in Theorem 2.3 and Condition (C2), we find that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty} \lambda_n = 0$. Item (d) in Theorem 2.3 says that, if $(\|x_{n+1} - x_n\|)_{n\in\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}}$ satisfy the more restrictive condition, $\|x_{n+1} - x_n\| = o(\lambda_n)$, i.e., $\lim_{n\to\infty} \|x_{n+1} - x_n\|/\lambda_n = 0$, all $(x_n^{(i)})_{n\in\mathbb{N}}$s converge to $x^\star$ in $\mathrm{VI}(\bigcap_{i\in I} \mathrm{Fix}(T^{(i)}), \nabla(\sum_{i\in I} f^{(i)}))$ even when the $f^{(i)}$s are nonconvex. User K can verify whether $\|x_n^{(K)} - x_{n-1}^{(K)}\| = \|x_{n+1} - x_n\| = o(\lambda_n)$ is satisfied or not.[‖] Hence, it would be reasonable that user K determines a slowly diminishing $(\lambda_n)_{n\in\mathbb{N}}$ (e.g., $\lambda_n := 1/(n+1)^{1.01}$ or $1/(n+1)^{1.001}$) to satisfy $\|x_{n+1} - x_n\| = o(\lambda_n)$ as much as possible. From such a viewpoint, it would be desirable to set user K as an operator who manages the networked system.

References [12, 13] present centralized fixed point optimization algorithms for solving Problem 1.1 when $f^{(i)}$s are nonconvex, and these algorithms can be applied to power control [12] and network bandwidth allocation [13] problems. For convenience, we shall write the algorithm in [12] for Problem 1.1 as follows:

$$\begin{cases} y_n := \prod_{i\in I} T^{(i)} \left(x_n - \lambda_n \nabla \left(\sum_{i\in I} f^{(i)} \right)(x_n) \right), \\ x_{n+1} := P_C\left(\alpha_n x_n + (1-\alpha_n) y_n\right) \ (n \in \mathbb{N}), \end{cases} \tag{2.2}$$

where C $(\subset \mathbb{R}^N)$ is closed and convex, and $(\lambda_n)_{n\in\mathbb{N}}$ and $(\alpha_n)_{n\in\mathbb{N}}$ satisfy Conditions (C1) and (C2). Theorem 6 and Remark 7 (c) in [12] guarantee that $(x_n)_{n\in\mathbb{N}}$ in Algorithm (2.2) converges to a solution to Problem 1.1 if $\|x_n - y_n\| = o(\lambda_n)$. Numerical examples in [12, 13] indicated that the algorithms [12, 13] with $\lambda_n := 1/(n+1)^{1.01}$ satisfy $\|x_n - y_n\| = o(\lambda_n)$, while the algorithms with $\lambda_n := 1/(n+1)^2$ do not always satisfy $\|x_n - y_n\| = o(\lambda_n)$. Algorithm (2.2) requires us to use the explicit forms of all $T^{(i)}$s and $f^{(i)}$s. However, it

[‖] User K can obtain, for each $n > 1$, $x_n^{(K)}$, $x_{n-1}^{(K)}$, and λ_n and then compute $z_n := \|x_n^{(K)} - x_{n-1}^{(K)}\|/\lambda_n$ and $w_n := z_{n-1} - z_n$. If $w_n \geq 0$ and z_n is small for each n, the condition will be satisfied.

would be difficult to apply Algorithm (2.2) to Problem 1.1 in networked systems. This is because each user in such systems cannot get the explicit forms of other users' objective functions and nonexpansive mappings. Meanwhile, Algorithm 2.1 can be implemented under the assumption that each user knows its own private objective function and nonexpansive mapping.

2.1 Proof of Theorem 2.3

(a) The definition of $(x_n^{(i)})_{n\in\mathbb{N}}$ $(i \in I)$ and the boundedness of $X^{(i)}$ $(i \in I)$ guarantee the boundedness of $(x_n^{(i)})_{n\in\mathbb{N}}$ $(i \in I)$. Condition (C3) ensures the existence of $n_1 \in \mathbb{N}$ such that $\beta_n \leq 1/2$ for all $n \geq n_1$. We put $M_1^{(i)} := \sup\{\|\nabla f^{(i)}(x_n^{(i-1)})\| \colon n \in \mathbb{N}\}$, $\bar{M}_1^{(i)} := \max\{M_1^{(i)}, \|d_{n_1}^{(i)}\|\}$ $(i \in I)$, and $M_1 := \max_{i\in I} \bar{M}_1^{(i)}$ ($M_1 < \infty$ is guaranteed by the boundedness of $(\nabla f^{(i)}(x_n^{(i-1)}))_{n\in\mathbb{N}}$ $(i \in I)$). We then have that, for all $n \geq n_1$ and for all $i \in I$, $\|d_{n+1}^{(i)}\| \leq \|\nabla f^{(i)}(x_{n+1}^{(i-1)})\| + \beta_{n+1}\|d_n^{(i)}\| \leq M_1 + (1/2)\|d_n^{(i)}\|$. Let us fix $i \in I$ and assume that $\|d_n^{(i)}\| \leq 2M_1$ for some $n \geq n_1$. We find that $\|d_{n+1}^{(i)}\| \leq M_1 + (1/2)\|d_n^{(i)}\| \leq M_1 + (1/2)2M_1 = 2M_1$. Induction guarantees that $\|d_n^{(i)}\| \leq 2M_1$ for all $i \in I$ and for all $n \geq n_1$, which implies that $(d_n^{(i)})_{n\in\mathbb{N}}$ $(i \in I)$ is bounded.

Choose $y \in \bigcap_{i\in I} \mathrm{Fix}(T^{(i)})$ arbitrarily and put $M_2 := \max_{i\in I}(\sup\{2|\langle x_n^{(i-1)} - y, d_n^{(i)}\rangle| + \lambda_n\|d_n^{(i)}\|^2 \colon n \in \mathbb{N}\})$ ($M_2 < \infty$ is guaranteed by the boundedness of $(x_n^{(i)})_{n\in\mathbb{N}}$ and $(d_n^{(i)})_{n\in\mathbb{N}}$ $(i \in I)$). The nonexpansivity of $T^{(i)}$ means that, for all $n \in \mathbb{N}$ and for all $i \in I$,

$$\begin{aligned} \left\|y_n^{(i)} - y\right\|^2 &= \left\|T^{(i)}\left(x_n^{(i-1)} + \lambda_n d_n^{(i)}\right) - T^{(i)}(y)\right\|^2 \leq \left\|\left(x_n^{(i-1)} + \lambda_n d_n^{(i)}\right) - y\right\|^2 \\ &= \left\|x_n^{(i-1)} - y\right\|^2 + 2\lambda_n\left\langle x_n^{(i-1)} - y, d_n^{(i)}\right\rangle + \lambda_n^2\left\|d_n^{(i)}\right\|^2 \\ &\leq \left\|x_n^{(i-1)} - y\right\|^2 + M_2\lambda_n. \end{aligned} \tag{2.3}$$

Inequality (2.3) and the boundedness of $(x_n^{(i)})_{n\in\mathbb{N}}$ $(i \in I)$ ensure that $(y_n^{(i)})_{n\in\mathbb{N}}$ $(i \in I)$ is bounded. Moreover, the nonexpansivity of $P_{X^{(i)}}$ $(i \in I)$, Assumption (A2) ($\mathrm{Fix}(T^{(i)}) \subset X^{(i)} = \mathrm{Fix}(P_{X^{(i)}})$), and the convexity of $\|\cdot\|^2$ guarantee that, for all $n \in \mathbb{N}$ and for all $i \in I$,

$$\begin{aligned} \left\|x_n^{(i)} - y\right\|^2 &= \left\|P_{X^{(i)}}\left(\alpha_n x_n^{(i-1)} + (1-\alpha_n)y_n^{(i)}\right) - P_{X^{(i)}}(y)\right\|^2 \\ &\leq \left\|\left(\alpha_n x_n^{(i-1)} + (1-\alpha_n)y_n^{(i)}\right) - y\right\|^2 \\ &= \left\|\alpha_n\left(x_n^{(i-1)} - y\right) + (1-\alpha_n)\left(y_n^{(i)} - y\right)\right\|^2 \\ &\leq \alpha_n\left\|x_n^{(i-1)} - y\right\|^2 + (1-\alpha_n)\left\|y_n^{(i)} - y\right\|^2 \\ &\leq \alpha_n\left\|x_n^{(i-1)} - y\right\|^2 + (1-\alpha_n)\left\{\left\|x_n^{(i-1)} - y\right\|^2 + M_2\lambda_n\right\} \\ &\leq \left\|x_n^{(i-1)} - y\right\|^2 + M_2\lambda_n. \end{aligned} \tag{2.4}$$

Accordingly, we find that, for all $n \in \mathbb{N}$,

$$\|x_{n+1} - y\|^2 = \left\|x_n^{(K)} - y\right\|^2 \leq \left\|x_n^{(K-1)} - y\right\|^2 + M_2\lambda_n$$
$$\leq \left\|x_n^{(0)} - y\right\|^2 + KM_2\lambda_n = \|x_n - y\|^2 + KM_2\lambda_n,$$

which means that, for all $m, n \in \mathbb{N}$,

$$\|x_{n+m+1} - y\|^2 \leq \|x_{n+m} - y\|^2 + KM_2\lambda_{n+m} \leq \|x_m - y\|^2 + KM_2 \sum_{i=m}^{n+m} \lambda_i$$
$$\leq \|x_m - y\|^2 + KM_2 \sum_{i=m}^{\infty} \lambda_i.$$

Thus, for all $m \in \mathbb{N}$, $\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_{n+m+1} - y\|^2 \leq \|x_m - y\|^2 + KM_2 \sum_{i=m}^{\infty} \lambda_i$. Therefore, Condition (C2) leads us to

$$\limsup_{n\to\infty} \|x_n - y\|^2 \leq \liminf_{m\to\infty} \left\{ \|x_m - y\|^2 + KM_2 \sum_{i=m}^{\infty} \lambda_i \right\} = \liminf_{m\to\infty} \|x_m - y\|^2,$$

which implies that $\lim_{n\to\infty} \|x_n - y\|$ exists for all $y \in \bigcap_{i\in I} \mathrm{Fix}(T^{(i)})$.

(b) We shall prove that $\lim_{n\to\infty} \|x_n^{(i-1)} - y_n^{(i)}\| = 0$ $(i \in I)$. From the firm nonexpansivity of $T^{(i)}$ and $\langle x, y\rangle = (1/2)(\|x\|^2 + \|y\|^2 - \|x - y\|^2)$ $(x, y \in \mathbb{R}^N)$, we have that, for all $n \in \mathbb{N}$, for all $i \in I$, and for all $y \in \bigcap_{i\in I} \mathrm{Fix}(T^{(i)})$,

$$\begin{aligned}
&\left\|y_n^{(i)} - y\right\|^2 = \left\|T^{(i)}\left(x_n^{(i-1)} + \lambda_n d_n^{(i)}\right) - T^{(i)}(y)\right\|^2 \\
&\leq \left\langle \left(x_n^{(i-1)} + \lambda_n d_n^{(i)}\right) - y, y_n^{(i)} - y \right\rangle \\
&= \frac{1}{2}\left\{ \left\|\left(x_n^{(i-1)} - y\right) + \lambda_n d_n^{(i)}\right\|^2 + \left\|y_n^{(i)} - y\right\|^2 - \left\|\left(x_n^{(i-1)} - y_n^{(i)}\right) + \lambda_n d_n^{(i)}\right\|^2 \right\},
\end{aligned}$$

which means that

$$\begin{aligned}
&\left\|y_n^{(i)} - y\right\|^2 \leq \left\|\left(x_n^{(i-1)} - y\right) + \lambda_n d_n^{(i)}\right\|^2 - \left\|\left(x_n^{(i-1)} - y_n^{(i)}\right) + \lambda_n d_n^{(i)}\right\|^2 \\
&= \left\|x_n^{(i-1)} - y\right\|^2 + 2\lambda_n \left\langle x_n^{(i-1)} - y, d_n^{(i)} \right\rangle + \lambda_n^2 \left\|d_n^{(i)}\right\|^2 \\
&\quad - \left\|x_n^{(i-1)} - y_n^{(i)}\right\|^2 + 2\lambda_n \left\langle y_n^{(i)} - x_n^{(i-1)}, d_n^{(i)} \right\rangle - \lambda_n^2 \left\|d_n^{(i)}\right\|^2 \\
&= \left\|x_n^{(i-1)} - y\right\|^2 + 2\lambda_n \left\langle y_n^{(i)} - y, d_n^{(i)} \right\rangle - \left\|x_n^{(i-1)} - y_n^{(i)}\right\|^2 \\
&\leq \left\|x_n^{(i-1)} - y\right\|^2 + M_3\lambda_n - \left\|x_n^{(i-1)} - y_n^{(i)}\right\|^2,
\end{aligned} \tag{2.5}$$

where $M_3 := \max_{i\in I}(\sup\{2|\langle y_n^{(i)} - y, d_n^{(i)}\rangle| \colon n \in \mathbb{N}\}) < \infty$. Hence, Inequality (2.4) ensures that, for all $n \in \mathbb{N}$, for all $i \in I$, and for all $y \in \bigcap_{i\in I} \mathrm{Fix}(T^{(i)})$,

$$\begin{aligned}
&\left\| x_n^{(i)} - y \right\|^2 \leq \alpha_n \left\| x_n^{(i-1)} - y \right\|^2 + (1-\alpha_n) \left\| y_n^{(i)} - y \right\|^2 \\
&\leq \alpha_n \left\| x_n^{(i-1)} - y \right\|^2 + (1-\alpha_n) \left\{ \left\| x_n^{(i-1)} - y \right\|^2 + M_3 \lambda_n - \left\| x_n^{(i-1)} - y_n^{(i)} \right\|^2 \right\} \\
&\leq \left\| x_n^{(i-1)} - y \right\|^2 + M_3 \lambda_n - (1-\alpha_n) \left\| x_n^{(i-1)} - y_n^{(i)} \right\|^2 .
\end{aligned}$$

Therefore, for all $n \in \mathbb{N}$ and for all $y \in \bigcap_{i\in I} \mathrm{Fix}(T^{(i)})$,

$$\begin{aligned}
\|x_{n+1} - y\|^2 &= \left\| x_n^{(K)} - y \right\|^2 \leq \left\| x_n^{(K-1)} - y \right\|^2 + M_3 \lambda_n - (1-\alpha_n) \left\| x_n^{(K-1)} - y_n^{(K)} \right\|^2 \\
&\leq \left\| x_n^{(0)} - y \right\|^2 + K M_3 \lambda_n - (1-\alpha_n) \sum_{i\in I} \left\| x_n^{(i-1)} - y_n^{(i)} \right\|^2 \\
&= \|x_n - y\|^2 + K M_3 \lambda_n - (1-\alpha_n) \sum_{i\in I} \left\| x_n^{(i-1)} - y_n^{(i)} \right\|^2 ,
\end{aligned}$$

and hence,

$$(1-\alpha_n) \sum_{i\in I} \left\| x_n^{(i-1)} - y_n^{(i)} \right\|^2 \leq \|x_n - y\|^2 - \|x_{n+1} - y\|^2 + K M_3 \lambda_n .$$

The existence of $\lim_{n\to\infty} \|x_n - y\|$ ($y \in \bigcap_{i\in I} \mathrm{Fix}(T^{(i)})$), $\lim_{n\to\infty} \lambda_n = 0$ (by Condition (C2)), and Condition (C1) lead one to deduce that $\lim_{n\to\infty} \sum_{i\in I} \|x_n^{(i-1)} - y_n^{(i)}\| = 0$; i.e.,

$$\lim_{n\to\infty} \left\| x_n^{(i-1)} - y_n^{(i)} \right\| = 0 \ (i \in I). \tag{2.6}$$

Next, we shall prove that $\lim_{n\to\infty} \|x_n^{(i)} - y_n^{(i)}\| = 0$ ($i \in I$). The firm nonexpansivity of $P_{X^{(i)}}$ and $\langle x, y\rangle = (1/2)(\|x\|^2 + \|y\|^2 - \|x-y\|^2)$ ($x, y \in \mathbb{R}^N$) imply that, for all $n \in \mathbb{N}$, for all $i \in I$, and for all $y \in \bigcap_{i\in I} \mathrm{Fix}(T^{(i)}) \subset X^{(j)} = \mathrm{Fix}(P_{X^{(j)}})$ ($j \in I$),

$$\begin{aligned}
&\left\| x_n^{(i)} - y \right\|^2 = \left\| P_{X^{(i)}} \left(\alpha_n x_n^{(i-1)} + (1-\alpha_n) y_n^{(i)} \right) - P_{X^{(i)}}(y) \right\|^2 \\
&\leq \left\langle \left(\alpha_n x_n^{(i-1)} + (1-\alpha_n) y_n^{(i)} \right) - y, x_n^{(i)} - y \right\rangle \\
&= \frac{1}{2} \left\{ \left\| \alpha_n \left(x_n^{(i-1)} - y \right) + (1-\alpha_n) \left(y_n^{(i)} - y \right) \right\|^2 + \left\| x_n^{(i)} - y \right\|^2 \right. \\
&\quad \left. - \left\| \alpha_n \left(x_n^{(i-1)} - x_n^{(i)} \right) + (1-\alpha_n) \left(y_n^{(i)} - x_n^{(i)} \right) \right\|^2 \right\},
\end{aligned}$$

which means that

$$\begin{aligned}
\left\| x_n^{(i)} - y \right\|^2 \leq &\left\| \alpha_n \left(x_n^{(i-1)} - y \right) + (1-\alpha_n) \left(y_n^{(i)} - y \right) \right\|^2 \\
&- \left\| \alpha_n \left(x_n^{(i-1)} - x_n^{(i)} \right) + (1-\alpha_n) \left(y_n^{(i)} - x_n^{(i)} \right) \right\|^2 .
\end{aligned}$$

Accordingly, the convexity of $\|\cdot\|^2$ and the equality, $\|\alpha x + (1-\alpha) y\|^2 = \alpha \|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$ ($x, y \in \mathbb{R}^N, \alpha \in [0,1]$), ensure that, for all $n \in \mathbb{N}$, for all $i \in I$, and for all $y \in \bigcap_{i\in I} \mathrm{Fix}(T^{(i)})$,

$$\left\|x_n^{(i)} - y\right\|^2 \leq \alpha_n \left\|x_n^{(i-1)} - y\right\|^2 + (1-\alpha_n)\left\|y_n^{(i)} - y\right\|^2 - \alpha_n \left\|x_n^{(i-1)} - x_n^{(i)}\right\|^2 - (1-\alpha_n)\left\|y_n^{(i)} - x_n^{(i)}\right\|^2 + \alpha_n(1-\alpha_n)\left\|x_n^{(i-1)} - y_n^{(i)}\right\|^2.$$

Since Inequality (2.5) implies that $\|y_n^{(i)} - y\|^2 \leq \|x_n^{(i-1)} - y\|^2 + M_3\lambda_n$ $(n \in \mathbb{N}, i \in I, y \in \bigcap_{i\in I} \mathrm{Fix}(T^{(i)}))$, we find that

$$\begin{aligned}\left\|x_n^{(i)} - y\right\|^2 &\leq \alpha_n \left\|x_n^{(i-1)} - y\right\|^2 + (1-\alpha_n)\left\{\left\|x_n^{(i-1)} - y\right\|^2 + M_3\lambda_n\right\} \\ &- \alpha_n \left\|x_n^{(i-1)} - x_n^{(i)}\right\|^2 - (1-\alpha_n)\left\|y_n^{(i)} - x_n^{(i)}\right\|^2 + \alpha_n(1-\alpha_n)\left\|x_n^{(i-1)} - y_n^{(i)}\right\|^2 \\ &\leq \left\|x_n^{(i-1)} - y\right\|^2 + M_3\lambda_n - \alpha_n \left\|x_n^{(i-1)} - x_n^{(i)}\right\|^2 - (1-\alpha_n)\left\|y_n^{(i)} - x_n^{(i)}\right\|^2 \\ &+ \left\|x_n^{(i-1)} - y_n^{(i)}\right\|^2.\end{aligned}$$

Thus, for all $n \in \mathbb{N}$ and for all $y \in \bigcap_{i\in I} \mathrm{Fix}(T^{(i)})$,

$$\begin{aligned}\|x_{n+1} - y\|^2 &\leq \|x_n - y\|^2 + KM_3\lambda_n - \alpha_n \sum_{i\in I}\left\|x_n^{(i-1)} - x_n^{(i)}\right\|^2 \\ &- (1-\alpha_n)\sum_{i\in I}\left\|y_n^{(i)} - x_n^{(i)}\right\|^2 + \sum_{i\in I}\left\|x_n^{(i-1)} - y_n^{(i)}\right\|^2,\end{aligned}$$

which means that

$$\begin{aligned}(1-\alpha_n)\sum_{i\in I}\left\|y_n^{(i)} - x_n^{(i)}\right\|^2 &\leq \|x_n - y\|^2 - \|x_{n+1} - y\|^2 + KM_3\lambda_n \\ &+ \sum_{i\in I}\left\|x_n^{(i-1)} - y_n^{(i)}\right\|^2, \\ \alpha_n \sum_{i\in I}\left\|x_n^{(i-1)} - x_n^{(i)}\right\|^2 &\leq \|x_n - y\|^2 - \|x_{n+1} - y\|^2 + KM_3\lambda_n \\ &+ \sum_{i\in I}\left\|x_n^{(i-1)} - y_n^{(i)}\right\|^2.\end{aligned}$$

The existence of $\lim_{n\to\infty}\|x_n - y\|$ $(y \in \bigcap_{i\in I} \mathrm{Fix}(T^{(i)}))$, $\lim_{n\to\infty}\lambda_n = 0$, Condition (C1), and Equation (2.6) lead one to deduce that $\lim_{n\to\infty}\sum_{i\in I}\|y_n^{(i)} - x_n^{(i)}\| = 0$ and $\lim_{n\to\infty}\sum_{i\in I}\|x_n^{(i-1)} - x_n^{(i)}\| = 0$; i.e.,

$$\lim_{n\to\infty}\left\|y_n^{(i)} - x_n^{(i)}\right\| = 0 \ (i \in I), \tag{2.7}$$

$$\lim_{n\to\infty}\left\|x_n^{(i-1)} - x_n^{(i)}\right\| = 0 \ (i \in I). \tag{2.8}$$

Since the triangle inequality means that, for all $n \in \mathbb{N}$,

$$\|x_n - x_{n+1}\| = \left\| x_n^{(0)} - x_n^{(K)} \right\| \leq \sum_{j=1}^{K} \left\| x_n^{(j-1)} - x_n^{(j)} \right\|,$$

Equation (2.8) guarantees that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Let us prove that $\lim_{n\to\infty} \|x_n - x_n^{(i-1)}\| = 0$ $(i \in I)$. Fix $i \in I$ arbitrarily. The triangle inequality guarantees that, for all $n \in \mathbb{N}$,

$$\left\| x_n - x_n^{(i-1)} \right\| \leq \sum_{j=1}^{i-1} \left\| x_n^{(j-1)} - x_n^{(j)} \right\| \leq \sum_{j=1}^{i-1} \left(\left\| x_n^{(j-1)} - y_n^{(j)} \right\| + \left\| y_n^{(j)} - x_n^{(j)} \right\| \right).$$

Hence, Equations (2.6) and (2.7) ensure that

$$\lim_{n\to\infty} \left\| x_n - x_n^{(i-1)} \right\| = 0 \ (i \in I). \tag{2.9}$$

Moreover, we have from $\|x_n - y_n^{(i)}\| \leq \|x_n - x_n^{(i-1)}\| + \|x_n^{(i-1)} - y_n^{(i)}\|$ $(i \in I, n \in \mathbb{N})$, and Equations (2.6) and (2.9) that

$$\lim_{n\to\infty} \left\| x_n - y_n^{(i)} \right\| = 0 \ (i \in I). \tag{2.10}$$

The nonexpansivity of $T^{(i)}$ guarantees that, for all $n \in \mathbb{N}$ and for all $i \in I$, $\|y_n^{(i)} - T^{(i)}(x_n)\| = \|T^{(i)}(x_n^{(i-1)} + \lambda_n d_n^{(i)}) - T^{(i)}(x_n)\| \leq \|(x_n^{(i-1)} + \lambda_n d_n^{(i)}) - x_n\| \leq \|x_n^{(i-1)} - x_n\| + \lambda_n \|d_n^{(i)}\|$. Accordingly, Equation (2.9), $\lim_{n\to\infty} \lambda_n = 0$, and the boundedness of $(d_n^{(i)})_{n\in\mathbb{N}}$ $(i \in I)$ imply that

$$\lim_{n\to\infty} \left\| y_n^{(i)} - T^{(i)}(x_n) \right\| = 0 \ (i \in I). \tag{2.11}$$

Since $\|x_n - T^{(i)}(x_n)\| \leq \|x_n - y_n^{(i)}\| + \|y_n^{(i)} - T^{(i)}(x_n)\|$ $(i \in I, n \in \mathbb{N})$, Equations (2.10) and (2.11) lead us to deduce that

$$\lim_{n\to\infty} \left\| x_n - T^{(i)}(x_n) \right\| = 0 \ (i \in I). \tag{2.12}$$

(c) The boundedness of $(x_n)_{n\in\mathbb{N}}$ guarantees the existence of an accumulation point of $(x_n)_{n\in\mathbb{N}}$. Let $x^* \in \mathbb{R}^N$ be an arbitrary accumulation point of $(x_n)_{n\in\mathbb{N}}$; i.e., there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ $(\subset (x_n)_{n\in\mathbb{N}})$ converging to x^*. We shall prove that $x^* \in \bigcap_{\in I} \mathrm{Fix}(T^{(i)})$. Fix $i \in I$ arbitrarily. The continuity of $T^{(i)}$ and Equation (2.12) ensure that

$$0 = \lim_{k\to\infty} \left\| x_{n_k} - T^{(i)}(x_{n_k}) \right\| = \left\| x^* - T^{(i)}(x^*) \right\|,$$

which implies that $x^* \in \mathrm{Fix}(T^{(i)})$. Therefore, we have $x^* \in \bigcap_{i\in I} \mathrm{Fix}(T^{(i)})$.

Let $x_* \in \mathbb{R}^N$ be an accumulation point of $(x_n)_{n\in\mathbb{N}}$. Then, $(x_{n_l})_{l\in\mathbb{N}}$ $(\subset (x_n)_{n\in\mathbb{N}})$ exists such that $(x_{n_l})_{l\in\mathbb{N}}$ converges to x_*. A discussion similar to the one above leads us to

$x_* \in \bigcap_{i \in I} \mathrm{Fix}(T^{(i)})$. Assume $x^* \neq x_*$. Accordingly, the existence of $\lim_{n\to\infty} \|x_n - y\|$ ($y \in \bigcap_{i \in I} \mathrm{Fix}(T^{(i)})$) and Opial's condition** mean that

$$\begin{aligned} 0 &= \lim_{k\to\infty} \|x_{n_k} - x^*\| < \lim_{k\to\infty} \|x_{n_k} - x_*\| = \lim_{n\to\infty} \|x_n - x_*\| \\ &= \lim_{l\to\infty} \|x_{n_l} - x_*\| = 0. \end{aligned}$$

This is a contradiction; i.e., $x^* = x_*$. This guarantees that $(x_n)_{n\in\mathbb{N}}$ $(= (x_{n-1}^{(K)})_{n\in\mathbb{N}})$ converges to $x^* \in \bigcap_{i \in I} \mathrm{Fix}(T^{(i)})$. Equation (2.9) and the convergence of $(x_n)_{n\in\mathbb{N}}$ to x^* lead us to that $(x_n^{(i-1)})$ ($i \in I$) also converges to $x^* \in \bigcap_{i \in I} \mathrm{Fix}(T^{(i)})$. Therefore, we can conclude that $(x_n^{(i)})_{n\in\mathbb{N}}$ ($i \in I$) converges to $x^* \in \bigcap_{i \in I} \mathrm{Fix}(T^{(i)})$.

(d) Inequality (2.3) guarantees that, for all $n \in \mathbb{N}$, for all $i \in I$, and for all $y \in \bigcap_{i \in I} \mathrm{Fix}(T^{(i)})$,

$$\begin{aligned} \left\|y_n^{(i)} - y\right\|^2 &\leq \left\|x_n^{(i-1)} - y\right\|^2 + 2\lambda_n \left\langle x_n^{(i-1)} - y, d_n^{(i)} \right\rangle + \lambda_n^2 \left\|d_n^{(i)}\right\|^2 \\ &\leq \left\|x_n^{(i-1)} - y\right\|^2 + 2\lambda_n \left\langle x_n^{(i-1)} - y, -\nabla f^{(i)}\left(x_n^{(i-1)}\right) + \beta_n d_{n-1}^{(i)} \right\rangle + \lambda_n^2 \left\|d_n^{(i)}\right\|^2 \\ &\leq \left\|x_n^{(i-1)} - y\right\|^2 + 2\lambda_n \left\langle y - x_n^{(i-1)}, \nabla f^{(i)}\left(x_n^{(i-1)}\right) \right\rangle + M_4 \lambda_n \beta_n + M_5 \lambda_n^2, \end{aligned}$$

where $M_4 := \max_{i\in I}(\sup\{2|\langle x_n^{(i-1)} - y, d_{n-1}^{(i)}\rangle| \colon n \in \mathbb{N}\}) < \infty$ and $M_5 := \max_{i \in I}(\sup\{\|d_n^{(i)}\|^2 \colon n \in \mathbb{N}\}) < \infty$. So, Inequality (2.4) implies that, for all $n \in \mathbb{N}$, for all $i \in I$, and for all $y \in \bigcap_{i \in I} \mathrm{Fix}(T^{(i)})$,

$$\begin{aligned} \left\|x_n^{(i)} - y\right\|^2 &\leq \alpha_n \left\|x_n^{(i-1)} - y\right\|^2 + (1-\alpha_n) \left\|y_n^{(i)} - y\right\|^2 \\ &\leq \alpha_n \left\|x_n^{(i-1)} - y\right\|^2 + (1-\alpha_n) \bigg\{ \left\|x_n^{(i-1)} - y\right\|^2 + 2\lambda_n \left\langle y - x_n^{(i-1)}, \nabla f^{(i)}\left(x_n^{(i-1)}\right) \right\rangle \\ &\quad + M_4 \lambda_n \beta_n + M_5 \lambda_n^2 \bigg\} \\ &\leq \left\|x_n^{(i-1)} - y\right\|^2 + 2(1-\alpha_n)\lambda_n \left\langle y - x_n^{(i-1)}, \nabla f^{(i)}\left(x_n^{(i-1)}\right) \right\rangle + M_4 \lambda_n \beta_n + M_5 \lambda_n^2, \end{aligned}$$

which means that

$$\begin{aligned} \|x_{n+1} - y\|^2 \leq \|x_n - y\|^2 + 2(1-\alpha_n)\lambda_n \sum_{i \in I} \left\langle y - x_n^{(i-1)}, \nabla f^{(i)}\left(x_n^{(i-1)}\right) \right\rangle \\ + K M_4 \lambda_n \beta_n + K M_5 \lambda_n^2, \end{aligned}$$

**Suppose that $(x_n)_{n\in\mathbb{N}}$ $(\subset \mathbb{R}^N)$ converges to $x^* \in \mathbb{R}^N$ and $x^* \neq x_*$. Then the following condition, called Opial's condition [23], is satisfied: $\lim_{n\to\infty} \|x_n - x^*\| < \lim_{n\to\infty} \|x_n - x_*\|$.

and hence,

$$
\begin{aligned}
0 &\leq \frac{\|x_n - y\|^2 - \|x_{n+1} - y\|^2}{\lambda_n} + 2(1-\alpha_n) \sum_{i \in I} \left\langle y - x_n^{(i-1)}, \nabla f^{(i)} \left(x_n^{(i-1)} \right) \right\rangle \\
&\quad + KM_4 \beta_n + KM_5 \lambda_n \\
&= \frac{(\|x_n - y\| + \|x_{n+1} - y\|)(\|x_n - y\| - \|x_{n+1} - y\|)}{\lambda_n} \\
&\quad + 2(1-\alpha_n) \sum_{i \in I} \left\langle y - x_n^{(i-1)}, \nabla f^{(i)} \left(x_n^{(i-1)} \right) \right\rangle + KM_4 \beta_n + KM_5 \lambda_n \\
&\leq \frac{(\|x_n - y\| + \|x_{n+1} - y\|) \|x_n - x_{n+1}\|}{\lambda_n} \\
&\quad + 2(1-\alpha_n) \sum_{i \in I} \left\langle y - x_n^{(i-1)}, \nabla f^{(i)} \left(x_n^{(i-1)} \right) \right\rangle + KM_4 \beta_n + KM_5 \lambda_n.
\end{aligned}
$$

The convergence of $(x_n^{(i)})_{n \in \mathbb{N}}$ $(i \in I)$ to $x^* \in \bigcap_{i \in I} \mathrm{Fix}(T^{(i)})$, the continuity of $\nabla f^{(i)}$ $(i \in I)$, Conditions (C1), (C2), and (C3), and $\|x_{n+1} - x_n\| = o(\lambda_n)$ ensure that, for all $y \in \bigcap_{i \in I} \mathrm{Fix}(T^{(i)})$,

$$
\begin{aligned}
0 &\leq \sum_{i \in I} \left\langle y - x^*, \nabla f^{(i)}(x^*) \right\rangle = \left\langle y - x^*, \sum_{i \in I} \left(\nabla f^{(i)}(x^*) \right) \right\rangle \\
&= \left\langle y - x^*, \nabla \left(\sum_{i \in I} f^{(i)} \right)(x^*) \right\rangle;
\end{aligned}
$$

that is, x^* is a solution to Problem 1.1. Therefore, Item (c) guarantees that $(x_n^{(i)})_{n \in \mathbb{N}}$ $(i \in I)$ converges to a solution x^* to Problem 1.1. □

3 Broadcast Iterative Method for Nonmonotone Variational Inequality and Its Convergence Analysis

This section presents the following algorithm:

Algorithm 3.1 (Broadcast Fixed Point Optimization Algorithm)**.**

Step 0. User i $(i \in I)$ sets $(\alpha_n)_{n \in \mathbb{N}}$, $(\lambda_n)_{n \in \mathbb{N}}$, and $(\beta_n)_{n \in \mathbb{N}}$, transmits an arbitrary chosen $x_0^{(i)} \in \mathbb{R}^N$ to the all users, and computes $x_0 := (1/K) \sum_{i \in I} x_0^{(i)}$. User i sets $d_0^{(i)} := -\nabla f^{(i)}(x_0)$.

Step 1. Given $x_n, d_n^{(i)} \in \mathbb{R}^N$ $(i \in I)$, user i computes $x_{n+1}^{(i)} \in \mathbb{R}^N$ by

$$
\begin{cases}
y_n^{(i)} := T^{(i)} \left(x_n + \lambda_n d_n^{(i)} \right), \\
x_{n+1}^{(i)} := P_{X^{(i)}} \left(\alpha_n x_n + (1-\alpha_n) y_n^{(i)} \right)
\end{cases}
$$

and transmits $x_{n+1}^{(i)}$ to all users.

Step 2. User i computes $x_{n+1} \in \mathbb{R}^N$ and $d_{n+1}^{(i)} \in \mathbb{R}^N$ by

$$
\begin{aligned}
x_{n+1} &:= \frac{1}{K} \sum_{i \in I} x_{n+1}^{(i)}, \\
d_{n+1}^{(i)} &:= -\nabla f^{(i)}(x_{n+1}) + \beta_{n+1} d_n^{(i)}.
\end{aligned}
$$

Put $n := n+1$, and go to Step 1.

In this section, we assume the following:

Assumption 3.1. $(\nabla f^{(i)}(x_n))_{n\in\mathbb{N}}$ $(i \in I)$ is bounded.

The boundedness of $X^{(i)}$ (Assumption (A1)) implies that $(x_n^{(i)})_{n\in\mathbb{N}}$ $(i \in I)$ is bounded. Hence, $(x_n)_{n\in\mathbb{N}}$ generated by $x_n := (1/K)\sum_{i\in I} x_n^{(i)}$ $(n \in \mathbb{N})$ is also bounded. Therefore, the same discussion as in Assumption 2.2 shows that, if $\nabla f^{(i)}$ $(i \in I)$ is Lipschitz continuous, Assumption 3.1 is satisfied.

The following convergence analysis is presented for Algorithm 3.1:

Theorem 3.2. *Suppose that Assumptions 1.1, 2.1, and 3.1 are satisfied. Then,* $(x_n)_{n\in\mathbb{N}}$ *in Algorithm 3.1 satisfies the following properties:*

(a) $(x_n)_{n\in\mathbb{N}}$ *is bounded and* $\lim_{n\to\infty}\|x_n - y\|$ *exists for all* $y \in \bigcap_{i\in I}\mathrm{Fix}(T^{(i)})$.

(b) $\lim_{n\to\infty}\|x_n - T^{(i)}(x_n)\| = 0$ $(i \in I)$.

(c) $(x_n)_{n\in\mathbb{N}}$ *converges to a common fixed point of* $T^{(i)}$*s.*

(d) *If* $\|x_{n+1} - x_n\| = o(\lambda_n)$, $(x_n)_{n\in\mathbb{N}}$ *converges to a solution to Problem 1.1.*

Item (c) in Theorem 3.2 ensures that Algorithm 3.1 under Assumptions 2.1 and 3.1 can find a feasible point in Problem 1.1. Moreover, the convergence of $(x_n)_{n\in\mathbb{N}}$ to $x^\star \in \bigcap_{i\in I}\mathrm{Fix}(T^{(i)})$ guarantees that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Item (d) in Theorem 3.2 guarantees that, if $\|x_{n+1} - x_n\| = o(\lambda_n)$, $(x_n)_{n\in\mathbb{N}}$ in Algorithm 3.1 converges in not only $\bigcap_{i\in I}\mathrm{Fix}(T^{(i)})$ but also $\mathrm{VI}(\bigcap_{i\in I}\mathrm{Fix}(T^{(i)}), \nabla(\sum_{i\in I} f^{(i)}))$. When user i $(i \in I)$ has $x_n^{(i)}$ in Algorithm 3.1, each point is broadcast to all users. Accordingly, all users have $(x_n)_{n\in\mathbb{N}} := ((1/K)\sum_{i\in I} x_n^{(i)})_{n\in\mathbb{N}}$, which implies that all users can verify whether $\|x_{n+1} - x_n\| = o(\lambda_n)$ is satisfied or not. On the other hand, in Algorithm 2.1, only user K can verify whether the convergence condition is satisfied or not (see Section 2).

3.1 Proof of Theorem 3.2

(a) The same discussion as in the proof of Theorem 2.3 (a) and Assumption 3.1 guarantee that $(x_n)_{n\in\mathbb{N}}$, $(x_n^{(i)})_{n\in\mathbb{N}}$, $(d_n^{(i)})_{n\in\mathbb{N}}$, $(y_n^{(i)})_{n\in\mathbb{N}}$ $(i \in I)$ are bounded.

We shall prove the existence of $\lim_{n\to\infty}\|x_n - y\|$ $(y \in \bigcap_{i\in I}\mathrm{Fix}(T^{(i)}))$. By replacing $x_n^{(i-1)}$ in Inequality (2.3) with x_n, we have that, for all $n \in \mathbb{N}$ and for all $i \in I$,

$$\begin{aligned}\left\|y_n^{(i)} - y\right\|^2 &\le \|x_n - y\|^2 + 2\lambda_n\left\langle x_n - y, d_n^{(i)}\right\rangle + \lambda_n^2\left\|d_n^{(i)}\right\|^2\\ &\le \|x_n - y\|^2 + N_1\lambda_n,\end{aligned} \tag{3.1}$$

where $N_1 := \max_{i\in I}(\sup\{2|\langle x_n - y, d_n^{(i)}\rangle| + \lambda_n\|d_n^{(i)}\|^2 \colon n \in \mathbb{N}\}) < \infty$. Moreover, by replacing $x_n^{(i-1)}$ in Inequality (2.4) with x_n and replacing $x_n^{(i)}$ in Inequality (2.4) with $x_{n+1}^{(i)}$, we find that

$$\begin{aligned}\left\|x_{n+1}^{(i)} - y\right\|^2 &\le \alpha_n\|x_n - y\|^2 + (1-\alpha_n)\left\|y_n^{(i)} - y\right\|^2\\ &\le \alpha_n\|x_n - y\|^2 + (1-\alpha_n)\left\{\|x_n - y\|^2 + N_1\lambda_n\right\}\\ &\le \|x_n - y\|^2 + N_1\lambda_n.\end{aligned} \tag{3.2}$$

On the other hand, the convexity of $\|\cdot\|^2$ means that, for all $n \in \mathbb{N}$,

$$\|x_{n+1} - y\|^2 = \left\| \frac{1}{K} \sum_{i \in I} \left(x_{n+1}^{(i)} - y \right) \right\|^2 \leq \frac{1}{K} \sum_{i \in I} \left\| x_{n+1}^{(i)} - y \right\|^2. \tag{3.3}$$

Therefore, summing Inequality (3.2) over all i ensures that, for all $n \in \mathbb{N}$,

$$\|x_{n+1} - y\|^2 \leq \|x_n - y\|^2 + N_1 \lambda_n.$$

So, the same discussion as in the proof of Theorem 2.3 (a) and Condition (C2) guarantee that

$$\limsup_{n \to \infty} \|x_n - y\|^2 \leq \liminf_{m \to \infty} \left\{ \|x_m - y\|^2 + N_1 \sum_{i=m}^{\infty} \lambda_i \right\} = \liminf_{n \to \infty} \|x_m - y\|^2;$$

that is, there exists $\lim_{n\to\infty} \|x_n - y\|$ $(y \in \bigcap_{i \in I} \mathrm{Fix}(T^{(i)}))$.

(b) We shall prove that $\lim_{n\to\infty} \|x_n - y_n^{(i)}\| = 0$. Replacing $x_n^{(i-1)}$ in Inequality (2.5) with x_n implies that, for all $n \in \mathbb{N}$, for all $i \in I$, and for all $y \in \bigcap_{i \in I} \mathrm{Fix}(T^{(i)})$,

$$\begin{aligned} \left\| y_n^{(i)} - y \right\|^2 &\leq \|x_n - y\|^2 + 2\lambda_n \left\langle y_n^{(i)} - y, d_n^{(i)} \right\rangle - \left\| x_n - y_n^{(i)} \right\|^2 \\ &\leq \|x_n - y\|^2 + N_2 \lambda_n - \left\| x_n - y_n^{(i)} \right\|^2, \end{aligned}$$

where $N_2 := \max_{i \in I}(\sup\{2|\langle y_n^{(i)} - y, d_n^{(i)} \rangle| \colon n \in \mathbb{N}\}) < \infty$. Hence, Inequality (3.2) implies that, for all $n \in \mathbb{N}$, for all $i \in I$, and for all $y \in \bigcap_{i \in I} \mathrm{Fix}(T^{(i)})$,

$$\begin{aligned} \left\| x_{n+1}^{(i)} - y \right\|^2 &\leq \alpha_n \|x_n - y\|^2 + (1 - \alpha_n) \left\| y_n^{(i)} - y \right\|^2 \\ &\leq \alpha_n \|x_n - y\|^2 + (1 - \alpha_n) \left\{ \|x_n - y\|^2 + N_2 \lambda_n - \left\| x_n - y_n^{(i)} \right\|^2 \right\} \\ &\leq \|x_n - y\|^2 + N_2 \lambda_n - (1 - \alpha_n) \left\| x_n - y_n^{(i)} \right\|^2. \end{aligned}$$

Summing this inequality over all i and Inequality (3.3) mean that, for all $n \in \mathbb{N}$ and for all $y \in \bigcap_{i \in I} \mathrm{Fix}(T^{(i)})$,

$$\|x_{n+1} - y\|^2 \leq \|x_n - y\|^2 + N_2 \lambda_n - \frac{1 - \alpha_n}{K} \sum_{i \in I} \left\| x_n - y_n^{(i)} \right\|^2,$$

which implies that

$$\frac{1 - \alpha_n}{K} \sum_{i \in I} \left\| x_n - y_n^{(i)} \right\|^2 \leq \|x_n - y\|^2 - \|x_{n+1} - y\|^2 + N_2 \lambda_n.$$

Therefore, Conditions (C1) and (C2), and the existence of $\lim_{n\to\infty} \|x_n - y\|$ $(y \in \bigcap_{i \in I} \mathrm{Fix}(T^{(i)}))$ lead one to deduce that $\lim_{n\to\infty} \sum_{i \in I} \|x_n - y_n^{(i)}\| = 0$; i.e.,

$$\lim_{n \to \infty} \left\| x_n - y_n^{(i)} \right\| = 0 \ (i \in I). \tag{3.4}$$

Moreover, the nonexpansivity of $T^{(i)}$ guarantees that, for all $n \in \mathbb{N}$ and for all $i \in I$, $\|y_n^{(i)} - T^{(i)}(x_n)\| = \|T^{(i)}(x_n + \lambda_n d_n^{(i)}) - T^{(i)}(x_n)\| \leq \|(x_n + \lambda_n d_n^{(i)}) - x_n\| \leq \lambda_n \|d_n^{(i)}\|$. Accordingly, the boundedness of $(d_n^{(i)})_{n\in\mathbb{N}}$ $(i \in I)$ and $\lim_{n\to\infty} \lambda_n = 0$ (by Condition (C2)) imply that

$$\lim_{n\to\infty} \left\| y_n^{(i)} - T^{(i)}(x_n) \right\| = 0 \ (i \in I). \tag{3.5}$$

Since $\|x_n - T^{(i)}(x_n)\| \leq \|x_n - y_n^{(i)}\| + \|y_n^{(i)} - T^{(i)}(x_n)\|$ $(n \in \mathbb{N}, i \in I)$, Equations (3.4) and (3.5) ensure that

$$\lim_{n\to\infty} \left\| x_n - T^{(i)}(x_n) \right\| = 0 \ (i \in I). \tag{3.6}$$

(c) The boundedness of $(x_n)_{n\in\mathbb{N}}$ guarantees the existence of an accumulation point of $(x_n)_{n\in\mathbb{N}}$. Let $x^* \in \mathbb{R}^N$ be an arbitrary accumulation point of $(x_n)_{n\in\mathbb{N}}$. Then, the same discussion as in the proof of Theorem 2.3 (c) and Equation (3.6) imply that $x^* \in \mathrm{Fix}(T^{(i)})$ $(i \in I)$, i.e., $x^* \in \bigcap_{i\in I} \mathrm{Fix}(T^{(i)})$.

Let $x_* \in \mathbb{R}^N$ be an accumulation point of $(x_n)_{n\in\mathbb{N}}$. Then, a discussion similar to the proof of Theorem 2.3 (c) leads us to $x_* \in \bigcap_{i\in I} \mathrm{Fix}(T^{(i)})$ with $x^* = x_*$. This guarantees that $(x_n)_{n\in\mathbb{N}}$ converges to $x^* \in \bigcap_{i\in I} \mathrm{Fix}(T^{(i)})$.

(d) Inequality (3.1) and the definition of $d_n^{(i)}$ $(n \in \mathbb{N}, i \in I)$ imply that, for all $n \in \mathbb{N}$, for all $i \in I$, and for all $y \in \bigcap_{i\in I} \mathrm{Fix}(T^{(i)})$,

$$\begin{aligned}
\left\| y_n^{(i)} - y \right\|^2 &\leq \|x_n - y\|^2 + 2\lambda_n \left\langle x_n - y, d_n^{(i)} \right\rangle + \lambda_n^2 \left\| d_n^{(i)} \right\|^2 \\
&\leq \|x_n - y\|^2 + 2\lambda_n \left\langle x_n - y, -\nabla f^{(i)}(x_n) + \beta_n d_{n-1}^{(i)} \right\rangle + \lambda_n^2 \left\| d_n^{(i)} \right\|^2 \\
&\leq \|x_n - y\|^2 + 2\lambda_n \left\langle y - x_n, \nabla f^{(i)}(x_n) \right\rangle + N_3 \lambda_n \beta_n + N_4 \lambda_n^2,
\end{aligned}$$

where $N_3 := \max_{i\in I}(\sup\{2|\langle x_n - y, d_{n-1}^{(i)}\rangle| \colon n \in \mathbb{N}\}) < \infty$ and $N_4 := \max_{i\in I}(\sup\{\|d_n^{(i)}\|^2 \colon n \in \mathbb{N}\}) < \infty$. Hence, Inequality (3.2) means that

$$\begin{aligned}
&\left\| x_{n+1}^{(i)} - y \right\|^2 \leq \alpha_n \|x_n - y\|^2 + (1 - \alpha_n) \left\| y_n^{(i)} - y \right\|^2 \\
&\leq \alpha_n \|x_n - y\|^2 + (1 - \alpha_n) \Big\{ \|x_n - y\|^2 + 2\lambda_n \left\langle y - x_n, \nabla f^{(i)}(x_n) \right\rangle \\
&\quad + N_3 \lambda_n \beta_n + N_4 \lambda_n^2 \Big\} \\
&\leq \|x_n - y\|^2 + 2(1 - \alpha_n)\lambda_n \left\langle y - x_n, \nabla f^{(i)}(x_n) \right\rangle + N_3 \lambda_n \beta_n + N_4 \lambda_n^2.
\end{aligned}$$

Accordingly, summing this inequality over all i and Inequality (3.3) guarantee that, for all $n \in \mathbb{N}$ and for all $y \in \bigcap_{i\in I} \mathrm{Fix}(T^{(i)})$,

$$\begin{aligned}
\|x_{n+1} - y\|^2 \leq \|x_n - y\|^2 &+ \frac{2(1 - \alpha_n)\lambda_n}{K} \sum_{i\in I} \left\langle y - x_n, \nabla f^{(i)}(x_n) \right\rangle \\
&+ N_3 \lambda_n \beta_n + N_4 \lambda_n^2.
\end{aligned}$$

Therefore, we find that, for all $n \in \mathbb{N}$ and for all $y \in \bigcap_{i\in I} \mathrm{Fix}(T^{(i)})$,

$$\begin{aligned}0 \leq &\frac{\|x_n - y\|^2 - \|x_{n+1} - y\|^2}{\lambda_n} + \frac{2(1-\alpha_n)}{K} \sum_{i\in I} \left\langle y - x_n, \nabla f^{(i)}(x_n) \right\rangle \\ &+ N_3\beta_n + N_4\lambda_n \\ =& \frac{(\|x_n - y\| + \|x_{n+1} - y\|)(\|x_n - y\| - \|x_{n+1} - y\|)}{\lambda_n} \\ &+ \frac{2(1-\alpha_n)}{K} \sum_{i\in I} \left\langle y - x_n, \nabla f^{(i)}(x_n) \right\rangle + N_3\beta_n + N_4\lambda_n \\ \leq& \frac{(\|x_n - y\| + \|x_{n+1} - y\|)\|x_n - x_{n+1}\|}{\lambda_n} \\ &+ \frac{2(1-\alpha_n)}{K} \sum_{i\in I} \left\langle y - x_n, \nabla f^{(i)}(x_n) \right\rangle + N_3\beta_n + N_4\lambda_n.\end{aligned}$$

The convergence of $(x_n)_{n\in\mathbb{N}}$ to $x^* \in \bigcap_{i\in I} \mathrm{Fix}(T^{(i)})$, the continuity of $\nabla f^{(i)}$ $(i \in I)$, Conditions (C1), (C2), and (C3), and $\|x_{n+1} - x_n\| = o(\lambda_n)$ ensure that, for all $y \in \bigcap_{i\in I} \mathrm{Fix}(T^{(i)})$,

$$\begin{aligned}0 \leq \sum_{i\in I} \left\langle y - x^*, \nabla f^{(i)}(x^*) \right\rangle &= \left\langle y - x^*, \sum_{i\in I} \left(\nabla f^{(i)}(x^*)\right) \right\rangle \\ &= \left\langle y - x^*, \nabla \left(\sum_{i\in I} f^{(i)} \right)(x^*) \right\rangle.\end{aligned}$$

This completes the proof. □

4 Numerical Examples

Let us apply Algorithms 2.1 and 3.1 to the network bandwidth allocation problem. The objective of utility-based bandwidth allocation is to share the available bandwidth among traffic sources so as to maximize the overall utility subject to the capacity constraints [26, Chapter 2]. In this section, we shall discuss a nonconcave utility bandwidth allocation problem [13] which can be expressed as a nonmonotone variational inequality with the gradient of a nonconcave, differentiable, step utility function. We assume that source i has its own private nonconvex $f^{(i)} := -\mathcal{U}^{(i)}$ and $C^{(i)}$ with the capacity constraints for links used by source i.

Consider the following problem on a network [26, Fig.2.2] (Figure 1 in the paper) that consists of three links and four sources:

$$\text{Find } x^\star \in \mathrm{VI}\left(\bigcap_{i\in I} C^{(i)}, -\nabla \left(\sum_{i\in I} \mathcal{U}^{(i)} \right) \right), \tag{4.1}$$

where $I := \{1,2,3,4\}$, $\mathcal{U}^{(i)}(x) := x + \sin x$ $(i \in I, x \in \mathbb{R}_+)$, $D^{(1)} := \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \colon x_1 + x_3 \leq c_1\}$,†† $D^{(2)} := \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \colon x_2 + x_3 \leq c_2\}$, $D^{(3)} := \{(x_1,x_2,x_3,x_4) \in$

††The projection onto $D := \{x \in \mathbb{R}^N \colon \langle a, x\rangle \leq b\}$, where $a\ (\neq 0) \in \mathbb{R}^N$ and $b \in \mathbb{R}$, is expressed as follows [1, p.406], [2, Subchapter 28.3]: $P_D(x) := x + [(b - \langle a, x\rangle)/\|a\|^2]a$ $(x \notin D)$, or x $(x \in D)$.

$\mathbb{R}^4 \colon x_2 + x_4 \leq c_3\}$, $B := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \colon x_i \in [0, c]\ (i \in I)\}\ (c > 0)$[‡‡], $C^{(1)} := B \cap D^{(1)}$, $C^{(2)} := B \cap D^{(2)} \cap D^{(3)}$, $C^{(3)} := B \cap D^{(1)} \cap D^{(2)}$, and $C^{(4)} := B \cap D^{(3)}$.

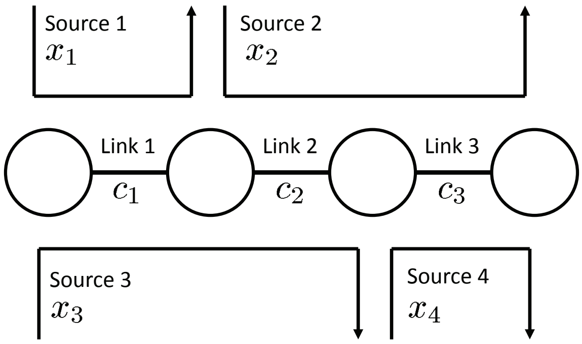

Figure 1: Network with three links and four sources [26, Fig.2.2]

Since $C^{(i)}$ $(i \in I)$ in problem (4.1) satisfies $C^{(i)} \subset B$, B is bounded, and P_B can be easily computed, source i $(i \in I)$ can set $X^{(i)}$ $(i \in I)$ in Assumption 1.1 (A1) and (A2) by B.

Let us define $T^{(i)} \colon \mathbb{R}^4 \to \mathbb{R}^4$ $(i \in I)$ by

$$T^{(1)} := \frac{1}{2}\left(\mathrm{Id} + P_B P_{D^{(1)}}\right),\ T^{(2)} := \frac{1}{2}\left(\mathrm{Id} + P_B P_{D^{(2)}} P_{D^{(3)}}\right),$$
$$T^{(3)} := \frac{1}{2}\left(\mathrm{Id} + P_B P_{D^{(1)}} P_{D^{(2)}}\right),\ T^{(4)} := \frac{1}{2}\left(\mathrm{Id} + P_B P_{D^{(3)}}\right).$$

Then, we find that

$$\begin{aligned}
&\bigcap_{i \in I} \mathrm{Fix}\left(T^{(i)}\right) \\
=&B \cap \mathrm{Fix}\left(P_{D^{(1)}}\right) \cap \mathrm{Fix}\left(P_{D^{(2)}} P_{D^{(3)}}\right) \cap \mathrm{Fix}\left(P_{D^{(1)}} P_{D^{(2)}}\right) \cap \mathrm{Fix}\left(P_{D^{(3)}}\right) \\
=&B \cap D^{(1)} \cap \left(D^{(2)} \cap D^{(3)}\right) \cap \left(D^{(1)} \cap D^{(2)}\right) \cap D^{(3)} \\
=&B \cap \bigcap_{i=1}^{3} D^{(i)} = \bigcap_{i \in I} C^{(i)} \neq \emptyset.
\end{aligned}$$

From $\bigcap_{i \in I} \mathrm{Fix}(T^{(i)}) = B \cap \bigcap_{i=1}^{3} D^{(i)}$, any point in $\bigcap_{i \in I} \mathrm{Fix}(T^{(i)})$ satisfies the capacity constraints for all links.

We use

$$\lambda_n := \frac{\mu}{(n+1)^a},\ \alpha_n := \frac{1}{2},\ \beta_n := \frac{1}{(n+1)^{0.01}}\ (n \in \mathbb{N}),$$

[‡‡]Each source can use a box constraint set B with a constant c (e.g., $c := \max\{c_1, c_2, c_3\}$ or c is a large enough positive number) because the network in Figure 1 has a resource with finite capacity and supports a finite number of sources.

where $\mu = 10^{-2}, 1$, and $a = 1.01, 2, 3, 10$, satisfying Conditions (C1)–(C3). Theorems 2.3 and 3.2 guarantee that Algorithm 2.1 (IFPOA) and Algorithm 3.1 (BFPOA) with the λ_n, α_n, and β_n converge to a solution to Problem (4.1) if $\|x_{n+1} - x_n\| = o(\mu/(n+1)^a)$. We set $c_1 := 5$, $c_2 := 4$, $c_3 := 5$, $c := 100$, chose ten random initial points, and executed Algorithms 2.1 and 3.1 for any point. The graphs in this section plot the mean values of the tenth execution. The computer used in the experiment had an Intel Boxed Core i7 i7-870 2.93 GHz 8 M CPU and 8 GB of memory. The language was MATLAB 7.13.

To check whether Algorithms 2.1 and 3.1 converge in $\bigcap_{i \in I} C^{(i)} = \bigcap_{i \in I} \mathrm{Fix}(T^{(i)})$, we employed the evaluation function*

$$D_n := \sum_{i \in I} \left\| x_n - T^{(i)}(x_n) \right\| \quad (n \in \mathbb{N}).$$

$(x_n)_{n \in \mathbb{N}}$ converges in $\bigcap_{i \in I} \mathrm{Fix}(T^{(i)})$ if and only if $(D_n)_{n \in \mathbb{N}}$ converges to 0. We also employed the following:

$$X_n := \frac{\|x_{n+1} - x_n\|}{\lambda_n} = \frac{(n+1)^a}{\mu} \|x_{n+1} - x_n\| \quad (n \in \mathbb{N}),$$

where $\mu = 10^{-2}, 1$ and $a = 1.01, 2, 3, 10$. Algorithms 2.1 and 3.1 satisfy the convergence condition, $\|x_{n+1} - x_n\| = o(\lambda_n)$, if $(X_n)_{n \in \mathbb{N}}$s generated by Algorithms 2.1 and 3.1 converge to 0.

Figure 2 plots the behaviors of D_n ($n = 1, 2, \ldots, 1000$) for Algorithms 2.1 and 3.1 with $\mu = 10^{-2}$ and $a = 1.01, 2$. This figure shows that $(D_n)_{n \in \mathbb{N}}$s generated by Algorithms 2.1 and 3.1 converge to 0; i.e., the algorithms converge in $\bigcap_{i \in I} C^{(i)} = \bigcap_{i \in I} \mathrm{Fix}(T^{(i)})$, as promised by Item (c) in Theorem 2.3 and Item (c) in Theorem 3.2. Figure 2 shows that $(D_n)_{n \in \mathbb{N}}$s in Algorithms 2.1 and 3.1 with $\lambda_n := 10^{-2}/(n+1)^2$ converge quickly and $(D_n)_{n \in \mathbb{N}}$s in Algorithms 2.1 and 3.1 with $\lambda_n := 10^{-2}/(n+1)^{1.01}$ converge slowly. This implies that Algorithms 2.1 and 3.1 with fast diminishing step-size sequences converge in $\bigcap_{i \in I} \mathrm{Fix}(T^{(i)})$ faster than ones with slowly diminishing step-size sequences.

Figure 3 indicates the behaviors of X_n ($n = 1, 2, \ldots, 1000$) for Algorithms 2.1 and 3.1 with $\mu = 10^{-2}$ and $a = 1.01, 2$. We can see from this figure that $(X_n)_{n \geq 100}$s in Algorithms 2.1 and 3.1 decrease monotonically and converge to 0, which means that Algorithms 2.1 and 3.1 with $\lambda_n = 10^{-2}/(n+1)^a$ ($a = 1.01, 2$) satisfy the convergence condition $\|x_{n+1} - x_n\| = o(\lambda_n)$. Therefore, Item (d) in Theorem 2.3 and Item (d) in Theorem 3.2 guarantee that Algorithms 2.1 and 3.1 with $\lambda_n = 10^{-2}/(n+1)^a$ ($a = 1.01, 2$) converge to a solution to Problem (4.1).

The behaviors of $\mathcal{U}(x_n)$ generated by Algorithms 2.1 and 3.1 with $\mu = 10^{-2}$ and $a = 1.01, 2$ are presented in Figure 4. This figure indicates that Algorithms 2.1 and 3.1 are stable in the early stages. Figures 2–4 and Theorems 2.3 and 3.2 ensure that Algorithms 2.1 and 3.1 find the optimal bandwidth for all sources; i.e., each source can find its own optimal bandwidth ($x_1^\star \approx 2.7786$, $x_2^\star \approx 2.0531$, $x_3^\star \approx 1.9468$, $x_4^\star \approx 2.8851$) by using the incremental and broadcast fixed point optimization algorithms.

Figure 3 indicates that Algorithms 2.1 and 3.1 with $\mu = 10^{-2}$ and $a = 1.01, 2$ satisfy the convergence condition $\|x_{n+1} - x_n\| = o(\lambda_n)$. To see whether Algorithms 2.1 and 3.1 with faster diminishing step-size sequences such as $\lambda_n = 10^{-2}/(n+1)^a$ ($a = 3, 10$) satisfy the convergence condition, we checked the behaviors of X_n when $a = 3, 10$. Figure 5 shows that $(X_n)_{n \in \mathbb{N}}$s in Algorithms 2.1 and 3.1 with $\mu = 10^{-2}$ and $a = 3, 10$ converge to 0; however, $(X_n)_{n \in \mathbb{N}}$s in Algorithms 2.1 and 3.1 with $\mu = 10^{-2}$ and $a = 10$ are unstable in the early

*$x \in \mathbb{R}^4$ satisfies $\sum_{i \in I} \|x - T^{(i)}(x)\| = 0$ if and only if $x \in \mathrm{Fix}(T^{(i)})$ ($i \in I$), i.e., $x \in \bigcap_{i \in I} \mathrm{Fix}(T^{(i)}) = B \cap \bigcap_{i=1}^{3} D^{(i)} = \bigcap_{i \in I} C^{(i)}$.

stage. Since the algorithms ought to be stable from the implementation viewpoint, it would be useful to use the algorithms with $\lambda_n = 10^{-2}/(n+1)^a$ $(a = 1.01, 2)$. Figure 6 plots the behaviors of X_n $(n = 1, 2, \ldots, 1000)$ for Algorithms 2.1 and 3.1 with $\mu = 1$ and $a = 1.01, 2$. This figure shows that $(X_n)_{n\in\mathbb{N}}$s in Algorithms 2.1 and 3.1 with $\mu = 1$ and $a = 1.01$ converge to 0, while $(X_n)_{n\in\mathbb{N}}$s in Algorithms 2.1 and 3.1 with $\mu = 1$ and $a = 2$ do not converge to 0; i.e., Algorithms 2.1 and 3.1 with $\mu = 1$ and $a = 2$ do not satisfy the convergence condition. This implies that the algorithms with $\mu = 1$ and $a = 2$ are not good ways for solving Problem (4.1). The above observations suggest that Algorithms 2.1 and 3.1 with a slowly diminishing step-size sequence such as $\lambda_n = \mu/(n+1)^{1.01}$ $(\mu = 10^{-2}, 1)$ should be used for nonconcave utility bandwidth allocation problems.

5 Conclusion

We discussed a nonmonotone variational inequality over the intersection of the fixed point sets of nonexpansive mappings and presented two distributed fixed point optimization algorithms for solving it. One algorithm is based on conventional incremental subgradient methods, and the other is a broadcast type of distributed iterative method. We gave the convergence analyses showing that the algorithms converge to a solution to the nonmonotone variational inequality under certain assumptions. We also provided numerical examples for the bandwidth allocation. The analyses and numerical examples suggested that the algorithms with slowly diminishing step-size sequences converge to a solution to the nonmonotone variational inequality.

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References

[1] H.H. Bauschke and J.M. Borwein, On projection algorithms for solving convex feasibility problems, *SIAM Review* 38 (1996) 367–426.

[2] H.H. Bauschke and P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, New York, 2011.

[3] D.P. Bertsekas, A. Nedić and A.E. Ozdaglar, *Convex Analysis and Optimization*, Athena Scientific, Belmont, 2003.

[4] D. Blatt, A.O. Hero and H. Gauchman, A convergent incremental gradient method with a constant step size, *SIAM J. Optim.* 18 (2007) 29–51.

[5] P.L. Combettes, Iterative construction of the resolvent of a sum of maximal monotone operators, *J. Convex Anal.* 16 (2009) 727–748.

[6] P.L. Combettes and J.C. Pesquet, A proximal decomposition method for solving convex variational inverse problems, *Inverse Problems* 24 (2008) article ID 065,014.

[7] F. Facchinei and J.S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems I*, Springer, New York, 2003.

[8] F. Facchinei and J.S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems II*, Springer, New York, 2003.

[9] K. Goebel and W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1990.

[10] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Dekker, New York, 1984.

[11] H. Iiduka, Fixed point optimization algorithm and its application to network bandwidth allocation, *J. Comp. Appl. Math.* 236 (2012) 1733–1742.

[12] H. Iiduka, Fixed point optimization algorithm and its application to power control in CDMA data networks, *Math. Program.* 133 (2012) 227–242.

[13] H. Iiduka, Iterative algorithm for triple-hierarchical constrained nonconvex optimization problem and its application to network bandwidth allocation, *SIAM J. Optim.* 22 (2012) 862–878.

[14] H. Iiduka, Fixed point optimization algorithms for distributed optimization in networked systems, *SIAM J. Optim.* 23 (2013) 1–26.

[15] H. Iiduka and M. Uchida, Fixed point optimization algorithms for network bandwidth allocation problems with compoundable constraints, *IEEE Communications Letters* 15 (2011) 596–598.

[16] H. Iiduka and I. Yamada, A use of conjugate gradient direction for the convex optimization problem over the fixed point set of a nonexpansive mapping, *SIAM J. Optim.* 19 (2009) 1881–1893.

[17] B. Johansson, M. Rabi and M. Johansson, A randomized incremental subgradient method for distributed optimization in networked systems, *SIAM J. Optim.* 20 (2009) 1157–1170.

[18] K.C. Kiwiel, Convergence of approximate and incremental subgradient methods for convex optimization, *SIAM J. Optim.* 14 (2004) 807–840.

[19] S. Low and D.E. Lapsley, Optimization flow control, i: Basic algorithm and convergence, *IEEE/ACM Trans. Networking* 7 (1999) 861–874.

[20] P. Maillé and L. Toka, Managing a peer-to-peer data storage system in a selfish society, *IEEE J. Selected Areas in Commun.* 26 (2008) 1295–1301.

[21] A. Nedić and D.P. Bertsekas, Incremental sugradient methods for nondifferentiable optimization, *SIAM J. Optim* 12 (2001) 109–138.

[22] J. Nocedal and S.J. Wright, *Numerical Optimization*, 2nd edn. Springer Series in Operations Research and Financial Engineering, Springer, New York, 2006.

[23] Z. Opial, Weak convergence of the sequence of successive approximation for nonexpansive mappings, *Bulletin Amer. Math. Soc.* 73 (1967) 591–597.

[24] R.T. Rockafellar and R.J.B. Wets, *Variational Analysis*, 3rd edn. Springer, Berlin, 2010.

[25] S. Sharma and D. Teneketzis, An externalities-based decentralized optimal power allocation algorithm for wireless networks. *IEEE/ACM Transactions on Networking* 17 (2009) 1819–1831.

[26] R. Srikant, *The Mathematics of Internet Congestion Control*, Birkhauser, Boston, 2004.

[27] E. Zeidler, *Nonlinear Functional Analysis ans Its Applications III*, Variational Methods and Optimization, Springer, New York, 1985.

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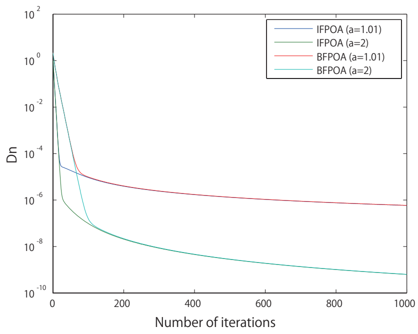

Figure 2: Behavior of $D_n := \sum_{i \in I} \|x_n - T^{(i)}(x_n)\|$ for Algorithm 2.1 (IFPOA) and Algorithm 3.1 (BFPOA) with $\mu = 10^{-2}$ and $a = 1.01, 2$

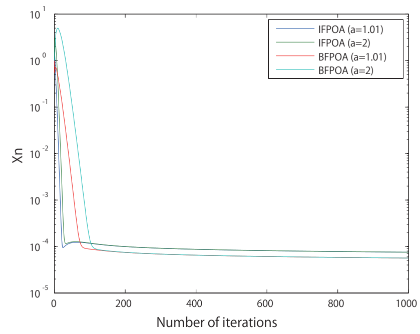

Figure 3: Behavior of $X_n := ((n+1)^a/10^{-2})\|x_{n+1} - x_n\|$ for Algorithm 2.1 (IFPOA) and Algorithm 3.1 (BFPOA) with $\mu = 10^{-2}$ and $a = 1.01, 2$

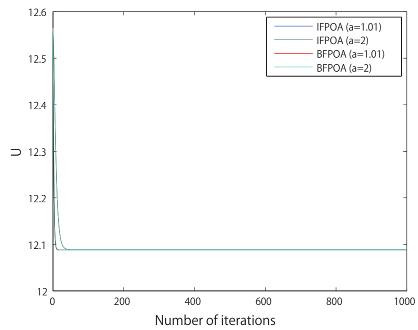

Figure 4: Behavior of $\mathcal{U}(x_n)$ for Algorithm 2.1 (IFPOA) and Algorithm 3.1 (BFPOA) with $\mu = 10^{-2}$ and $a = 1.01, 2$

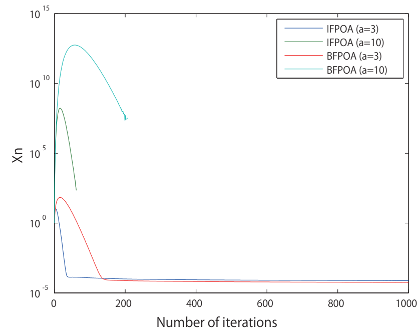

Figure 5: Behavior of $X_n := ((n+1)^a/10^{-2})\|x_{n+1} - x_n\|$ for Algorithm 2.1 (IFPOA) and Algorithm 3.1 (BFPOA) with $\mu = 10^{-2}$ and $a = 3, 10$

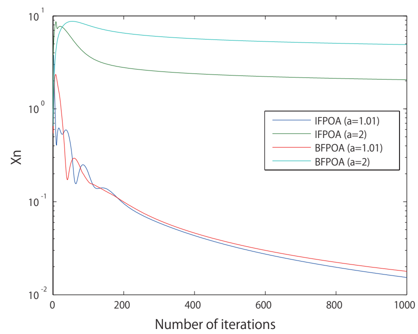

Figure 6: Behavior of $X_n := (n+1)^a\|x_{n+1} - x_n\|$ for Algorithm 2.1 (IFPOA) and Algorithm 3.1 (BFPOA) with $\mu = 1$ and $a = 1.01, 2$