Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 10, No. 4, pp. 667–689, October 2014

DIRECTIONAL DERIVATIVES, SUBDIFFERENTIALS AND OPTIMALITY CONDITIONS FOR SET-VALUED CONVEX FUNCTIONS

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Abstract: A new directional derivative and a new subdifferential for set-valued convex functions are constructed and a set-valued version of the "max formula" is proven. The new concepts are used to provide optimality conditions for convex minimization problems with a set-valued objective. As a major tool, residuation operations are used which act on spaces of closed convex, but not necessarily bounded subsets of a topological linear space. The residuation serves as a substitute for the inverse addition and is intimately related to the Minkowski or geometric difference of convex sets. The results, when specialized, even extend those for extended real-valued convex functions since the improper case is included.

Key words: *directional derivative, subdifferential, set-valued function, optimality condition, max-formula, residuation*

Mathematics Subject Classification: *90C46, 90C48, 52A41*

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1 Introduction

In this note, we introduce new notions of directional derivatives and subdifferentials for set-valued convex functions, we prove the so-called max formula, a result of 'exceptional importance' in the scalar case [33, p. 90], and characterize solutions of set-valued optimization problems in terms of the new derivatives. The latter topic sheds some new light on what should actually be understood by a solution of a convex optimization problem with a set-valued objective. In particular, we supplement the solution concept given in [19] by a new one and show that a solution set can be reduced to a singleton via a generalized translation.

There exist basically three different (but partially overlapping) approaches for defining derivatives for set-valued functions. One approach starts by picking a point in the graph of the set-valued function and assigns to it another set-valued function whose graph is some kind of tangent cone to the graph of the original function at the point in question. The book [1] gives a prestigious account of such concepts, and Mordukhovich's coderivative [25] is of the same nature. The second approach selects a class of 'simple' set-valued functions, the elements of which shall serve as approximation for a general set-valued function, and then defines what is actually understood by 'approximation'. A representative for this approach is [23]. The third approach embeds the class of set-valued functions under consideration into a linear space and operates with classical derivative concepts. The reader may consult [11] for more references and a more complete account of the three basic approaches described above. Note, however, that the last two approaches very often are restricted to set-valued functions with compact convex values, and to finite dimensional [11] or even one-dimensional pre-image spaces [23].

On the other hand, it turned out that it is a hard task to generalize basic results in convex analysis from extended real- to vector- or even set-valued functions. The max formula may feature as an example which is relevant for the present paper: Under some qualifying conditions, the directional derivative of a convex function at a given point is the support function of the subdifferential at the same point. Since this implies the non-emptiness of the subdifferential, this result is counted among the 'core results of the convex analysis' [6, p. 122]. The difficulties which arise when passing from one-dimensional to more general image spaces are brought out, for example, in [3, Theorem 6.1]: The pre-image space must be a 'Minkowski differentiability space', the image space must be ordered by a closed normal cone and enjoy the so-called monotone sequence (= greatest lower bound) property.

Our approach is more in the spirit of traditional derivative concepts which rely on increments of a function at a point in some direction. Using a residuation instead of a difference (an inverse group operation which is not available in relevant subsets of the power set of a linear space) we are able to define difference quotients and their limits even for set-valued functions. In fact, it seems to be natural to "skip" the vector-valued case by embedding it into the set-valued one. The residuation is defined on carefully selected subsets of the power set of the (linear) image space; these subsets carry the order structure of a complete lattice (= every subset has an infimum and supremum) and the algebraic structure of a semi-module over the semi-ring $\mathbb{R}_+$. It turns out that the old concept of the Minkowski (or geometric) difference of convex sets [12] can be identified with the residuation in these spaces of sets; even this seems to be a new contribution (see also [16]) although residuations have been used before in (convex) analysis, see for example [7], [10] and the references therein.

The following point should be emphasized. The theory developed in this paper makes it apparent that a function with closed convex sets as values can equivalently be represented as a family of scalar functions. This simply is a consequence of a separation theorem. Concepts like the directional derivative and the subdifferential as introduced below are defined "scalarization-wise", and many results (e.g. the convexity of the set-valued function) can be characterized using the family of scalarizations. However, in order to obtain e.g. optimality conditions one has to resort to the whole family of scalarizations, not just a particular one (see (2.11) and Theorem 5.10 below). Therefore, the message of this paper may also be summarized as follows: In the set-valued case, one does not need to do more than to apply scalar convex analysis simultaneously to a family of extended real-valued functions. Three major reasons for preferring a set-valued formulation are these: Firstly, scalarizing functions quickly become improper, and one has to either exclude them case by case or incorporate them into the calculus. We preferred to do the latter because this can be done naturally in a set-valued setting. Secondly, the set-valued calculus produces formulas and results which strikingly resemble their scalar little sisters. Thirdly, the additional dual variable which represents the scalarization and the classical one are in an adjoint convex process type of relationship as pointed out in Section 4.

The dual variables in our theory are simple set-valued functions generated by pairs of continuous linear functionals instead of continuous linear operators as, for instance, in [3], [5]. Moreover, no restrictive assumptions to the ordering cone in the underlying (linear) image space are imposed such as normality, pointedness, non-empty interior, generating a lattice order etc. These features make the theory presented in this note much more adequate for applications. The interested reader is referred to [15] for a financial application where the ordering cone is not pointed, in general, and has "many" generating vectors.

In the next section, basics about set-valued functions and their image spaces are introduced. Section 3 contains the definitions of directional derivatives and subdifferentials for set-valued convex functions and the main results. Section 4 presents the link between 'adjoint process duality' (Borwein [4]), Mordukhovich's coderivative and our derivative concepts. In the final section, set-valued optimization problems are discussed: For the first time, an optimality condition of the type "zero belongs to the subdifferential" is presented which characterizes infimizing sets rather than points.

2 Preliminaries

2.1 Image spaces

Let Z be a separated locally convex, topological linear space and $C \subsetneq Z$ a convex cone with $0 \in C$. Here, C is understood to be a cone if $z \in C$ and $t > 0$ imply $tz \in C$, and we define $\operatorname{cone} D = \{tz \mid t > 0,\ z \in D\}$ to be the conical hull of a set $D \subseteq Z$. We write $z_1 \leq_C z_2$ for $z_2 - z_1 \in C$ with $z_1, z_2 \in Z$ which defines a reflexive and transitive relation (a preorder). The topological dual space of Z is denoted by Z^*, the (negative) dual cone of C by $C^- = \{z^* \in Z^* \mid \forall z \in C \colon z^*(z) \leq 0\}$. Note that $C^- \neq \{0\}$ if, and only if, $\operatorname{cl} C \neq Z$ which is assumed throughout the paper. A corresponding notation is used in other spaces.

Let $\mathcal{P}(Z)$ be the power set of Z, i.e. the set of all subsets of Z including the empty set $\emptyset$. The Minkowski (element-wise) addition for non-empty subsets of Z is extended to $\mathcal{P}(Z)$ by

$$\emptyset + A = A + \emptyset = \emptyset$$

for $A \in \mathcal{P}(Z)$. We shall also write $z + A$ for $\{z\} + A$ and $z - A$ for $z + (-1)A$ with $-A = \{-a \mid a \in A\}$. Obviously,

$$z_1 \leq_C z_2 \ \Leftrightarrow\ z_2 \in z_1 + C \ \Leftrightarrow\ z_1 \in z_2 - C,$$

and these relationships can be used to extend $\leq_C$ to $\mathcal{P}(Z)$ in two ways:

$$A \preccurlyeq_C B \ :\Leftrightarrow\ B \subseteq A + C \quad \text{and} \quad A \curlyeqprec_C B \ :\Leftrightarrow\ A \subseteq B - C$$

for $A, B \in \mathcal{P}(Z)$. The following facts are immediate (see, e.g., [22], [13]): (a) Both $\preccurlyeq_C$ and $\curlyeqprec_C$ are reflexive and transitive relations on $\mathcal{P}(Z)$. Moreover, they are not antisymmetric in general, and they do not coincide; (b) $A \preccurlyeq_C B \Leftrightarrow -B \curlyeqprec_C -A \Leftrightarrow B \preccurlyeq_{-C} A$; (c) $A \preccurlyeq_C B \Leftrightarrow A + C \supseteq B + C$ and $A \curlyeqprec_C B \Leftrightarrow A - C \subseteq B - C$.

The first relationship in (c) makes it clear that the set $\{A \in \mathcal{P}(Z) \mid A = A + C\}$ can be identified with the set of equivalence classes of the following equivalence relation on $\mathcal{P}(Z)$: Closedness and convexity issues give rise $A \sim_C B :\Leftrightarrow A + C = B + C$. Including closedness and convexity issues this gives rise to consider the following subsets of $\mathcal{P}(Z)$:

$$\mathcal{F}(Z, C) = \{A \in \mathcal{P}(Z) \mid A = \operatorname{cl}(A + C)\},$$
$$\mathcal{G}(Z, C) = \{A \in \mathcal{P}(Z) \mid A = \operatorname{cl}\operatorname{co}(A + C)\}.$$

Elements of $\mathcal{F}(Z, C)$ are sometimes called upper closed ([24, Definition 1.50]) with respect to C. We shall abbreviate $\mathcal{F}(Z, C)$ and $\mathcal{G}(Z, C)$ to $\mathcal{F}(C)$ and $\mathcal{G}(C)$, respectively. A similar procedure is possible for $\curlyeqprec_C$, but this note is restricted to constructions starting with $\preccurlyeq_C$ which is appropriate having "minimization" and "convexity" in mind.

We define an associative and commutative binary operation $\oplus \colon \mathcal{F}(C) \times \mathcal{F}(C) \to \mathcal{F}(C)$ by

$$A \oplus B = \operatorname{cl}(A + B) \tag{2.1}$$

for $A, B \in \mathcal{F}(C)$. The element-wise multiplication of a set $A \subseteq Z$ with a (non-negative) real number is extended by

$$0 \cdot A = \operatorname{cl} C, \quad t \cdot \emptyset = \emptyset$$

for all $A \in \mathcal{F}(C)$ and $t > 0$. In particular, $0 \cdot \emptyset = \operatorname{cl} C$ by definition, and we will drop the $\cdot$ in most cases.

The triple $(\mathcal{F}(C), \oplus, \cdot)$ is a conlinear space with neutral element $\operatorname{cl} C$, and, obviously, $(\mathcal{G}(C), \oplus, \cdot)$ is a conlinear subspace of it. The concept of a 'conlinear space' has been introduced in [13], see also [14], [16]. It basically means that $(\mathcal{F}(C), \oplus)$ is a commutative monoid, and a multiplication of elements of $\mathcal{F}(C)$ with those of $\mathbb{R}_+$ is defined and satisfies some obvious requirements, but not, in general, the second distributivity law $(s+t) \cdot A = s \cdot A \oplus t \cdot A$ for $s, t \in \mathbb{R}_+$, $A \in \mathcal{F}(C)$. The elements of $\mathcal{F}(C)$ which do satisfy this law are precisely those of $\mathcal{G}(C)$, thus $(\mathcal{G}(C), \oplus, \cdot)$ is a semi-module over the semi-ring $\mathbb{R}_+$.

On $\mathcal{F}(C)$ and $\mathcal{G}(C)$, $\supseteq$ is a partial order which is compatible (in the usual sense) with the algebraic operations just introduced. By construction, $A \preccurlyeq_C B$ is equivalent to $A \supseteq B$ for all $A, B \in \mathcal{F}(C)$. Thus, $(\mathcal{F}(C), \oplus, \cdot, \supseteq)$ and $(\mathcal{G}(C), \oplus, \cdot, \supseteq)$ are partially ordered, conlinear spaces in the sense of [13], [14]. Note that this is true without any further assumptions to C. In particular, C is not required to generate a partial order, a fact, which will be used later on.

We will abbreviate $\mathcal{F}^\vartriangle = (\mathcal{F}(C), \oplus, \cdot, \supseteq)$ and $\mathcal{G}^\vartriangle = (\mathcal{G}(C), \oplus, \cdot, \supseteq)$, and we will write $A \in \mathcal{G}^\vartriangle$ and $\mathcal{A} \subseteq \mathcal{G}^\vartriangle$ in order to denote an element $A \in \mathcal{G}(C)$ and a subset $\mathcal{A} \subseteq \mathcal{G}(C)$, respectively.

Moreover, $(\mathcal{F}(C), \supseteq)$ and $(\mathcal{G}(C), \supseteq)$ are complete lattices with greatest (top) element $\emptyset$ and least (bottom) element Z. For a subset $\mathcal{A} \subseteq \mathcal{G}^\vartriangle$, the infimum and the supremum of $\mathcal{A}$ are given by

$$\inf \mathcal{A} = \operatorname{cl} \operatorname{co} \bigcup_{A \in \mathcal{A}} A, \qquad \sup \mathcal{A} = \bigcap_{A \in \mathcal{A}} A \tag{2.2}$$

where we agree upon $\inf \mathcal{A} = \emptyset$ and $\sup \mathcal{A} = Z$ whenever $\mathcal{A} = \emptyset$. Finally, for all $\mathcal{A} \subseteq \mathcal{G}^\vartriangle$ and $B \in \mathcal{G}^\vartriangle$,

$$B \oplus \inf \mathcal{A} = \inf (B \oplus \mathcal{A}) \tag{2.3}$$

where $B \oplus \mathcal{A} = \{B \oplus A \mid A \in \mathcal{A}\}$. It follows that $\mathcal{G}^\vartriangle$ is an inf-residuated space (see [16] for more details). The inf-residuation will serve as a substitute for the inverse addition and is defined as follows: For $A, B \in \mathcal{G}^\vartriangle$, set

$$A -\!\!\!\cdot\, B = \inf \left\{D \in \mathcal{G}^\vartriangle \mid B + D \subseteq A\right\} = \{z \in Z \mid B + z \subseteq A\}. \tag{2.4}$$

Note that, for $A \in \mathcal{G}^\vartriangle$, the set on the right hand side of (2.4) is indeed closed and convex since

$$\{z \in Z \mid B + z \subseteq A\} = \bigcap_{b \in B} \{z \in Z \mid b + z \in A\}$$

which is an intersection of closed convex sets whenever A is closed and convex.

Sometimes, the right hand side of (2.4) is called the geometric difference [26], star difference [32] or Minkowski difference [12] of the two sets A and B, and H. Hadwiger should probably be credited for its introduction. The relationship with residuation theory (see, for instance, [2], [9]) has been established in [16]. At least, we do not know an earlier reference.

Example 2.1. Let us consider $Z = \mathbb{R}$, $C = \mathbb{R}_+$. Then $\mathcal{G}(C) = \{[r, +\infty) \mid r \in \mathbb{R}\} \cup \{\mathbb{R}\} \cup \{\emptyset\}$, and $\mathcal{G}^\vartriangle$ can be identified (with respect to the algebraic and order structures which turn $\mathcal{G}(\mathbb{R}_+)$ into an ordered conlinear space and a complete lattice admitting an inf-residuation) with $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ using the 'inf-addition' $\dot{+}$ (see [27], [16]). The inf-residuation on $\overline{\mathbb{R}}$ is given by

$$r \dot{-} s = \inf \left\{ t \in \mathbb{R} \mid r \leq s \dot{+} t \right\}$$

for all $r, s \in \overline{\mathbb{R}}$, compare [16] for further details.

Historically, it is interesting to note that R. Dedekind [8] introduced the residuation concept and used it in order to construct the real numbers as 'Dedekind sections' of rational numbers. The construction above is in this line of ideas, but in a rather abstract setting.

Remark 2.2. The inf-residuation can be defined on $\mathcal{F}^\vartriangle$ and even other subspaces of $\mathcal{P}(Z)$, but we only need the construction in $\mathcal{G}^\vartriangle$ in this paper. Likewise, in $\mathcal{G}^\triangledown = (\mathcal{G}(-C), \oplus, \cdot, \subseteq)$, a sup-residuation can be defined such that the whole theory becomes symmetric. The interested reader is referred to [16].

In many cases, the set $A \dot{-} B$ is "too small", even empty: Consider $Z = \mathbb{R}^2$, $A = C = \mathbb{R}^2_+$, $B = \left\{ z \in \mathbb{R}^2 \mid \varepsilon z_1 + z_2 \geq 0,\ z_1 + \varepsilon z_2 \geq 0 \right\}$. Then $A \dot{-} B = \emptyset$ for each $\varepsilon > 0$. Therefore, we modify the inf-residuation in $\mathcal{G}^\vartriangle$ as follows. Take $z^* \in C^- \backslash \{0\}$ and let $H(z^*) = \{z \in Z \mid z^*(z) \leq 0\}$ be the homogenous half-space with normal z^*. We set

$$A -_{z^*} B = (A \oplus H(z^*)) \dot{-} B = \{z \in Z \mid B + z \subseteq A \oplus H(z^*)\}. \tag{2.5}$$

The operation $-_{z^*}$ can be expressed using the inf-residuation in $\overline{\mathbb{R}}$ and support functions, see [16, Proposition 5.20] and therefore, it would be interesting to study the relationships to the Demyanov and Rubinov difference [29, p. 180 and p. 182, respectively]. However, our construction is tailor-made for non-compact convex sets. Therefore, a particular difficulty had to be dealt with since the support functions of elements of $\mathcal{G}(C)$ may quickly attain non-finite values.

By definition, $A -_{z^*} B = Z$ if $A \oplus H(z^*) = Z$ or $B = \emptyset$, and $A -_{z^*} B = \emptyset$ if $A \oplus H(z^*) \neq Z$, $B \oplus H(z^*) = Z$ and if $A = \emptyset$, $B \neq \emptyset$. In all other cases, $A -_{z^*} B$ is a non-empty closed half-space parallel to $H(z^*)$. The relationship

$$A -_{z^*} B = (A \oplus H(z^*)) -_{z^*} B = A -_{z^*} (B \oplus H(z^*)) = (A \oplus H(z^*)) -_{z^*} (B \oplus H(z^*))$$

for all $A, B \in \mathcal{G}^\vartriangle$ is immediate from the definition of $-_{z^*}$. The next proposition shows that $H(z^*)$ can be understood as a "generalized zero." In fact, it is the zero element of $\mathcal{G}(H(z^*))$ and, of course, $\mathcal{G}(H(z^*)) \subseteq \mathcal{G}(C)$ for all $z^* \in C^- \backslash \{0\}$. One may consider the collection of all $H(z^*) -_{z^*} A$ for $z^* \in C^- \backslash \{0\}$ (which also are elements of $\mathcal{G}(C)$) as a substitute for $-A$ (which is not in $\mathcal{G}(C)$ and not the inverse element of A with respect to addition, in general).

Proposition 2.3. *Let $A, B \in \mathcal{G}^\vartriangle$ and $z^* \in C^- \backslash \{0\}$. Then (a)*

$$A \oplus H(z^*) \supseteq B \oplus H(z^*) \quad \Leftrightarrow \quad H(z^*) -_{z^*} B \supseteq H(z^*) -_{z^*} A,$$

and (b)

$$A -_{z^*} B \notin \{Z, \emptyset\} \quad \Leftrightarrow \quad A \oplus H(z^*), B \oplus H(z^*) \notin \{Z, \emptyset\}.$$

Proof. (a) "$\Rightarrow$": We have

$$\begin{aligned} H(z^*) -_{z^*} A = H(z^*) -_{z^*} (A \oplus H(z^*)) &= \{z \in Z \mid A \oplus H(z^*) + z \subseteq H(z^*)\} \\ &\subseteq \{z \in Z \mid B \oplus H(z^*) + z \subseteq H(z^*)\} = H(z^*) -_{z^*} B \end{aligned}$$

since $B \oplus H(z^*) \subseteq A \oplus H(z^*)$.

"$\Leftarrow$": This implication is certainly true if $A \oplus H(z^*) = Z$. If $A \oplus H(z^*) = \emptyset$, then $H(z^*) -_{z^*} A = Z$, hence $H(z^*) -_{z^*} B = Z$ by assumption which in turn implies $B = \emptyset$. Finally, assume $A \oplus H(z^*) = z_A + H(z^*)$ for some $z_A \in Z$. Then

$$-z_A \in H(z^*) -_{z^*} A \subseteq H(z^*) -_{z^*} B,$$

hence $B \oplus H(z^*) \subseteq z_A + H(z^*) \subseteq A \oplus H(z^*)$.

(b) is a straightforward consequence of the definition of $-_{z^*}$. □

The following calculus rules for $-_{z^*}$ apply and will be used frequently.

Proposition 2.4. *Let $A, B, D \subseteq \mathcal{G}^\vartriangle$ and $z^* \in C^- \setminus \{0\}$. Then*

(a) $A \supseteq B \Rightarrow A -_{z^*} D \supseteq B -_{z^*} D$ *and* $D -_{z^*} B \supseteq D -_{z^*} A$.

(b) $A -_{z^*} A = H(z^*)$ *if and only if,* $A \oplus H(z^*) \notin \{Z, \emptyset\}$, *and* $A -_{z^*} A = Z$ *if, and only if,* $A \oplus H(z^*) \in \{Z, \emptyset\}$.

(c) $s_1, s_2 \geq 0$, $s_1 + s_2 = 1 \Rightarrow$

$$(s_1 A \oplus s_2 B) -_{z^*} D \supseteq s_1 (A -_{z^*} D) \oplus s_2 (B -_{z^*} D).$$

(d) $(A \oplus B) -_{z^*} D \supseteq (A -_{z^*} D) \oplus B$. *The strict inclusion applies if, and only if,* $B = D = \emptyset$, *or* $B \oplus H(z^*) = D \oplus H(z^*) = Z$ *and* $A \oplus H(z^*) \notin \{Z, \emptyset\}$.

(e1) $(B -_{z^*} B) \oplus A \supseteq A \oplus H(z^*)$. *The strict inclusion applies if, and only if,* $A \oplus H(z^*) \notin \{Z, \emptyset\}$ *and* $B \in \{Z, \emptyset\}$.

(e2) $(A \oplus B) -_{z^*} B \supseteq (B -_{z^*} B) \oplus A$. *The strict inclusion applies if, and only if,* $A = B = \emptyset$.

(e3) $A \oplus H(z^*) \supseteq (A -_{z^*} B) \oplus B$. *The strict inclusion applies if, and only if,* $A \neq B = \emptyset$, *or* $A \oplus H(z^*) \notin \{Z, \emptyset\}$ *and* $B \oplus H(z^*) = Z$.

(f) $H(z^*) \supseteq A -_{z^*} B \Rightarrow B \oplus H(z^*) \supseteq A$.

Proof. (a) - (c) are elementary using the definition of $-_{z^*}$.

(d) If $(A -_{z^*} D) \oplus B = \emptyset$, then the inclusion is trivially true. Otherwise, $B \neq \emptyset$, $(A -_{z^*} D) \neq \emptyset$, and for each $z \in (A -_{z^*} D)$ it holds $D + z \subseteq A \oplus H(z^*)$ which implies $D + B + z \subseteq A \oplus B \oplus H(z^*)$ which in turn gives $B + z \subseteq (A \oplus B) -_{z^*} D$. The inclusion follows.

If both sides are neither Z nor $\emptyset$, then equality holds true. Indeed, in this case (see Proposition 2.3 (b)) $A \oplus H(z^*), B \oplus H(z^*) \notin \{Z, \emptyset\}$, hence there are $z_A, z_B \in Z$ such that $A \oplus H(z^*) = z_A + H(z^*)$ and $B \oplus H(z^*) = z_B + H(z^*)$. This gives

$$\begin{aligned}(A -_{z^*} D) \oplus B &= \operatorname{cl}\{z + b \mid z \in Z,\, b \in B,\, D + z \subseteq z_A + H(z^*)\} \\ &= \{z + z_B \mid z \in Z,\, D + z \subseteq z_A + H(z^*)\} \oplus H(z^*) \\ &= \{z \in Z \mid D + z - z_B \subseteq z_A + H(z^*)\} = (A \oplus B) -_{z^*} D.\end{aligned}$$

This leaves two cases for strict inclusion: The first is $(A -_{z^*} D) \oplus B = \emptyset$ and $(A \oplus B) -_{z^*} D \neq \emptyset$, the second $(A -_{z^*} D) \oplus B \notin \{Z, \emptyset\}$ and $(A \oplus B) -_{z^*} D = Z$.

The second case can not occur as a straightforward analysis shows.

In the first, we can have $B = \emptyset$ in which case $(A \oplus B) -_{z^*} D = \emptyset -_{z^*} D \in \{Z, \emptyset\}$, so $D = \emptyset$ is necessary and sufficient for strict inclusion in this case. Or we can have $A -_{z^*} D = \emptyset$ which produces the strict inclusion precisely when $B \oplus H(z^*) = D \oplus H(z^*) = Z$ and $A \oplus H(z^*) \notin \{Z, \emptyset\}$.

(e1) This follows from (b) since $(B -_{z^*} B) \in \{Z, H(z^*)\}$.

(e2) Replacing A by B, B by A and D by B in (d) we obtain the inclusion. From the corresponding cases in (d) one may deduce that the inclusion is strict if, and only if, $A = B = \emptyset$.

(e3) The inclusion is a direct consequence of the definition of $-_{z^*}$. Again, a discussion of the two possibilities for strict inclusion, namely $(A -_{z^*} B) \oplus B = \emptyset$ and $A \oplus H(z^*) \neq \emptyset$, or $(A -_{z^*} B) \oplus B \notin \{Z, \emptyset\}$ and $A \oplus H(z^*) = Z$, produce the corresponding conditions.

(f) First, note that $H(z^*) \supseteq A -_{z^*} B$ implies $A \oplus H(z^*) \neq Z$ and $B \neq \emptyset$.

If $B \oplus H(z^*) = Z$ or $A = \emptyset$, then $A -_{z^*} B = \emptyset$, hence the assumption is trivially satisfied, and, also trivially, $A \subseteq B \oplus H(z^*)$.

Now, assume that $A \subseteq B \oplus H(z^*)$ is not true. Then, $B \oplus H(z^*) \neq Z$ and there exist $z_B \in Z$ and $a \in A$ such that $B \oplus H(z^*) = z_B + H(z^*)$, $a \notin B \oplus H(z^*) = z_B + H(z^*)$ and finally $z^*(a - z_B) > 0$. Then $z = a - z_B$ satisfies

$$B + z \subseteq B \oplus H(z^*) + z = z_B + H(z^*) + a - z_B = a + H(z^*) \subseteq A \oplus H(z^*),$$

hence $z \in A -_{z^*} B$ contradicting the assumption since $z^*(z) > 0$. □

Lemma 2.5. *Let $A, B \in \mathcal{G}^{\vartriangle}$ and $z^* \in C^- \setminus \{0\}$. Then*

$$H(z^*) -_{z^*} B \subseteq (A -_{z^*} B) -_{z^*} A. \tag{2.6}$$

The strict inclusion applies if, and only if, $A \oplus H(z^*) \in \{Z, \emptyset\}$ and $B \neq \emptyset$.

Proof. From the definition of $-_{z^*}$ one obtains

$$\begin{aligned}(A -_{z^*} B) -_{z^*} A &= \{z \in Z \mid A + z \subseteq A -_{z^*} B\} \\ &= \{z \in Z \mid B + A + z \subseteq A \oplus H(z^*)\} \\ &= \{z \in Z \mid B + z \subseteq A -_{z^*} A\} = (A -_{z^*} A) -_{z^*} B.\end{aligned}$$

Proposition 2.4 (a), (b) produce (2.6).

Again, the strict inclusion applies if either $H(z^*) -_{z^*} B = \emptyset$ and $(A -_{z^*} B) -_{z^*} A \neq \emptyset$, or $H(z^*) -_{z^*} B \notin \{Z, \emptyset\}$ and $(A -_{z^*} B) -_{z^*} A = Z$. A case study produces the claimed condition. □

We remark that most of the calculus rules for the inf-residuation follow from general rules within the framework of residuated lattices, see e.g. [2] and [9].

2.2 $\mathcal{G}^{\vartriangle}$-valued functions and their scalarizations

Let X be another locally convex, topological linear space with dual X^*. A function $f\colon X \to \mathcal{G}^{\vartriangle}$ is called convex if

$$\forall x_1, x_2 \in X,\ \forall t \in (0,1) : f(tx_1 + (1-t)x_2) \supseteq tf(x_1) \oplus (1-t) f(x_2). \tag{2.7}$$

It is an exercise (see, for instance, [14]) to show that f is convex if, and only if, the set

$$\operatorname{graph} f = \{(x, z) \in X \times Z \mid z \in f(x)\}$$

is convex. A $\mathcal{G}^{\vartriangle}$-valued function f is called positively homogeneous if

$$\forall t > 0, \forall x \in X : f(tx) \supseteq tf(x),$$

and it is called sublinear if it is positively homogeneous and convex. Another exercise shows that f is sublinear if, and only if, $\operatorname{graph} f$ is a convex cone.

A function $f\colon X \to \mathcal{G}^\vartriangle$ is called lower semi-continuous (l.s.c. for short) at $x_0 \in X$ if $f(x_0) \supseteq \liminf_{x\to x_0} f(x)$ where

$$\liminf_{x\to x_0} f(x_0) = \inf_{U\in\mathcal{N}_X(0)} \sup_{x\in x_0+U} f(x) = \bigcap_{U\in\mathcal{N}_X(0)} \operatorname{cl}\operatorname{co} \bigcup_{x\in x_0+U} f(x) \tag{2.8}$$

and $\mathcal{N}_X(0)$ is a neighborhood base of $0 \in X$. The function $f\colon X \to \mathcal{G}^\vartriangle$ is called closed if it is l.s.c. at every $x \in X$. One can show that a convex function $f\colon X \to \mathcal{G}^\vartriangle$ is closed if, and only if, $\operatorname{graph} f \subseteq X \times Z$ is a closed set with respect to the product topology. A more general definition of lower semicontinuity can be given for functions mapping into $\mathcal{F}^\vartriangle$ such that closedness of the graph corresponds to closedness of the function even for non-convex ones, see [24, Proposition 2.34].

The function $(\operatorname{cl}\operatorname{co} f) : X \to \mathcal{G}^\vartriangle$ defined by

$$\operatorname{graph}\, (\operatorname{cl}\operatorname{co} f) = \operatorname{cl}\operatorname{co}\, (\operatorname{graph} f) \tag{2.9}$$

is the (point-wise in $\mathcal{G}^\vartriangle$) greatest closed convex minorant of a function $f\colon X \to \mathcal{G}^\vartriangle$; it is called the closed convex hull of f.

Remark 2.6. A more common convexity concept for functions $F\colon X \to \mathcal{P}(Z)$ is the following (compare, for instance, [21, Definition 14.6]): F is called C-convex if

$$t \in [0,1], x_1, x_2 \in X \;\Rightarrow\; F(tx_1 + (1-t)x_2) + C \supseteq tF(x_1) + (1-t)F(x_2).$$

It is easily seen that C-convexity of F implies that the function $x \mapsto f(x) = F(x) + C$ maps into

$$\{D \in \mathcal{P}(Z) \mid D = \operatorname{co}\,(D + C)\}$$

and has a convex graph which coincides with

$$\operatorname{epi} F = \{(x,z) \in X \times Z \mid z \in F(x) + C\}$$

(see [21, Definition 14.7]). Moreover, if $\operatorname{epi} F = \operatorname{graph} f$ is additionally closed, then the values of f are elements of $\mathcal{G}(C)$.

Finally, note that it does not make sense to distinguish between the graph and the epigraph of a $\mathcal{G}(C)$-valued function since the two sets coincide.

A function $f\colon X \to \mathcal{G}(C)$ is called proper, if its domain

$$\operatorname{dom} f = \{x \in X \mid f(x) \neq \emptyset\}$$

is nonempty and f does not attain the value Z. A $\mathcal{G}(C)$-valued function is called C-proper if $(f(x) - C) \backslash f(x) \neq \emptyset$ for all $x \in \operatorname{dom} f$. A function is called z^*-proper for $z^* \in C^-\backslash\{0\}$ if the function $x \mapsto f(x) \oplus H(z^*)$ is proper. Of course, if f is z^*-proper for at least one $z^* \in C^-\backslash\{0\}$, then it is proper. Vice versa, if f is a closed convex proper function, then there is at least one $z^* \in C^-\backslash\{0\}$ such that f is z^*-proper. The latter fact follows, for example, from [14, Theorem 1].

Example 2.7. Let $x^* \in X^*$ and $z^* \in Z^*\backslash\{0\}$ be given. The function $S_{(x^*,z^*)}\colon X \to \mathcal{P}(Z)$ defined through

$$S_{(x^*,z^*)}(x) = \{z \in Z \mid x^*(x) + z^*(z) \leq 0\}$$

maps into $\mathcal{G}(C)$ if, and only if, $z^* \in C^-$. Moreover, it is positively homogeneous and additive. Therefore, if $z^* \in C^-$ the function $x \mapsto S_{(x^*,z^*)}(x)$ is $\mathcal{G}(C)$-valued and convex. It is z^*-proper if, and only if, it is sz^*-proper for all $s > 0$ (since $H(sz^*) = H(z^*)$ for all $s > 0$) if, and only if, $z^* \in C^- \setminus \{0\}$. Finally, $S_{(x^*,z^*)}(0) = H(z^*) = \{z \in Z \mid z^*(z) \le 0\}$ is a homogeneous closed half space with normal z^* if $z^* \neq 0$ and $S_{(x^*,0)}(0) = Z$. In particular, $S_{(x^*,z^*)}(0) \neq \{0\}$ for all $z^* \in C^-$.

For $z^* \in C^-$, the useful relation

$$\forall x \in X,\ \forall t > 0\colon S_{(x^*,tz^*)}(x) = S_{\left(\frac{1}{t}x^*,z^*\right)}(x)$$

immediately follows from the definition of $S_{(x^*,z^*)}(x)$. If $Z = \mathbb{R}$ and $C = R_+$, then $S_{(x^*,-s)}(x) = -\frac{1}{s}x^*(x) + \mathbb{R}_+$ for $s \in C^- \setminus \{0\} = -\mathbb{R}_+ \setminus \{0\}$ while $S_{(x^*,0)}(x) \in \{Z, \emptyset\}$.

Let a function $f\colon X \to \mathcal{G}^\triangle$ be given. The family of extended real-valued functions $\varphi_{f,z^*}\colon X \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\varphi_{f,z^*}(x) = \inf\{-z^*(z) \mid z \in f(x)\},\ z^* \in C^- \setminus \{0\}, \tag{2.10}$$

is called the family of (linearly generated) scalarizations for f. The function f is convex if, and only if, the scalarizing function φ_{f,z^*} is convex for each $z^* \in C^- \setminus \{0\}$. A closed convex function f is proper if, and only if, there is $z^* \in C^- \setminus \{0\}$ such that φ_{f,z^*} is proper (in the usual sense of classical convex analysis), and this is the case if, and only if, the function $x \mapsto f(x) \oplus H(z^*)$ is proper. A standard separation argument shows

$$\forall x \in X\colon f(x) = \bigcap_{z^* \in C^- \setminus \{0\}} \{z \in Z \mid \varphi_{f,z^*}(x) + z^*(z) \le 0\}. \tag{2.11}$$

With some effort, one can show that for a closed convex proper function $f\colon X \to \mathcal{G}^\triangle$ it suffices to run the intersection in the above formula over the set of those $z^* \in C^- \setminus \{0\}$ which generate a closed proper (and convex) scalarization φ_{f,z^*}, see [30] and [31, Corollary 3.4]. The previous two formulas provide a one-to-one relationship between a $\mathcal{G}^\triangle$-valued functions f and the family $\{\varphi_{f,z^*}\}_{z^* \in C^- \setminus \{0\}}$ of extended real-valued functions which turns out to be crucial for the developments below.

3 Directional Derivatives and Subdifferentials of $\mathcal{G}^\triangle$-valued Functions

Definition 3.1. Let $f\colon X \to \mathcal{G}^\triangle$ be a convex function. The directional derivative of f with respect to $z^* \in C^- \setminus \{0\}$ at $x_0 \in X$ in direction $x \in X$ is given by

$$f'_{z^*}(x_0, x) = \liminf_{t \downarrow 0} \frac{1}{t}\left[f(x_0 + tx) -_{z^*} f(x_0)\right] = \bigcap_{s>0} \operatorname{cl} \bigcup_{0<t<s} \frac{1}{t}\left[f(x_0 + tx) -_{z^*} f(x_0)\right]. \tag{3.1}$$

Note that one can drop the convex hull involved in the infimum in $\mathcal{G}^\triangle$ since the union of closed half spaces with the same normal (or Z, or $\emptyset$) automatically is convex. If $f(x_0) = \emptyset$ then $f'_{z^*}(x_0, x) = Z$ for all $x \in X$. Therefore, we can restrict the analysis to the case $x_0 \in \operatorname{dom} f$. The main tool will be the directional difference quotient of f at $x_0 \in X$ which is defined to be the function $g_{z^*}\colon \mathbb{R} \times X \to \mathcal{G}^\triangle$ given by

$$g_{z^*}(t, x) = \frac{1}{t}\left[f(x_0 + tx) -_{z^*} f(x_0)\right].$$

The next lemma demonstrates the monotonicity of the difference quotient.

Lemma 3.2. *Let* $f\colon X \to \mathcal{G}^{\triangle}$ *be convex and* $x_0, x \in X$*. If* $0 < t \leq s$ *then*

$$H(z^*) -_{z^*} g_{z^*}(s, -x) \supseteq H(z^*) -_{z^*} g_{z^*}(t, -x) \quad \text{and} \quad g_{z^*}(t, x) \supseteq g_{z^*}(s, x). \tag{3.2}$$

If, additionally, $f(x_0) \oplus H(z^*) \notin \{Z, \emptyset\}$*, then*

$$H(z^*) -_{z^*} g_{z^*}(t, -x) \supseteq g_{z^*}(t, x). \tag{3.3}$$

Proof. Since $0 < \frac{t}{s} \leq 1$, $x_0 + tx = \frac{t}{s}(x_0 + sx) + \frac{s-t}{s}x_0$ is well-defined. The convexity of f produces

$$f(x_0 + tx) \supseteq \frac{t}{s} f(x_0 + sx) + \frac{s-t}{s} f(x).$$

The rules (a), (c) and (b) of Proposition 2.4 produce

$$\begin{aligned} g_{z^*}(t, x) &\supseteq \frac{1}{t}\left[\left(\frac{t}{s} f(x_0 + sx) + \frac{s-t}{s} f(x_0)\right) -_{z^*} f(x_0)\right] \\ &\supseteq \frac{1}{t}\left[\frac{t}{s}\left(f(x_0 + sx) -_{z^*} f(x_0)\right) + \frac{s-t}{s}\left(f(x_0) -_{z^*} f(x_0)\right)\right] \\ &\supseteq \frac{1}{s}\left[f(x_0 + sx) -_{z^*} f(x_0)\right] \oplus H(z^*) = g_{z^*}(s, x). \end{aligned}$$

Hence $g_{z^*}(t, x) \supseteq g_{z^*}(s, x)$. Replacing x by $-x$ we obtain $g_{z^*}(t, -x) \supseteq g_{z^*}(s, -x)$. Proposition 2.4, (a) produces

$$H(z^*) -_{z^*} g_{z^*}(s, -x) \supseteq H(z^*) -_{z^*} g_{z^*}(t, -x).$$

It remains to demonstrate the inequality (3.3). Since $x_0 = \frac{1}{2}(x_0 + tx) + \frac{1}{2}(x_0 - tx)$, convexity of f implies

$$f(x_0) \supseteq \frac{1}{2} f(x_0 + tx) + \frac{1}{2} f(x_0 - tx),$$

and from Proposition 2.4, (a), (c) and $f(x_0) -_{z^*} f(x_0) = H(z^*)$ we obtain

$$\begin{aligned} H(z^*) &\supseteq \left[\frac{1}{2} f(x_0 + tx) \oplus \frac{1}{2} f(x_0 - tx)\right] -_{z^*} f(x_0) \\ &\supseteq \frac{1}{2}\left[f(x_0 + tx) -_{z^*} f(x_0)\right] \oplus \frac{1}{2}\left[f(x_0 - tx) -_{z^*} f(x_0)\right] \\ &= t \cdot \left[\frac{1}{2} g_{z^*}(t, x) \oplus \frac{1}{2} g_{z^*}(t, -x)\right]. \end{aligned}$$

Since $H(z^*)$ is a cone, the above relation can be divided by $\frac{t}{2} > 0$ without changing $H(z^*)$ on the very left side. Now,

$$H(z^*) -_{z^*} g_{z^*}(t, -x) \supseteq g_{z^*}(t, x).$$

immediately follows from the definition of $-_{z^*}$. □

Lemma 3.3. *Let* $f\colon X \to \mathcal{G}^{\triangle}$ *be convex,* $x_0 \in X$ *and* $z^* \in C^- \setminus \{0\}$*. Then*

$$\forall x \in X\colon f'_{z^*}(x_0, x) = \inf_{t>0} \frac{1}{t}\left[f(x_0 + tx) -_{z^*} f(x_0)\right], \tag{3.4}$$

and the function

$$x \mapsto f'_{z^*}(x_0, x)$$

is sublinear as a function from X *into* $\mathcal{G}^{\vartriangle}$. *If* $x_0 \in \operatorname{dom} f$, *then* $\operatorname{dom} f'_{z^*}(x_0, \cdot) = \operatorname{cone}(\operatorname{dom} f - x_0)$. *Moreover,*

$$f'_{z^*}(x_0, 0) = \begin{cases} H(z^*) & : \; f(x_0) \oplus H(z^*) \notin \{Z, \emptyset\} \\ Z & : \; f(x_0) \oplus H(z^*) \in \{Z, \emptyset\} \end{cases} .$$

Proof. The monotonicity of the difference quotient as proven in Lemma 3.2 yields

$$\bigcup_{0<t<s} \frac{1}{t}\left[f(x_0 + tx) -_{z^*} f(x_0)\right] = \bigcup_{t>0} \frac{1}{t}\left[f(x_0 + tx) -_{z^*} f(x_0)\right],$$

and this implies (3.4). In turn, (3.4) immediately yields the positive homogeneity of $f'_{z^*}(x_0, \cdot)$. Take $x_1, x_2 \in X$. By convexity of f

$$\begin{aligned} f(x_0 + t(x_1 + x_2)) &= f\left(\frac{1}{2}(x_0 + 2tx_1) + \frac{1}{2}(x_0 + 2tx_2)\right) \\ &\supseteq \frac{1}{2} f(x_0 + 2tx_1) + \frac{1}{2} f(x_0 + 2tx_2). \end{aligned}$$

Proposition 2.4, (c) gives

$$\frac{1}{t}\left[f(x_0 + t(x_1 + x_2)) -_{z^*} f(x_0)\right] \supseteq \\ \frac{1}{2t}\left[f(x_0 + 2tx_1) -_{z^*} f(x_0)\right] + \frac{1}{2t}\left[f(x_0 + 2tx_2) -_{z^*} f(x_0)\right]. \quad (3.5)$$

Fix $s > 0$ and choose $t > 0$ such that $2t \le s$. Then, by the monotonicity of the difference quotient (see (3.2) in Lemma 3.2)

$$\frac{1}{2t}\left[f(x_0 + 2tx_1) -_{z^*} f(x_0)\right] + \frac{1}{2t}\left[f(x_0 + 2tx_2) -_{z^*} f(x_0)\right] \supseteq \\ \frac{1}{2t}\left[f(x_0 + 2tx_1) -_{z^*} f(x_0)\right] + \frac{1}{s}\left[f(x_0 + sx_2) -_{z^*} f(x_0)\right].$$

This implies

$$\operatorname{cl} \bigcup_{t>0} \left[\frac{1}{2t}\left[f(x_0 + 2tx_1) -_{z^*} f(x_0)\right] + \frac{1}{2t}\left[f(x_0 + 2tx_2) -_{z^*} f(x_0)\right]\right] \supseteq \\ \operatorname{cl} \bigcup_{0<2t\le s} \frac{1}{2t}\left[f(x_0 + 2tx_1) -_{z^*} f(x_0)\right] + \frac{1}{s}\left[f(x_0 + sx_2) -_{z^*} f(x_0)\right]$$

for all $s > 0$ which produces, together with (3.5), the desired subadditivity.

We have $f'_{z^*}(x_0, x) = \emptyset$ if, and only if,

$$\forall t > 0 \colon f(x_0 + tx) -_{z^*} f(x_0) = \emptyset.$$

Since, by assumption, $f(x_0) \neq \emptyset$, this is true if and only if $f(x_0 + tx) = \emptyset$ for all $t > 0$ (see definition of $-_{z^*}$). This proves $\operatorname{dom} f'_{z^*}(x_0, \cdot) = \operatorname{cone}(\operatorname{dom} f - x_0)$.

Finally, one easily checks the formula for $f'_{z^*}(x_0, 0)$. □

Example 3.4. A function $\varphi\colon X \to \overline{\mathrm{IR}}$ can be identified with the function $f\colon X \to \mathcal{G}(\mathrm{IR}, \mathrm{IR}_+)$ defined by $f(x) = \varphi(x) + \mathrm{IR}_+$ where it is understood that $f(x) = \mathrm{IR}$ whenever $\varphi(x) = -\infty$ and $f(x) = \emptyset$ whenever $\varphi(x) = +\infty$ (compare Example 2.1). Vice versa, if $f\colon X \to \mathcal{G}(\mathrm{IR}, \mathrm{IR}_+)$ is given, one obtains a function $\varphi\colon X \to \overline{\mathrm{IR}}$ by $\varphi(x) = \inf\{r \in \mathrm{IR} \mid r \in f(x)\}$ with $\varphi(x) = -\infty$ for $f(x) = \mathrm{IR}$ and $\varphi(x) = +\infty$ for $f(x) = \emptyset$. Obviously, φ is convex if, and only if, f is convex. In this case, we obtain a directional derivative f' (one and the same for all $z^* = s \in (\mathrm{IR}_+)^- \setminus \{0\} = -\mathrm{IR}_+ \setminus \{0\}$) for f by means of Definition 3.1. We set for $x \in X$

$$\varphi'(x_0, x) = \inf\{r \in \mathrm{IR} \mid r \in f'(x_0, x)\}, \tag{3.6}$$

and this definition is an extension of the classical definition for the directional derivative of proper extended real-valued functions. One can show (see [16]) that for convex f

$$\forall x \in X\colon \varphi'(x_0, x) = \inf_{t>0} \frac{1}{t}\left[\varphi(x_0 + tx) \dot{-} \varphi(x_0)\right].$$

Remark 3.5. Let $f\colon X \to \mathcal{G}^\vartriangle$. The function $\psi\colon X \to \overline{\mathrm{IR}}$ defined by

$$\psi(x) = \inf\{-z^*(z) \mid z \in f'_{z^*}(x_0, x)\}$$

is the scalarization of $x \mapsto f'_{z^*}(x_0, x)$. Since $f'_{z^*}(x_0, \cdot)$ maps into $\mathcal{G}(H(z^*))$, it is an exercise to prove that for all $x \in X$

$$\psi(x) = \varphi'_{f,z^*}(x_0, x)$$

and

$$f'_{z^*}(x_0, x) = \{z \in Z \mid \psi(x) + z^*(z) \le 0\}.$$

Example 3.6. Let $X = Z = \mathrm{IR}^2$ and $C = \mathrm{IR}^2_+$. Consider $f\colon X \to \mathcal{G}(C)$ defined by

$$f(x) = \begin{cases} \{z \in \mathrm{IR}^2 \mid z_1 \ge -x_1 + x_2,\ z_2 \ge -x_1 - x_2,\ z_1 + z_2 \ge x_1\} & : \quad x_1 \ge 0 \\ \emptyset & : \quad \text{otherwise} \end{cases}$$

The function f is convex since graph f is convex. Since $f(x)$ is given through three linear inequalities it can have at most two vertexes which depend linearly on x:

$$V^1(x) = \begin{pmatrix} -x_1 + x_2 \\ 2x_1 - x_2 \end{pmatrix} \quad \text{and} \quad V^2(x) = \begin{pmatrix} 2x_1 + x_2 \\ -x_1 - x_2 \end{pmatrix}.$$

Moreover, $V^2(x) - V^1(x) = 3x_1(1, -1)^T$, and the recession cone of $f(x)$ is IR^2_+. This shows that for $w \in C^- = -\mathrm{IR}^2_+$

$$f(x) \oplus H(w) = \begin{cases} V^1(x) + H(w) & : \quad w_1 \ge w_2 \\ V^2(x) + H(w) & : \quad w_1 \le w_2 \end{cases}$$

for all $x \in \operatorname{dom} f$. Hence

$$\frac{1}{t}\left[f(x + ty) -_{z^*} f(x)\right] = \begin{cases} V^1(y) + H(w) & : \quad w_1 \ge w_2 \\ V^2(y) + H(w) & : \quad w_1 < w_2 \end{cases}$$

whenever $x_1 > 0$ or $y_1 \ge 0$. Since this "difference quotient" does not depend on t, it equals the directional derivative. We obtain for $x_1 > 0$ or $y_1 \ge 0$

$$f'_w(x, y) = \begin{cases} V^1(y) + H(w) & : \quad w_1 \ge w_2 \\ V^2(y) + H(w) & : \quad w_1 < w_2, \end{cases}$$

and $f'_w(x,y) = \emptyset$ whenever $x_1 = 0$ and $y_1 < 0$. Since $z \in V^i(y) + H(w)$, if and only if, $w^T\left(z - V^i(y)\right) \leq 0$, $i = 1,2$,

$$V^1(y) + H(w) = \left\{z \in \mathrm{I\!R}^2 \mid y_1(w_1 - 2w_2) + y_2(-w_1 + w_2) + w^T z \leq 0\right\},$$
$$V^2(y) + H(w) = \left\{z \in \mathrm{I\!R}^2 \mid y_1(-2w_1 + w_2) + y_2(-w_1 + w_2) + w^T z \leq 0\right\}.$$

Finally, $f'_w(x,y) = Z$ for $x \notin \operatorname{dom} f$, i.e. $x_1 < 0$.

The following result tells us when the directional derivative has only "finite" values. As usual, we denote by $\operatorname{core} M$ the algebraic interior of a set $M \subseteq X$.

Theorem 3.7. *Let $f\colon X \to \mathcal{G}^{\vartriangle}$ be convex and $x_0 \in \operatorname{core}(\operatorname{dom} f)$. If f is proper (C-proper), then there exists $z^* \in C^-\backslash\{0\}$ ($z^* \in C^-\backslash -C^-$) such that $f'_{z^*}(x_0,x) \notin \{Z,\emptyset\}$ for all $x \in X$.*

Proof. Since f is proper (C-proper), we have $f(x_0) \neq Z$. Hence there is $z^* \in C^-\backslash\{0\}$ ($z^* \in C^-\backslash -C^-$) such that $f(x_0) \oplus H(z^*) \neq Z$ (a separation argument). This implies $f'_{z^*}(x_0,0) = H(z^*)$.

We have to show that $f'_{z^*}(x_0,x) \neq Z$ for all $x \in X$. Since $x_0 \in \operatorname{core}(\operatorname{dom} f)$ there is $t_0 > 0$ such that

$$\forall t \in [0,t_0] : x_0 \pm tx \in \operatorname{dom} f,$$

hence $\pm x \in \operatorname{dom} f'_{z^*}(x_0,\cdot)$ (see Lemma 3.3). Sublinearity of $f'_{z^*}(x_0,\cdot)$ implies

$$H(z^*) = f'_{z^*}(x_0,0) \supseteq f'_{z^*}(x_0,x) \oplus f'_{z^*}(x_0,-x) \neq \emptyset,$$

and the latter inclusion implies $f'_{z^*}(x_0,x), f'_{z^*}(x_0,-x) \neq Z$. □

Baptizing the functions $x \mapsto S_{(x^*,z^*)}(x)$ as "conlinear" we use conlinear minorants of the sublinear directional derivative in order to define elements of the subdifferential.

Definition 3.8. Let $f\colon X \to \mathcal{G}^{\vartriangle}$ be convex, $x_0 \in X$ and $z^* \in C^-\backslash\{0\}$. The set

$$\partial_{z^*} f(x_0) = \left\{x^* \in X^* \mid \forall x \in X\colon S_{(x^*,z^*)}(x) \supseteq f'_{z^*}(x_0,x)\right\}$$

is called the z^*-subdifferential of f at x_0.

Note that

$$\forall s > 0\colon s\partial_{z^*} f(x) = \partial_{sz^*} f(x)$$

by virtue of $f'_{sz^*}(x_0,x) = f'_{z^*}(x_0,x)$. This relationship will be used in the next section to establish a link to adjoints of convex processes. The directional derivative $f'_{z^*}(x_0,\cdot)$ is improper if, and only if, $\partial_{z^*} f(x_0) = \emptyset$. Elements of the z^*-subdifferential can also be characterized by the subdifferential inequality.

Proposition 3.9. *Let $f\colon X \to \mathcal{G}^{\vartriangle}$ be convex and $x_0 \in X$. The following statements are equivalent for $x^* \in X^*$, $z^* \in C^-\backslash\{0\}$:*

(a) $\forall x \in X\colon S_{(x^*,z^*)}(x) \supseteq f'_{z^*}(x_0,x)$,

(b) $\forall x \in X\colon S_{(x^*,z^*)}(x - x_0) \supseteq f(x) -_{z^*} f(x_0)$.

If, additionally, $\operatorname{dom} f \neq \emptyset$ *and $f(x_0) \oplus H(z^*) \neq Z$, then (a) and (b) are equivalent to*

(c) $\forall x \in X\colon f(x_0) \oplus S_{(x^*,z^*)}(-x_0) \supseteq f(x) \oplus S_{(x^*,z^*)}(-x)$ *and*

(d) $\forall x \in X\colon S_{(x^*,z^*)}(x) -_{z^*} f(x) \supseteq S_{(x^*,z^*)}(x_0) -_{z^*} f(x_0)$.

Proof. (a) $\Rightarrow$ (b): Choosing $t = 1$ in the definition of the directional derivative and replacing x by $x - x_0$ we obtain (b) from (a).

(b) $\Rightarrow$ (a): Use $x = x_0 + ty$ for $t > 0$, $y \in X$ and $S_{(x^*,z^*)}(ty) = tS_{(x^*,z^*)}(y)$.

Next, note that if $x_0 \notin \operatorname{dom} f$ and $\operatorname{dom} f \neq \emptyset$, then there is no pair (x^*, z^*) which satisfies either one of the conditions (a), (b), (c), (d). So, assume $x_0 \in \operatorname{dom} f$ in the remaining part of the proof.

(b) $\Rightarrow$ (c): Since $x_0 \in \operatorname{dom} f$, the additional assumptions give $f(x_0) \oplus H(z^*) \notin \{Z, \emptyset\}$. Adding $f(x_0)$ to both sides of the condition in (b), using the additivity of the $S_{(x^*,z^*)}$ and applying the equality case of Proposition 2.4 (e3) to the right hand side of the obtained inclusion with $A = f(x)$ and $B = f(x_0)$ we arrive at the condition in (c).

(c) $\Rightarrow$ (b): Adding $S_{(x^*,z^*)}(x)$ to both sides of the condition in (c) and using the additivity of $S_{(x^*,z^*)}$ we obtain

$$f(x_0) \oplus S_{(x^*,z^*)}(x - x_0) \supseteq f(x) \oplus H(z^*).$$

Next, we "z^*-residuate" $f(x_0)$ on both sides of the latter inclusion. Since we can apply the equality case of Proposition 2.4 (d) with $A = D = f(x_0)$ and $B = S_{(x^*,z^*)}(x - x_0)$ on the left hand side and Proposition 2.4 (b) on the right hand side of the resulting inclusion we arrive at the condition in (b).

(c) $\Leftrightarrow$ (d): This follows with the help of Proposition 2.3 and

$$\begin{aligned} H(z^*) -_{z^*} \left(f(x) \oplus S_{(x^*,z^*)}(-x)\right) &= \left\{z \in Z \mid f(x) + S_{(x^*,z^*)}(-x) + z \subseteq H(z^*)\right\} \\ &= \left\{z \in Z \mid f(z) + z \subseteq S_{(x^*,z^*)}(x)\right\} = S_{(x^*,z^*)}(x) -_{z^*} f(x) \end{aligned}$$

which completes the proof. □

To complete the picture, we establish the relationship with the subdifferential of the scalarizations in the next proposition. Note that because we do not a priori exclude the improper case we cannot just use the known scalar results.

Using (3.6) we define

$$\partial\varphi_{f,z^*}(x_0) = \left\{x^* \in X^* \mid \forall x \in X \colon x^*(x) \leq \varphi'_{f,z^*}(x_0, x)\right\}$$

which, according to [16, Proposition 5.5], coincides with the definition given in [16, Definition 5.4]. Now, if $f \colon X \to \mathcal{G}^\vartriangle$ is convex, $x_0 \in X$ and $x^* \in X^*$, $z^* \in C^- \backslash \{0\}$, then

$$\partial_{z^*} f(x_0) = \partial\varphi_{f,z^*}(x_0). \tag{3.7}$$

Indeed, this follows from Remark 3.5 and Example 2.7.

The first main result of the paper is the following set-valued extension of the max formula.

Theorem 3.10. *Let $f \colon X \to \mathcal{G}^\vartriangle$ be a convex function, $x_0 \in \operatorname{dom} f$ and $z^* \in C^- \backslash \{0\}$ such that the function $x \mapsto f(x) \oplus H(z^*)$ is proper and the function $\varphi_{f,z^*} \colon X \to \mathrm{I\!R} \cup \{+\infty\}$ is upper semi-continuous at x_0. Then $\partial_{z^*} f(x_0) \neq \emptyset$ and it holds*

$$\forall x \in X \colon f'_{z^*}(x_0, x) = \bigcap_{x^* \in \partial_{z^*} f(x_0)} S_{(x^*,z^*)}(x). \tag{3.8}$$

Moreover, for each $x \in X$ there exists $x_0^* \in \partial_{z^*} f(x_0)$ such that

$$f'_{z^*}(x_0, x) = S_{(x_0^*,z^*)}(x). \tag{3.9}$$

Proof. We have $\operatorname{dom}\left[f(\cdot)\oplus H(z^*)\right] = \operatorname{dom}\varphi_{f,z^*}$. Since the extended real-valued convex proper function φ_{f,z^*} is continuous at x_0, $\partial\varphi_{f,z^*}(x_0) \neq \emptyset$ (see Theorem 2.4.9 in [33]),

$$\forall x \in X \colon \varphi'_{f,z^*}(x_0, x) = \sup_{x^* \in \partial\varphi_{f,z^*}(x_0)} x^*(x),$$

and for each $x \in X$ there is $x_0^* \in \partial\varphi_{f,z^*}(x_0)$ such that $\varphi'_{f,z^*}(x_0, x) = x^*(x)$.

The scalarization formulas for the directional derivative (see Remark 3.5) produce the desired result. □

Remark 3.11. The regularity assumption in the previous theorem is concerned with the scalarization φ_{f,z^*} of the set-valued function f. This might not seem appropriate. However, the assumption used seems to be the weakest possible. For example, it is implied by the following interior point condition: there is $z_0 \in Z$ such that $(x_0, z_0) \in \operatorname{int}(\operatorname{graph} f)$. For this and further details about continuity concepts for set-valued functions, compare [20], for example Proposition 3.25 and Theorem 4.4.

Finally, we shall link the subdifferential with Fenchel conjugates for set-valued functions as introduced in [14] and [30]. In [14] the function $-f^* \colon X^* \times C^-\setminus\{0\} \to \mathcal{G}^\triangle$ defined by

$$(-f^*)(x^*, z^*) = \inf_{x\in X}\left[f(x) + S_{(x^*,z^*)}(-x)\right] = \operatorname{cl}\bigcup_{x\in X}\left[f(x) + S_{(x^*,z^*)}(-x)\right]$$

for $x^* \in X^*$ and $z^* \in C^-\setminus\{0\}$ has been called the (negative) Fenchel conjugate of f, and in [30] the function $f^* \colon X^* \times C^-\setminus\{0\} \to \mathcal{G}^\triangle$ defined by

$$f^*(x^*, z^*) = \sup_{x\in X}\left[S_{(x^*,z^*)}(x) -_{z^*} f(x)\right] = \bigcap_{x\in X}\left[S_{(x^*,z^*)}(x) -_{z^*} f(x)\right]$$

for $x^* \in X^*$ and $z^* \in C^-\setminus\{0\}$ has been called the (positive) Fenchel conjugate of f.

Corollary 3.12. *For $x_0 \in \operatorname{dom} f$, $x^* \in X^*$ and $z^* \in C^-\setminus\{0\}$ with $f(x_0) \oplus H(z^*) \neq Z$ the following statements are equivalent:*
(a) $x^* \in \partial_{z^*} f(x_0)$,
(b) $-f^*(x^*, z^*) = f(x_0) \oplus S_{(x^*,z^*)}(-x_0)$,
(c) $f^*(x^*, z^*) = S_{(x^*,z^*)}(x_0) -_{z^*} f(x_0)$.

Proof. Apply Proposition 3.9 (a), (c) and (d). □

4 $\mathcal{G}^\triangle$-valued Functions and Convex Processes

A convex process F defined on X and with values in Z is a set-valued map whose graph is a convex cone, and a closed convex process is a convex process whose graph is closed, compare, for example [1, Definition 2.1.1]. That is, convex processes and sublinear functions mapping into $\mathcal{G}(Z, \{0\})$ represent the same concept. Note that the choice $C = \{0\}$ admits to include the case where no cone is available a priori.

An important concept in the theory of convex processes is the notion of the adjoint process. If $F \colon X \to \mathcal{G}(Z, \{0\})$ is a convex process, its adjoint is $F^\diamond \colon Z^* \to \mathcal{P}(X^*)$ defined by

$$F^\diamond(z^*) = \{x^* \in X^* \mid \forall (x, z) \in \operatorname{graph} F \colon x^*(x) \le z^*(z)\}$$

(see [1, Definition 2.5.1]). This is,

$$(z^*, x^*) \in \operatorname{graph} F^\diamond \quad \Leftrightarrow \quad (x^*, -z^*) \in (\operatorname{graph} F)^-.$$

The definition of $F^\diamond$ readily implies

$$x^* \in F^\diamond(-z^*) \quad \Leftrightarrow \quad \forall x \in X\colon F(x) \subseteq S_{(x^*,z^*)}(x).$$

Thus, $x^* \in F^\diamond(-z^*)$ if, and only if, $S_{(x^*,z^*)}$ is a conlinear minorant of F. The collection of linear minorants of a scalar or even vector-valued sublinear function is sometimes called the support set of the sublinear function (see, for example, [28, p. 119]). Since in our framework the functions $S_{(x^*,z^*)}\colon X \to \mathcal{G}(Z,\{0\})$ replace continuous linear operators, graph $F^\diamond$ can be identified with the support set of F. Moreover, if the sublinear function $F\colon X \to \mathcal{G}^\vartriangle$ is closed and proper, then it is the pointwise supremum of its conlinear minorants (compare [14, Proposition 14]), and this produces

$$F(x) = \bigcap_{z^* \in C^- \setminus \{0\}} \left[\bigcap_{x^* \in F^\diamond(-z^*)} S_{(x^*,z^*)}(x) \right].$$

The following proposition summarizes the situation for a convex set-valued function.

Proposition 4.1. *Let $f\colon X \to \mathcal{G}^\vartriangle$ be convex, $x_0 \in \operatorname{dom} f$ and $F(x) = f'_{z^*}(x_0,x)$ for $x \in X$. Then*

$$F^\diamond(u^*) = \begin{cases} \emptyset & : \; u^* \notin \operatorname{cone}\{-z^*\} \\ (\operatorname{cone}(\operatorname{dom} f - x_0))^- & : \; u^* = 0 \\ s\partial_{z^*} f(x_0) & : \; u^* = -sz^*, \; s > 0 \end{cases}$$

Proof. By definition of the adjoint process,

$$\begin{aligned} F^\diamond(u^*) &= \{x^* \in X^* \mid \forall (x,z) \in \operatorname{graph} F\colon x^*(x) \le u^*(z)\} \\ &= \left\{x^* \in X^* \mid \forall x \in X\colon f'_{z^*}(x_0,x) \subseteq S_{(x^*,-u^*)}(x)\right\}. \end{aligned}$$

Since $f'_{z^*}(x_0,x)$ and $S_{(x^*,-u^*)}(x)$ are both closed half spaces with normals z^* and $-u^*$, respectively, $F^\diamond(u^*) = \emptyset$ if $u^* \notin \mathbb{R}_+\{-z^*\}$. If $u^* = 0$, then $x^* \in F^\diamond(u^*)$ if, and only if,

$$\forall x \in \operatorname{dom} f'_{z^*}(x_0,\cdot) : x^*(x) \le 0.$$

Lemma 3.3 yields the second case. Finally, if $u^* = -sz^*$ for some $s > 0$, then $x^* \in F^\diamond(u^*)$ if, and only if,

$$\forall x \in X\colon f'_{z^*}(x_0,x) \subseteq S_{(x^*,sz^*)}(x) = S_{\left(\frac{1}{s}x^*,z^*\right)}(x),$$

therefore $\frac{1}{s}x^* \in \partial_{z^*} f(x_0)$. Definition 3.8 produces the result in this case. □

The above proposition shows that the directional derivative and the subdifferential of a convex $\mathcal{G}^\vartriangle$-function f are related via convex process duality. In particular,

$$\partial_{z^*} f(x_0) = \left(f'_{z^*}(x_0,\cdot)\right)^\diamond(-z^*).$$

In [25, Proposition 1.37], it is proven that the coderivative of a convex-graph multifunction $f\colon X \to \mathcal{P}(Z)$ at some point $(x_0,z_0) \in \operatorname{graph} f$ coincides with the function $\hat{D}^* f(x_0,z_0) : Z^* \to \mathcal{P}(X^*)$ given by

$$\hat{D}^* f(x_0,z_0)(-z^*) = \left\{x^* \in X^* \mid (x^*,z^*) \in (\operatorname{cone}(\operatorname{graph} f - (x_0,z_0)))^-\right\},$$

and it is pointed out 'that the coderivatives under consideration can be viewed as proper set-valued generalizations of the adjoint linear operator to the classical derivative at the point in question' ([25, p. 46]). This construction is closely related to our subdifferentials.

If $f\colon X \to \mathcal{G}^\vartriangle$ is a convex function and $(x_0, z_0) \in \operatorname{graph} f$, then the normal cone of $\operatorname{graph} f$ at $(x_0, z_0) \in \operatorname{graph} f$

$$\mathcal{N}_{\operatorname{graph} f}(x_0, z_0) = (\operatorname{cone}(\operatorname{graph} f - (x_0, z_0)))^-$$

consists exactly of those elements $(x^*, z^*) \in X^* \times C^-$ with $z^* = 0$ and $x^* \in (\operatorname{cone}(\operatorname{dom} f - x_0))^-$, or $z^* \in C^- \setminus \{0\}$, $-z^*(z_0) = \varphi_{f,z^*}(x_0)$ and $S_{(x^*,z^*)}(x_0) -_{z^*} f(x_0) = f^*(x^*, z^*)$. From Corollary 3.12 we may conclude

$$\hat{D}^* f(x_0, z_0)(-z^*) = \begin{cases} \partial_{z^*} f(x_0) & : \quad z^* \in C^- \setminus \{0\} \wedge -z^*(z_0) = \varphi_{f,z^*}(x_0) \\ (\operatorname{cone}(\operatorname{dom} f - x_0))^- & : \quad z^* = 0 \\ \emptyset & : \quad \text{otherwise} \end{cases}$$

Hence for all $z^* \in C^- \setminus \{0\}$

$$\hat{D}^* f(x_0, z_0)(-sz^*) = \begin{cases} \left(f'_{z^*}(x_0, \cdot)\right)^\diamond(-sz^*) & : \quad s \geq 0 \text{ and } -sz^*(z_0) = s\varphi_{f,z^*}(x_0) \\ \emptyset & : \quad \text{otherwise} \end{cases}$$

where the convention $0 \cdot (\pm\infty) = 0$ is essential.

Finally, it may happen that the Mordukhovich coderivative is empty whereas the z^*-subdifferential is non-empty. For example, if $f\colon \mathrm{I\!R} \to \mathcal{G}\left(\mathrm{I\!R}^2, \mathrm{I\!R}_+\right)$ given by $f(x) = \left\{(z_1, z_2) \in \mathrm{I\!R}^2 \mid z_2 \geq \frac{1}{z_1}\right\}$ for all $x \in \mathrm{I\!R}$ and $z^* = (-1, 0)$, then $\partial_{z^*} f(x) = \{0\}$, and for all $z \in f(x)$ we have $-z^*(z) > \varphi_{z^*}(x) = 0$.

The relationship between Mordukhovich's coderivative and our directional derivative/subdifferential via convex process duality may be roughly summarized as follows. At least for convex set-valued functions, the set-valued directional derivative provides the "primal concept" whose adjoint is a slight extension of Mordukhovich's coderivative.*

5 $\mathcal{G}^\vartriangle$-valued Optimization Problems

We are interested in the problem

$$\text{minimize} \quad f \quad \text{subject to} \quad x \in X \tag{P}$$

where f is a $\mathcal{G}^\vartriangle$-valued function. The minimization is understood as looking for the infimum in $\mathcal{G}^\vartriangle$, that is $\inf_{x\in X} f(x) = \operatorname{cl}\operatorname{co} \cup_{x\in X} f(x)$, and subsets of X in which this infimum is attained. This approach, initiated in [19], is different from most other approaches in set optimization, see for example [21, Definition 14.2], [17], [18] and the references therein. Note that the convex hull in the definition of the infimum above can be dropped if f is convex.

Definition 5.1. Let $f\colon X \to \mathcal{G}^\vartriangle$ be a function. A set $M \subseteq X$ is called an infimizer of f if

$$\inf_{x\in M} f(x) = \inf_{x\in X} f(x).$$

*This provides some evidence for the statement 'Primal is primal and always possible.' repeatedly given by J.-P. Penot at the Spring School on Variational Analysis, Paseky na Jizerou, 2003.

Note that dom f is always an infimizer of f which amounts to the necessity to introduce further requirements to a solution of (P). In particular, the values $f(x)$ for $x \in M$ should satisfy additional conditions, e.g. be minimal in some sense. See [19], [24] for further motivations and corresponding concepts.

Example 5.2. Let us again consider the function f given in Example 3.6 and define the two sets

$$M = \left\{x \in \mathbb{R}^2 \mid x_1 = 0\right\} \quad \text{and} \quad N = \operatorname{dom} f \backslash M.$$

Then, $\inf f = \left\{z \in \mathbb{R}^2 \mid z_1 + z_2 \geq 0\right\}$ and both M and N are infimizers. Indeed,

$$f[M] = \bigcup_{x_2 \in \mathbb{R}} \left\{z \in \mathbb{R}^2 \mid z_1 \geq x_2,\ z_2 \geq -x_2,\ z_1 + z_2 \geq 0\right\} = \left\{z \in \mathbb{R}^2 \mid z_1 + z_2 \geq 0\right\},$$

and by definition of f, $f(x) \subseteq \left\{z \in \mathbb{R}^2 \mid z_1 + z_2 \geq 0\right\}$ for all $x \in \mathbb{R}^2$. The infimizer property for N is a consequence of the closure operation which is part of the infimum in $\mathcal{G}(C)$.

In this note, we introduce another solution concept for problem (P) and provide characterizations in terms of the directional derivative and the subdifferential introduced above.

Definition 5.3. Let $f\colon X \to \mathcal{G}^\vartriangle$ be a function. A point $x_0 \in X$ is called a C^--minimizer of f if there is $z^* \in C^- \backslash \{0\}$ such that

$$f(x_0) \oplus H(z^*) = \left[\inf_{x \in X} f(x)\right] \oplus H(z^*).$$

A set $M \subseteq X$ is called a C^--solution of (P) if M is an infimizer of (P) and each $m \in M$ is a C^--minimizer of f.

The concept of C^--minimizers clearly generalizes the solution of a real-valued minimization problem as well as weak solutions in vector optimization (see [21], Theorem 5.13). Moreover, it is also clear that x_0 is a C^--minimizer of f if, and only if, it is a solution of the scalarized problem

$$\text{minimize} \quad \varphi_{f,z^*} \quad \text{subject to} \quad x \in X \tag{5.1}$$

for some $z^* \in C^- \backslash \{0\}$. We will call a C^--minimizer of f belonging to $z^* \in C^- \backslash \{0\}$ a z^*-solution of (P).

Example 5.4. We continue the discussion of Example 3.6. One may see that each $f(x)$ for $x \in \operatorname{dom} f$ is minimal with respect to $\supseteq$, i.e. $f(x) \supseteq f(y)$ implies $f(x) = f(y)$ (and even $x = y$). However, the only w-solutions for some $w \in C^- \backslash \{0\} = -\mathbb{R}^2_+ \backslash \{0\}$ with a "finite" value of f are the elements of M, i.e. $x \in \mathbb{R}^2$ with $x_1 = 0$ (see Example 5.2), and one has to choose w such that $w_1 = w_2 < 0$. In particular, this shows that there are infimizers which do not include a single w-solution, and, moreover, that there are infimizers which are solutions in the sense of [19], [24, Definition 2.8], but do not include a single w-solution. In fact, the set N (Example 5.2) is such an infimizer.

Proposition 5.5. *Let $f\colon X \to \mathcal{G}^\vartriangle$ be convex and $z^* \in C^- \backslash \{0\}$. The following statements are equivalent for $x_0 \in X$:*
(a) *x_0 is a z^*-solution of (P);*
(b) *$\operatorname{dom} f = \emptyset$, or $f(x_0) \oplus H(z^*) = Z$, or $H(z^*) \supseteq f'_{z^*}(x_0, x)$ for all $x \in X$;*
(c) *$\operatorname{dom} f = \emptyset$, or $f(x_0) \oplus H(z^*) = Z$, or $0 \in \partial_{z^*} f(x_0)$.*

Proof. Since the improper cases are trivial, let us assume $f(x_0) \oplus H(z^*) \notin \{Z, \emptyset\}$.

(a) $\Rightarrow$ (b): Since x_0 is a z^*-solution of (P) we have $f(x_0) \oplus H(z^*) = [\inf_{x \in X} f(x)] \oplus H(z^*) \neq Z$. Take $x \in X$, $t > 0$ and $z \in f(x_0 + tx) -_{z^*} f(x_0)$. Then

$$f(x_0 + tx) \oplus H(z^*) + z \subseteq f(x_0) \oplus H(z^*) + z \subseteq f(x_0 + tx) \oplus H(z^*).$$

This is only possible if either $f(x_0 + tx) \oplus H(z^*) = Z$ and hence $f(x_0) \oplus H(z^*) = Z$, or $z \in H(z^*)$. The former case is now excluded, so we may conclude (b).

(b) $\Rightarrow$ (a): (b) implies (choose $t = 1$ and use the definition of the infimum)

$$\forall x \in X \colon H(z^*) \supseteq f(x_0 + tx) -_{z^*} f(x_0),$$

hence

$$\forall y \in X \colon H(z^*) \supseteq f(y) -_{z^*} f(x_0).$$

Proposition 2.4, (f) implies

$$\forall y \in X \colon f(y) \subseteq f(x_0) \oplus H(z^*).$$

The equivalence of (b) and (c) is Proposition 3.9 for $x^* = 0$. □

In most cases, a single C^--minimizer does not characterize $\inf_{x \in X} f(x)$ too well. This is due to the obvious (well-known and sometimes annoying) fact that a fixed (linear) scalarization does not provide a good substitute for a convex vector/set optimization problem. This justifies the introduction of C^--solutions which are sets rather than single points.

The final task of the paper is to characterize infimizers in terms of directional derivatives and subdifferentials. In order to do so, we need one more new concept. Its introduction is motivated by two facts. First, an infimizer is not a singleton in general, hence one would need to determine, for instance, the subdifferential of a set-valued function "at an infimizer (set)" in order to establish a set-valued version of the condition "zero belongs to the subdifferential." This seems hard (and kind of artificial). Secondly, an infimizer does not necessarily consist of z^*-solutions, hence one cannot reduce the problem to solutions (of the family) of scalar problems as defined in (5.1). The following procedure can be understood as a way to reduce infimizers to singletons.

Definition 5.6. Let $f \colon X \to \mathcal{G}^\triangle$ be a function and $M \subseteq X$. We define the inf-translation of f by M to be the function $\hat{f}(\cdot; M) : X \to \mathcal{G}^\triangle$ given by

$$\hat{f}(x; M) = \inf_{m \in M} f(m + x) = \operatorname{cl} \operatorname{co} \bigcup_{m \in M} f(m + x). \tag{5.2}$$

The function $\hat{f}(\cdot; M)$ is nothing else than the canonical extension of f at $M + x$ as defined in [19].

Example 5.7. We compute $\hat{f}(x; M)$ for f and M as defined in Example 3.6 and 5.2. The definitions of f and M imply $\hat{f}(x; M) = \emptyset$ for $x \notin \operatorname{dom} f$. If $x \in \operatorname{dom} f$, then

$$\begin{aligned}
\hat{f}(x; M) &= \operatorname{cl} \bigcup_{m_1 = 0,\, m_2 \in \mathbb{R}} f(m + x) \\
&= \bigcup_{m_2 \in \mathbb{R}} \left\{ z \in \mathbb{R}^2 \mid z_1 \geq -x_1 + x_2 + m_2,\ z_2 \geq -x_1 - x_2 - m_2,\ z_1 + z_2 \geq x_1 \right\} \\
&= \left\{ z \in \mathbb{R}^2 \mid z_1 + z_2 \geq x_1 \right\}.
\end{aligned}$$

A few elementary properties of the inf-translation are collected in the following lemma.

Lemma 5.8. *Let $M \subseteq X$ be non-empty and $f\colon X \to \mathcal{G}^{\triangle}$ a function. Then*
(a) *if $M \subseteq N \subseteq X$ then $\hat{f}(x; M) \subseteq \hat{f}(x; N)$ for all $x \in X$;*
(b) $\inf_{x\in X} f(x) = \inf_{x\in X} \hat{f}(x; M)$*;*
(c) *if f and M are convex, so is $\hat{f}(\cdot; M) : X \to \mathcal{G}^{\triangle}$, and in this case* $\hat{f}(x; M) = \operatorname{cl} \bigcup_{m\in M} f(m + x)$.

Proof. (a) Immediate from the definition of $\hat{f}(\cdot; M)$.

(b) By definition of $\hat{f}(\cdot; M)$,

$$\inf_{x\in X} \hat{f}(x; M) = \inf_{x\in X} \inf_{m\in M} f(m + x) = \inf_{m\in M} \inf_{x\in X} f(m + x)$$
$$= \inf_{m\in M} \left[\inf_{m+x\in X} f(m + x) \right] = \inf_{m\in M} \left[\inf_{y\in X} f(y) \right] = \inf_{x\in X} f(x).$$

(c) Take $t \in (0, 1)$, $x_1, x_2 \in X$. Since M is convex, we have $M = tM + (1 - t) M$. This and the convexity of f yield

$$\begin{aligned}\hat{f}(tx_1 + (1 - t) x_2; M) &= \inf_{m\in M} f(tx_1 + (1 - t) x_2 + m) \\ &= \inf_{m_1,m_2\in M} f(t (x_1 + m_1) + (1 - t) (x_2 + m_2)) \\ &\supseteq \inf_{m_1,m_2\in M} tf(x_1 + m_1) \oplus (1 - t) f(x_2 + m_2) \\ &= t\hat{f}(x_1; M) \oplus (1 - t) \hat{f}(x_2; M).\end{aligned}$$

This completes the proof of the lemma. □

Proposition 5.9. *Let $f\colon X \to \mathcal{G}^{\triangle}$ be a convex function and $\emptyset \neq M \subseteq \operatorname{dom} f$. The following statements are equivalent:*
(a) *M is an infimizer for f;*
(b) *$\{0\} \subseteq X$ is an infimizer for $\hat{f}(\cdot; M)$;*
(c) *$\{0\}$ is an infimizer for $\hat{f}(\cdot; \operatorname{co} M)$ and $\hat{f}(0; M) = \hat{f}(0; \operatorname{co} M)$.*

Proof. The equivalence of (a) and (b) is immediate from $\hat{f}(0; M) = \inf_{m\in M} f(m)$ and Lemma 5.8 (b). The equivalence of (a) and (c) follows from $\hat{f}(0; \operatorname{co} M) = \inf_{m\in \operatorname{co} M} f(m)$ and Lemma 5.8 (b). □

It should be apparent that we need to consider $\hat{f}(\cdot; \operatorname{co} M)$: Since we want to characterize infimizers via directional derivatives and subdifferentials, a convex function is needed, and $\hat{f}(\cdot; M)$ is not convex in general even if f is convex. Obviously, an infimizer need not be a convex set; on the contrary, sometimes one prefers a nonconvex one, for example a collection of vertexes of a polyhedral set instead of a face.

Theorem 5.10. *Let $f\colon X \to \mathcal{G}^{\triangle}$ be a proper convex function with*

$$I(f) = \inf_{x\in X} f(x) \neq Z.$$

Let $\Gamma^-(f) = \{z^* \in C^- \backslash \{0\} \mid I(f) \oplus H(z^*) \neq Z\}$. The following statements are equivalent:
(a) *M is an infimizer for f;*
(b) *$\hat{f}(0; M) = \hat{f}(0; \operatorname{co} M)$ and*

$$\forall z^* \in \Gamma^-(f), \ \forall x \in X\colon H(z^*) \supseteq \hat{f}'_{z^*}(\cdot; \operatorname{co} M)(0, x);$$

(c) $\hat{f}(0; M) = \hat{f}(0; \operatorname{co} M)$ *and*

$$0 \in \bigcap_{z^* \in \Gamma^-(f)} \partial_{z^*} \hat{f}(\cdot; \operatorname{co} M)(0).$$

Proof. Since $\{0\}$ is a singleton infimizer of the function $x \mapsto \hat{f}(x; M)$, $0 \in X$ is a z^*-solution of (P) for $f(\cdot; M)$ for each $z^* \in \Gamma^-(f)$. Now, the result follows from Proposition 5.5 and Proposition 5.9. □

Theorem 5.10 highlights the use of the "z^*-wise" defined directional derivative/subdifferential. One needs to take into consideration all reasonable (= proper) scalarizations at the same time in order to characterize infimizers.

Example 5.11. We know that M from Example 5.2 is an infimizer of f defined in Example 3.6. Moreover,

$$\hat{f}(x; M) = \inf_{x \in \mathbb{R}^2} f(x) = \left\{z \in \mathbb{R}^2 \mid z_1 + z_2 \geq x_1\right\},$$

for $x \in \operatorname{dom} f$, see Example 5.7. Since $\hat{f}(x; M)$ is a closed half space with normal $(-1, -1)^T$ for all $x \in \operatorname{dom} f$ we obtain

$$\left(\hat{f}(\cdot; M)\right)'_w (0, y) = \begin{cases} \emptyset & : \quad y_1 < 0 \\ \left\{z \in Z \mid w^T z \leq w_1 y_1\right\} & : \quad w_1 = w_2,\ y_1 \geq 0 \\ Z & : \quad w_1 \neq w_2,\ y_1 \geq 0 \end{cases}$$

This relationship illustrates (b) in Theorem 5.10 since for $w \in C^- \setminus \{0\} = -\mathbb{R}^2_+ \setminus \{0\}$ with $w_1 = w_2$ we have $w_1 y_1 \leq 0$ and so $\left\{z \in Z \mid w^T z \leq w_1 y_1\right\} \subseteq H(w)$ whenever $y_1 \geq 0$. Obviously, for $w \in -\mathbb{R}^2_+ \setminus \{0\}$

$$\inf_{x \in X} f(x) \oplus H(w) \neq \mathbb{R}^2 \quad \Leftrightarrow \quad w_1 = w_2$$

since $\inf_{x \in X} f(x) = \{z \in Z \mid z_1 + z_2 \geq 0\}$ as shown in Example 5.2.

Acknowledgement

The function f for the "running example" discussed in Example 3.6, 5.2, 5.4, 5.7 and 5.11 has been constructed by Frank Heyde. Thank you, Frank. Moreover, we highly appreciated the comments and hints of two anonymous reviewers.

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Manuscript received 21 June 2013
revised 26 September 2013
accepted for publication 22 October 2013

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