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MINKOWSKI–RÅDSTRÖM–HÖRMANDER CONE*

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Abstract: In this paper we introduce and investigate the notion of the generalized Minkowski–Rådström–Hörmander cone giving a special attention to the order law of cancellation. In particular we embed the convex cone $\mathcal{C}(X)$ of nonempty closed convex subsets of real Hausdorff topological vector space and the convex cone $\mathcal{W}(\mathbb{R}^n)$ of nonempty convex subsets of $\mathbb{R}^n$ into respective Minkowski–Rådström–Hörmander cones. We also prove the existence of minimal representation of elements of the MRH cone $\mathcal{C}(X)\times\mathcal{K}(X)/_\sim$ ($\mathcal{K}(X)$ is the family of all compact convex subsets of X) and present reduction techniques for elements of the MRH cone $\mathcal{C}(X)\times\mathcal{B}(X)/_\sim$ ($\mathcal{B}(X)$ is the family of all bounded closed convex subsets of X).

Key words: *Minkowski–Rådström–Hörmander cone, closed convex sets*

Mathematics Subject Classification: *46B20, 52A05, 54B20*

1 Introduction

Let $\mathcal{C}(X)$ be the family of all nonemty closed convex subsets of real Hausdorff topological vector space X and $\mathcal{B}(X)$ be the subfamily of all bounded sets in $\mathcal{C}(X)$. From algebraic point of view the properties of the abstract convex cone $\mathcal{B}(X)$ and of its extension to a vector space (the MRH space) play a very important role in the study of exhausters and quasidifferentials (see Appendix). In particular, the order law of cancellation is of great importance.

This paper aims at clarifying algebraic properties of semigroups $\mathcal{C}(X)$ and $\mathcal{B}(X)$ which enable us to embed $\mathcal{C}(X)$ into the generalized MRH cone $\mathcal{C}(X)\times\mathcal{B}(X)/_\sim$. In Section 2 we formulate the law of cancellation in a commutative semigroup S with respect to a subsemigroup T. In Section 3 we carefully investigate relations among a number of properties of ordered semigroup like order law of cancellation, translation of supremum, existence of abstract difference of two elements, which is a generalization of Minkowski difference. In Section 4 Theorem 4.2 presents embedding of a semigroup (abstract convex cone) S satisfying the law of cancellation with respect to a subsemigroup (convex subcone) T into the semigroup $\widetilde{S} = S\times T/_\sim$, which is a generalization of the MRH space. Theorem 4.5 gives formulas for the supremum and infimum of subsets of $\widetilde{S}$ in terms of elements of S.

In section 5 new and very short proofs of the law of cancellation in $\mathcal{C}(X)$ with respect to $\mathcal{B}(X)$ (compare [16], [20]) and in $\mathcal{W}(\mathbb{R}^n)$ (family of all nonempty convex subsets of $\mathbb{R}^n$)

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with respect to $\mathcal{B}(\mathbb{R}^n)$ are given. At the 1026th AMS Meeting in Hoboken, 2007, V. Soltan informed one of the authors that the later law holds true. Theorem 5.7 gives formulas for the supremum an infimum of subsets of $\widetilde{X}$ =MRH(X). In Section 6 we prove the existence of a minimal representation of elements of $\mathcal{C}(X) \times \mathcal{K}(X)/_\sim$, where $\mathcal{K}(X)$ is the family of all nonempty compact convex subsets of X. In Section 7 convex pairs (A, B) of closed convex sets (i.e. such that $A \cup B$ is convex) are studied. A criterion for the quotient cone $\mathcal{S} \times \mathcal{B}(X)/_\sim$ being a lattice is given. In Section 8 we present reduction techniques for elements of MRH$(X) = \mathcal{C}(X) \times \mathcal{B}(X)/_\sim$.

2 Commutative Semigroups. Law of Cancellation.

Let $S = (S, +)$ be a commutative semigroup. Assume that $T \subset S$ is a subsemigroup of S. We say that S satisfies the *cancellation law with respect to* T if for all $b \in T, a, c \in S$ we have

$$a + b = c + b \text{ implies } a = c. \tag{lc$T$}$$

Notice that if S satisfies the cancellation law with respect to finite subsemigroup T then T is a subgroup.

Example 2.1. Let $\mathbb{Z}_2$ be an additive group containing two elements 0 and 1. Let $\overline{\mathbb{Z}}_2$ contain 0, 1, ∞ and be a comutative semigroup with $a + \infty = \infty$ for all $a \in \overline{\mathbb{Z}}_2$. Then $\overline{\mathbb{Z}}_2$ satisfies the cancellation law with respect to $\mathbb{Z}_2$ (lc$\mathbb{Z}_2$).

Semigroup S satisfies the *cancellation law* (lc) if it satisfies the cancellation law with respect to S.

Theorem 2.2. *Let G be a subgroup of a commutative semigroup S. Then semigroup S satisfies the cancellation law with respect to subgroup G (lc$G$) if and only if zero 0_G of the subgroup G is a zero of the semigroup S.*

Proof. Let S satisfy (lc$G$). Then $a + 0_G = a + 0_G + 0_G$ and by (lc$G$) we get $a = a + 0_G$. Conversely if 0_G is a zero of the semigroup S and $a+b = b+c$ with $b \in G$, then $a = a+0_G = 0_G + c = c$. □

3 Ordered Commutative Semigroups

Let $S = (S, +, \leq)$ be a commutative ordered semigroup. By definition of an ordered semigroup the addition in S is *isotonic*

$$a \leq b \text{ implies } a + x \leq b + x \text{ for all } x \in S. \tag{i}$$

For $a, b \in S$ let $a \vee b = \sup\{a, b\}$, $a \wedge b = \inf\{a, b\}$, if they exist. Semigroup S satisfies property (v) if

$$\text{for any } a, b \in S \text{ there exists } a \vee b \tag{v}$$

and the *order law of cancellation with respect to* $T \subset S$ if for all $b \in T, a, c \in S$ we have

$$a + b \leq c + b \text{ implies } a \leq c. \tag{olc$T$}$$

Let G be a subset of the semigroup S. We say that S satisfies the property of *translation of supremum with respect to* G if for all $x \in G$ and any $\{a_i\}_{i\in I} \subset S$ such that $\sup_{i\in I} a_i$ exists in S we have

$$\sup_{i\in I}(x + a_i) = x + \sup_{i\in I} a_i. \qquad \text{(tsup}G\text{)}$$

We say that S satisfies the *law of distribution of supremum with respect to adding elements of* G if for all $x \in G$ and any $a, b \in S$ such that $a \vee b$ exists we have

$$x + a \vee b = (x + a) \vee (x + b). \qquad \text{(tv}G\text{)}$$

We write (tsup) instead of (tsup$S$) and (tv) instead of (tv$S$). Obviously (tsup$G$) implies (tv$G$).

Proposition 3.1. (lc$T$ implies olc$T$). *The cancellation law with respect to* T (lc$T$) *and the distributive law* (tv$T$) *imply the order law of cancellation with respect to* T (olc$T$).

Proof. Assume $a + b \le c + b$. By (tv$T$), $a \vee c + b = (a + b) \vee (c + b) = c + b$. By (lc$T$), $a \vee c = c$. Hence $a \le c$. □

Obviously, the order law of cancellation implies the cancellation law in S with respect to elements of T.

According to Example 1.3 in [7] for a commutative ordered semigroup the conditions (olc) and (v) do not imply the distribution law (tv).

Example 3.2. Let $(\mathbb{Z}_+, +)$ be a commutative semigroup of non-negative integer numbers. Let $a \preceq b$ if and only if $a \le b$ and $(a, b) \ne (0, 1)$. Then $(\mathbb{Z}_+, +, \preceq)$ is a commutative ordered semigroup with zero satisfying the cancellation law and the property (v). However, since $0 + s \preceq 1 + s$ for all $s \ge 1$ and $0 \not\preceq 1$, the semigroup $(\mathbb{Z}_+, +, \preceq)$ satisfies the order law of cancellation (olc$T$) with respect to no subsemigroup T other than trivial subgroup $\{0\}$. By Proposition 3.1 the semigroup $(\mathbb{Z}_+, +, \preceq)$ does not satisfy the distributive law (tv). For example $0 \vee 1 + 1 = 2 + 1 = 3 \ne 2 = 1 \vee 2 = (0 + 1) \vee (1 + 1)$.

An abstract difference has been recently introduced in [13]. Let $a, b \in S$ be two elements of a commutative ordered semigroup S. Then *abstract difference* of a and b is the greatest element (if it exists) of the set $\mathcal{D}(a, b) = \{x \mid x + b \le a\}$ and it is denoted by $a \dot{-} b$.

Proposition 3.3. *Let* S *be a commutative ordered semigroup. Then:*

(d1) *If* $a \dot{-} b$ *exists then* $(a \dot{-} b) + b \le a$.

(d2) *If* $0 \in S$ *and* $a \dot{-} a$ *exists then* $0 \le a \dot{-} a$.

(d3) *If* $0 \in S$, $a \le b$ *and for some* $c \in S$ *abstract differences* $a \dot{-} c$ *and* $b \dot{-} c$ *exist then* $a \dot{-} c \le b \dot{-} c$.

Let S *additionally satisfy the order law of cancellation with respect to subsemigroup* T. *Then:*

(d4) *If* $a = b + c$ *and* $b \in T$ *then* $c = a \dot{-} b$.

(d5) *If* $0 \in S$ *then for every* $a \in T$ *we have* $a \dot{-} a = 0$.

(d6) *If* $a \dot{-} c$ *exists then for every* $b \in T$, $(a+b) \dot{-} (c+b) = a \dot{-} c$.

The proof of Proposition 3.3 is based on definitions and is omitted.

Let $G \subset S$. Now we introduce the following conditions:

$$\text{If } \mathcal{D}(a,b) \neq \emptyset \text{ and } b \in G, \quad \text{then } \sup \mathcal{D}(a,b) \text{ exists.} \tag{s*G}$$

$$\text{If } \mathcal{D}(a,b) \neq \emptyset \text{ and } b \in G, \quad \text{then } a \dot{-} b \text{ exists.} \tag{sG}$$

In the later case obviously $a \dot{-} b = \sup \mathcal{D}(a,b) \in \mathcal{D}(a,b)$. Instead of (s*$S$) and (s$S$) we write (s*) and (s).

Lemma 3.4. *Let G be a subgroup of a commutative ordered semigroup $S = (S, +, \leq)$. If $a \in S, b \in G$ then:*

(a) $\mathcal{D}(a,b) \subset \mathcal{D}(a-b, 0_G) = \mathcal{D}(a+0_G, b)$.

(b) $a - b \in \mathcal{D}(a-b, 0_G)$.

Proof. (a) If $x+b \leq a$ then $x+0_G \leq a-b$. Now let $y+0_G \leq a-b$. Then $y+b = y+0_G+b \leq a+0_G$. From $z+b \leq a+0_G$ it follows $z+0_G \leq a+0_G-b = a-b$.
(b) By the equality $a-b+0_G = a-b$, we have $a-b \in \mathcal{D}(a-b, 0_G)$. □

Proposition 3.5. *Let G be a subgroup of a commutative ordered semigroup $S = (S, +, \leq)$. Then for all $a \in S$ the following conditions are equivalent:*

(a) $\mathcal{D}(a+0_G, b) \subset \mathcal{D}(a,b)$, *for all* $b \in G$.

(b) $\mathcal{D}(a+0_G, b) = \mathcal{D}(a,b)$, *for all* $b \in G$.

(c) $a + 0_G \leq a$.

Proof. (a)⇒(c) Let $\mathcal{D}(a+0_G, b) \subset \mathcal{D}(a,b)$ for all $b \in G$. In particular, $\mathcal{D}(a+0_G, 0_G) \subset \mathcal{D}(a, 0_G)$. From the equality $a+0_G+0_G = a+0_G$ we have $a+0_G \in \mathcal{D}(a+0_G, 0_G)$. Hence $a+0_G \leq a$.
(c)⇒(a) From $x+b \leq a+0_G \leq a$ we get $x \in \mathcal{D}(a,b)$.
(a)⇒(b) By Lemma 3.4 we have $\mathcal{D}(a,b) \subset \mathcal{D}(a+0_G, b)$. □

Proposition 3.6. *Let G be a subgroup of a commutative ordered semigroup $S = (S, +, \leq)$ and $b \in G$. If $a \leq a + 0_G$ for every $a \in S$ and $a \dot{-} b$ exists then $a \dot{-} b \leq a - b$.*

Proof. Let $x = a \dot{-} b$. Then $b + x \leq a$ and $x \leq x + 0_G \leq a + (-b)$. □

Example 3.7. Let $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{\infty\}$ and $S = (\overline{\mathbb{Z}}, +, \leq)$ be a commutative ordered semigroup. Then $G = \{\infty\}$ is a trivial subgroup of S and ∞ is a zero of subgroup G but not a zero of S. Moreover S does not satisfy the cancellation law with respect to G but $a \leq a + \infty$ for every $a \in S$.

Theorem 3.8. *Let G be a subgroup of a commutative ordered semigroup $S = (S, +, \leq)$. Then the following statements are equivalent:*

(a) *Zero* 0_G *of the subgroup* G *is a zero of the semigroup* S.

(b) *We have* $a \dot{-} b = a - b$ *for every* $a \in S, b \in G$.

(c) *We have* $a \dot{-} 0_G = a + 0_G$ *for every* $a \in S$.

Proof. (a) $\Rightarrow$ (b) From $x + b \le a$ we have $x = x + 0_G \le a - b$. Now by Proposition 3.5 we get $a \dot{-} b = a - b$.

(c) $\Rightarrow$ (a) Let $a \dot{-} 0_G = a + 0_G$. Then $a + 0_G = a + 0_G + 0_G \le a$ and we have $a \in \mathcal{D}(a, 0_G)$. Therefore $a \le a \dot{-} 0_G = a + 0_G$ and $a + 0_G = a$. □

Example 3.9. Let $S = \mathbb{Z} \times \mathbb{Z}_2$, $(m, s) + (n, t) = (m + n, st)$, $(m, s) \le (n, t)$ if $m < n$ and $(m, s) \le (m, t)$ if $s \ge t$. Then $(S, +, \le)$ is a commutative ordered semigroup, $G = \mathbb{Z} \times \{0\}$ is a subgroup of S and $0_G = (0, 0) \ne 0_S = (0, 1)$. Notice that $(m, 1) \dot{-} (n, 0) = (m - n - 1, 0) < (m - n, 0) = (m, 1) + (-n, 0) = (m, 1) - (n, 0)$. Semigroup S satisfies the condition (s) and the assumption of Proposition 3.5 (c). However, subgroup G does not satisfy conditions of Theorem 3.8.

The following proposition connects different conditions that were introduced earlier.

Proposition 3.10. *Let* $S = (S, +, \le)$ *be a commutative ordered semigroup,* G *be a subset of* S *and* $\{a_i\}_{i \in I} \subset S$. *Then:*

(a) *If semigroup* S *satisfies the condition* (v) *then* S *satisfies the condition* (tv$G$) *if and only if for every finite the set* I *and* $x \in G$ *we have*

$$\sup_{i \in I}(a_i + x) = \sup_{i \in I} a_i + x.$$

(b) *Semigroup* S *satisfies the condition* (sG) *with respect to a subset* G *if and only if* S *satisfies the conditions* (s*G) *and* (tsup$G$).

(c) *If a subset* G *is a subgroup of semigroup* S *and* 0_G *is a zero of* S *then* S*satisfies the conditions* (tsup$G$).

Proof. (a) It is obvious from the condition (tv$G$). (b) Let $a = \sup_{i \in I} a_i$. Since $a_i \le a$ for $i \in I$, $a_i + x \le a + x$. Now let $a_i + x \le b$, for $i \in I$ then by the condition (sG) we have $a_i \le b \dot{-} x$, which implies $a \le b \dot{-} x$. Hence $a + x \le (b \dot{-} x) + x \le b$ and $\sup_{i \in I}(a_i + x) = a + x$.

Conversely, if $\mathcal{D}(a, b) \ne \emptyset$, $b \in G$ then $\sup \mathcal{D}(a, b) \in S$. For $x \in \mathcal{D}(a, b)$ we have $x + b \le a$. Now by (tsup$G$) we get $\sup(\mathcal{D}(a, b) + b) = \sup \mathcal{D}(a, b) + b \le a$. Hence $\sup \mathcal{D}(a, b) \in \mathcal{D}(a, b)$.

Statement (c) follows from (b) and Theorem 3.8. □

In view of Proposition 3.10 (b) properties (v) and (s) imply (tv).

Example 3.11. Let $\underline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty\}$ and $S = (\underline{\mathbb{Z}}, +, \le)$ be a commutative ordered semigroup. Then $G = \{-\infty\}$ is a trivial subgroup of S and $-\infty$ is a zero of subgroup G but not a zero of S. Moreover for $\{a_i\}_{i \in I} \subset S$, $\sup_{i \in I} a_i \in S$ if and only if a set $\{a_i\}_{i \in I}$ is bounded. If $\sup_{i \in I} a_i \in S$ exists then $-\infty + \sup_{i \in I} a_i = \sup_{i \in I}(-\infty + a_i) = -\infty$. Hence S satisfies the condition (tsup$G$). Semigroup S also satisfies the condition (v) but it does not satisfy the condition (s*G).

Semigroup $\mathbb{Z}_+$ from Example 3.2 satisfies conditions (v) and (s*G) but it does not satisfy the condition (tv$G$) for a subset $G = \{1\}$.

Example 3.12. Let $X = \mathbb{R}^2$ and S be a family of all closed balls in Euclidean norm. The family S with Minkowski addition and inclusion is an ordered semigroup. In S the property (s) and order law of cancellation (olc) are satisfied. However, if a ball A is not contained in a ball B, and $B \not\subset A$ then supremum $A \vee B$ does not exist. In Figure 2.1 no ball containing balls M and N contains balls A and B.

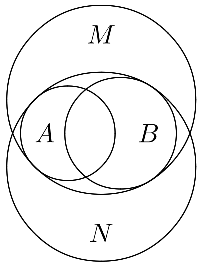

Figure 2.1

A commutative partially ordered semigroup S with zero which satisfies the conditions (olc$T$), (v) and (tv$T$) is called *q-semigroup with respect to T*. If $S = T$ then semigroup S is called *q-semigroup.*

A commutative partially ordered q-semigroup S with respect to T which satisfies condition (s), is called a *$\dot{q}$-semigroup with respect to T*. If $S = T$ then semigroup S is called *$\dot{q}$-semigroup.*

For a general representation theorem of q-semigroups we refer also to the paper of H. Ratschek and G. Schröder [17].

4 Embedding of A Commutative Semigroup

Let $S = (S, +)$ be a commutative semigroup satisfying the law of cancellation (lc$T$) with respect to its subsemigroup T. We define the relation of equivalence "$\sim$" in $S \times T$ by

$$(a, b) \sim (c, d) \ \text{ if and only if } \ a + d = b + c.$$

For $(a, b) \in S \times T$ let us denote

$$[a, b] = \{(c, d) \in S \times T |\, (c, d) \sim (a, b)\},$$

and $\widetilde{S} = S \times T/_{\sim}$. For $a \in S$ let $j(a) = [a + b, b]$, $b \in T$.

We define an addition in $\widetilde{S}$ by

$$[a, b] + [c, d] = [a + c, b + d].$$

We say that a commutative semigroup S is an *abstract convex cone* if there is a given mapping $(t,a)\mapsto ta$ of $\mathbb{R}_+\times S$ into S such that

$$1a=a,\quad t(sa)=(ts)a,\quad t(a+b)=ta+tb,\quad (t+s)a=ta+sa$$

for all $a,b\in S$ and $t,s\in\mathbb{R}_+=[0,\infty)$.

We say that an abstract convex cone $S=(S,+,\cdot)$ satisfies the *law of cancellation* (lc$T$) *with respect to its abstract convex subcone* T if S as a commutative semigroup satisfies the law of cancellation (lc) with respect to T. Since $0b=0b+0b$, $b\in T$, we have $a+0b=a+0b+0b$, $a\in S$ and by the law of cancellation with respect to T, $a=a+0b$. Then $0b$, $b\in T$ is a neutral element in S with respect to the addition.

We introduce multiplication in $\widetilde{S}$ as a mapping $(t,[a,b])\mapsto[ta,tb]$ of $\mathbb{R}_+\times\widetilde{S}$ into $\widetilde{S}$.

Example 4.1. (a) Let $S=\{a,b\}$ be a commutative group where the addition is defined by the following equalities: $a+a=b+b=b$, $a+b=a$. In fact S is an additive group of integers modulo 2. Let multiplication by scalars be defined by equalities: $ta=tb=b$, $t\in\mathbb{R}_+$. In S last three conditions for abstract convex cone are fulfilled. However, $1a=b\neq a$.

(b) Let $\overline{\mathbb{R}}=\mathbb{R}\cup\{\infty\}$ be a commutative group where $a+\infty=\infty$ for $a\in\overline{\mathbb{R}}$. Let multiplication by non-negative scalars satisfy $0\cdot\infty=0$ and $t\cdot\infty=\infty$ for $t>0$. Then $\overline{\mathbb{R}}$ is an abstract convex cone satisfying the law of cancellation with respect to $\mathbb{R}$.

The following theorem follows immediately.

Theorem 4.2. ([12], [20]) (a) *If S is a commutative semigroup satisfying the law of cancellation* (lc$T$) *with respect to its subsemigroup T, then the pair $(\widetilde{S},+)$ is a commutative semigroup with neutral element $\widetilde{0}=[b,b], b\in T$, the subset $\widetilde{T}=T^2/_\sim$ is a subgroup of $\widetilde{S}$ with $-[a,b]=[b,a]$ and j defines an isomorphic embedding of S in $\widetilde{S}$; $\widetilde{T}=j(T)-j(T)$. Furthermore, if G is a commutative group such that T is embedded in G in an isomorphic way then $\widetilde{T}$ is isomorphic to some subgroup of G.*

(b) *If, additionally, S is an abstract convex cone and T is an abstract convex subcone, then the triplet $(\widetilde{S},+,\cdot)$ is an abstract convex cone, the subset $\widetilde{T}=T^2/_\sim$ is both a subgroup and an abstract convex subcone of $\widetilde{S}$ and j defines an isomorphic embedding of cone S in $\widetilde{S}$. Furthermore, if V is a vector space such that T is embedded in V in an isomorphic way then $\widetilde{T}$ with multiplication extended to negative scalars t by $t[a,b]=[(-t)b,(-t)a]$ is isomorphic to some subspace of V.*

Let S be a commutative ordered semigroup satisfying the order law of cancellation (olc$T$) with respect to its subsemigroup T. We introduce an ordering in $\widetilde{S}$ in the following way:

$$[a,b]\le[c,d]\ \text{ if and only if }\ a+d\le b+c.$$

Notice, that $[a,b]\le[c,d]\le[e,f]$ imply the following inequalities: $a+d+f\le c+b+f=c+f+b\le e+d+b$. Cancelling $d\in T$, we obtain $[a,b]\le[e,f]$. It is easy to observe that the definition does not depend on the choice of representatives.

Immediately by Proposition 3.10 (c) we have the following proposition.

Proposition 4.3. *Let S be a commutative ordered semigroup satisfying the order law of cancellation* (olc$T$) *with respect to its subsemigroup T. Then $\tilde{S}$ satisfies the condition* (tsup$\tilde{T}$), *i.e. if $(\tilde{x}_i)_{i\in I} \subset \tilde{S}$ and $\sup_{i\in I} \tilde{x}_i \in \tilde{S}$, then for every $\tilde{x} \in \tilde{T}$,*

$$\sup_{i\in I}(\tilde{x}_i + \tilde{x}) = \sup_{i\in I} \tilde{x}_i + \tilde{x}.$$

Lemma 4.4. *Let $S = (S, +, \le)$ be a commutative ordered semigroup satisfying the order law of cancellation* (olc$T$) *with respect to its subsemigroup T and $(c_i)_{i\in I} \subset S$, $\sup_{i\in I} c_i \in S$. If*

(i) *S satisfies the condition* (sT) *or*

(ii) *S satisfies the conditions* (v) *and* (tv$T$) *and the set of indices I is finite*

then for $\tilde{x}_i = [c_i + t, t], i \in I, t \in T$

$$\sup_{i\in I} \tilde{x}_i = \left[\sup_{i\in I} c_i + t, t\right].$$

Proof. Let $a = \sup_{i\in I} c_i$, $\tilde{x} = [a+t, t]$. Since $c_i \le a$ for $i \in I$, $\tilde{x}_i \le \tilde{x}$. Now let $\tilde{x}_i \le \tilde{y} = [c, d]$, then $c_i + d \le c$ for $i \in I$ and by Proposition 3.9 we have $\sup_{i\in I}(c_i + d) = a + d \le c$, which imply $\tilde{x} \le \tilde{y}$. Hence $\tilde{x} = \sup_{i\in I} \tilde{x}_i$. □

It is easy to observe that for $\tilde{x} \in \tilde{T}, \tilde{y} \in \tilde{S}$ the equality $\tilde{y} \dot{-} \tilde{x} = \tilde{y} - \tilde{x}$ is satisfied.

Theorem 4.5. *Let $S = (S, +, \le)$ be a commutative ordered semigroup satisfying the order law of cancellation* (olc$T$) *with respect to its subsemigroup T and $(\tilde{x}_i)_{i\in I} \subset \tilde{S}$, $\tilde{x}_i = [c_i + c, d], i \in I, [c, d] \in \tilde{T}$, $\sup_{i\in I} c_i \in S$. If*

(i) *S satisfies the condition* (sT) *or*

(ii) *S satisfies the conditions* (v) *and* (tv$T$) *and the set of indices I is finite*

then

$$\sup_{i\in I} \tilde{x}_i = [\sup_{i\in I} c_i + c, d].$$

If, additionally, $\sup_{i\in I} c_i \in T$ *then*

$$\inf_{i\in I}(-\tilde{x}_i) = [d, \sup_{i\in I} c_i + c].$$

Proof. Denote $\tilde{y}_i = [c_i + t, t]$ and $\tilde{x} = [c, d]$. Then by Proposition 4.3 and Lemma 4.4, we obtain

$$\sup_{i\in I} \tilde{x}_i = \sup_{i\in I}(\tilde{y}_i + \tilde{x}) = \sup_{i\in I} \tilde{y}_i + \tilde{x} = [\sup_{i\in I} c_i + c, d].$$

From the equations $\inf_{i\in I} \tilde{x}_i = -\sup_{i\in I} -\tilde{x}_i$, we obtain the second formula. □

Corollary 4.6. *Let $S = (S, +, \leq)$ be a commutative ordered semigroup satisfying the order law of cancellation* (olc$T$) *with respect to its subsemigroup T, S satisfy the conditions* (v) *and* (tv$T$) *and the set of indices I be finite. If $([a_i, b_i])_{i\in I} \subset \tilde{S}$ then*

$$\sup_{i\in I}[a_i, b_i] = \left[\bigvee_{i\in I}\left(a_i + \sum_{k\in I\setminus\{i\}} b_k\right), \sum_{i\in I} b_i\right].$$

If $([a_i, b_i])_{i\in I} \subset \widetilde{T}$ then

$$\inf_{i\in I}[a_i, b_i] = \left[\sum_{i\in I} a_i, \bigvee_{i\in I}\left(b_i + \sum_{k\in I\setminus\{i\}} a_k\right)\right].$$

Proof. We observe that $\tilde{x}_i = [a_i, b_i] = [a_i + \sum_{k\in I\setminus\{i\}} b_k, \sum_{i\in I} b_i]$. Then by Theorem 4.5 we obtain the first formula.

Since $-\tilde{x}_i = [b_i, a_i] = [b_i + \sum_{k\in I\setminus\{i\}} a_k, \sum_{i\in I} a_i]$. From the equations $\inf_{i\in I} \tilde{x}_i = -\sup_{i\in I} -\tilde{x}_i$, we obtain the second formula. □

Corollary 4.7. *Let $S = (S, +, \leq)$ be a commutative ordered semigroup satisfying the order law of cancellation* (olc$T$) *with respect to its subsemigroup T and S satisfy the conditions* (v) *and* (tv$T$)*. Then $\widetilde{S} = (\widetilde{S}, +, \leq)$ is a lattice such that for $\tilde{x} = [a, b], \tilde{y} = [c, d] \in \widetilde{S}$ we have*

$$\tilde{x} \vee \tilde{y} = \sup\{\tilde{x}, \tilde{y}\} = [(a + d) \vee (b + c), b + d].$$

If $\tilde{x}, \tilde{y} \in \widetilde{T}$ then

$$\tilde{x} \wedge \tilde{y} = \inf\{\tilde{x}, \tilde{y}\} = [a + b, (a + d) \vee (b + c)].$$

5 Applications to the Minkowski–Rådström–Hörmander Cone

Let $X = (X, \tau)$ be a Hausdorff topological vector space over the field $\mathbb{R}$ and $\mathcal{W}(X) \supset \mathcal{C}(X) \supset \mathcal{B}(X) \supset \mathcal{K}(X)$ be the families of all nonempty and, respectively, convex, closed convex, bounded closed convex and compact convex subsets of X.

For $A, B, C, D \subset X$ we have
$A + B = \{a + b \mid a \in A, b \in B\}$, $A \dot{+} B = \overline{A + B}$,
$A \mathring{\vee} B = \operatorname{conv}(A \cup B)$, $A \vee B = \overline{A \mathring{\vee} B}$, $A \dot{-} B = \{x \in X \mid x + B \subset A\}$.

The following proposition is well known [20]. We provide a short proof of it.

Proposition 5.1. *If A, B and C are subsets of X, B is nonempty bounded and C is closed and convex, then*

$$A + B \subset B \dot{+} C \implies A \subset C.$$

Proof. Let $b \in B$. The inclusion $A + B \subset B \dot{+} C$ is equivalent to the inclusion $A + (B - b) \subset (B - b) \dot{+} C$. Further we may assume that $0 \in B$. Let $\mathcal{U}$ be a base of neighbourhoods of 0 in

X. Let $U \in \mathcal{U}$ and $V \in \mathcal{U}$ such that $V + V \subset U$. From $A + B \subset B \dot{+} C$ and by the convexity of C it follows that for every $n \in \mathbb{N}$ we have $nA + B \subset B \dot{+} nC$. Hence

$$A \subset A + \frac{1}{n}B \subset \frac{1}{n}B \dot{+} C \text{ for any } n \in \mathbb{N}.$$

By boundedness of B for sufficiently large n, we have $A \subset V \dot{+} C \subset V + C + V \subset C + U$. Thus $A \subset C + U$ for all $U \in \mathcal{U}$. Then $A \subset C$. □

The following proposition holds true even if C is not a closed set.

Proposition 5.2. *A, B and C are subsets of $\mathbb{R}^n$, B is nonempty compact and C is convex, then*

$$A + B \subset B + C \implies A \subset C.$$

Proof. Suppose that $A + B \subset B + C$ and $a \in A \setminus C$. Then $0 \notin C - a$. Since $C - a$ is convex, there exists a system of coordinates in $\mathbb{R}^n$ such that $x \prec 0$ for all $x \in C - a$, where '$\prec$' is the lexicographic ordering in $\mathbb{R}^n$. Since B is compact, the ordering attains its supremum at some point $b \in B$. Then by $A + B \subset B + C$ we have $B \subset B + (C - a)$, while $y + x \precneqq \sup B + 0 = b \in B$ for all $y \in B$ and $x \in C - a$. □

Remark 5.3. Proposition 5.1 is equivalent to the following equation $(B \dot{+} C) \dot{-} B = C$, for all $C \in \mathcal{C}(X)$ and bounded $B \subset X$. In a similar way, Proposition 5.2 is equivalent to the equation $(B + C) \dot{-} B = C$, for all $C \in \mathcal{W}(\mathbb{R}^n)$ and compact $B \subset \mathbb{R}^n$.

Proposition 5.2 does not hold true in the case of $\dim X = \infty$.

Example 5.4. Let $X = l^1$ with $*-$weak topology generated by c_0. Then the unit ball B of the space l^1 is compact in the $*-$weak topology. There exists a functional f on l^1 such that $\|f\|_1 = 1$ and f does not attain supremum on B. Now we consider a convex set $C = \{x \in l^1 \mid f(x) < 0\}$. Then $B + C = \{x \in l^1 \mid f(x) < 1\}$, $B \subset B + C$ but $0 \notin C$. We obtain similar example in Hilbert space l^2 by embedding the sets B and C with help of operator $F : l^1 \longrightarrow l^2$ defined by $F((a_n)_{n\in\mathbb{N}}) = (2^{-n}a_n)_{n\in\mathbb{N}}$.

Corollary 5.5. *The order law of cancellation* (olc$T$ from section 3) *holds true in an ordered cone* $(\mathcal{C}(X), \dot{+}, \subset)$ *for* $T = \mathcal{B}(X)$ *and in the cone* $(\mathcal{W}(\mathbb{R}^n), +, \subset)$ *for* $T = \mathcal{K}(\mathbb{R}^n)$.

For abstract convex cones $(\mathcal{C}(X), \dot{+}, \cdot)$ and $(\mathcal{W}(\mathbb{R}^n), +, \cdot)$ let us denote the following quotient sets:

$$\widetilde{X}_{cb} = \mathcal{C}(X) \times \mathcal{B}(X)/_\sim, \widetilde{X}_{ck} = \mathcal{C}(X) \times \mathcal{K}(X)/_\sim$$
$$\text{and } \widetilde{\mathbb{R}^n}_{wk} = \mathcal{W}(\mathbb{R}^n) \times \mathcal{K}(\mathbb{R}^n)/_\sim.$$

For abstract convex cones $(\mathcal{B}(X), \dot{+}, \cdot)$ and $(\mathcal{K}(X), +, \cdot)$ let us denote the following quotient sets:

$$\widetilde{X}_b = \mathcal{B}^2(X)/_\sim, \widetilde{X}_k = \mathcal{K}^2(X)/_\sim.$$

Last two quotient sets are vector spaces called Minkowski–Rådström–Hörmander spaces (MRH) which were studied in a number of papers (e.g. [11], [16] and [20]).

By Theorem 4.2, Propositions 5.1 and 5.2 the following theorem holds true.

Theorem 5.6. *The triplets* $(\widetilde{X}_{cb}, +, \cdot), (\widetilde{X}_{ck}, +, \cdot)$ *and* $(\widetilde{\mathbb{R}^n}_{wk}, +, \cdot)$ *are abstract convex cones having, respectively,* $\widetilde{X}_b, \widetilde{X}_k$ *and* $\widetilde{\mathbb{R}^n}_k$ *as a vector subspace with multiplication defined for negative numbers by* $t[A, B] = [(-t)B, (-t)A]$.

By Proposition 4.3, 4.5 and Corollary 4.6 the following theorem holds true.

Theorem 5.7. *Let* X *be a Hausdorff topological vector space. Then* $\widetilde{X}_{cb}, \widetilde{X}_{ck}$ *and* $\widetilde{\mathbb{R}^n}_{wk}$ *are ordered cones where ordering* $\leq$ *satisfies the following properties:*

(a) *If* $\sup_{i\in I}[A_i, B_i] = [A, B]$ *then*
$$\sup_{i\in I}[A_i \dot{+} C, B_i \dot{+} D] = [A \dot{+} C, B \dot{+} D]$$
for every $[C, D]$ (translation of supremum).

(b) *If* $C \in \mathcal{B}(X)$ *then*
$$\sup_{i\in I}[C_i \dot{+} C, D] = \Big[\bigvee_{i\in I} C_i \dot{+} C, D\Big].$$

(c) *If* $\bigvee_{i\in I} C_i \in \mathcal{B}(X), C \in \mathcal{B}(X)$ *then*
$$\inf_{i\in I}[D, C_i \dot{+} C] = \Big[D, \bigvee_{i\in I} C_i \dot{+} C\Big].$$

(d) *If* $|I| < \infty$ *then*
$$\sup_{i\in I}[A_i, B_i] = \Big[\bigvee_{i\in I} \Big(A_i \dot{+} \sum_{k\neq i} B_k\Big), \sum_{i\in I} B_i\Big].$$

(e) *If* $|I| < \infty, A_i \in \mathcal{B}(X)$ *then*
$$\inf_{i\in I}[A_i, B_i] = \Big[\sum_{i\in I} A_i, \bigvee_{i\in I} \Big(B_i \dot{+} \sum_{k\neq i} A_k\Big)\Big].$$

6 Existence of Minimal Representative of Elements in the Minkowski–Rådström–Hörmander Cone

Lemma 6.1. *Let* X *be a topological vector space and subsets* A *and* B_i, $i \in I$ *of* X *be closed. Moreover, let* A *be compact or* B_i *be compact for some* $i \in I$. *If one of the following conditions holds true*

(i) *the family* $\{B_i\}_{i\in I}$ *is directed with respect to inclusion "*$\supset$*"*

(ii) *all the sets* A, B_i *are convex and the family* $\{B_i\}_{i\in I}$ *is a family of pairwise convex sets (that is for all* $i, j \in I$ *the union* $B_i \cup B_j$ *is convex)*

then
$$A + \bigcap_{i\in I} B_i = \bigcap_{i\in I}(A + B_i).$$

Proof. We have $x \in \bigcap_{i\in I}(A+B_i)$ if and only if $0 \in A-x+B_i$ for all $i \in I$ but it is equivalent to $(x-A)\cap B_i \neq \emptyset$ for all $i \in I$. Since the family of compact sets $\{(x-A)\cap B_i\}_{i\in I}$ has the finite intersection property, we have $\bigcap_{i\in I}((x-A)\cap B_i) \neq \emptyset$. Hence $(x-A)\cap\bigcap_{i\in I} B_i \neq \emptyset$ and $x \in A + \bigcap_{i\in I} B_i$. □

In Lemma 6.1 (i) it is enough to assume that X is a Hausdorff topological group. Moreover, from Lemma 6.1 (i) it follows that if A is compact and B is a closed subset of X then $A+B$ is closed.

Example 6.2. Let $X = \mathbb{R}^2$. Now we consider $B_1 = (-1,0) \vee (1,0), B_2 = (0,-1) \vee (0,1)$ and the unit ball $A = \mathbb{B}((0,0),1)$. Then $B_1 \cap B_2 = \{(0,0)\}$ and $A + B_1 \cap B_2 = A$. But $(A+B_1) \cap (A+B_2) = A + B = (-1,1) \vee (-1,-1) \vee (1,-1) \vee (1,1)$. Hence Lemma 6.1 does not hold true for the family $\{B_1, B_2\}$ of compact convex subsets of X. The family $\{B_1, B_2\}$ has the finite intersection property but is not a family of pairwise convex sets [5] (see Figure 6.1).

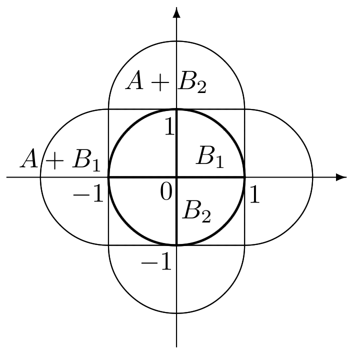

Figure 6.1

Theorem 6.3. (Existence of minimal representative in $\widetilde{X}_{ck}$). *Let X be a real Hausdorff topological vector space. Then every class $[A,B] \in \mathcal{C}(X) \times \mathcal{K}(X)/_\sim$ contains a minimal pair (C,D) such that $(C,D) \le (A,B)$.*

Proof. Let $\{(A_i,B_i)\}_{i\in I} \subset [A,B]$ be a chain. Since the family $\{B_i\}_{i\in I}$ is a chain, $B_0 = \bigcap_{i\in I} B_i \neq \emptyset$. By Lemma 6.1 (a) we have

$$A + \bigcap_{i\in I} B_i = \bigcap_{i\in I}(A + B_i) = \bigcap_{i\in I}(B + A_i) = B + \bigcap_{i\in I} A_i,$$

hence $A_0 = \bigcap_{i\in I} A_i \neq \emptyset$ and $(A_0,B_0) \in [A,B]$. Moreover $(A_0,B_0) \le (A_i,B_i)$ for any $i \in I$. By the Kuratowski-Zorn Lemma the class $[A,B]$ has minimal element $(C,D) \le (A,B)$. □

Theorem 6.3 generalizes the result proved in [14] for MRH space $\widetilde{X}_k$.

Corollary 6.4. (Existence of minimal representative in $\widetilde{X}_{cb}$). *Let X be a reflexive locally convex topological vector space. Then every class $[A,B] \in \mathcal{C}(X) \times \mathcal{B}(X)/_\sim$ contains a minimal pair (C,D) such that $(C,D) \le (A,B)$.*

Proof. Let $X_\tau = (X,\tau)$ be a reflexive locally convex space. By $\sigma = \sigma(X,X^*)$ we denote the weak topology on X. Then $\mathcal{B}_\tau(X) = \mathcal{K}_\sigma(X)$ and $\mathcal{C}_\tau(X) = \mathcal{C}_\sigma(X)$. Hence $\mathcal{C}_\tau(X) \times \mathcal{B}_\tau(X) = \mathcal{C}_\sigma(X) \times \mathcal{K}_\sigma(X)$. □

Corollary 6.4 generalizes a result proved in [8] for MRH space $\widetilde{X}_b$. Similar theorem cannot be proved for all $\widetilde{X}_{cb}$ since for MRH space $\widetilde{l^\infty}_b$ we have an example of a class $[A,B]$ with no minimal element [8]. Similar theorem cannot be proved for $\widetilde{\mathbb{R}}_{wk}$ due to the following example. Let $A = (0,1) \in \mathcal{W}(\mathbb{R})$ and $B = [0,1]$. Then $[A,B] = \{((a,b),[a,b])|a < b\}$. Obviously, the class $[A,B]$ contains no minimal elements.

7 Lattices Contained in the Minkowski–Rådström–Hörmander Cone

Let A, B, S be nonempty subsets of a vector space X. Recall that S *separates* the sets A and B if $[a, b] \cap S \neq \emptyset$ for every $a \in A$ and $b \in B$. The following theorem characterizes separation of sets with the help of convexity of the union of sets. Though this convexity seems very restrictive, it gives a good enumerate of extreme points of symmetric interval in MRH space [9]. It is also useful in characterizing an infimum of two elements of MRH space in Lemma 7.3 and Theorem 7.4.

Theorem 7.1. *Let X be a real Hausdorff topological vector space and $A, B \in \mathcal{C}(X)$. Then the following statements are equivalent:*

(a) *The set $A \cup B$ is convex.*

(b) *The set $A \cap B$ separates the sets A and B.*

(c) $C \dot{+} A \cap B = (A \dot{+} C) \cap (B \dot{+} C)$ *for every segment C in X.*

(d) $C \dot{+} A \cap B = (A \dot{+} C) \cap (B \dot{+} C)$ *for every $C \in \mathcal{W}(X)$.*

(e) $C + A \cap B = (A + C) \cap (B + C)$ *for every $C \in \mathcal{W}(X)$.*

Proof. Equivalence of the conditions (a) and (b) can be proved similarly as in paper [21]. Now let the condition (c) be satisfied. Let $a \in A$ and $b \in B$. Then for $C = a \vee b$ we have

$$a + b \in (A + a \vee b) \cap (B + a \vee b) = a \vee b + A \cap B.$$

Therefore there exists $\alpha \in [0, 1]$ such that $a + b = \alpha a + (1 - \alpha)b + c$ for a some $c \in A \cap B$. Hence $A \cap B$ separates the sets A and B.

Now, let $C \in \mathcal{W}(X)$ and $x \in (A \dot{+} C) \cap (B \dot{+} C)$. Given any neighbourhood $U \in \mathcal{U}$ and balanced $V \in \mathcal{U}$ such that $V + V \subset U$. Then $x \in (A+C+V) \cap (B+C+V)$ and $x = a+c_1+v_1 = b+c_2+v_2$ for some $a \in A, b \in B, c_1, c_2 \in C$ and $v_1, v_2 \in V$. Hence $a = x - c_1 - v_1, b = x - c_2 - v_2$. If $A \cap B$ separates the sets A and B then $\alpha x - \alpha c_1 - \alpha v_1 + \beta x - \beta c_2 - \beta v_2 \in A \cap B$ for some $\alpha + \beta = 1, \alpha, \beta \geq 0$. Hence $x \in A \cap B + C + V + V \subset A \cap B + C + U$ this implies $x \in A \cap B \dot{+} C$. Hence $(A \dot{+} C) \cap (B \dot{+} C) \subset A \cap B \dot{+} C$. Therefore, (b) implies (d).

In a similar and even simpler way (b) implies (e). Independently, (d) and (e) implies (c). □

Example 7.2. In general for $A, B \in \mathcal{C}(X)$ and $C \in \mathcal{W}(X)$ the equality $C \dot{+} A \cap B = (A \dot{+} C) \cap (B \dot{+} C)$ does not imply the equality $C + A \cap B = (A + C) \cap (B + C)$. Take for example $A = [0, 1] \times [0, 1]$, $B = [-1, 0] \times [-1, 0]$ and $C = (0, 1) \times (0, 1) \cup \{(0, 0)\}$. Neither the latter equality implies the first one. Take for example $A = [0, 1]$, $B = [2, 3]$ and $C = (0, 1)$. However, here the intersection $A \cap B$ is empty. The following example shows that even with nonempty intersection $A \cap B$ the implication does not hold true. Let $A \in \mathcal{B}(\mathbb{R}^3)$ be the epigraph of the following convex function $f : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$:

$$f(x, y) = \begin{cases} \frac{1}{xy} & \text{if} \quad xy \geq 3, \\ 1 - \frac{2\sqrt{xy}}{3\sqrt{3}} & \text{if} \quad xy \leq 3. \end{cases}$$

Let $B = \{(x, y, z) \in \mathbb{R}^3 | (-x, y, z) \in A\}$ and C be the straight line $x = z = 0$. Then the set $(A + C) \cap (B + C) = A \cap B + C = A \cap B \dot{+} C$ is a proper subset of $(A \dot{+} C) \cap (B \dot{+} C)$.

In [15], [21], it was shown that for $A, B \in \mathcal{B}(X)$ the union $A \cup B$ is a convex set if and only if $A \vee B$ is a summand of $A \dot{+} B$. However, for unbounded closed convex sets this condition is not sufficient. For instance, let A and B are two nonparallel lines in $\mathbb{R}^2$. Then $A + B = A \vee B = \mathbb{R}^2$. But $A \cup B$ is not convex set.

In general the convex hull "$\overset{\circ}{\vee}$" of two closed convex sets does not have to be closed. For example, let A be x-axis in $\mathbb{R}^2$ and $B = \{(0,1)\}$. Then $A \overset{\circ}{\vee} B = \mathbb{R} \times [0,1) \cup B$, which is not a closed set.

Lemma 7.3. *Let X be a real Hausdorff topological vector space and $A, B \in \mathcal{C}(X)$. Moreover let $\tilde{x} = [A, \{0\}]$, $\tilde{y} = [B, \{0\}]$. Then $\tilde{x} \wedge \tilde{y} = [C, D] \in \widetilde{X}_{cb}$ if and only if the set $(A \dot{+} D) \cup (B \dot{+} D)$ is convex and $C = (A \dot{+} D) \cap (B \dot{+} D)$.*

Proof. Since $[C, D] \le \tilde{x}$ and $[C, D] \le \tilde{y}$, we have $[C, D] \le [(A \dot{+} D) \cap (B \dot{+} D), D] \le [A, \{0\}] \wedge [B, \{0\}] = \tilde{x} \wedge \tilde{y} = [C, D]$. Hence $C = (A \dot{+} D) \cap (B \dot{+} D)$. Also for every $C' \in \mathcal{B}(X)$ we have $[C \dot{+} C', D \dot{+} C'] = \tilde{x} \wedge \tilde{y}$. Therefore for every $C' \in \mathcal{B}(X)$ we get $[(A \dot{+} D \dot{+} C') \cap (B \dot{+} D \dot{+} C'), D \dot{+} C'] = \tilde{x} \wedge \tilde{y}$ and $(A \dot{+} D \dot{+} C') \cap (B \dot{+} D \dot{+} C') = (A \dot{+} D) \cap (B \dot{+} D) \dot{+} C'$. By Theorem 7.1 the set $(A \dot{+} D) \cup (B \dot{+} D)$ is convex.

Obviously, $[C, D] = [(A \dot{+} D) \cap (B \dot{+} D), D] \le \tilde{x}$ and $[C, D] =\le \tilde{y}$. Let $[E, F] \le \tilde{x}$ and $[E, F] \le \tilde{y}$. Hence $E \subset (A \dot{+} F) \cap (B \dot{+} F)$. By Theorem 7.1 we have $[C, D] = [(A \dot{+} D \dot{+} F) \cap (B \dot{+} D \dot{+} F), D \dot{+} F]$. Since $(A \dot{+} F) \cap (B \dot{+} F) \dot{+} D \subset (A \dot{+} D \dot{+} F) \cap (B \dot{+} D \dot{+} F)$, we have $[E, F] \le [(A \dot{+} F) \cap (B \dot{+} F), F] \le [(A \dot{+} D \dot{+} F) \cap (B \dot{+} D \dot{+} F), D \dot{+} F] = [C, D]$. Then $[C, D] = \tilde{x} \wedge \tilde{y}$. □

Theorem 7.4. *Let X be a real Hausdorff topological vector space and $\mathcal{B}(X) \subset \mathcal{S} \subset \mathcal{C}(X)$ be a cone. Then the cone $(\mathcal{S} \times \mathcal{B}(X)/_\sim, +, \cdot)$ is a lattice if and only if for every $A, B \in \mathcal{S}$ there exists a set $D \in \mathcal{B}(X)$ such that $(A \dot{+} D) \cup (B \dot{+} D)$ is convex.*

Proof. Suppose that $\mathcal{S} \times \mathcal{B}(X)/_\sim$ is a lattice. Let $A, B \in \mathcal{C}(X)$ and $\tilde{x} = [A, \{0\}], \tilde{y} = [B, \{0\}]$ then there exists $\tilde{x} \wedge \tilde{y} = [C, D]$. Hence by Lemma 7.3 we have that $(A \dot{+} D) \cup (B \dot{+} D)$ is a convex set.

Conversely, let $\tilde{x} = [A, B], \tilde{y} = [C, D] \in \mathcal{S} \times \mathcal{B}(X)/_\sim$. Then we have $\tilde{x} = [A \dot{+} D, B \dot{+} D], \tilde{y} = [B \dot{+} C, B \dot{+} D]$ and there exists $M \in \mathcal{B}(X)$ such that $(A \dot{+} D \dot{+} M) \cup (B \dot{+} C \dot{+} M)$ is convex. By Lemma 7.3 we have $[A \dot{+} D, \{0\}] \wedge [B \dot{+} C, \{0\}] = [(A \dot{+} D \dot{+} M) \cap (B \dot{+} C \dot{+} M), M]$. Hence $\tilde{x} \wedge \tilde{y} = [(A \dot{+} D \dot{+} M) \cap (B \dot{+} C \dot{+} M), B \dot{+} D \dot{+} M]$. □

In the case $X = \mathbb{R}$ it is easy to characterize minimal pairs of the cone $\widetilde{\mathbb{R}}_{ck}$. We observe that for every pair $(A, B) \in \widetilde{\mathbb{R}}_{ck}$ the interval A is a summand of B or vice versa. Then the pair $(A, B) \in \widetilde{\mathbb{R}}_{ck}$ is minimal if and only if A or B is a singleton. From this characterization immediately follows that minimal pairs in $\widetilde{\mathbb{R}}_{ck}$ have a translation property. For example $(\mathbb{R}, \{a\})_{a \in \mathbb{R}}$ is a family of equivalent minimal pairs.

For any pair of intervals $A, B \subset \mathbb{R}$ there exists a bounded interval $D \subset \mathbb{R}$ such that $(A + D) \cup (B + D)$ is convex. Hence, by Theorem 7.4 the cone $\widetilde{\mathbb{R}}_{ck}$ is a lattice. In fact the following proposition links the cone $\widetilde{\mathbb{R}}_{ck}$ with the cone $\overline{\mathbb{R}}$ defined in Example 4.1(b).

Proposition 7.5. *The lattice* $\widetilde{\mathbb{R}}_{ck}$ *is isomorphic to* $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$.

Proof. Let $i : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \longrightarrow \widetilde{\mathbb{R}}_{ck}$ be a mapping given by $i(1,0) = [[0,1],\{0\}]$, $i(0,1) = [[-1,0],\{0\}]$, $i(a,b) = a \cdot i(1,0) + b \cdot i(0,1)$, where $a, b \in \overline{\mathbb{R}}$, $\infty \cdot i(1,0) = [[0,\infty],\{0\}]$ and $\infty \cdot i(0,1) = [[-\infty,0],\{0\}]$. The mapping i establishes an isomorphism of abstract convex cones. □

For every $A, B \in \mathcal{C}(X)$ the set $(A \dot{+} A \vee B) \cup (B \dot{+} A \vee B) = 2(A \vee B)$ is convex. Hence, by Theorem 7.4 the MRH space $\widetilde{X}_b$ is a lattice.

Example 7.6. Let $X = \mathbb{R}^2$ and $A = \mathbb{R} \times \{0\}$, $B_i = [(-i,-1),(i,1)], i \in \mathbb{R}$. Then the family $(A, B_i) \in \mathcal{C}(X) \times \mathcal{K}(X)$ forms a family of equivalent minimal pairs which are not connected by translations (see Figure 7.1).

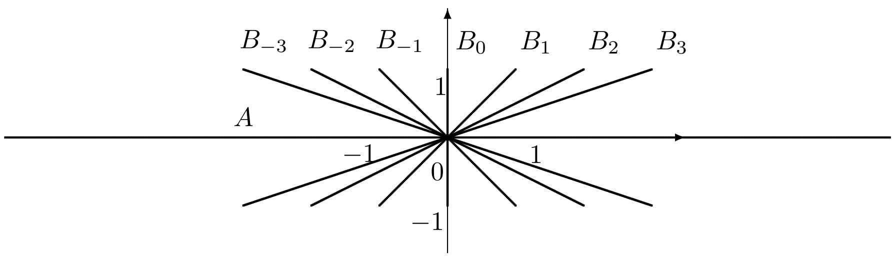

Figure 7.1

8 Reduction Techniques

In short we extend two known basic techniques of reducing a pair $(A, B) \in \mathcal{B}^2(X)$ to reducing a pair $(A, B) \in \mathcal{C}(X) \times \mathcal{B}(X)$ [15].

Proposition 8.1. (Reduction by summand) *If the sets* $A \in \mathcal{C}(X), B \in \mathcal{B}(X)$ *have a common summand* $C \in \mathcal{B}(X)$ *then* $(A \dot{-} C, B \dot{-} C) \in [A, B]$.

Similarly as in [23] we can prove the following lemma.

Lemma 8.2. *Let* A, B *be convex subsets of* X *and* $A \cup B$ *be convex. Then*

$$A \dot{+} B = A \cup B \dot{+} A \cap B.$$

Proposition 8.3. *Let* $(A, B) \in \mathcal{C}(X) \times \mathcal{B}(X)$, $P \in \mathcal{B}(X)$ *and let* $A \cup P, B \cup P$ *be convex,* $A \cap P = B \cap P$. *Then* $(A \cup P, B \cup P) \sim (A, B)$.

Proof. By Lemma 6.2 we obtain $A \dot{+} P = A \cup P \dot{+} A \cap P$ and $B \dot{+} P = B \cup P \dot{+} B \cap P$. Hence

$$A \dot{+} P \dot{+} B \cup P \dot{+} B \cap P = B \dot{+} P \dot{+} A \cup P \dot{+} A \cap P$$

and by the law of cancellation we get $(A \cup P, B \cup P) \sim (A, B)$. □

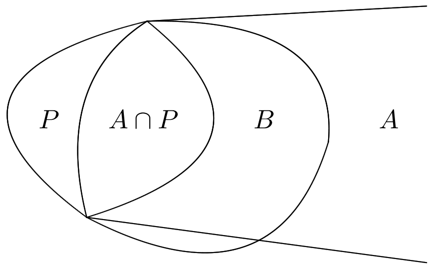

Figure 8.1

Corollary 8.4. (Cutting by a hyperplane) *Let* $A \in \mathcal{C}(X), B \in \mathcal{B}(X)$ *and* H *be a closed hyperplane dividing the space* X *into two closed halfspaces* H^+ *and* H^- *such that* $A \cap H$ *is nonempty. Let* $A \cap H^+ = B \cap H^+$ *and* $A_0 = A \cap H^-, B_0 = B \cap H^-$. *Then the pairs* (A, B) *and* (A_0, B_0) *are equivalent.*

9 Appendix. Upper Exhauster and Quasidifferential

Let us consider Dini or Hadamard directionally differentiable functions. Directional derivative is a positively homogenous (p.h.) function. If $(X, \|\cdot\|)$ is a normed space and p.h. function $h : X \longrightarrow \mathbb{R}$ is upper semicontinous (u.s.c.) then h can be represented in the form

$$h(x) = \inf_{C \in E^*} p_C(x),$$

where p_C is a support function of C and $E^* = E^*(h)$ is a subfamily of the family $\mathcal{B}(X^*)$ of all nonempty *-weakly bounded closed convex subsets of dual space X^*. Such family $E^*(h)$ is called an *upper exhauster* of the function h. Analogously if h is lower semicontinous (l.s.c.) then it can be represented as

$$h(x) = \sup_{C \in E^*} (-p_{-C}(x)) = -\inf_{C \in E^*} p_{-C}(x).$$

If the function h is Lipschitz then $h(x) = \inf_{C \in E^*} p_C(x)$, where the family E^* is totally bounded. Let us notice that upper exhauster is not unique. Moreover, the example given in [7] shows that in general a finite minimal exhauster is not unique.

The theory of exhausters was introduced by V. F. Demyanov and A. M. Rubinov [4] in quasidifferential calculus and was subsequently studied in a series of papers by V. F. Demyanov, A. M. Rubinov and V. Roshchina ([2], [3], [19] and others) and in the framework of lattices and semigroups by J. Grzybowski, D. Pallaschke and R. Urbański [7].

If a function f is directionally differentiable and the directional derivative h at some point can be represented as

$$h(x) = p_A(x) - p_B(x),$$

that is as a difference of support functions of nonempty *-weakly bounded closed convex sets $A, B \in \mathcal{B}(X^*)$ then f is called *quasidifferentiable* at the given point. The pair (A, B) is called the *quasidifferential* of f at the given point. The quasidifferential is never unique.

Moreover, in general minimal quasidifferential is not unique up to translation for $\dim X \geq 3$ [6].

Until now, most of the study was dedicated to directional derivatives assuming finite values. However, in optimization some directional derivatives of directionally differentiable functions assume the value $+\infty$ [18]. In such a case we need to extend our study to p.h. functions represented as

$$h(x) = \inf_{C \in E^*} p_C(x),$$

where E^* is a subfamily of $\mathcal{C}(X^*)$, the family of all nonempty *-weakly closed convex subsets of X^*.

E. Caprari and J. P. Penot in [1] investigated quasidifferentiable functions with directional derivative h represented as

$$h(x) = p_A(x) - p_B(x),$$

where $A \in \mathcal{C}(X^*)$ and $B \in \mathcal{B}(X^*)$.

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