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AN IMPROVED INFEASIBLE INTERIOR-POINT METHOD WITH FULL-NEWTON STEP FOR LINEAR OPTIMIZATION*

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Abstract: We present an improved infeasible interior-point method for solving linear optimization problems, which is obtained by modifying the search directions of the method introduced by Roos (SIAM J. Optim. 16(4):1110-1136, 2006). The method contains two types of full-Newton steps, the feasibility step and the centering step. For the feasibility step, the improved method targets the small neighborhood of the central path with respect to the next pair of perturbed problems. As for the centering steps, the improved method uses a new search direction, which can give a large neighborhood for the feasibility step. To simplify the complexity analysis, we adopt a simple proximity measure. The iteration bound of the algorithm is $\mathcal{O}\left(n\log\frac{n}{\epsilon}\right)$, which is the best-known for IIPMs.

Key words: *linear optimization, full-Newton step, iteration bound, infeasible interior-point method*

Mathematics Subject Classification: *17C99, 90C25, 90C51*

1 Introduction

In this paper, we consider the linear optimization (LO) problem in the standard form

$$(P) \qquad \min\{c^T x : Ax = b, x \geq 0\},$$

with its dual problem

$$(D) \qquad \max\{b^T y : A^T y + s = c, s \geq 0\},$$

where $A \in R^{m\times n}$, $\text{rank}(A) = m,\ b \in R^m,\ c \in R^n$.

Interior-point Methods (IPMs) are now among the most effective methods for solving LO problems. For a survey, we refer to recent books [11, 12] on the subject. One may distinguish different IPMs according to whether they are *feasible* IPMs or *infeasible* IPMs (IIPMs). Feasible IPMs start with a strictly feasible interior point and maintain feasibility during the solution process. IIPMs start with an infeasible point and feasibility is reached as optimality is approached. Readers who are interested may refer to [4, 8, 9] for further reading.

Recently, a new IIPM has been proposed in [10] to solve the above LO problems. A special feature of the new method is that it uses only full-Newton steps. Two types of

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full-Newton steps are used, one is feasibility step and the other is centering step. Starting at strictly feasible iterates of a perturbed pair (perturbation of the given problem and its dual problem), which is close to its central path, feasibility step serves to generate strictly feasible iterate for the next perturbed pair. By accomplishing a few centering steps for the new perturbed pair, it obtains strictly feasible iterates close enough to the central path of the new perturbed pair. The iteration bound for getting an ϵ-solution of the problem method is $O(n \log(n/\epsilon))$, which coincides with the best-known bound for IIPM.

It worths noting that, the search direction used in [10] for feasibility step is designed for keeping the iterates feasible for the next perturbed pair, moreover, the iterates target the central path of the old perturbed pair. Mansouri et al., [6, 5] change the direction by targeting the old iterate pair (the iterate point before feasibility step). Later on, Gu et al., [2, 3] improve the method by targeting the central path of the new perturbed pair. All the above methods target some special points.

Different with all the existed variants of IIPM, our motivations come from the two fact: let $\delta(x, s; \mu)$ be a quantity that measures proximity of the feasible triple (x, y, s) to the μ-center $(x(\mu), y(\mu), s(\mu))$, then one has the following results [11]:

1. Denote (x^+, s^+) as the iterates obtained from one classic-Newton step, and if $\delta := \delta(x, s; \mu) < 1$, then the primal-dual Newton step is strictly feasible. Moreover, if $\delta := \delta(x, s; \mu) \leq 1/\sqrt{2}$, then $\delta(x^+, s^+; \mu) \leq \delta^2$, i.e., quadratically convergent.
2. Denote $\rho(\delta) = \delta + \sqrt{1 + \delta^2}$, one has $1/\rho(\delta) \leq \sqrt{x_i s_i/\mu} \leq \rho(\delta),\ i = 1, \ldots, n$.

The first result illustrates a gap between the feasibility condition and the quadratically convergent condition, i.e., $\delta < 1$ and $\delta < 1/\sqrt{2}$, respectively. The second result shows that to obtain the lower-upper bound for the composite element $\sqrt{x_i s_i/\mu}$, one needs to solve an inequality which is quite complicated.

Considering that, for the full-Newton step IIPM, there are three key issues, the feasibility step, the centering step and the proximity measure. In this paper, we relax the searching target for feasibility step, and make it targeting a small neighborhood of the central path with respect to the next perturbed pair. As the aim of feasibility step is to calculate a feasible solution for the next pair of perturbed problems, which should also lie in a small neighborhood to its new center, our search direction for feasibility step is more intuitive. Moreover, we adopt a new system for centering steps, by which the neighborhood for the feasibility steps can be determined. The feasibility condition and the quadratically convergent condition obtained from this new centering step system keep consistent, thus the gap vanishs. Besides these, by using a simple proximity measure, the analysis for the improved method is much simplified, largely because the lower-upper bound for the composite element can easily be obtained according to the simple proximity measure. We derive the complexity for the algorithm and obtained the iteration bound which is as good as the best-known for IIPMs.

The paper is organized as follows. As a preparation for the rest of the paper, in section 2 we first recall some basic results for full-Newton step IIPM, which include central path, perturbed problems and the description for an iteration of full Newton step IIPM. Then, in Section 3, we specify our new search directions for full-Newton step IIPM in details. Section 4 deals with the analysis of our new IIPM. The analysis is much simplified, one of the main reasons is that we adopt a simple proximity measure, by which the element of vector can be easily estimated. We end the paper with a conclusion in Section 5.

In the whole paper: $\| \cdot \|$ denotes the 2-norm of a vector, $\| \cdot \|_1$ and $\| \cdot \|_\infty$ denote the 1-norm and infinity-norm, respectively. Furthermore, $x_{\min}$ denotes the smallest and $x_{\max}$

denotes the largest value of the components of x, respectively. xs is the coordinate-wise product of the vectors x and s, i.e., $xs = [x_1 s_1; x_2 s_2; \ldots; x_n s_n]$. Similarly, we use $\frac{x}{s}$ to denote $\left[\frac{x_1}{s_1}; \ldots; \frac{x_n}{s_n}\right]$, for each vector x and s such that $s_i \neq 0,\ i = 1, \ldots, n$.

2 Preliminaries

In preparation for dealing with our IIPM, we recall briefly the notions of central path and of perturbed problems. Furthermore, we describe and present the full-Newton step IIPM for LO.

2.1 The central path

It is well known that finding an optimal solution of (P) and (D) is equivalent to solving the following system:

$$\begin{array}{rcll} Ax & = & b, & x \geq 0, \\ A^T y + s & = & c, & s \geq 0, \\ xs & = & 0, & \end{array}$$

where the last equation is called complementarity condition.

The basic idea of primal-dual IPMs is to replace the complementarity condition by the parameterized equation $xs = \mu e$, with $\mu > 0$. This yields the system:

$$\begin{array}{rcl} Ax & = & b,\ x \geq 0, \\ A^T y + s & = & c,\ s \geq 0, \\ xs & = & \mu e. \end{array} \tag{2.1}$$

Surprisingly enough, if system (2.1) has a solution for some $\mu > 0$, then a solution exists for every $\mu > 0$, and this solution is unique [11]. This happens if and only if problems (P) and (D) satisfy the interior-point condition (IPC), i.e., there exists (x^0, s^0, y^0) such that

$$Ax^0 = b,\ x^0 > 0 \text{ and } A^T y^0 + s^0 = c,\ s^0 > 0.$$

If the IPC is satisfied, then the solution of (2.1) is denoted by $(x(\mu), y(\mu), s(\mu))$ and called the μ-center of (P) and (D). The set of all μ-centers gives a homotopy path, which is called the central path of (P) and (D). If $\mu \to 0$, then the limit of the central path exists and since the limit points satisfy the complementarity condition, the limit yields optimal solutions for (P) and (D).

If we are given a positive feasible pair (x, s), and some $\mu > 0$, our aim is to define search directions $(\triangle x, \triangle y, \triangle s)$ that move in the direction of the small neighborhood of the μ-center $(x(\mu), y(\mu), s(\mu))$. In fact, we want the new iterates $x + \triangle x$, $y + \triangle y$ and $s + \triangle s$ to satisfy system (2.1) and be positive with respect to μ. After substitution this yields the following conditions on $(\triangle x, \triangle y, \triangle s)$

$$\begin{array}{rcll} A\,(x + \triangle x) & = & b, & x + \triangle x > 0, \\ A^T (y + \triangle y) + (s + \triangle s) & = & c, & s + \triangle s > 0, \\ (x + \triangle x)\,(s + \triangle s) & = & \mu e. & \end{array}$$

If we neglect for the moment the inequality constraints and the quadratic term $\triangle x \triangle s$, then, since $Ax = b$ and $A^T y + s = c$, this system can be rewritten as follows

$$\begin{array}{rcl} A\triangle x & = & 0, \\ A^T\triangle y + \triangle s & = & 0, \\ x\triangle s + s\triangle x & = & \mu v - xs. \end{array} \tag{2.2}$$

Since A has full row rank, the above system uniquely defines a search direction $(\triangle x, \triangle y, \triangle s)$ for any $x > 0$ and $s > 0$. This Newton direction are called as centering step in the implementations of full-Newton step IIPM.

$\boxed{2.2}$ The perturbed problems

In the case of an IIPM, if the 2-norms of the residual vectors $b - Ax$ and $c - A^T y - s$ do not exceed ϵ and the duality gap satisfies $x^T s \leq \epsilon$, then we call the triple (x, y, s) an ϵ-optimal solution of (P) and (D) .

Denote that the initial residual vectors r_b^0 and r_c^0, respectively, as

$$\begin{array}{l} r_b^0 = b - Ax^0, \\ r_c^0 = c - A^T y^0 - s^0, \end{array}$$

where x^0 and s^0 are positive point. For any ν with $0 < \nu \leq 1$, we consider the following perturbed problem,

$$(P_\nu) \quad \min\{(c - \nu r_c^0)^T x : Ax = b - \nu r_b^0, \quad x \geq 0\}, \tag{2.3}$$

and its dual problem (D_ν)

$$(D_\nu) \quad \max\{(b - \nu r_b^0)^T y : A^T y + s = c - \nu r_c^0, \quad s \geq 0\}. \tag{2.4}$$

Note that if $\nu = 1$ then $x = x^0$ yields a strictly feasible solution of (P_ν), and $(y, s) = (y^0, s^0)$ a strictly feasible solution of (D_ν), which implies that if $\nu = 1$, then (P_ν) and (D_ν) satisfy the IPC. Furthermore, we point out the following result.

Lemma 2.1 ([12, Theorem 5.13])**.** *The original problems, (P) and (D), are feasible if and only if for each ν satisfying $0 < \nu \leq 1$ the perturbed problems (P_ν) and (D_ν) satisfy the IPC.*

Assuming that problems (P) and (D) are feasible. It follows from Lemma 2.1 that the perturbed problem pair (P_ν) and (D_ν) satisfy the IPC, for each $\nu \in (0, 1]$, and hence their central paths exist. This means that the system

$$\begin{array}{rcl} Ax & = & b - \nu r_b^0, \quad x \geq 0, \\ A^T y + s & = & c - \nu r_c^0, \quad s \geq 0, \\ xs & = & \mu e, \end{array} \tag{2.5}$$

has a unique solution for every $\mu > 0$. Denoting this unique solution in the sequel as $(x(\nu), y(\nu), s(\nu))$. As a consequence, $(x(\nu), y(\nu), s(\nu))$ is the μ-center of (P_ν) and (D_ν).

If we want to find the new iterates x^f, y^f and s^f that satisfy (2.5) with ν replaced by $\nu^+ := (1 - \theta)\nu$, as it will be seen, by taking θ small enough this can be realized by one feasibility step. That is

$$x^f = x + \Delta^f x, \; y^f = y + \Delta^f y, \; s^f = s + \Delta^f s,$$

the search directions $\Delta^f x$, $\Delta^f y$ and $\Delta^f s$ which are (uniquely) defined by the system

$$\begin{aligned} A\Delta^f x &= \theta\nu r_b^0, \\ A^T\Delta^f y + \Delta^f s &= \theta\nu r_c^0, \\ s\Delta^f x + x\Delta^f s &= \mu e - xs. \end{aligned} \tag{2.6}$$

It can be easily understood that if (x, y, s) is feasible for the perturbed problems (P_ν) and (D_ν), then after the feasibility step the iterates satisfy the feasibility conditions for (P_ν^+) and (D_ν^+), provided that they satisfy the nonnegativity conditions.

As usual for IIPMs, let us assume that the initial iterates

$$x^0 = s^0 = \xi e, \quad y^0 = 0 \text{ and } \mu^0 = \xi^2, \tag{2.7}$$

μ^0 is the initial dual gap and $\xi > 0$ is such that

$$\|x^* + s^*\|_\infty \le \xi, \tag{2.8}$$

for some optimal solutions (x^*, y^*, s^*). Due to this notation, by taking $\nu = 1$, one has

$$(x(1), y(1), s(1)) = (x^0, y^0, s^0) = (\xi e, 0, \xi e).$$

We measure proximity of iterates (x, y, s) to the μ-center of the perturbed problems (P_ν) and (D_ν) by $\sigma(x, s; \mu)$ as follows:

$$\sigma(x, s; \mu) = \sigma(v) = \|e - v\|, \text{ where } v = \sqrt{\frac{xs}{\mu}}. \tag{2.9}$$

Thus, initially $x^0 = s^0 = \xi e$ and $\mu^0 = \xi^2$, whence $v^0 = e$ and $\sigma(x^0, s^0; \mu) = 0$. In the sequel, we assume $\sigma(x, s; \mu)$ is smaller than or equal to a (small) threshold value $\tau > 0$ at the start of each iteration. So this is certainly true at the start of the first iteration.

2.3 One main iteration of the full-Newton step IIPM

Now we describe one main iteration of full-Newton step IIPM. Each main iteration consists of a feasibility step, a μ-update, and a few centering steps, respectively. Suppose that for some $\nu \in (0, 1]$, the iterates (x, s) satisfy the first and the second equation of system (2.5), $x^T s \le n\mu$ and $\sigma(x, s; \mu) \le \tau$, where $\mu = \nu\xi^2$ and τ is a small threshold.

With ν replaced by $\nu^+ = (1 - \theta)\nu$, the feasibility step serves to get iterates (x^f, y^f, s^f) that are strictly feasible for (P_{ν^+}) and (D_{ν^+}) and close to their μ^+-centers $(x(\nu^+), y(\nu^+), s(\nu^+))$. In fact, the feasibility step is designed in such a way that $\sigma(x^f, s^f; \mu^+) < \eta$, where $\eta \le 1$ is a bound for keeping the iteration quadratic convergence, i.e., (x^f, y^f, s^f) lies in the quadratic convergence neighborhood with respect to the μ^+-center of (P_ν^+) and (D_ν^+). Then, just by performing a few centering steps starting at (x^f, y^f, s^f) and targeting at the μ^+-centers of (P_ν^+) and (D_ν^+), one can easily get iterates (x^+, y^+, s^+) that are strictly feasible for (P_ν^+) and (D_ν^+) and such that $\sigma(x^+, s^+; \mu^+) \le \tau$.

Now we give a more formal description of the algorithm.

Generic full-Newton step IIPM for LO

Input:

A threshold parameter $\tau > 0$;

an accuracy parameter $\varepsilon > 0$;
a fixed barrier update parameter $\theta, 0 < \theta < 1$;
bound parameter ξ;
begin
$x := \xi e; y := 0; s := \xi e; \mu := \xi^2; \nu := 1$;
while $\max\{x^T s,\ \|b - Ax\|,\ \|c - A^T y - s\|\} \geq \varepsilon$ do
begin
feasibility step: solving (2.6) and obtain
$(x, y, s) := (x, y, s) + (\Delta^f x, \Delta^f y, \Delta^f s)$;
μ-update: $\mu := (1 - \theta)\mu,\ \nu := (1 - \theta)\nu$;
centering steps:
while $\sigma(x, s; \mu) > \tau$ do
solving (2.2) and obtain
$(x, y, s) := (x, y, s) + (\Delta x, \Delta y, \Delta s)$;
end
end
end

Note that after each iteration the residuals and the duality gap are reduced by a factor $1 - \theta$. The algorithm stops if the norms of the residuals and the duality gap are less than the accuracy parameter ϵ.

3 A New Full-Newton Step IIPM

In this section, we give a new full-Newton step IIPM, which has different feasibility step and different centering step with the method in [10, 6, 5, 2, 3].

3.1 The new feasibility step

Suppose we have strictly feasible iterate (x, y, s) for (P_ν) and (D_ν). This means that (x, y, s) satisfies the first two equations of system (2.5). With ν replaced by $\nu^+ = (1 - \theta)\nu$, denoting

$$x^f = x + \Delta^f x,\ y^f = y + \Delta^f y,\ s^f = s + \Delta^f s,$$

to find new iterates (x^f, y^f, s^f) feasible for (P_{ν^+}) and (D_{ν^+}), we need search directions $\Delta^f x$, $\Delta^f y$ and $\Delta^f s$ satisfy the first two equations in the following system.

$$\begin{array}{rcl} A\Delta^f x &=& \theta\nu r_b^0, \\ A^T\Delta^f y + \Delta^f s &=& \theta\nu r_c^0, \\ s\Delta^f x + x\Delta^f s &=& (1 - \theta)\mu^{\frac{1}{2}}(xs)^{\frac{1}{2}} - xs. \end{array} \tag{3.1}$$

Since matrix A has full row rank, for any $x > 0$ and $s > 0$, the coefficient matrix in the linear system (3.1) is nonsingular, the system uniquely defines $(\Delta^f x, \Delta^f y, \Delta^f s)$.

We specify our motivation for the third equation of (3.1) in detail. The following lemma shows that the components of the vector v can be estimated simply by the proximity measure $\sigma(v)$.

Lemma 3.1. *Let $\sigma(v)$ be defined as (2.9). Then one has*

$$1 - \sigma(v) \leq v_i \leq 1 + \sigma(v),\ i = 1, \ldots, n.$$

Proof. Since
$$|1 - v_i| \le \|e - v\| = \sigma(v), \ i = 1, \ldots, n,$$
the result easily follows. □

Remember that at the start of one main iteration, the condition $\sigma(v) \le \tau$ must be satisfied, where τ is (much) smaller than 1. By Lemma 3.1, one has

$$(1-\tau)e \le v \le (1+\tau)e. \tag{3.2}$$

Since the third equation in system (3.1) is the linearization of $x^f s^f = (1-\theta)\mu v = \mu^+ v$, i.e., neglecting the quadratic term $\Delta^f x \Delta^f s$. By (3.2), one has

$$(1-\tau)\mu^+ e \le x^f s^f \le (1+\tau)\mu^+ e,$$

which means the new iterates are now targeting a small neighborhood of the μ^+-center of (P_{ν^+}) and (D_{ν^+}).

Remark 3.2. For the third equation in system (3.1), the linearization of $x^f s^f = \mu e$ is used in [10] (targeting the μ-center). While in [6, 5], the linearization of $x^f s^f = xs$ is used (targeting the old xs) and in [2, 3, 7], the linearization of $x^f s^f = \mu^+ e$ is used (targeting the μ^+-center).

We conclude that after the feasibility step we have iterates (x^f, y^f, s^f) which satisfy the affine equations of the new perturbed problem pair (P_{ν^+}) and (D_{ν^+}). In the analysis we should also guarantee that x^f and s^f are positive and belong to the region of quadratic convergence to their μ^+-centers. On this occasion, we adopt the full-Newton step system of Darvay [1], which will be given in the next subsection.

3.2 The new centering step

After the feasibility step, if necessary, the new IIPM proceeds to operate centering steps. Starting from (x^f, y^f, s^f), several full Newton steps are needed to restore the condition $\sigma(x, s; \mu) \le \tau$. On this occasion, we use the following system for centering step search direction.

$$\begin{aligned} A\triangle x &= 0, \\ A^T\triangle y + \triangle s &= 0, \\ x\triangle s + s\triangle x &= 2\left(\mu^{\frac{1}{2}}\,(xs)^{\frac{1}{2}} - xs\right). \end{aligned} \tag{3.3}$$

Since A has full rank, and the vectors x and s are positive, one may easily verify that the coefficient matrix in the linear system (3.3) is nonsingular. Hence this system uniquely defines the search directions $\triangle x$, $\triangle y$ and $\triangle s$.

Remark 3.3. In [1], system (3.3) is represented as

$$\begin{aligned} \bar{A} d_x &= 0, \\ \bar{A}^T \tfrac{\triangle y}{\mu} + d_s &= 0, \\ d_x + d_s &= 2(e - v), \end{aligned}$$

where
$$d_x = \frac{v\triangle x}{x}\ , d_s = \frac{v\triangle s}{s}, \ \bar{A} = AV^{-1}X, \ V = \mathrm{diag}(v) \text{ and } X = \mathrm{diag}(x).$$

Let

$$x^+ = x + \triangle x,\ y^+ = y + \triangle y,\ s^+ = s + \triangle s$$

be the vectors obtained after a full-Newton step. In the following lemma Darvay [1] investigates the effect of the full Newton step on the duality gap.

Lemma 3.4 ([1, Lemma 5.3])**.** *Let $\sigma = \sigma(x, s; \mu)$ and suppose that the vectors x^+ and s^+ are obtained after a full-Newton step. We have*

$$(x^+)^T s^+ = \mu(n - \sigma^2),$$

hence $(x^+)^T s^+ \leq \mu n$.

Darvay [1] also gives a condition which guarantees the feasibility of the full-Newton step by the following lemma.

Lemma 3.5 ([1, Lemma 5.1])**.** *Let $\sigma = \sigma(x, s; \mu) < 1$. Then*

$$x^+ > 0 \quad \textit{and} \quad s^+ > 0,$$

thus the full-Newton step is strictly feasible.

The quadratically convergent property of the full-Newton step (3.3) is presented as follows.

Lemma 3.6 ([1, Lemma 5.2])**.** *Let $\sigma = \sigma(x, s; \mu) < 1$. Then*

$$\sigma(x^+, s^+; \mu) \leq \frac{\sigma^2}{1 + \sqrt{1 - \sigma^2}}.$$

Immediately, we have the following corollary.

Corollary 3.7. *If $\sigma = \sigma(x, s; \mu) < 1$, then $\sigma(x^+, s^+; \mu) \leq \sigma^2$.*

Proof. It directly follows from the fact that $\dfrac{\sigma^2}{1 + \sqrt{1 - \sigma^2}}$ is a monotonically decreasing function. □

Remark 3.8. Corollary 3.7 defines a neighborhood of the μ-center where the quadratic convergence occurs, namely $\sigma(x, s; \mu) < 1$.

After the feasibility step, assuming an upper bound for $\sigma(x^f, s^f; \mu^+)$ can be obtained and denoting this bound as η, we perform the centering steps in order to get the iterates (x^+, y^+, s^+) that satisfy $(x^+)^T s^+ \leq n\mu^+$ and $\sigma(x^+, s^+; \mu^+) \leq \tau$. By Corollary 3.7, the required number of centering steps can easily be obtained.

Indeed, assuming we have $\sigma(x^f, s^f; \mu^+) \leq \eta < 1$ after the feasibility step, hence after k centering steps the iterates (x, s) satisfy

$$\sigma(x, s; \mu^+) \leq \eta^{2^k}.$$

Therefore, $\sigma(x^+, s^+; \mu^+) \leq \tau$ will hold if k satisfies

$$\eta^{2^k} \leq \tau.$$

This implies that at most

$$\left\lceil \log_2 \left(\frac{\log_2 \tau}{\log_2 \eta} \right) \right\rceil \tag{3.4}$$

centering steps are needed.

4 Analysis of the New IIPM

As we have seen in section 3, the feasibility step generates new iterates (x^f, y^f, s^f) that satisfy the feasibility conditions for (P_{ν^+}) and (D_{ν^+}). Except the nonnegativity constraints, another crucial thing in the analysis is to show that after the feasibility step $\sigma(x^f, s^f; \mu^+) < 1$, i.e., the iterate (x^f, y^f, s^f) is within the neighborhood where the Newton process targeting the μ^+-centers of (P_{ν^+}) and (D_{ν^+}) is quadratically convergent.

4.1 The effect of the feasibility step

Let x, y and s denote the iterates at the start of an iteration. We define

$$d_x^f = \frac{v\Delta^f x}{x} \text{ and } d_s^f = \frac{v\Delta^f s}{s}, \tag{4.1}$$

where v is defined as (2.9). One has

$$\triangle^f x \triangle^f s = \mu d_x^f d_s^f.$$

We may write

$$\begin{aligned} x^f s^f &= (x + \triangle^f x)(s + \triangle^f s) \\ &= xs + (s\triangle^f x + x\triangle^f s) + \triangle^f x \triangle^f s \\ &= (1-\theta)\mu^{\frac{1}{2}}(xs)^{\frac{1}{2}} + \triangle^f x \triangle^f s \\ &= \mu((1-\theta)v + d_x^f d_s^f). \end{aligned} \tag{4.2}$$

The condition for feasibility step is as follows.

Lemma 4.1. *The new iterates (x^f, y^f, s^f) are strictly feasible if and only if $(1-\theta)v + d_x^f d_s^f > 0$.*

Proof. The "only if" part of the statements in the lemma follows immediately from (4.2). For the proof of the converse implication we introduce a step length $\alpha \in [0, 1]$, and define

$$x^\alpha = x + \alpha\triangle^f x,\ y^\alpha = y + \alpha\triangle^f y,\ s^\alpha = s + \alpha\triangle^f s.$$

We then have $x^0 = x$, $x^1 = x^f$ and $s^0 = s$, $s^1 = s^f$. Hence, we have $x^0 s^0 = xs > 0$. We write

$$x^\alpha s^\alpha = (x + \alpha\triangle^f x)(s + \alpha\triangle^f s) = xs + \alpha(x\triangle^f s + s\triangle^f x) + \alpha^2\triangle^f x\triangle^f s.$$

Using $s\Delta^f x + x\Delta^f s = (1-\theta)\mu^{\frac{1}{2}}(xs)^{\frac{1}{2}} - xs$ and (4.1), we obtain

$$\begin{aligned} x^\alpha s^\alpha &= xs + \alpha((1-\theta)\mu^{\frac{1}{2}}(xs)^{\frac{1}{2}} - xs) + \alpha^2\triangle^f x\triangle^f s \\ &= \mu[(1-\alpha)v^2 + \alpha(1-\theta)v + \alpha^2 d_x^f d_s^f]. \end{aligned}$$

Suppose $(1-\theta)v + d_x^f d_s^f > 0$, i.e., $d_x^f d_s^f > -(1-\theta)v$, substitution gives

$$\begin{aligned} x^\alpha s^\alpha &> \mu[(1-\alpha)v^2 + \alpha(1-\theta)v - \alpha^2(1-\theta)v] \\ &= \mu(1-\alpha)[v^2 + \alpha(1-\theta)v]. \end{aligned}$$

Since v^2 and v are positive it follows that $x^\alpha s^\alpha > 0$ for $0 \le \alpha \le 1$. Hence, none of the entries of x^α and s^α vanish for $0 \le \alpha \le 1$. Since x^0 and s^0 are positive, and x^α and s^α depend linearly on α, this implies that $x^\alpha > 0$ and $s^\alpha > 0$ for $0 \le \alpha \le 1$. Hence, the vectors x^1 and s^1 must be positive. This completes the "if" part of the statement. $\square$

Corollary 4.2. *The new iterates* (x^f, y^f, s^f) *are strictly feasible if*

$$\|d_x^f d_s^f\|_\infty < (1-\theta)v_{\min}.$$

Proof. By Lemma 4.1, x^f and s^f are strictly feasible if and only if $(1-\theta)v + d_x^f d_s^f > 0$. Since the inequality holds if $\|d_x^f d_s^f\|_\infty < (1-\theta)v_{\min}$, the corollary follows. □

In the sequel we denote

$$\omega(v) = \frac{1}{2}\sqrt{\|d_x^f\|^2 + \|d_s^f\|^2}, \tag{4.3}$$

this implies $\|d_x^f\| \le 2\omega(v)$ and $\|d_s^f\| \le 2\omega(v)$, and moreover,

$$\|d_x^f d_s^f\| \le \|d_x^f\|\|d_s^f\| \le \frac{1}{2}(\|d_x^f\|^2 + \|d_s^f\|^2) = 2\omega(v)^2, \tag{4.4}$$

$$\|d_x^f d_s^f\|_\infty \le \|d_x^f\|\|d_s^f\| \le \frac{1}{2}(\|d_x^f\|^2 + \|d_s^f\|^2) = 2\omega(v)^2. \tag{4.5}$$

Lemma 4.3. *The new iterates* (x^f, y^f, s^f) *are strictly feasible if*

$$\omega(v)^2 < \frac{(1-\theta)}{2}(1-\sigma(v)). \tag{4.6}$$

Proof. By Lemma 3.1, Corollary 4.2 and (4.5), the result easily follows. □

4.2 An upper bound of $\sigma(v^f)$

Theorem 4.4. *Assume that the new iterates* (x^f, y^f, s^f) *are strictly feasible, and denote* $v^f = \sqrt{\frac{x^f s^f}{\mu^+}}$. *One has*

$$\sigma(v^f) \le \sigma(v) + \frac{2\omega(v)^2}{(1-\theta)}.$$

Proof.

$$(v^f)^2 = \frac{x^f s^f}{\mu^+} = \frac{\mu((1-\theta)v + d_x^f d_s^f)}{\mu^+} = v + \frac{d_x^f d_s^f}{1-\theta}.$$

Hence

$$\begin{aligned}\sigma(v^f) &= \|e - v^f\| \le \|(e-v^f)(e+v^f)\| \\ &= \|e-(v^f)^2\| = \|e - v - \frac{d_x^f d_s^f}{1-\theta}\| \\ &\le \|e-v\| + \frac{1}{1-\theta}\|d_x^f d_s^f\| \\ &\le \sigma(v) + \frac{2\omega(v)^2}{1-\theta},\end{aligned}$$

the last inequality follows from (4.4). This completes the proof. □

We hope the new iterate (x^f, y^f, s^f) is within the neighborhood where the Newton process targeting the μ^+-centers of (P_{ν^+}) and (D_{ν^+}) is quadratically convergent, ie., $\sigma(v^f) < 1$, according to Theorem 4.4, it suffices if

$$\sigma(v) + \frac{2\omega(v)^2}{1-\theta} < 1,$$

which gives

$$\omega(v)^2 < \frac{(1-\theta)}{2}(1-\sigma(v)). \tag{4.7}$$

Lemma 4.5. *If the iterate (x^f, y^f, s^f) is within the neighborhood where the Newton process targeting the μ^+-centers of (P_{ν^+}) and (D_{ν^+}) is quadratically convergent, then it is strictly feasible.*

Proof. Comparing Lemma 4.3 and (4.7), the result follows. $\square$

We proceed by considering the value $\omega(v)$ in more details.

$\boxed{4.3}$ An upper bound for $\omega(v)$

Defining

$$\bar{A} = AV^{-1}X, \text{ where } V = \mathrm{diag}(v),\ X = \mathrm{diag}(x), \tag{4.8}$$

one may easily check that the system (3.1), which defines the search directions $(\Delta^f x, \Delta^f y, \Delta^f s)$, can be expressed as follows.

$$\begin{array}{rcl} \bar{A}d_x^f & = & \theta\nu r_b^0, \\ \bar{A}^T \frac{\Delta^f y}{\mu} + d_s^f & = & \theta v \nu s^{-1} r_c^0, \\ d_s^f + d_x^f & = & (1-\theta)e - v. \end{array} \tag{4.9}$$

Denoting the null space of the matrix $\bar{A}$ as $\mathcal{L}$, so

$$\mathcal{L} := \{\xi \in R^n : \bar{A}\xi = 0\}.$$

Obviously, the affine space $\{\xi \in R^n : \bar{A}\xi = \theta\nu r_b^0\}$ equals $d_x^f + \mathcal{L}$. Note that due to a well-known result from Linear Algebra the row space of $\bar{A}$ equals the orthogonal complement $\mathcal{L}^\perp$ of $\mathcal{L}$. Therefore, $d_s^f \in \theta v \nu s^{-1} r_c^0 + \mathcal{L}^\perp$. Also note that $\mathcal{L} \cap \mathcal{L}^T = 0$, and as a consequence the affine spaces $d_x^f + \mathcal{L}$ and $d_s^f + \mathcal{L}^T$ meet in a unique point. This point is denoted by q.

Lemma 4.6. *Let q be the (unique) point in the intersection of the affine spaces $d_x^f + \mathcal{L}$ and $d_s^f + \mathcal{L}^\perp$. Then*

$$2\omega(v) \leq \sqrt{\|q\|^2 + (\|q\| + \sigma(v) + \theta\sqrt{n})^2}.$$

Proof. To simplify the notation, we denote $r = (1-\theta)e - v$. Since $\mathcal{L}^\perp + \mathcal{L} = \mathcal{R}^n$, there exists $q_1,\ r_1 \in \mathcal{L}$ and $q_2,\ r_2 \in \mathcal{L}^\perp$ such that

$$q = q_1 + q_2,\ r = r_1 + r_2.$$

On the other hand, since $d_x^f - q \in \mathcal{L}$ and $d_s^f - q \in \mathcal{L}^\perp$, there must exist $l_1 \in \mathcal{L}$ and $l_2 \in \mathcal{L}^\perp$ such that

$$d_x^f = q + l_1, \quad d_s^f = q + l_2.$$

From the third equation of (4.9), it follows that $r = 2q + l_1 + l_2$, which implies

$$(2q_1 + l_1) + (2q_2 + l_2) = r_1 + r_2.$$

Since the decomposition $\mathcal{L}^\perp + \mathcal{L} = \mathcal{R}^n$ is unique, we conclude that

$$l_1 = r_1 - 2q_1, \quad l_2 = r_2 - 2q_2.$$

Hence we obtain

$$d_x^f = q + r_1 - 2q_1 = (r_1 - q_1) + q_2,$$

$$d_s^f = q + r_2 - 2q_2 = (r_2 - q_2) + q_1.$$

Since the spaces $\mathcal{L}$ and $\mathcal{L}^\perp$ are orthogonal we conclude that

$$4\omega(v)^2 = \|d_x^f\|^2 + \|d_s^f\|^2 = \|r_1 - q_1\|^2 + \|q_2\|^2 + \|q_1\|^2 + \|r_2 - q_2\|^2 = \|q - r\|^2 + \|q\|^2.$$

Assuming $q \neq 0$, as the right-hand side of $\|r\| = \|(1-\theta)e - v\|$ reaches its maximal when $r = -\frac{\|(1-\theta)e-v\|q}{\|q\|}$, thus we obtain

$$\begin{aligned} 4\omega(v)^2 &\leq \left\| \left(1 + \frac{\|(1-\theta)e - v\|}{\|q\|}\right) q \right\|^2 + \|q\|^2 \\ &= \left(\|q\| + \|(1-\theta)e - v\|\right)^2 + \|q\|^2 \\ &\leq \|q\|^2 + (\|q\| + \sigma(v) + \theta\sqrt{n})^2, \end{aligned}$$

which implies the inequality in the lemma if $q \neq 0$. Since the inequality in the lemma holds with equality if $q = 0$, this completes the proof. □

$\boxed{4.4}$ An upper bound for $\|q\|$

We proceed to derive an upper bound for $\|q\|$. Before doing this, we choose the initial point in the usual way as defined in (2.7).

Recall Lemma 4.6 that q is the (unique) solution of the system

$$\begin{aligned} \bar{A}q &= \theta\nu r_b^0, \\ \bar{A}^T\xi + q &= \theta\nu v s^{-1} r_c^0, \end{aligned}$$

which is the same as the condition of Lemma 2.6 by Mansouri in [7].

We state the following lemma without further proof.

Lemma 4.7 ([7, Lemma 2.6]). *Let (x^0, y^0, s^0) be an initial point as defined in* (2.7) *and* (2.8)*, we have*

$$\|q\| \leq \frac{\theta}{\xi v_{\min}} \left(\|x\|_1 + \|s\|_1\right).$$

The next Lemma is a part of the proof for Lemma 4.3 in [3], we also give a simple proof here.

Lemma 4.8. *Let x and (y, s) be feasible for the perturbed problems (P_ν) and (D_ν) respectively and (x^0, y^0, s^0) as defined in* (2.7)*. Then for any primal-dual optimal solution (x^*, y^*, s^*), one has*

$$\begin{aligned} &\nu(x^T s^0 + s^T x^0) \\ = \;& s^T x + \nu^2 (s^0)^T x^0 + \nu(1-\nu)\left((s^0)^T x^* + (x^0)^T s^*\right) - (1-\nu)(s^T x^* + x^T s^*). \end{aligned}$$

Proof. Let (x^*, y^*, s^*) be an optimal solution satisfying (2.8). Then from the first two equation of system (2.5), ie., feasibility conditions of the perturbed problems (P_ν) and (D_ν), it is easily seen that

$$\begin{aligned} A[x - \nu x^0 - (1-\nu)x^*] &= 0, \\ A^T[y - \nu y^0 - (1-\nu)y^*] + [s - \nu s^0 - (1-\nu)s^*] &= 0. \end{aligned}$$

This implies that $x - \nu x^0 - (1-\nu)x^*$ and $s - \nu s^0 - (1-\nu)s^*$ belong to the null space and row space of A, respectively. Thus,

$$[x - \nu x^0 - (1-\nu)x^*]^T [s - \nu s^0 - (1-\nu)s^*] = 0.$$

By expanding the above equality and using the fact that $(x^*)^T s^* = 0$, the result easily follows. □

Now we give an upper bound for $\|x\|_1 + \|s\|_1$ in terms of $\sigma(v)$ as follows.

Lemma 4.9. *Let x and (y, s) be feasible for the perturbed problems (P_ν) and (D_ν) respectively, $\sigma(v)$ as defined in (2.9) and $x^0 = s^0 = \xi e$, where $\xi > 0$ is a constant such that $\|x^* + s^*\|_\infty \le \xi$ for some primal-dual optimal solution (x^*, y^*, s^*). Then we have*

$$\|x\|_1 + \|s\|_1 \le \left((1+\sigma(v))^2 + 1\right) n\xi.$$

Proof. Since x, s, x^* and s^* are nonnegative, $\nu \le 1$, Lemma 4.8 implies that

$$x^T s^0 + s^T x^0 \le \frac{s^T x}{\nu} + \nu (s^0)^T x^0 + (1-\nu)\left((s^0)^T x^* + (x^0)^T s^*\right). \tag{4.10}$$

Since $x^0 = s^0 = \xi e$ and $\|x^* + s^*\|_\infty \le \xi$, we have

$$(x^0)^T s^* + (s^0)^T x^* = \xi e^T (x^* + s^*) \le \xi e^T (\|x^* + s^*\|_\infty) e = \xi \|x^* + s^*\|_\infty (e^T e) \le n\xi^2.$$

Also by using $(x^0)^T s^0 = n\xi^2$ in (4.10), we get

$$x^T s^0 + s^T x^0 \le \frac{s^T x}{\nu} + n\xi^2 = \frac{\mu(e^T v^2)}{\nu} + n\xi^2 = \xi^2 (e^T v^2) + n\xi^2,$$

where for the last equality we used $\nu = \frac{\mu}{\mu_0}$ and $\mu_0 = \xi^2$. By the right part of inequality in Lemma 3.1, we obtain

$$x^T s^0 + s^T x^0 \le \left((1+\sigma(v))^2 + 1\right) n\xi^2. \tag{4.11}$$

Substitution $x^0 = s^0 = \xi e$, we have

$$x^T s^0 + s^T x^0 = \xi(e^T x + e^T s) = \xi(\|x\|_1 + \|s\|_1).$$

Hence, by (4.11), it follows that

$$\|x\|_1 + \|s\|_1 \le \left((1+\sigma(v))^2 + 1\right) n\xi,$$

which proves the lemma. □

We conclude this part by the following lemma.

Lemma 4.10. *One has*

$$\|q\| \le \frac{\left(1 + (1+\sigma(v))^2\right) n\theta}{1 - \sigma(v)}. \tag{4.12}$$

Proof. Combining the Lemma 3.1, Lemma 4.7 and Lemma 4.9, the result easily follows. □

4.5 Fixing θ

From the analysis for inequality (4.7) in subsection 4.2, we know that if the inequality (4.7) is satisfied, then $\sigma(v^f) < 1$ certainly holds, which means that after the feasibility step, the iterate (x^f, y^f, s^f) is strictly feasible and within the neighborhood where the Newton process targeting the μ^+-centers of (P_{ν^+}) and (D_{ν^+}) is quadratically convergent.

The inequality (4.7) certainly holds by Lemma 4.6, if $\|q\|$ satisfies

$$\frac{\|q\|^2 + (\|q\| + \sigma(v) + \theta\sqrt{n})^2}{4} < \frac{(1-\theta)}{2}(1 - \sigma(v)). \tag{4.13}$$

Since the left side of the inequality (4.13) is monotonically increasing with respect to $\sigma(v)$, while the right side is monotonically decreasing with respect to $\sigma(v)$. Given a threshold τ, for $\sigma(v) \le \tau < 1$, one has

$$\frac{\|q\|^2 + (\|q\| + \sigma(v) + \theta\sqrt{n})^2}{4} \le \frac{\|q\|^2 + (\|q\| + \tau + \theta\sqrt{n})^2}{4}$$

and

$$\frac{(1-\theta)}{2}(1-\tau) \le \frac{(1-\theta)}{2}(1-\sigma(v)).$$

To keep the new iterates (x^f, y^f, s^f) feasible and within the quadratic convergent region, by (4.13), it suffices for

$$\frac{\|q\|^2 + (\|q\| + \tau + \theta\sqrt{n})^2}{4} < \frac{(1-\theta)}{2}(1-\tau). \tag{4.14}$$

At this stage, we assume

$$\tau = \frac{1}{8}, \quad \theta = \frac{\alpha}{2\sqrt{n}} \text{ and } \alpha \le 1. \tag{4.15}$$

Substitute (4.15) into (4.14), which gives

$$\|q\|^2 + (\|q\| + \frac{1}{8} + \frac{\alpha}{2})^2 < \frac{7}{4}(1-\theta). \tag{4.16}$$

We give an upper bound for $\|q\|$, by Lemma 4.10 and (4.15), one has

$$\|q\| \le \frac{\left(1 + (1+\sigma(v))^2\right) n\theta}{1-\sigma(v)} \le \frac{\left(1 + (1+\frac{1}{8})^2\right) n\theta}{1-\frac{1}{8}} = 2.5893n\theta. \tag{4.17}$$

Combining (4.17) with the inequality (4.16), to make the new iterate (x^f, y^f, s^f) within the neighborhood where the Newton process targeting the μ^+-centers of (P_{ν^+}) and (D_{ν^+}) is quadratically convergent, it suffices for

$$6.7045n^2\theta^2 + (2.5893n\theta + 0.125 + 0.5\alpha)^2 < \frac{7}{4}(1-\theta). \tag{4.18}$$

Substitute $\theta = \dfrac{\alpha}{2\sqrt{n}}$ into (4.18), we have

$$3.3523n\alpha^2 + 0.3237\sqrt{n}\alpha + 1.2947\sqrt{n}\alpha^2 + 0.125\alpha + 0.25\alpha^2 < 0.86. \tag{4.19}$$

Let $\alpha = \frac{1}{2\sqrt{2n}}$, then an upper bound for the left side of inequality (4.19) is 0.77, which satisfies (4.19). According to (4.15), we obtain

$$\theta = \frac{1}{4\sqrt{2}n}. \tag{4.20}$$

4.6 Analysis of the centering step

In the previous subsection, we have fixed $\theta = \frac{1}{4\sqrt{2}n}$ and $\tau = \frac{1}{8}$. To complete the algorithm, we need to calculate the number of centering steps which are needed to restore the condition $\sigma(x, s; \mu^+) \le \tau$. Remember that the first centering step starts from the new iterate point (x^f, y^f, s^f), thus an upper bound for $\sigma(v^f)$ is needed.

We specify an upper bound for $\|q\|$ first. From (4.17) and (4.20), one has

$$\|q\| \leq 2.589n\theta = 0.4576.$$

Thus an upper bound for $\omega(v)^2$ can be obtained by Lemma 4.6 as follows,

$$\omega(v)^2 \leq \frac{1}{4}\left(\|q\|^2 + \left(\|q\| + \sigma(v) + \frac{1}{4\sqrt{2n}}\right)^2\right) \leq 0.1965. \tag{4.21}$$

Then, after one feasibility step, we can derive an upper bound for $\sigma(v^f)$ by Theorem 4.4, (4.20) and (4.21), one has

$$\sigma(v^f) \leq \sigma(v) + \frac{2\omega(v)^2}{(1-\theta)} \leq 0.6024. \tag{4.22}$$

Then, according to (3.4), with $\tau = 0.125$ and $\eta = 0.6024$, after the feasibility step at most

$$\left\lceil \log_2\left(\frac{\log_2 \tau}{\log_2 \eta}\right)\right\rceil = 3 \tag{4.23}$$

centering steps suffice to get an iterate (x^+, y^+, s^+) that satisfies $\sigma(x, s; \mu^+) \leq \tau$.

$\boxed{4.7}$ The iteration bound

In the previous sections, we have found that if at the start of an iteration, the iterates satisfy $\sigma(x, s; \mu) \leq \tau$ with τ and θ as defined in (4.15) and (4.20), then after the feasibility step and the μ-update are taken, the iterates satisfy $\sigma(v^f) \leq 0.6024$.

According to (4.23), at most three centering steps suffice to get iterates that satisfy $\sigma(x, s; \mu^+) \leq \tau$. So each iteration consists of one feasibility step and three centering steps.

In each iteration, both the duality gap and the norms of the residual vectors are reduced by the factor $1-\theta$. Hence, by $(x^0)^T s^0 = n\xi^2$, the total number of main iterations are bounded above by

$$\frac{1}{\theta}\log\frac{\max\{n\xi^2,\ \|r_b^0\|,\ \|r_c^0\|\}}{\epsilon}.$$

Since $\theta = \dfrac{1}{4\sqrt{2n}}$, the total number of inner iterations are bounded above by

$$16\sqrt{2}n\log\frac{\max\{n\xi^2,\ \|r_b^0\|,\ \|r_c^0\|\}}{\epsilon}.$$

In the following we state our main result without further proof.

Theorem 4.11. *If (P) and (D) have optimal solutions x^* and (y^*, s^*) such that $\|x^* + s^*\|_\infty \leq \xi$, then after at most*

$$16\sqrt{2}n\log\frac{\max\{n\xi^2,\ \|r_b^0\|,\ \|r_c^0\|\}}{\epsilon}$$

iterations the algorithm finds an ϵ-solution of (P) and (D).

5 Conclusions

We have presented a new full-Newton step IIPM for LO. In each main iteration, the method operates one feasibility step and at most three centering steps. The iteration bound of the method is as good as the best-known iteration results for IIPMs.

Our further research line may focus on two aspects. One is to design the IIPM with large-updated feasibility step and full centering step, and the other is improve the algorithm by adaptive-updating strategy.

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