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A FAMILY OF GENERALIZED NCP-FUNCTIONS AND A SMOOTHING REFORMULATION OF D-EIGENVALUES

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Abstract: Diffusion kurtosis imaging (DKI) is a new Magnetic Resonance Imaging(MRI) model to characterize the non-Gaussian diffusion behavior in tissues. D-eigenvalues of diffusion kurtosis tensors provide important characterizations of DKI. In this paper, we first investigate some properties of a family of generalized nonlinear complementarity problem functions (NCP-functions, for short), such as smoothness, coerciveness, etc. Then, using this class of NCP-functions, we reformulate the D-eigenvalue problem of diffusion kurtosis tensor as a system of generalized non-smooth equations. A smoothing Newton method for solving such equations is proposed, and it is shown that the proposed method is globally convergent under suitable assumptions. Some preliminary numerical results based on randomly generated data are reported at the end of this paper, which show the proposed method is promising.

Key words: *complementarity problem, NCP-function, diffusion kurtosis tensor, D-eigenvalue.*

Mathematics Subject Classification: *90C33, 65K10*

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1 Introduction

Magnetic Resonance Imaging (MRI) [3], especially Diffusion Tensor Imaging (DTI) [1, 2], has great applications in medical engineering and other fields. However, DTI suffers a great challenge due to its limitation of dealing with non-Gaussian diffusion. Then, Diffusion Kurtosis Imaging (DKI), a new medical imaging model, was proposed [15,18]. Such a model has the ability to deal with non-Gaussian diffusion [7,10,15,18,22,27]. One of the key issues in DKI is to handle the diffusion kurtosis tensor in that model [10, 15, 18, 22]. A diffusion kurtosis tensor W is a fourth-order three-dimensional fully symmetric tensor. It has fifteen independent elements W_{ijkl} with W_{ijkl} being invariant for any permutation of its indices $i, j, k, l = 1, 2, 3$. The diffusion kurtosis tensor comes from the following extended Stejskal and Tanner equation [25]:

$$ln[S(b)] = ln[S(0)] - bD_{app} + \frac{1}{6}b^2 D_{app}^2 K_{app}, \tag{1.1}$$

where K_{app} is the apparent kurtosis coefficient (AKC) at the direction $x = (x_1, x_2, x_3)^T \in \Re^3$:

$$K_{app} = \frac{M_D^2}{D_{app}^2} W x^4,$$

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and

$$\begin{aligned} D_{app} &:= Dx^2 := \sum_{i,j=1}^{3} D_{ij}x_ix_j; \\ M_D &:= D_{11} + D_{22} + D_{33}; \\ Wx^4 &:= \sum_{i,j,k,l=1}^{3} W_{ijkl}x_ix_jx_kx_l. \end{aligned}$$

Qi, Wang and Wu [22] proposed the concept of D-eigenvalues for diffusion kurtosis tensor W in the following way: a real number λ, together with a real vector $x \in \Re^3$, is called a D-eigenvalue of W, and the real vector x is called a D-eigenvector of W associated with the D-eigenvalue λ, if x is a critical point of the following optimization problem and $\frac{1}{2}\lambda$ is the associated Lagrange multiplier:

$$\begin{array}{rl} \max & Wx^4 \\ s.t. & Dx^2 = 1. \end{array} \tag{1.2}$$

The following theorem is established in [22].

Theorem 1.1 ([22, Theorem 1]). *D-eigenvalues always exist for any given diffusion kurtosis tensor W. If x is a D-eigenvector associated with a D-eigenvalue λ, then*

$$\lambda = Wx^4.$$

The largest AKC value is equal to $M_D^2\lambda_{max}$, and the smallest AKC value is equal to $M_D^2\lambda_{min}$, where λ_{max} and λ_{min} are the largest and the smallest D-eigenvalues of W respectively.

Since the extreme AKC values are important clinical indicators, the extreme D-eigenvalues of the diffusion kurtosis tensor are of great significance. As the D-eigenvalues are defined through the optimality conditions of problem (1.2), we can resolve it by algorithms and techniques for nonlinear complementarity problems (NCPs) [14, 21]. In the fashion of recent investigations [4, 8, 9, 11], in this paper, we first introduce a new family of generalized NCP-functions. Many favorite properties of this new family of NCP-functions are investigated. This new family of NCP-functions includes many existing NCP-functions as special cases. By the numerical experiments of [4, 8, 11], it is expected that algorithms for the reformulations of NCPs based on this family of NCP-functions perform very well. So, we reformulate the D-eigenvalue problem of the diffusion kurtosis tensor as a system of non-smooth equations, and then a Newton method is applied to such a system of equations. Such a method is faster than the method proposed in [22]. While a fast algorithm is a basic requirement for clinical applications, it deserves investigation.

Definition 1.2. A fourth-order n-dimensional tensor W is called a nonsingular (NS) tensor, if given any $0 \neq x \in \Re^n$, there is $1 \leq i \leq n$, satisfying $x_i(Wx^3)_i \neq 0$.

Remark 1.3. It is easy to see that the concept of NS tensor is not vacuous. For example, it contains the class of positive definite tenors defined in [23], i.e., W is called positive definite if and only if $Wx^4 > 0$ for all nonzero x.

Assumption 1.4. We assume in the following that the fourth-order three-dimensional tensor W is an NS tensor.

In fact, we find that the assumption is reasonable and always holds in practice.

The rest of this paper is organized as follows: a family of generalized NCP-functions is proposed in the next section, and its many properties are discussed. In Section 3, we reformulate the D-eigenvalue problem of diffusion kurtosis tensor as a system of non-smooth equations, and then introduce a smoothing Newton method to solve such a system of equations based on the proposed NCP-functions in Section 2. Some preliminary numerical results for randomly generated data are presented in Section 4. Some final remarks are given in the last section.

$\boxed{2}$ A Family of Generalized NCP-Functions

In this section, we propose a family of generalized NCP-functions, and discuss its many properties which are important for smoothing algorithms.

First, we notice that for any given $x \in \Re^n$ and $1 < p_1 < p_2$, we have $\|x\|_{p_2} \le \|x\|_{p_1} \le n^{(\frac{1}{p_1}-\frac{1}{p_2})}\|x\|_{p_2}$, and $\|x\|_1 \le \sqrt{n}\|x\|_2$, where we denote $\|\cdot\|_p$ the p-norm of vector for any given $p > 1$. Then, we state the following lemma.

Lemma 2.1. *For every $1 < p \le 2, \theta \in (0, 2]$ and $(a, b) \in \Re^2$, the following inequality holds:*

$$\theta(|a|^p + |b|^p) + (1-\theta)|a+b|^p \ge 0.$$

Proof. If $1 < p \le 2$, and $\theta \in (0, 1]$, then the result is obvious. Now, we assume that $1 < p \le 2$ and $\theta \in (1, 2]$. Let $\gamma = \theta - 1$, then $\gamma \in (0, 1]$, and $\theta(|a|^p + |b|^p) + (1-\theta)|a+b|^p$ is reduced as

$$(1+\gamma)(|a|^p + |b|^p) - \gamma|a+b|^p.$$

With the observations before this lemma, we have

$$|a+b|^p \le (\sqrt{2}\|(a,b)\|_2)^p \le (\sqrt{2}\|(a,b)\|_p)^p \le 2\|(a,b)\|_p^p = 2(|a|^p + |b|^p).$$

Hence,

$$\begin{aligned}(1+\gamma)(|a|^p + |b|^p) - \gamma|a+b|^p &\ge 2\gamma(|a|^p + |b|^p) - \gamma|a+b|^p \\ &\ge 2\gamma(|a|^p + |b|^p) - 2\gamma(|a|^p + |b|^p) \\ &= 0.\end{aligned}$$

Here, the first inequality follows from $\gamma \in (0, 1]$. The proof is complete. □

For the convenience of the subsequent analysis, we define

$$\eta_{\theta p}(a,b) := \sqrt[p]{\theta(|a|^p + |b|^p) + (1-\theta)|a+b|^p}, \quad p \in (1,2], \quad \theta \in (0,2), \ (a,b) \in \Re^2. \tag{2.1}$$

Definition 2.2. A function is called an NCP-function, if for all $(a, b) \in \Re^2$, we have that

$$\phi(a,b) = 0 \quad \Longleftrightarrow \quad a \ge 0, \ b \ge 0, \ ab = 0.$$

We now introduce a family of generalized NCP-functions as follows:

$$\begin{aligned}\phi_{\theta p}(a,b) &= \sqrt[p]{\theta(|a|^p + |b|^p) + (1-\theta)|a+b|^p} - a - b, \\ &\quad p \in (1,2], \quad \theta \in (0,2], \ (a,b) \in \Re^2.\end{aligned} \tag{2.2}$$

Remark 2.3. We note that the same formula of $\phi_{\theta p}$ in the framework of second order cone is proposed in [9], while parameters are chosen as $\theta \in (0,2), p = 2$ or $\theta \in (0,1], p \in (1,2)\bigcup(2,+\infty)$ in that paper. Consequently, the function defined by (2.2) is different from that of [9] even if the latter reduces to the framework of $\Re^n_+$. We also note that only smoothness of $\phi_{\theta p}$ in [9] is investigated, so the properties discussed in the subsequence are new even for the cases $\theta \in (0,1], p \in (1,2]$.

Proposition 2.4. *The function $\phi_{\theta p}$ defined by* (2.2) *is an NCP-function.*

Proof. According to the definition of the NCP-function, we divide the proof into two parts.

Firstly, we assume that $a \geq 0, b \geq 0, ab = 0$. Without loss of generality, we assume that $a = 0$. With the expression of $\phi_{\theta p}$, we can get $\phi_{\theta p} = 0$ easily.

Secondly, we assume that $\phi_{\theta p} = 0$, i.e., $\sqrt[p]{\theta(|a|^p + |b|^p) + (1-\theta)|a+b|^p} = a + b$. Hence, $a + b \geq 0$. Furthermore, we get

$$\theta(|a|^p + |b|^p) + (1-\theta)|a+b|^p = (a+b)^p,$$

that is to say

$$\theta(|a|^p + |b|^p) = \theta(a+b)^p.$$

Since $\theta \in (0,2]$, we have $|a|^p + |b|^p = (a+b)^p$. From this and [4, Proposition 3.1], we get $a \geq 0, b \geq 0, ab = 0$.

Consequently, the function $\phi_{\theta p}$ is an NCP-function for all $1 < p \leq 2, \theta \in (0,2]$. $\square$

Moreover, applying the Minikowski inequality, the following relation

$$\begin{aligned}
& \eta_{\theta p}((a,b)+(c,d)) \\
= & \sqrt[p]{\theta(|a+c|^p + |b+d|^p) + (1-\theta)|a+c+b+d|^p} \\
= & \sqrt[p]{|\theta^{\frac{1}{p}} a + \theta^{\frac{1}{p}} c|^p + |\theta^{\frac{1}{p}} b + \theta^{\frac{1}{p}} d|^p + |(1-\theta)^{\frac{1}{p}}(a+b) + (1-\theta)^{\frac{1}{p}}(c+d)|^p} \\
\leq & \sqrt[p]{|\theta^{\frac{1}{p}} a|^p + |\theta^{\frac{1}{p}} b|^p + |(1-\theta)^{\frac{1}{p}}(a+b)|^p} + \sqrt[p]{|\theta^{\frac{1}{p}} c|^p + |\theta^{\frac{1}{p}} d|^p + |(1-\theta)^{\frac{1}{p}}(c+d)|^p} \\
= & \sqrt[p]{\theta(|a|^p + |b|^p) + (1-\theta)|(a+b)|^p} + \sqrt[p]{\theta(|c|^p + |d|^p) + (1-\theta)|(c+d)|^p} \\
= & \eta_{\theta p}(a,b) + \eta_{\theta p}(c,d)
\end{aligned}$$

holds for any $(a,b),(c,d) \in \Re^2$, and $1 < p \leq 2, \theta \in (0,1]$. Thus, it is easy to see that $\eta_{\theta p}$ is a norm on $\Re^2$ for all $1 < p \leq 2, \theta \in (0,1]$.

The concept of semismoothness of functions is somewhat the core of generalized Newton method or smoothing Newton method. It was first introduced by Miffin [19] for functions, and then extended to vector valued functions by Qi and Sun [21].

Definition 2.5. A locally Lipschitz continuous function $F : \Re^n \to \Re^n$ is semismooth (respectively, strongly semismooth) at $x \in \Re^n$, if it satisfies

(i) F is directionally differentiable at $x \in \Re^n$.

(ii) For all $V \in \partial F(x+h)$, $F(x+h) - F(x) - Vh = o(\|h\|_2)(respectively, O(\|h\|_2^2))$, where $\partial F(\cdot)$ denotes the generalized Jacobian in the sense of Clarke [5].

Theorem 2.6. *For all $1 < p \leq 2, \theta \in (0,1]$, we have*

(i) *$\phi_{\theta p}$ is sub-additive, i.e., for all $(a,b),(c,d) \in \Re^2$, it holds $\phi_{\theta p}((a,b)+(c,d)) \leq \phi_{\theta p}(a,b) + \phi_{\theta p}(c,d)$.*

(ii) $\phi_{\theta p}$ *is positive homogenous, i.e., for all* $(a,b) \in \Re^2$ *and* $\alpha > 0$, *it holds* $\phi_{\theta p}(\alpha(a,b)) = \alpha\phi_{\theta p}(a,b)$.

(iii) $\phi_{\theta p}$ *is a convex function.*

(iv) $\phi_{\theta p}$ *is Lipschitz continuous on* $\Re^2$.

Proof. By the fact that $\phi_{\theta p}(a,b) = \eta_{\theta p}(a,b) - (a+b)$ and $\eta_{\theta p}(a,b)$ is a norm, we can obtain that the results (i), (ii), and (iii) hold immediately.

We now consider the result (iv). Since $\eta_{\theta p}$ is a norm on $\Re^2$ and any two norms in finite dimensional space are equivalent, there exists a positive constant κ such that

$$\eta_{\theta p}(a,b) \leq \kappa\|(a,b)\|_2, \quad \forall (a,b) \in \Re^2,$$

Hence, for any $(a,b),(c,d) \in \Re^2$,

$$\begin{aligned}
|\phi_{\theta p}(a,b) - \phi_{\theta p}(c,d)| &= |(\eta_{\theta p}(a,b) - (a+b)) - (\eta_{\theta p}(c,d) - (c+d))| \\
&\leq |\eta_{\theta p}(a,b) - \eta_{\theta p}(c,d)| + |(a-c) + (b-d)| \\
&\leq \eta_{\theta p}(a-c, b-d) + |a-c| + |b-d| \\
&\leq \kappa\|(a-c, b-d)\|_2 + \sqrt{2}\|(a-c, b-d)\|_2 \\
&= (\kappa + \sqrt{2})\|(a-c, b-d)\|_2.
\end{aligned}$$

Hence, $\phi_{\theta p}$ is Lipschitz continuous, and the Lipschitz constant is $\kappa + \sqrt{2}$. Thus, the result (iv) holds. The proof is complete. □

Theorem 2.7. *For all* $1 < p \leq 2, \theta \in (0,2)$, $\phi_{\theta p}$ *satisfies*

(i) $\phi_{\theta p}$ *is continuously differentiable on* $\Re^2 \backslash \{(0,0)\}$.

(ii) $\phi_{\theta p}$ *is locally Lipschitz continuous.*

(iii) $\phi_{\theta p}$ *is strongly semismooth on* $\Re^2$.

Proof. (i) When $(a,b) \neq (0,0)$, we get $\eta_{\theta p}(a,b) > 0$. By a direct calculation, we have

$$\frac{\partial \phi_{\theta p}(a,b)}{\partial a} = \frac{\theta sgn(a)|a|^{p-1} + (1-\theta)sgn(a+b)|a+b|^{p-1}}{\eta_{\theta p}^{p-1}(a,b)} - 1; \tag{2.3}$$

$$\frac{\partial \phi_{\theta p}(a,b)}{\partial b} = \frac{\theta sgn(b)|b|^{p-1} + (1-\theta)sgn(a+b)|a+b|^{p-1}}{\eta_{\theta p}^{p-1}(a,b)} - 1, \tag{2.4}$$

where $sgn(\cdot)$ is the signum function. Now, it is obvious from (2.3) and (2.4) that the result (i) holds.

(ii)$\phi_{\theta p}$ is continuously differentiable on $\Re^2 \backslash \{(0,0)\}$, so $\phi_{\theta p}$ is locally Lipschitz continuous. Consequently, it is sufficient to prove that $\phi_{\theta p}$ is locally Lipschitz continuous at $(0,0)$.

For all $1 < p \leq 2, \theta \in (0,2)$, $\forall (h,k) \in \Re^2 \backslash \{(0,0)\}$,

$$\begin{aligned}
|\phi_{\theta p}(h,k) - \phi_{\theta p}(0,0)| &= |\sqrt[p]{\theta(|h|^p + |k|^p) + (1-\theta)|h+k|^p} - (h+k)| \\
&\leq |\sqrt[p]{\theta(|h|^p + |k|^p) + |1-\theta||h+k|^p}| + |h+k| \\
&\leq \sqrt[p]{\theta(2^{1/p-1/2}\|(h,k)\|_2)^p + |1-\theta|(\sqrt{2}\|(h,k)\|_2)^p} + \|(h,k)\|_2 \\
&\leq \kappa\|(h,k)\|_2 \\
&= \kappa\|(h,k) - (0,0)\|_2,
\end{aligned}$$

where κ is a positive constant. So, it is easy to get that $\phi_{\theta p}$ is locally Lipschitz continuous at $(0,0)$.

(iii)Based on the results of (i) and (ii), noticing that $\phi_{\theta p}$ is continuously differentiable except $(0,0)$, it is easy to see that $\phi_{\theta p}$ is strongly smeismooth on $\Re^2\backslash\{(0,0)\}$. Consequently, it is sufficient to prove that $\phi_{\theta p}$ is strongly semismooth at $(0,0)$. For any $(h,k)\in\Re^2\backslash\{(0,0)\}$, $\phi_{\theta p}$ is differentiable at (h,k), and hence,

$$\nabla\phi_{\theta p}(h,k)=(\partial_a\phi_{\theta p}(h,k),\partial_b\phi_{\theta p}(h,k)),$$

So, we have

$$\begin{aligned}
& \phi_{\theta p}((0,0)+(h,k))-\phi_{\theta p}(0,0)-(\partial_a\phi_{\theta p}(h,k),\partial_b\phi_{\theta p}(h,k))(h,k)^T\\
= & \sqrt[p]{\theta(|h|^p+|k|^p)+(1-\theta)|h+k|^p}-(h+k)\\
& -(\frac{\theta sgn(h)|h|^{p-1}+(1-\theta)sgn(h+k)|h+k|^{p-1}}{\eta_{\theta p}^{p-1}(h,k)}-1)h\\
& -(\frac{\theta sgn(k)|k|^{p-1}+(1-\theta)sgn(h+k)|h+k|^{p-1}}{\eta_{\theta p}^{p-1}(h,k)}-1)k\\
= & \sqrt[p]{\theta(|h|^p+|k|^p)+(1-\theta)|h+k|^p}\\
& -\frac{\theta(sgn(h)|h|^{p-1}h+sgn(k)|k|^{p-1}k)+(1-\theta)sgn(h+k)|h+k|^{p-1}(h+k)}{\eta_{\theta p}^{p-1}(h,k)}\\
= & \sqrt[p]{\theta(|h|^p+|k|^p)+(1-\theta)|h+k|^p}-\frac{\theta(|h|^p+|k|^p)+(1-\theta)|h+k|^p}{\eta_{\theta p}^{p-1}(h,k)}\\
= & \frac{\eta_{\theta p}^{p}(h,k)-\theta(|h|^p+|k|^p)-(1-\theta)|h+k|^p)}{\eta_{\theta p}^{p-1}(h,k)}\\
= & 0=O(\|(h,k)\|_2^2).
\end{aligned}$$

Thus, we obtain that $\phi_{\theta p}$ is strongly semismooth at $(0,0)$. The proof is complete. □

The following theorems deal with the related coerciveness property of $\phi_{\theta p}$ and $\Phi_{\theta p}$ defined below, which is important to analyze the convergence of smoothing algorithms [12].

Theorem 2.8. *Let $\phi_{\theta p}$ be defined by (2.2) for $p\in(1,2]$ and $\theta\in(0,2)$. Then, $|\phi_{\theta p}(a^k,b^k)|\to+\infty$ if the sequence $\{(a^k,b^k)\}\subseteq\Re^2$ satisfies one of the following conditions:*
(i) $a^k\to-\infty$; (ii) $b^k\to-\infty$;
(iii) $a^k\to+\infty$, $b^k\to+\infty$ *and* $\lim_{k\to+\infty}\frac{|a^k|^p+|b^k|^p}{|a^k+b^k|^p}<1$.

Proof. (i) Suppose that $a^k\to-\infty$. If there exists a constant M such that $b^k\le M$ for sufficiently large k, then $|\phi_{\theta p}(a^k,b^k)|\to+\infty$ is obvious. Now, we suppose $b^k\to+\infty$.

a) When $\{a^k+b^k\}$ is bounded from above, it is easy to see from the definition of $\phi_{\theta p}$ that $|\phi_{\theta p}(a^k,b^k)|\to+\infty$.

b) When $a^k+b^k\to+\infty$, it follows that $|a^k+b^k|^p\le|a^k-b^k|^p$ for sufficiently large k.

If $\theta\in(0,1]$, then

$$\begin{aligned}
\phi_{\theta p}(a^k,b^k) &= \sqrt[p]{\theta(|a^k|^p+|b^k|^p)+(1-\theta)|a^k+b^k|^p}-(a^k+b^k)\\
&\le \sqrt[p]{\theta(|a^k|^p+|b^k|^p)+(1-\theta)|a^k-b^k|^p}-(a^k+b^k).
\end{aligned}$$

By [8, Proposition 2.4], we know that $\sqrt[p]{\theta(|a^k|^p + |b^k|^p) + (1-\theta)|a^k - b^k|^p} - (a^k + b^k) \to -\infty$ and then $|\phi_{\theta p}(a^k, b^k)| \to +\infty$.

If $\theta \in (1,2)$, let $\gamma = \theta - 1 \in (0,1)$, then $\theta = \gamma + 1$. Since $|a^k + b^k|^p \le |a^k|^p + |b^k|^p$, or $-\gamma|a^k + b^k|^p \ge -\gamma(|a^k|^p + |b^k|^p)$, we have

$$\begin{aligned}\phi_{\theta p}(a^k, b^k) &= \sqrt[p]{\theta(|a^k|^p + |b^k|^p) + (1-\theta)|a^k + b^k|^p} - (a^k + b^k) \\ &= \sqrt[p]{(\gamma+1)(|a^k|^p + |b^k|^p) - \gamma|a^k + b^k|^p} - (a^k + b^k) \\ &\ge \sqrt[p]{|a^k|^p + |b^k|^p} - (a^k + b^k) \ge |b^k| - (a^k + b^k) = -a^k \to +\infty.\end{aligned}$$

Hence, $|\phi_{\theta p}(a^k, b^k)| \to +\infty$.

Thus, when $a \to -\infty, b \to +\infty$, and $a^k + b^k \to +\infty$, it follows that $|\phi_{\theta p}(a^k, b^k)| \to +\infty$.

(ii) For the case of $b^k \to -\infty$, the proof is similar to that of (i).
(iii) Suppose that $a^k \to +\infty$ and $b^k \to +\infty$ as $k \to +\infty$, then $|a^k + b^k| \to +\infty$.

a) When $\theta \in (0,1]$, if $\frac{a^k}{b^k} \to +\infty$ or $\frac{a^k}{b^k} \to 0$, it is easy to see that these contradict the condition that $\lim_{k \to +\infty} \frac{|a^k|^p + |b^k|^p}{|a^k + b^k|^p} < 1$. Hence, $\frac{a^k}{b^k} \to c > 0$, and then $\frac{|a^k|^p + |b^k|^p}{|a^k + b^k|^p} \to \frac{c^p + 1}{(c+1)^p} \in (0,1)$. Thus, we get that

$$\left\{ \left| \sqrt[p]{\theta \cdot \frac{|a^k|^p + |b^k|^p}{|a^k + b^k|^p} + (1-\theta)} - 1 \right| \right\}$$

has a positive bounded from below. Furthermore, it follows

$$|\phi_{\theta p}(a^k, b^k)| = |a^k + b^k| \left| \sqrt[p]{\theta \cdot \frac{|a^k|^p + |b^k|^p}{|a^k + b^k|^p} + (1-\theta)} - 1 \right| \to +\infty$$

as $k \to +\infty$.

b) Let $\theta \in (1,2)$, let $\gamma = \theta - 1 \in (0,1)$, then $\theta = \gamma + 1$. Since $|a^k + b^k|^p \ge |a^k|^p + |b^k|^p$, or $-\gamma|a^k + b^k|^p \le -\gamma(|a^k|^p + |b^k|^p)$, we have

$$\begin{aligned}\phi_{\theta p}(a^k, b^k) &= \sqrt[p]{\theta(|a^k|^p + |b^k|^p) + (1-\theta)|a^k + b^k|^p} - (a^k + b^k) \\ &= \sqrt[p]{(\gamma+1)(|a^k|^p + |b^k|^p) - \gamma|a^k + b^k|^p} - (a^k + b^k) \\ &\le \sqrt[p]{|a^k|^p + |b^k|^p} - (a^k + b^k).\end{aligned}$$

By [8, Proposition 2.4], we know that $\sqrt[p]{|a^k|^p + |b^k|^p} - (a^k + b^k) \to -\infty$ when (iii) is satisfied and $k \to +\infty$. Therefore,

$$|\phi_{\theta p}(a^k, b^k)| \to +\infty \quad \text{as} \quad k \to +\infty.$$

The proof is complete. □

Definition 2.9. Let $\phi : \Re^n \to \Re^n$ be a vector valued function.

- ϕ is said to be strongly monotone with modulus $\mu > 0$ if $(x-y)^T(\phi(x)-\phi(y)) \geq \mu\|x-y\|_2^2$ for all $x, y \in \Re^n$.
- ϕ is said to be a uniform P-function with modulus $\mu > 0$ if $\max_{1\leq i\leq n}(x_i - y_i)(\phi_i(x) - \phi_i(y)) \geq \mu\|x-y\|_2^2$ for all $x, y \in \Re^n$.
- ϕ is said to be coercive, if $\lim_{\|x\|_2\to+\infty} \|\phi(x)\|_2 = +\infty$.

Define the following function:

$$\Phi_{\theta p}(x) := \begin{bmatrix} \phi_{\theta p}(x_1, F_1(x)) \\ \phi_{\theta p}(x_2, F_2(x)) \\ \vdots \\ \phi_{\theta p}(x_n, F_n(x)) \end{bmatrix}, \tag{2.5}$$

where $F : \Re^n \to \Re^n, 1 < p \leq 2, \theta \in (0,2), \phi_{\theta p}$ is defined by (2.2).

Theorem 2.10. *Let $\Phi_{\theta p}$ be defined by* (2.5) *with $1 < p \leq 2, \theta \in (0,2)$. If the continuous function F is a uniform P-function with modulus $\mu > 0$. and satisfies that*

$$\lim_{\|x^k\|_2\to+\infty} \max_{1\leq i\leq n} \frac{|F_i(x^k)|}{\|x^k\|_2} < +\infty$$

for any $\{x^k\} \subset \Re^n$, and $\|x^k\|_2 \to +\infty$. Then, $\Phi_{\theta p}$ is coercive.

Proof. Assume that $\{x^k\} \subset \Re^n$ satisfies $\|x^k\|_2 \to +\infty$. Set $J := \{i : |x_i^k| \to +\infty\}$, then, J is nonempty. Let $\{y^k\} \subset \Re^n$ be defined as

$$y_i^k := \begin{cases} x_i^k, & i \notin J \\ 0, & i \in J \end{cases} \qquad \forall k \in \{1, 2, \dots\}.$$

Then, $\{y^k\}$ is bounded. Since F is a continuous and uniform P-function with modulus $\mu > 0$, we have

$$\begin{aligned} \mu \sum_{i\in J}(x_i^k)^2 &= \mu\|x^k - y^k\|_2^2 \\ &\leq \max_{1\leq i\leq n}(x_i^k - y_i^k)(F_i(x^k) - F_i(y^k)) \\ &= \max_{i\in J}(x_i^k - y_i^k)(F_i(x^k) - F_i(y^k)) \\ &= \max_{i\in J} x_i^k(F_i(x^k) - F_i(y^k)) \\ &= x_{i_0}^k(F_{i_0}(x^k) - F_{i_0}(y^k)) \\ &\leq |x_{i_0}^k| \cdot |F_{i_0}(x^k) - F_{i_0}(y^k)| \\ &\leq \sqrt{\sum_{i\in J}(x_i^k)^2}|F_{i_0}(x^k) - F_{i_0}(y^k)|. \end{aligned}$$

As $\sum_{i\in J}(x_i^k)^2 \to +\infty$, $\{y^k\}$ is bounded, and F is continuous, we can get that $\|F(x^k)\|_2 \to +\infty$. Now, Furthermore, since F is a uniform P-function with modulus $\mu > 0$, we have that $\lim_{\|x^k\|_2\to+\infty} \max_{1\leq i\leq n} \frac{|F_i(x^k)|}{\|x^k\|_2} \geq \mu > 0$. Now, combining it with

$\lim_{\|x^k\|_2\to+\infty} \max_{1\le i\le n} \frac{|F_i(x^k)|}{\|x^k\|_2} < +\infty$, it is easy to get that (iii) in Theorem 2.8 holds. Hence, we get $\lim_{k\to+\infty} \|\Phi_{\theta p}\|_2 = +\infty$, that is , $\Phi_{\theta p}$ is coercive. □

By a similar analysis of the proof in Theorem 2.10, we could get the following result.

Theorem 2.11. *Suppose that $\Phi_{\theta p}$ is defined as (2.5). If the continuous function $F : \Re^n \to \Re^n$ is strongly monotone with modulus $\mu > 0$ for all $1 < p \le 2$ and $\theta \in (0, 2)$, and satisfies $\lim_{\|x^k\|_2\to+\infty} \max_{1\le i\le n} \frac{|F_i(x^k)|}{\|x^k\|_2} < +\infty$ for any $\{x^k\} \subset \Re^n$ and $\|x^k\|_2 \to +\infty$. Then, $\Phi_{\theta p}$ is coercive.*

3 Smoothing Algorithm

We restate the D-eigenvalue problem:

$$\begin{array}{ll} \max & Wx^4 \\ s.t. & Dx^2 = 1, \end{array} \tag{3.1}$$

then, a D-eigenvalue λ and the corresponding D-eigenvector x of W satisfy the optimality conditions of (3.1) as follows:

$$Wx^3 = \lambda Dx; \quad Dx^2 = 1. \tag{3.2}$$

For the optimization problem (3.1), although it has a simple form, it is highly nonlinear and nonconvex. Actually, without the restriction of the dimension of tensor W to be three, such a problem is NP-hard [16, 26]. While what we need is the critical points of programming (3.1) in the applications of medical imaging [22], it may be more suitable to solve the KKT-system of some variants of the above programming. In the following, we construct a familiar complementarity problem from the optimality conditions of (3.1) to get the D-eigenvalue pairs. At first, we present the following results.

Theorem 3.1. *Suppose that Assumption 1.4 holds. Then the D-eigenvalue λ and the corresponding D-eigenvector x of W can be obtained from the optimality conditions of the following problem:*

$$\begin{array}{ll} \max & Wx^4 \\ s.t. & Dx^2 \le 1, \\ & Dx^2 \ge 0.5. \end{array} \tag{3.3}$$

Proof. The optimality conditions of (3.3) are to find $x \in \Re^3$ and $\lambda := (\lambda_1, \lambda_2) \in \Re^2$ such that

$$\begin{array}{c} 2Wx^3 + \lambda_1 Dx - \lambda_2 Dx = 0; \\ 1 - Dx^2 \ge 0; \;\; Dx^2 - 0.5 \ge 0; \;\; \lambda_1 \ge 0; \;\; \lambda_2 \ge 0; \\ \lambda_1(1 - Dx^2) = 0; \;\; \lambda_2(Dx^2 - 0.5) = 0, \end{array} \tag{3.4}$$

where Wx^3 denotes a vector whose i-th component is $\sum_{j,k,l=1}^{3} W_{ijkl}x_jx_kx_l$.

Suppose that $(x^*, \lambda_1^*, \lambda_2^*) \in \Re^3 \times \Re \times \Re$ is a solution to (3.4). The proof is divided into three parts.

Part 1. Suppose that $1 - D(x^*)^2 = 0$. In this case, it follows from (3.4) that $\lambda_1^* \ge 0$ and $\lambda_2^* = 0$. By the first equality of (3.4), we have

$$2W(x^*)^3 + \lambda_1^* Dx^* = 0.$$

Thus, $(x^*, -\frac{1}{2}\lambda_1^*)$ solves the optimality conditions (3.2) of (3.1).

Part 2. Suppose that $D(x^*)^2 - 0.5 = 0$. In this case, it follows from (3.4) that $D(\frac{x^*}{\sqrt{0.5}})^2 = 1$ and $\lambda_2^* \geq 0, \lambda_1^* = 0$. By the first equality of (3.4), we have

$$2W(x^*)^3 - \lambda_2^* Dx^* = 0,$$

So, $W(\frac{x^*}{\sqrt{0.5}})^3 = \frac{\lambda_2^*}{2} \cdot \frac{1}{\sqrt{0.5}^2} D(\frac{x^*}{\sqrt{0.5}}) = \lambda_2^* D(\frac{x^*}{\sqrt{0.5}})$. Thus, $(\frac{x^*}{\sqrt{0.5}}, \lambda_2^*)$ is a solution of (3.2).

Part 3. Suppose that $1 - Dx^2 > 0$ and $Dx^2 - 0.5 > 0$, then $x^* \neq 0$ and $\lambda_1^* = 0, \lambda_2^* = 0$. Therefore, $W(x^*)^3 = 0$, but $0 \neq x^* \in \Re^3$, which contradicts Assumption 1.4.

Combining Part 1-Part 3, we complete the proof. □

Let $F : \Re^n \to \Re^n$ be a nonlinear mapping, the nonlinear complementarity problem (NCP for short) is to find $x \in \Re^n$ such that

$$x \geq 0; \quad F(x) \geq 0 \ \text{ and } \ x^T F(x) = 0.$$

For two functions $G : \Re^n \times \Re^m \to \Re^n$ and $H : \Re^n \times \Re^m \to \Re^m$, the mixed nonlinear complementarity problem (MNCP for short) is to find $x \in \Re^n$, $y \in \Re^m$ such that

$$G(x, y) = 0; \quad y \geq 0; \quad H(x, y) \geq 0 \ \text{ and } \ y^T H(x, y) = 0.$$

Therefore, let $G(x, \lambda) = 2Wx^3 + \lambda_1 Dx - \lambda_2 Dx, H_1(x, \lambda) = 1 - Dx^2, H_2(x, \lambda) = Dx^2 - 0.5, n = 3, m = 2$, then the system (3.4) is an MNCP. Furthermore, if $x \in \Re^3$ is the optimal solution of (3.1), then there is a $\lambda \in \Re^2$ satisfying (3.4).

Theorem 3.2. *Suppose that D is positive definite. Then the solution set of* (3.4) *is nonempty and bounded.*

Proof. The object function of Theorem 3.1 is continuous and its feasible set is nonempty and bounded. Hence, the solution set of Theorem 3.1 is nonempty from the classical results of analysis. Consequently, the solution set of (3.4) is nonempty. From $1 - Dx^2 \geq 0, Dx^2 - 0.5 \geq 0$ and the positive definiteness of D, we have that the set of all x satisfying (3.4) is nonempty and bounded. Therefore, the sequence of λ satisfying $2Wx^3 + \lambda_1 Dx - \lambda_2 Dx = 0$ is bounded. The proof is complete. □

Since D is always positive definite in practice, we assume in the sequel that D is positive definite, and hence the solution set of (3.4) is always nonempty and bounded.

Definition 3.3. A function $H^\varepsilon : \Re^n \to \Re^m$ is called a smooth approximation of a function $H : \Re^n \to \Re^m$, if for any $\varepsilon > 0$, H^ε is continuous and differentiable, and for any $x \in \Re^n$, when $\varepsilon \downarrow 0$,

$$\|H^\varepsilon - H\|_2 \to 0.$$

Theorem 3.4. *For all $1 < p \leq 2, \theta \in (0, 2)$, the function defined as following*

$$\phi_{\theta p}^\varepsilon(a, b) := \sqrt[p]{\theta(|a|^p + |b|^p) + (1 - \theta)|a + b|^p + 4\varepsilon} - a - b, \quad \forall (a, b) \in \Re^2, \forall \varepsilon > 0 \qquad (3.5)$$

is a smooth approximation of $\phi_{\theta p}$.

Proof. From Definition 3.3, the desired result can be easily checked. We omit the details. □

In general, we call ε in (3.5) a smoothing parameter, which can be regarded as a variable similar to the argument x in some numerical algorithms. Now, we definite a function as following:

$$\phi_{\theta p}(\varepsilon, a, b) := \sqrt[p]{\theta(|a|^p + |b|^p) + (1-\theta)|a+b|^p + 4\varepsilon} - a - b, \quad \forall (a,b) \in \Re^2, \forall \varepsilon > 0, \tag{3.6}$$

where $1 < p \le 2$, and $\theta \in (0, 2)$.

In the following, we reformulate the MNCP (3.4) into a parameterized smooth equations. First of all, define

$$H(\varepsilon, x, s, \lambda) := \begin{bmatrix} 2Wx^3 + \lambda_1 Dx - \lambda_2 Dx \\ s_1 - 1 + Dx^2 \\ s_2 - Dx^2 + 0.5 \\ \phi_{\theta p}(\varepsilon, s, \lambda) - \varepsilon s \\ \varepsilon \end{bmatrix} \tag{3.7}$$

and $\Psi(\varepsilon, x, s, \lambda) := \|H(\varepsilon, x, s, \lambda)\|_2^2$. Then, it is easy to verify that for all $1 < p \le 2, \theta \in (0, 2)$, $x \in \Re^3$ and $\lambda \in \Re^2$ are the solutions of (3.4) if and only if $H(\varepsilon, x, s, \lambda) = 0$.

We state some notations for the convenience of the subsequent analysis. For a given finite set $C := \{c_i : i = 1, 2, \dots, m\}$, we denote an m-dimension column vector with its i-th component being c_i as $vec\{c_i : i = 1, 2, \dots, m\}$, an $m \times m$ diagonal matrix with its i-th diagonal element being c_i as $\mathrm{diag}\{c_i : i = 1, 2, \dots, m\}$. Set $K := \{0, 1, 2, \dots\}$.

Theorem 3.5. *For all $1 < p \le 2, \theta \in (0, 2)$, and $\varepsilon > 0$, the function $H(\varepsilon, x, s, \lambda)$ defined by (3.7) is continuously differentiable. Let $z := (\varepsilon, x, s, \lambda)$, and H' denotes the Jacobian matrix of $H(z)$, then*

$$H'(z) = \begin{bmatrix} 0_{3\times1} & 6Wx^2 + \lambda_1 D - \lambda_2 D & 0_{3\times2} & A \\ 0_{2\times1} & 2A & I_{2\times2} & 0_{2\times2} \\ d(z) - s & 0_{2\times3} & e(z) - \varepsilon I_{2\times2} & f(z) \\ 1 & 0_{1\times3} & 0_{1\times2} & 0_{1\times2} \end{bmatrix}.$$

where

$$\begin{aligned} A &:= [Dx \quad -Dx], d(z) := vec\left\{\frac{4}{p\eta_{\theta p}^{p-1}(\varepsilon, s_i, \lambda_i)} : i = 1, 2\right\}, \\ e(z) &:= diag\left\{\frac{\theta sgn(s_i)|s_i|^{p-1} + (1-\theta)sgn(s_i+\lambda_i)|s_i+\lambda_i|^{p-1}}{\eta_{\theta p}^{p-1}(\varepsilon, s_i, \lambda_i)} - 1 : i = 1, 2\right\}, \\ f(z) &:= diag\left\{\frac{\theta sgn(\lambda_i)|\lambda_i|^{p-1} + (1-\theta)sgn(s_i+\lambda_i)|s_i+\lambda_i|^{p-1}}{\eta_{\theta p}^{p-1}(\varepsilon, s_i, \lambda_i)} - 1 : i = 1, 2\right\}, \\ \eta_{\theta p}(\varepsilon, s, \lambda) &:= \sqrt[p]{\theta(|s_i|^p + |\lambda_i|^p) + (1-\theta)|s_i + \lambda_i|^p + 4\varepsilon}. \end{aligned}$$

Proof. The results can be obtained by direct calculation. We omit the details. $\square$

Next, we propose a non-interior-point smoothing Newton algorithm to solve the restructuring model (3.7).

Algorithm 3.6 (A Non-interior-point Smoothing Newton Algorithm).

Step 0 Given real numbers $1 < p \le 2, \delta, \sigma \in (0, 1), \theta \in (0, 1]$, and $\varepsilon^0 > 0, x^0 \in \Re^3, s^0, \lambda^0 \in \Re^2$ are the initial parameters. Set $z^0 := (\varepsilon^0, x^0, s^0, \lambda^0)$. Choose $\beta > 1$ such that $\|H(z^0)\|_2 \le \beta\varepsilon^0$. Set $e^0 := (1, 0, 0, 0) \in \Re \times \Re^3 \times \Re^2 \times \Re^2$ and $k := 0$.

Step 1 If $\|H(z^k)\|_2 = 0$, stop.

Step 2 Calculate $\delta z^k := (\delta\varepsilon^k, \delta x^k, \delta s^k, \delta\lambda^k) \in \Re \times \Re^3 \times \Re^2 \times \Re^2$, satisfying

$$H(z^k) + H'(z^k)\delta z^k = \frac{1}{\beta}\|H(z^k)\|_2 e^0. \tag{3.8}$$

Step 3 Find the smallest positive integer m_k such that

$$\|H(z^k + \delta^{m_k}\delta z^k)\|_2 \leq (1 - \sigma(1 - \frac{1}{\beta})\delta^{m_k})\|H(z^k)\|_2. \tag{3.9}$$

Step 4 Set $l^k := \delta^{m_k}, z^{k+1} := z^k + l^k\delta z^k, k := k+1$ and go to step 1.

Remark 3.7. Since some properties for $\phi_{\theta p}$ are hard to get for all parameters, we use only $p \in (1, 2], \theta \in (0, 1]$ in the above algorithm.

Theorem 3.8. *For all* $1 < p \leq 2, \theta \in (0, 1], \varepsilon > 0, s, \lambda \in \Re$, *we have*

$$\begin{aligned}
\frac{\theta sgn(\lambda)|\lambda|^{p-1} + (1-\theta)sgn(s+\lambda)|s+\lambda|^{p-1}}{\eta_{\theta p}^{p-1}(\varepsilon, s, \lambda)} - 1 &< 0,\\
\frac{\theta sgn(s)|s|^{p-1} + (1-\theta)sgn(s+\lambda)|s+\lambda|^{p-1}}{\eta_{\theta p}^{p-1}(\varepsilon, s, \lambda)} - 1 &< 0.
\end{aligned}$$

Proof. The proof is similar to that of [8, Proposition 2.5]. □

Assumption 3.9. Set $N(\beta) := \{z \in \Re_{++} \times \Re^3 \times \Re^2 \times \Re^2 : \|H(z)\|_2 \leq \beta\varepsilon\}$, where β is a given constant by Algorithm 3.6. Suppose that for every $z \in N(\beta)$, $H'(z)$ is invertible.

Remark 3.10. It is easy to see from Theorem 3.5 that the invertibility of $H'(z)$ relates to the invertibility of the term $6Wx^2 + \lambda_1 D - \lambda_2 D$. However, the invertibility of $6Wx^2 + \lambda_1 D - \lambda_2 D$ is difficult to characterize at present. We see from the practical computation that Assumption 3.9 is always satisfied.

Definition 3.11. A function $H : \Omega \subseteq \Re^n \to \Re^n$ is called a weak-univalent function, if there is a continuous injective function sequence $\{H_k\}$ which is uniformly convergent to H on any bounded subset of Ω.

Thus, similar to the proof of [6, Theorem 7], we can get the following theorem.

Theorem 3.12. *Suppose that Assumption 3.9 holds, then for all* $\varepsilon > 0$, *the function* H *defined in* (3.7) *is a weak-univalent function.*

Theorem 3.13. *Suppose that Assumption 3.9 holds. If* $\{z^k\}$ *is a infinite sequence obtained by Algorithm 3.6, then*

(i) *The sequence* $\{\|H(z^k)\|_2\}$ *is monotone and decreasing.*

(ii) *For every* $k \in K$, $z^k \in N(\beta)$, *where* $N(\beta)$ *is defined in Assumption 3.9.*

(iii) *The sequence* $\{\varepsilon^k\}$ *is monotone and decreasing.*

Proof. The proof is similar to [12, Lemma 3.1], and we omit the details. □

Theorem 3.14. *Suppose that Assumption 3.9 holds, then Algorithm 3.6 is well-defined.*

Proof. First of all, we have

$$\varepsilon^{k+1} = \varepsilon^k + l^k \delta\varepsilon^k = (1 - l^k)\varepsilon^k + l^k \frac{1}{\beta}\|H(z^k)\|_2 > 0.$$

Hence, according to Theorem 3.5, the Jacobian of $H(z^k)$ can be obtained.

Secondly, we will show that the Newton equation is solvable. Similar to [17, Lemma 4.1], with the assumption, we can easily get the nonsingularity of $H(z^k)$. Thus, Step 2 of Algorithm 3.6 is well-defined.

Finally, the line search in Step 3 of Algorithm 3.6 is well-defined. In fact, set

$$R(z^k) := H(z^k + \alpha\delta z^k) - H(z^k) - \alpha H'(z^k)\delta z^k,$$

then, by Step 2 of Algorithm 3.6, we have

$$\begin{aligned} \|H(z^k + \alpha\delta z^k)\|_2 &= \|R(z^k) + H(z^k) + \alpha H'(z^k)\delta z^k\|_2 \\ &\le \|R(z^k)\|_2 + \|H(z^k)\|_2 - \alpha(1 - \frac{1}{\beta})\|H(z^k)\|_2 \\ &\le \|R(z^k)\|_2 + [1 - \alpha(1 - \frac{1}{\beta})]\|H(z^k)\|_2. \end{aligned}$$

Since for all $k \in K$, $\varepsilon^k > 0$ holds, we can get that H is continuous and differentiable, and then $\|R(z^k)\|_2 = o(\alpha)$ holds.

Thus, Algorithm 3.6 is well-defined. The proof is complete. □

Theorem 3.15. *For all $\{z^k\}$ satisfying $\|(s^i, \lambda^i)\|_2 \to +\infty$, and $\varepsilon^k \in [\varepsilon_1, \varepsilon_2]$, where ε_1 and ε_2 both are positive constant satisfying $\varepsilon_1 < \varepsilon_2$, there holds*

$$\Psi(z^k) \to +\infty.$$

Proof. The proof is similar to that of [13, Lemma 2.4], we omit it. □

Theorem 3.16. *Suppose that Assumption 3.9 holds, and suppose that $\{z^k\}$ is the infinite sequence generated by Algorithm 3.6, then the sequence $\{z^k\}$ is bounded, and its any accumulation point is a solution to* (3.4).

Proof. The proof is similar to that of [11, Theorem 4.1], see also the proof of [14, Theorem 2]. We omit it. □

4 Numerical Results

In this section, we report some preliminary numerical results of Algorithm 3.6 for randomly generated data to get the corresponding D-eigenvalues.

The randomly generated data are constructed in the following way: construct a 3-dimensional positive definite matrix D (a second-order 3-dimensional symmetric positive definite tensor) and a fourth-order 3-dimensional symmetric tensor W. We denote an $n \times m$ matrix whose each element is generated uniformly in the interval $[0, 1]$ as $rand(n, m)$. All the codes are written in Matlab, and all the tests are implemented on our PC with CPU being 2.4GHZ and RAM being 256MB. The parameters of the numerical implementation are selected as following:

$$\delta := 0.5, \sigma := 0.001, \varepsilon^0 := 0.1, \beta := \max\left\{2, \frac{\|H(z^0)\|_2}{\varepsilon^0}\right\}.$$

In the numerical experiments, we use $\|H(z^k)\|_2 \leq 10^{-6}$ as the the stopping rule. We implement Algorithm 3.6 with the following example:

Example 4.1. Let $D := (rand(3,3)^T rand(3,3) + I_{3\times 3} \times 0.5) \times 10^{-3}$ and select W's fifteen independent variables as $0.2 \times rand(15,1)$.

The initial points in each numerical computations are selected as the following:

$$x^0 := 300 \times rand(3,1), s_1^0 := 1 - D(x^0)^2, s_2^0 := D(x^0)^2 - 0.5, \lambda_1^0 := 0, \lambda_2^0 := 0.$$

The numerical results are listed in Table 1. In Table 1, we denote by θ and p the parameters specified for the corresponding case; x the found D-eigenvector; λ the found D-eigenvalue; **IT** the iteration number when the algorithm terminates; **CPU** the cpu time spent for the algorithm finding the corresponding D-eigenvalue. From Table 1, we have the following observations:

(i) For the randomly generated example, we can find a D-eigenvalue and the corresponding D-eigenvector effectively for every case ($\theta \in (0,1]$ and $p \in (1,2]$). And the proposed algorithm can find the solutions within a few iteration steps and short CPU time.

(ii) For different values of θ or p, the found D-eigenvalue usually varies for different cases, which indicates that the D-eigenvalue and the D-eigenvector of W are not unique. In [20], the number of the D-eigenvalue of a fourth-order 3-dimensional symmetric tensor is estimated to be no more than 13. However, not every tensor has exactly 13 D-eigenvalues. For example, only 12 D-eigenvalues for the example in [22].

(iii) From Table 1, we see that the algorithm works better when θ around 0.75.

(iv) It is easy to see that when $\theta = 1$ and $p = 2$, Algorithm 3.6 corresponds to the case of using classical FB function. In this case, we can also see from Table 1 that the number of the iterate is more than two times that of the case with $\theta = 0.9$ and $p = 1.1$, and the cpu time is more than two times that of the case with $\theta = 0.25$ and $p = 1.1$. In addition, many cases show that the classical FB function is not the best choice, which demonstrate that our proposed generalized NCP-functions are useful from the view of improving the performance of the algorithm.

A method to find out all the D-eigenvalues of a given tensor by polynomial rooting was proposed in [22]. We know that it is difficult to get all, or even one, roots/root of a high-degree polynomial, but the smoothing algorithm of this paper can deal with high order tensors efficiently. Hence, for higher order tensors like those in [24], the method in this paper is applicable.

5 Final Results

In this paper, a family of generalized NCP-functions was proposed, and many favorite properties for this class of NCP-functions were analyzed. Such a class of NCP-functions has potential applications in designing efficient numerical algorithms for smoothing reformulations of NCPs. Based on the proposed NCP-functions, we reformulated the D-eigenvalue problem of diffusion kurtosis tensor as a system of non-smooth equations. A smoothing Newton method was proposed then to try to find out the D-eigenvalues of a diffusion kurtosis tensor by varying the parameters of this family of generalized NCP-functions. The numerical results show that the proposed numerical method is promising. This method has the potential application in the computation of eigenvalues of higher order tensors, because the sizes of the problems will not be changed after we reformulated them.

References

[1] Y. Assaf and O. Pasternak, Diffusion tensor imaging (DTI)-based white matter mapping in brain research: a review, *J. Molecular Neuroscience* 34 (2008) 51–61.

[2] P.J. Basser and D.K. Jones, Diffusion-tensor MRI: theory,experimental design and data analysis: a technical review, *NMR in Biomedicine* 15 (2002) 456–467.

[3] P.T. Callaghan, *Principles of Nuclear Magnetic Resonance Microscopy*, Oxford University Press, Oxford, 1993.

[4] J.S. Chen and S.H. Pan, A family of NCP functions and a desent method for the nonlinear complementarity problem. *Comput. Optim. Appl.* 40 (2008) 389–404.

[5] F.H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.

[6] M.S. Gowda and J.J. Sznajder, Weak univalence and connectedness of inverse images of continuous functions. *Math. Oper. Res.* 24 (1999) 55–261.

[7] D. Han, L. Qi and E.X. Wu, Extreme diffusion values for non-Gaussian diffusions, *Optim. Methods Softw.* 23 (2008) 703–716.

[8] S.L. Hu, Z.H. Huang and J.S. Chen, Properties of a family of generalized NCP-functions and a derivative free algorithm for complementarity problems. *J. Comput. Appl. Math.* 230 (2009) 69–82.

[9] S.L. Hu, Z.H. Huang and N. Lu, Smoothness of a generalized merit function for the second-order cone complementarity problem, *Pac. J. Optim.* 6 (2010) 551–571.

[10] S.L. Hu, Z.H. Huang, H.Y. Ni and L. Qi, Positive definiteness of diffusion kurtosis imaging, *Inverse Probl. Imaging*, 6 (2012) 57–75.

[11] S.L. Hu, Z.H. Huang and P. Wang, A non-monotone smoothing Newton algorithm for solving nonlinear complementarity problems, *Optim. Methods Softw.* 24 (2009) 447–460.

[12] Z.H. Huang, Locating a maximally complementary solution of the monotone NCP by using non-interior-point smoothing algorithms, *Math. Methods Oper. Res.* 61 (2005) 41–55.

[13] Z.H. Huang, J.Y. Han and Z. Chen, A predictor-corrector smoothing Newton algorithm, based on a new smoothing function, for solving the nonlinear complementarity problem with a P_0 function, *J. Optim. Theory Appl.* 117 (2003) 39–68.

[14] Z.H. Huang, L. Qi and D. Sun, Sub-quadratic convergence of a smoothing Newton algorithm for the P_0 and monotone LCP, *Math. Program.* 99 (2004) 423–441.

[15] H.J. Jensen, J.A. Helpern, A. Ramani, H. Lu and K. Kaczynski, Diffusional kurtosis imaging: The quantification of non-Gaussian water diffusion by means of magnetic resonance imaging, *Magnetic Resonance in Medicine* 53 (2005) 1432–1440.

[16] S. Kim and M. Kojima, Exact solutions of some nonconvex quadratic optimization problems via SDP and SOCP relaxations, *Comput. Optim. Appl.* 26 (2003) 143–154.

[17] M. Kojima, N. Megiddo and T. Noma, A unified approach to interior point algorithm for linear complementarity problems, Lecture Notes in Computer Science, Vol. 538, Springer-Verlag, Berlin, 1991.

[18] H. Lu, H.J. Jensen, A. Ramani and J.A. Helpern, Three-dimensional characterization of non-Gaussian water diffusion in humans using diffusion kurtosis imaging, *NMR in Biomedicine* 19 (2006) 236–247.

[19] R. Mifflin, Semismooth and semiconvex functions in constrained optimization, *SIAM J. Control Optim.* 15 (1977) 957–972.

[20] G. Ni, L. Qi, F. Wang and Y.J. Wang, The degree of the E-characteristic polynomial of an even order tensor, *J. Math. Anal. Appl.* 329 (2007) 1218–1229.

[21] L. Qi and J. Sun, A nonsmooth version of Newton's method, *Math. Program.* 58 (1993) 353–367.

[22] L. Qi, Y.J. Wang and E.X. Wu, D-eigenvalues of diffusion kurtosis tensors, *J. Comput. Appl. Math.* 221 (2008) 150–157.

[23] L. Qi and Y. Ye, Space tensor conic programming, to appear in: *Computational Optimization and Applications.*

[24] L. Qi, G. Yu and E.X. Wu, Higher order positive semi-definite diffusion tensor imaging, *SIAM J. Imaging Sci.* 3 (2010) 416-4-33.

[25] E.O. Stejskal and J.E. Tanner, Spin diffusion measurements: spin echoes in the presence of a time-dependent field gradient, *J. Chemical Physics* 42 (1965) 288–292.

[26] Y. Ye, Approximating quadratic programming with bound and quadratic constraints, *Math. Program.* 84 (1999) 219–226.

[27] X. Zhang, C. Ling, L. Qi and E.X. Wu, The measure of diffusion skewness and kurtosis in magnetic resonance imaging, *Pac. J. Optim.* 6 (2010) 391–404.

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Table 1: Numerical results

p	θ	x	$\lambda \times 10^{-6}$	**IT**	**CPU**
	0.10	(11.91, -6.94,22.75)	0.0446	26	10.4844
	0.25	(-20.35, 9.69,16.32)	0.17763	22	0.3594
1.1	0.50	(-20.35, 9.69,16.32)	0.17763	19	0.3750
	0.75	(11.91,-6.94,22.75)	0.04462	20	0.5000
	0.90	(-20.35, 9.69,16.32)	0.17763	16	0.4531
	1.00	(12.24,31.63, 2.16)	0.39403	26	0.3906
	0.10	(11.91,-6.94,22.75)	0.04462	100	3.7031
	0.25	(11.91,-6.94,22.75)	0.04462	97	2.1875
1.5	0.50	(11.91,-6.94,22.75)	0.04462	26	0.6718
	0.75	(-20.35, 9.69,16.32)	0.17763	19	0.5469
	0.90	(11.91,-6.94,22.75)	0.04462	54	1.0469
	1.00	(11.91,-6.94,22.75)	0.04462	27	0.67196
	0.10	(11.91,-6.94,22.75)	0.04462	20	0.5313
	0.25	(11.91,-6.94,22.75)	0.04462	34	0.8906
2.0	0.50	(11.91,-6.94,22.75)	0.044618	39	1.0000
	0.75	(11.91,-6.94,22.75)	0.044618	27	0.8281
	0.90	(12.24,31.63, 2.16)	0.39403	51	1.1719
	1.00	(12.24,31.63, 2.16)	0.39403	37	0.7967