Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL
Pacific Journal of Optimization
Vol. 10, No. 3, pp. 593–611, July 2014

DETECTION OF A COPOSITIVE MATRIX OVER A *P*-TH ORDER CONE*

Jing Zhou, Zhibin Deng, Shu-Cherng Fang and Wenxun Xing†

Abstract: In this paper, we prove that detecting the copositivity of a given matrix over a p-th order cone is equivalent to the decision problem of a quadratic programming problem over a cross-section of the p-th order cone. Then we identify some polynomial-time solvable subclasses for the detection problem. Based on the linear matrix inequality representations of the cone of nonnegative quadratic forms over a union of second order cones, a conic approximation algorithm is presented for the problems not belonging to the polynomial-time solvable subclasses. Computational experiments are conducted to illustrate the efficiency of the proposed algorithm.

Key words: *copositive matrix, p-th order cone, linear conic programming, conic approximation*

Mathematics Subject Classification: *90C26, 90C34*

1 Introduction

For a given set $\mathscr{F} \subseteq \mathbb{R}^n$, where $\mathbb{R}^n$ denotes the n-dimensional real space, the cone of nonnegative quadratic forms over $\mathscr{F}$ is defined as $\mathscr{D}_{\mathscr{F}} = \{U \in \mathbb{S}^n | y^T U y \geq 0, \forall y \in \mathscr{F}\}$ [21], where $\mathbb{S}^n$ denotes the set of real symmetric matrices of order n. The dual cone of $\mathscr{D}_{\mathscr{F}}$ is $\mathscr{D}^*_{\mathscr{F}} = \text{cl cone}\{yy^T \in \mathbb{S}^n \mid y \in \mathscr{F}\}$, where "cl" stands for closure of a set and "cone" for the conic hull of a set. Here, the conic hull of a set $\mathscr{F} \subseteq \mathbb{R}^n$ is defined as $\text{cone}(\mathscr{F}) = \{\sum_{i=1}^{\tau} \alpha_i y^i \mid y^i \in \mathscr{F},\ \alpha_i \geq 0,\ \tau \in \mathbb{Z}_+\}$ with $\mathbb{Z}_+$ being the set of positive integers. A matrix $M \in \mathbb{S}^n$ is called copositive over $\mathscr{F}$ if $M \in \mathscr{D}_{\mathscr{F}}$.

Optimization problems over a cone of nonnegative quadratic forms and their dual problems have attracted attentions in recent years. For example, the quadratically constrained quadratic programming problem can be reformulated as the following linear conic programming problem over $\mathscr{D}^*_{\mathscr{F}}$ [21],

$$\begin{array}{ll} \min & U_0 \cdot Y \\ s.t. & U_i \cdot Y \leq 0, \quad i = 1, 2, \ldots, m, \\ & Y \in \mathscr{D}^*_{\mathscr{F}}, \end{array} \tag{1.1}$$

where $U_i \in \mathbb{S}^n$, $i = 0, 1, \ldots, m$, and "·" is the inner product of two matrices.

*This work has been partially supported by the National Natural Science Foundation of China Grant Numbers 11171177 and 11371216, and by the DARPA Adaptive Execution Office (Todd Hughes) through US Army Research Office Grant number W911NF-04-D-0003.

†Corresponding author.

If $Y \in \mathscr{D}^*_{\mathscr{F}}$ can be detected in polynomial time, then problem (1.1) can be solved in polynomial time. In fact, when $\mathscr{F}$ is n-dimensional real space, an ellipsoid or a second order cone, whether Y belongs to $\mathscr{D}^*_{\mathscr{F}}$ can be determined in polynomial time [9, 21]. However, in general, checking whether a matrix belongs to $\mathscr{D}_{\mathscr{F}}$ or $\mathscr{D}^*_{\mathscr{F}}$ is NP-hard. For example, Murty and Kabadi [18] proved that checking whether a given matrix is copositive over $\mathbb{R}^n_+$ is co-NP-complete. In order to solve the linear conic programming problems over $\mathscr{D}_{\mathscr{F}}$ or $\mathscr{D}^*_{\mathscr{F}}$, finding computable cones to approximate $\mathscr{D}_{\mathscr{F}}$ or $\mathscr{D}^*_{\mathscr{F}}$ becomes a promising research topic. Based on the fact that the cone of nonnegative quadratic forms over an ellipsoid is computable, Deng et al. [6] and Lu et al. [15] proposed conic approximation methods for detecting copositivity of a symmetric matrix and for the box constrained quadratic programming problem, respectively. Zhou et al. [24] solved the 0-1 quadratic knapsack problem based on that the cone of nonnegative quadratic forms over the intersection of an ellipsoid and a linear inequality is computable. Furthermore, Tian et al. [22] derived the computable representation of the cone of nonnegative quadratic forms over a general second-order cone for solving the completely positive programming problem. Inspired by these results and the decomposition methods introduced in [21], a union of second order cones is chosen to approximate the p-th order cone in this paper.

As a generalization of the second order cone programming, the p-th order cone constrained optimization problem has received increasing attention. Several papers [1, 14, 23] considered problems with p-th order cone constraints. Also, Burer [5] proposed a p-th order cone sequential relaxation procedure for solving 0-1 integer constrained linear programming problem. In practice, some problems are formulated into models with a quadratic objective function over p-th order cones. For instance, the problem proposed by Meinshausen and Bühlmann [17] in the learning sparse networks can be formulated as the p-th order cone constrained quadratic programming problem with $p = 1$. Furthermore, the l_p distance location-allocation problem [20] can be casted into a quadratic programming problem with several p-th order cones and linear constraints. By using the conic reformulation method, all these problems can be reformulated or relaxed as a linear conic programming problem over $\mathscr{D}_{\mathscr{F}}$ or $\mathscr{D}^*_{\mathscr{F}}$ where $\mathscr{F}$ is a p-th order cone. Therefore, efficiently detecting whether a matrix is copositive over a p-th order cone becomes critical.

By the definition of $\mathscr{D}_{\mathscr{F}}$, detection of a copositive matrix over a p-th order cone, which is briefly named the detection problem in this paper, can be formulated as:

$$\begin{aligned} V_{\text{detect-P}} = \min \quad & \begin{bmatrix} t \\ x \end{bmatrix}^T M \begin{bmatrix} t \\ x \end{bmatrix}, \\ s.t. \quad & \begin{bmatrix} t \\ x \end{bmatrix} \in \mathscr{F}_{\text{detect-P}}, \end{aligned} \tag{detect-P}$$

where

$$\mathscr{F}_{\text{detect-P}} = \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^n \middle| \|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p} \leq t \right\}$$

is a p-th order cone and $M = \begin{bmatrix} M_{11} & (M^{21})^T \\ M^{21} & M^{22} \end{bmatrix}$ with $M_{11} \in \mathbb{R}$, $M^{21} \in \mathbb{R}^n$ and $M^{22} \in \mathbb{S}^n$. Notice that $M \in \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$ if and only if $V_{\text{detect-P}} \geq 0$.

When $p = 2$, problem (detect-P) can be solved in polynomial time [9, 22], and thus we can detect whether $M \in \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$ in polynomial time. However, as far as we know, there is no known complexity result for $1 \leq p \neq 2$.

In this paper, we prove that $M \in \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$ if and only if the optimal value of the following problem

$$V_{\text{detect-CS}} = \min \quad \begin{bmatrix}1\\x\end{bmatrix}^T M \begin{bmatrix}1\\x\end{bmatrix}$$
$$s.t. \quad \begin{bmatrix}1\\x\end{bmatrix} \in \mathscr{F}_{\text{detect-CS}}, \tag{detect-CS}$$

where $\mathscr{F}_{\text{detect-CS}} = \left\{ \begin{bmatrix}1\\x\end{bmatrix} \in \mathbb{R}\times\mathbb{R}^n \middle| \|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p} \leq 1 \right\}$ is a cross-section of $\mathscr{F}_{\text{detect-P}}$ at $t=1$, is no less than 0. Problem (detect-CS) is NP-hard when $p=1$ [8], and it is still NP-hard [13] when $p>2$ under the "unique games conjecture" [11], which is a commonly used assumption in complexity theory. Hence we do not expect to solve the detection problem by solving problem (detect-CS) directly. Nevertheless, there are some polynomial-time solvable subclasses. For example, based on the results in [19], we can show that detecting whether $M \in \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$ can be solved in polynomial time when $M_{11}=0$ and $p>2$. In addition, we can identify other polynomial-time solvable subclasses of the detection problem by using the equivalence of different norms.

For the detection problems not belonging to the proposed polynomial-time solvable subclasses, a conic approximation algorithm is developed. Redundant constraints based on the special structure of the p-th order cone are introduced in order to improve the efficiency of approximation. Moreover, in every iteration of the proposed algorithm, one additional redundant constraint is generated based on the solution obtained in the previous iteration. All these redundant constraints could improve the tightness of the approximation. In order to guarantee the proposed algorithm to be terminated in finite iterations, the definition of ε-copositivity over a p-th order cone is given. The finiteness and validity of the proposed algorithm is proved in theory.

The following notations are adopted in this paper. Let $\mathbb{R}^n_+$ be the set of n dimensional nonnegative vectors, $\mathbb{S}^n_+$ be the set of all $n\times n$ symmetric positive semidefinite matrices, and $\mathbb{S}^n_{++}$ be the set of all $n\times n$ symmetric positive definite matrices. For a matrix $A\in\mathbb{S}^n$, $A\succeq 0$ means $A\in\mathbb{S}^n_+$ and $A\succ 0$ means $A\in\mathbb{S}^n_{++}$. For $A\succeq 0$, if $A=V\Lambda V^T$ where V is an orthogonal matrix and Λ is a diagonal matrix, then $A^{\frac{1}{2}} = V\Lambda^{\frac{1}{2}}V^T$. For a vector $x=\begin{bmatrix}x_1 & \dots & x_n\end{bmatrix}^T \in\mathbb{R}^n$, $Diag(x)$ represents the diagonal matrix with x_i being its i-th diagonal elements and $[x]^2 = \begin{bmatrix}x_1^2 & \dots & x_n^2\end{bmatrix}^T$. M_{ij} represents the element in the i-th row and the j-th column of an $n\times n$ real matrix M. For $n\times n$ real matrices $A=(A_{ij})$ and $B=(B_{ij})$, $A\cdot B = \text{trace}(A^TB) = \sum_{i,j=1}^n A_{ij}B_{ij}$. $\nabla\|\overline{x}\|_p$ means the gradient of $\|x\|_p$ at the point $\overline{x}\neq 0$. For a set $\mathscr{F}$, $int(\mathscr{F})$ means the interior of $\mathscr{F}$. For a constant c, $c\mathscr{F} = \{cy \mid y\in\mathscr{F}\}$. $1_n = [1,1,...,1]_n^T$ and $0_{m\times n}$ represents the $m\times n$ matrix with all the elements being zero. $a<b$ means $a_i<b_i$ for $i=1,\dots,n$ and $a,b\in\mathbb{R}^n$.

The layout of this paper is as follows. In Section 2, we prove the equivalence between the detection problem and the decision problem of a quadratic programming problem over a cross-section of the p-th order cone. Section 3 is devoted to presenting some polynomial-time solvable subclasses of the detection problem. Section 4 presents the results related to conic relaxations and decompositions. In Section 5, an adaptive approximation algorithm is designed for the detection problem when it cannot be classified into one of the polynomial-time solvable subclasses appeared in Section 3. In Section 6, some numerical examples and experiments are provided to demonstrate the validity and efficiency of the proposed algorithm. Conclusions are given in Section 7.

2 The Relationship between the Detection Problem and a Decision Problem

In this section, we prove the equivalence between detection of a copositive matrix over a p-th order cone and the decision problem of (detect-CS).

Theorem 2.1. *$M \in \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$ if and only if $V_{\text{detect-CS}} \geq 0$.*

Proof. Because $0_{(n+1)\times 1} \in \mathscr{F}_{\text{detect-P}}$, the optimal value $V_{\text{detect-P}} \leq 0$. Denote $y = \begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R}\times\mathbb{R}^n$. If there is a feasible solution $y \in \mathscr{F}_{\text{detect-P}}$ satisfying $y^T My < 0$, then $\lambda y \in \mathscr{F}_{\text{detect-P}}$, for all $\lambda \geq 0$. Therefore, when $\lambda \to +\infty$, we have $(\lambda y)^T M(\lambda y) \to -\infty$. In all, the optimal value of (detect-P) is either zero or $-\infty$. Hence, $M \in \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$ if and only if $V_{\text{detect-P}} = 0$ and $M \notin \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$ if and only if $V_{\text{detect-P}} = -\infty$.

On one hand, if $M \in \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$, then $V_{\text{detect-CS}} \geq V_{\text{detect-P}} = 0$, because of $\mathscr{F}_{\text{detect-CS}} \subseteq \mathscr{F}_{\text{detect-P}}$.

On the other hand, if $V_{\text{detect-CS}} \geq 0$, then $V_{\text{detect-P}} = 0$. Otherwise, there exists a feasible solution $\begin{bmatrix} t \\ x \end{bmatrix} \in \mathscr{F}_{\text{detect-P}}$ such that $\begin{bmatrix} t \\ x \end{bmatrix}^T M \begin{bmatrix} t \\ x \end{bmatrix} < 0$ with $t > 0$. Then, $\begin{bmatrix} 1 \\ \frac{x}{t} \end{bmatrix}^T M \begin{bmatrix} 1 \\ \frac{x}{t} \end{bmatrix} < 0$ and $\begin{bmatrix} 1 \\ \frac{x}{t} \end{bmatrix} \in \mathscr{D}_{\mathscr{F}_{\text{detect-CS}}}$. This contradicts with $V_{\text{detect-CS}} \geq 0$. □

Theorem 2.1 implies if we can solve problem (detect-CS) in polynomial time, then the detection problem can be solved efficiently. However, problem (detect-CS) is an NP-hard problem when $p = 1$ [8] and next lemma indicates that it is still an NP-hard problem under the "unique games conjecture" when $p > 2$. To our best knowledge, there is no known complexity result when $2 > p > 1$.

Lemma 2.2 ([13])**.** *Let $p > 2$ and $\delta > 0$ be constants satisfying $(1-\delta)\gamma_p^2 > 1$, where $\gamma_p = (\frac{2^{\frac{p}{2}}\Upsilon(\frac{p+1}{2})}{\sqrt{\pi}})^{\frac{1}{p}}$ and $\Upsilon(\frac{p+1}{2}) = \int_0^\infty t^{\frac{p+1}{2}-1}e^{-t}dt$. Then, under the "unique games conjecture", it is NP-hard to approximate the problem*

$$\max\left\{\sum_{i,j=1}^{n} a_{ij}x_ix_j \,\middle|\, x \in \mathbb{R}^n,\ \sum_{i=1}^{n} |x_j|^p \leq 1\right\} \tag{2.1}$$

within a factor $(1-\delta)\gamma_p^2$.

The "unique games conjecture" has broad applications in the theory of hardness of approximation [11, 13]. If it is true, then it is not only too hard to obtain an exact solution, but also too hard to obtain a good approximation for many problems.

Lemma 2.2 indicates that it is hard to detect whether $M \in \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$ by solving problem (detect-CS) directly. But Theorem 2.1 provides us an inspiration to seek some polynomial-time solvable subclasses for the detection problem. These subclasses are introduced in the next section.

3 Polynomial-Time Solvable Subclasses

Based on the equivalence between the detection problem and the decision problem of (detect-CS), some polynomial-time solvable subclasses are presented in this section.

Firstly, if $M^{22} \in \mathbb{S}^n_+$, then both the objective function and constraint of problem (detect-CS) are convex and problem (detect-CS) can be solved in polynomial time [2,3]. Furthermore, when $M_{11} < 0$, $M \notin \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$ because $[1, 0, \ldots, 0]^T \in \mathscr{F}_{\text{detect-P}}$ and

$$[1, 0, \ldots, 0]M[1, 0, \ldots, 0]^T = M_{11} < 0.$$

In order to discuss the polynomial-time solvable subclass when $M_{11} = 0$ and $p > 2$, we consider the following relaxation of problem (detect-CS),

$$\begin{aligned} V_1 = \min \quad & \begin{bmatrix} t \\ x \end{bmatrix}^T M \begin{bmatrix} t \\ x \end{bmatrix} \\ s.t. \quad & \|x\|_p \le 1, \\ & |t| \le 1. \end{aligned} \tag{3.1}$$

Theorem 3.1. *If $M_{11} = 0$, then problem* (detect-CS) *is equivalent to problem* (3.1) *in the sense that they have the same optimal value.*

Proof. It is obvious $V_1 \le V_{\text{detect-CS}}$. We need to prove at least one optimal solution $\begin{bmatrix} t \\ x \end{bmatrix}$ of problem (3.1) is achieved when $t = 1$.

If $\begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^n$ is a feasible solution of problem (3.1) when $-1 \le t \le 0$, then $\begin{bmatrix} -t \\ -x \end{bmatrix}$ is also a feasible solution of problem (3.1) satisfying $\begin{bmatrix} t \\ x \end{bmatrix}^T M \begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} -t \\ -x \end{bmatrix}^T M \begin{bmatrix} -t \\ -x \end{bmatrix}$. Hence, we can assume one optimal solution of problem (3.1) is $\begin{bmatrix} t_0 \\ x_0 \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^n$, where $\|x_0\|_p \le 1$ and $0 \le t_0 \le 1$. Without loss of generality, we can assume $(M^{21})^T x_0 \le 0$. Otherwise, $(-x_0)^T M^{22}(-x_0) + 2(M^{21})^T(-x_0)t_0 \le x_0^T M^{22} x_0 + 2(M^{21})^T x_0 t_0$. We can replace $\begin{bmatrix} t_0 \\ x_0 \end{bmatrix}$ with $\begin{bmatrix} t_0 \\ -x_0 \end{bmatrix}$ since $\begin{bmatrix} t_0 \\ -x_0 \end{bmatrix}$ is an optimal solution. Under these conditions, we have $x_0^T M^{22} x_0 + 2(M^{21})^T x_0 t_0 \ge x_0^T M^{22} x_0 + 2(M^{21})^T x_0$. Therefore, $\begin{bmatrix} 1 \\ x_0 \end{bmatrix}$ is also an optimal solution, which implies problem (detect-CS) is equivalent to problem (3.1) when $M_{11} = 0$ in the sense that they have the same optimal value. $\square$

Theorem 3.1 shows that, when $M_{11} = 0$, we can determine the sign of the optimal value of problem (3.1) instead of problem (detect-CS). In order to determine the sign of the optimal value of problem (3.1), we consider the following two problems,

$$\begin{aligned} V_2 = \min \quad & \begin{bmatrix} t \\ x \end{bmatrix}^T M \begin{bmatrix} t \\ x \end{bmatrix} \\ s.t. \quad & \left\| \begin{bmatrix} t \\ x \end{bmatrix} \right\|_p \le 1, \end{aligned} \tag{3.2}$$

$$\begin{aligned} V_3 = \min \quad & \|u\|_q \\ s.t. \quad & Diag(u) \succeq -M, \end{aligned} \tag{3.3}$$

where $p > 2$ and $q = \frac{p}{p-2} > 1$. Lemma 2.2 shows problem (3.2) is NP-hard under the "unique games conjecture". However, problem (3.3) can be solved in polynomial time because the objective and constraint functions are convex [2,3]. In fact, the following lemma shows that the signs of optimal values of problems (3.1) and (3.2) can be determined by the optimal value of problem (3.3).

Lemma 3.2. *If $p > 2$ and $M_{11} = 0$, then $2^{\frac{2}{p}} V_2 \leq V_1 \leq V_2$, $-V_3 \leq V_2 \leq -\frac{1}{p-1} V_3$ and $-2^{\frac{2}{p}} V_3 \leq V_1 \leq -\frac{1}{p-1} V_3$.*

Proof. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be the feasible set of problem (3.1) and (3.2), respectively. Then it is obvious that $\mathscr{F}_2 \subseteq \mathscr{F}_1$ and $V_1 \leq V_2$. $\left\| \begin{bmatrix} t \\ x \end{bmatrix} \right\|_p \leq 2^{\frac{1}{p}}$ for any $\begin{bmatrix} t \\ x \end{bmatrix} \in \mathscr{F}_1$, then $\mathscr{F}_1 \subseteq 2^{\frac{1}{p}} \mathscr{F}_2$. Let $\begin{bmatrix} t \\ x \end{bmatrix} = 2^{\frac{1}{p}} \begin{bmatrix} t' \\ x' \end{bmatrix}$, thus

$$\min \left\{ \begin{bmatrix} t \\ x \end{bmatrix}^T M \begin{bmatrix} t \\ x \end{bmatrix} \middle| \left\| \begin{bmatrix} t \\ x \end{bmatrix} \right\|_p \leq 2^{\frac{1}{p}} \right\} = \min \left\{ 2^{\frac{2}{p}} \begin{bmatrix} t' \\ x' \end{bmatrix}^T M \begin{bmatrix} t' \\ x' \end{bmatrix} \middle| \left\| \begin{bmatrix} t' \\ x' \end{bmatrix} \right\|_p \leq 1 \right\} = 2^{\frac{2}{p}} V_2,$$

so $2^{\frac{2}{p}} V_2 \leq V_1$. By the result $-V_3 \leq V_2 \leq -\frac{1}{p-1} V_3$ in [19], we have $-2^{\frac{2}{p}} V_3 \leq V_1 \leq -\frac{1}{p-1} V_3$. □

If $p > 2$ and $M_{11} = 0$, the sign of the optimal value of problem (3.1) is completely decided by problem (3.3). Then we have the following theorem,

Theorem 3.3. *When $p > 2$ and $M_{11} = 0$, detecting whether $M \in \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$ can be solved in polynomial time.*

Proof. The conclusion follows from Theorems 2.1, 3.1 and Lemma 3.2. □

Due to $\|x\|_b \leq \|x\|_a \leq n^{\frac{1}{a} - \frac{1}{b}} \|x\|_b$ for $0 < a < b$ and $x \in \mathbb{R}^n$ [10], we can derive some polynomial-time solvable subclasses for both $p > 2$ and $1 \leq p < 2$.

For $p > 2$, assigning $a = 2$ and $b = p$ leads to $\frac{1}{n^{\frac{1}{2} - \frac{1}{p}}} \|x\|_2 \leq \|x\|_p \leq \|x\|_2$. Consider the following two polynomial-time solvable problems [21],

$$\begin{aligned} V_4 = \min \quad & x^T M^{22} x + 2(M^{21})^T x + M_{11} \\ s.t. \quad & \|x\|_2 \leq 1, \end{aligned} \tag{3.4}$$

$$\begin{aligned} V_5 = \min \quad & x^T M^{22} x + 2(M^{21})^T x + M_{11} \\ s.t. \quad & \|x\|_2 \leq n^{\frac{1}{2} - \frac{1}{p}}. \end{aligned} \tag{3.5}$$

Theorem 3.4. *$V_5 \leq V_{detect\text{-}CS} \leq V_4$. If $V_5 \geq 0$ or $V_4 < 0$, then detecting whether $M \in \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$ can be solved in polynomial time when $p > 2$.*

Proof. Because $\left\{ x \in \mathbb{R}^n \middle| \|x\|_2 \leq 1 \right\} \subseteq \left\{ x \in \mathbb{R}^n \middle| \|x\|_p \leq 1 \right\} \subseteq \left\{ x \in \mathbb{R}^n \middle| \|x\|_2 \leq n^{\frac{1}{2} - \frac{1}{p}} \right\}$ when $p > 2$, $V_5 \leq V_{\text{detect-CS}} \leq V_4$. If $V_5 \geq 0$, then the optimal value of problem (detect-P) is 0 and $M \in \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$. If $V_4 < 0$, then the optimal value of problem (detect-P) is $-\infty$ and $M \notin \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$. □

For $1 \le p < 2$, assigning $a = p$ and $b = 2$ leads to $\|x\|_2 \le \|x\|_p \le n^{\frac{1}{p}-\frac{1}{2}}\|x\|_2$. The following problem can be solved in polynomial time [21],

$$\begin{aligned} V_6 = \min \quad & x^T M^{22} x + 2(M^{21})^T x + M_{11} \\ s.t. \quad & n^{\frac{1}{p}-\frac{1}{2}}\|x\|_2 \le 1 \end{aligned} \tag{3.6}$$

Similar to the proof for Theorem 3.4, the following theorem holds when $2 > p \ge 1$.

Theorem 3.5. *$V_4 \le V_{detect\text{-}CS} \le V_6$. If $V_4 \ge 0$ or $V_6 < 0$, then detecting whether $M \in \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$ can be solved in polynomial time when $2 > p \ge 1$.*

Remark 3.6. If $p = 2$, then

$$\left\{x \in \mathbb{R}^n \middle| n^{\frac{1}{p}-\frac{1}{2}}\|x\|_2 \le 1\right\} = \left\{x \in \mathbb{R}^n \middle| \|x\|_p \le 1\right\} = \left\{x \in \mathbb{R}^n \middle| \|x\|_2 \le 1\right\}.$$

Thus, problems (3.4), (3.6) and (detect-CS) are actually the same and they can be solved efficiently [21]. In this case, detecting whether $M \in \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$ can be solved in polynomial time.

However, not all instances belong to these polynomial-time solvable subclasses. A conic approximation scheme is designed to estimate the sign of the optimal value of problem (detect-CS) in the next two sections.

4 Conic Relaxations and Decompositions

In this section, we provide a conic reformulation for problem (detect-P). However, as far as we know, there is no polynomial-time algorithm to check whether a matrix belongs to $\mathscr{D}^*_{\mathscr{F}_{\text{detect-P}}}$. Nevertheless, Tian et al. [22] showed that checking whether a matrix belongs to $\mathscr{D}^*_{\mathscr{G}}$, where $\mathscr{G}$ is a union of several nontrivial second order cones, can be solved in polynomial time. The nontriviality means the cone contains at least one point other than origin point. Inspired by this result, we derive a conic relaxation for problem (detect-P) where $\mathscr{G}$ is a second order cone. In order to improve the conic relaxation, we change the cover $\mathscr{G}$ to a collection of second order cones and provide a decomposition of the optimal solution of the conic relaxation problem which is critical for the adaptive algorithm in Section 5.

Firstly, we will state six lemmas which will be used in the following conic approximation method.

Lemma 4.1 ([15, 21])**.** *If $\mathscr{F} \subseteq \mathscr{G}$, then $\mathscr{D}_{\mathscr{G}} \subseteq \mathscr{D}_{\mathscr{F}}$ and $\mathscr{D}^*_{\mathscr{F}} \subseteq \mathscr{D}^*_{\mathscr{G}}$.*

Lemma 4.2 ([22])**.** *Let $\mathscr{G} = \mathscr{G}_1 \cup \mathscr{G}_2 \cup \cdots \cup \mathscr{G}_m$, and each $\mathscr{G}_i$, $i = 1, 2, \ldots, m$, is nonempty, then $\mathscr{D}_{\mathscr{G}} = \mathscr{D}_{\mathscr{G}_1} \cap \mathscr{D}_{\mathscr{G}_2} \cap \cdots \cap \mathscr{D}_{\mathscr{G}_m}$ and $\mathscr{D}^*_{\mathscr{G}} = \mathscr{D}^*_{\mathscr{G}_1} + \mathscr{D}^*_{\mathscr{G}_2} + \cdots + \mathscr{D}^*_{\mathscr{G}_m}$.*

Lemma 4.3 ([22])**.** *Let $\mathscr{F} \subseteq \mathbb{R}^{n+1}$. If $\mathscr{F}$ has an interior point, then $\mathscr{D}_{\mathscr{F}}$ and $\mathscr{D}^*_{\mathscr{F}}$ are proper cones.*

Lemma 4.4 ([22])**.** *Suppose that $\mathscr{F} \subseteq \mathbb{R}^{n+1}$ is a closed convex set, then $\mathscr{D}_{\mathscr{F}} = \mathscr{D}_{cone(\mathscr{F})}$ and $\mathscr{D}^*_{\mathscr{F}} = \mathscr{D}^*_{cone(\mathscr{F})}$.*

Lemma 4.5 ([15])**.** *If $\mathscr{G}$ is a bounded closed set, then $\mathscr{D}^*_{\mathscr{G}} = cone\left\{yy^T \middle| y \in \mathscr{G}\right\}$.*

Lemma 4.6. *If $\mathscr{F}_{detect\text{-}CS} \subseteq \mathscr{H}_1 \cup \mathscr{H}_2 \cup \cdots \cup \mathscr{H}_m$, then $\mathscr{F}_{detect\text{-}P} \subseteq cone(\mathscr{H}_1) \cup cone(\mathscr{H}_2) \cup \ldots \cup cone(\mathscr{H}_m)$.*

Lemma 4.6 can be proved by the definition of cone and Lemma 4.4.

Because $\mathscr{F}_{\text{detect-P}} = \text{cone}(\mathscr{F}_{\text{detect-CS}})$, Lemma 4.4 implies $\mathscr{D}_{\mathscr{F}_{\text{detect-P}}} = \mathscr{D}_{\mathscr{F}_{\text{detect-CS}}}$ and $\mathscr{D}^*_{\mathscr{F}_{\text{detect-P}}} = \mathscr{D}^*_{\mathscr{F}_{\text{detect-CS}}}$. Given the following problem,

$$\begin{aligned} V_{\text{CR-P}} = \min \quad & M \cdot Y \\ s.t. \quad & Y_{11} = 1, \\ & Y \in \mathscr{D}^*_{\mathscr{F}_{\text{detect-P}}}, \end{aligned} \tag{CR-P}$$

we have the following theorem,

Theorem 4.7. *$M \in \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$ if and only if $V_{CR\text{-}P} \geq 0$.*

Proof. Because $\mathscr{D}^*_{\mathscr{F}_{\text{detect-P}}} = \mathscr{D}^*_{\mathscr{F}_{\text{detect-CS}}}$, problem (CR-P) is the conic reformulation of problem (detect-CS) and $V_{\text{CR-P}} = V_{\text{detect-CS}}$ [21]. Hence, by Theorem 2.1, we come to the conclusion. $\square$

However, checking whether $Y \in \mathscr{D}^*_{\mathscr{F}_{\text{detect-P}}}$ is a hard problem in general, here we will provide a conic relaxation for problem (CR-P):

$$\begin{aligned} V_{\text{RLB}} = \min \quad & M \cdot Y \\ s.t. \quad & Y_{11} = 1, \\ & Y \in \mathscr{D}^*_{\mathscr{G}}, \end{aligned} \tag{RLB}$$

where $\mathscr{G} \supseteq \mathscr{F}_{\text{detect-P}}$ and $\mathscr{D}_{\mathscr{G}}$, $\mathscr{D}^*_{\mathscr{G}}$ have linear matrix inequality representations.

In what follows, we will introduce how to find such $\mathscr{G}$. Because $\mathscr{F}_{\text{detect-P}} = cone(\mathscr{F}_{\text{detect-CS}})$ and $\mathscr{D}^*_{\mathscr{F}_{\text{detect-P}}} = \mathscr{D}^*_{\mathscr{F}_{\text{detect-CS}}}$, we just need to find an $\mathscr{H}$ to cover $\mathscr{F}_{\text{detect-CS}}$ such that the cone of nonnegative quadratic forms over $\mathscr{H}$ has the linear matrix inequality representations. Then $\mathscr{D}^*_{\mathscr{F}_{\text{detect-P}}} \subseteq \mathscr{D}^*_{\mathscr{H}}$. We know $\mathscr{F}_{\text{detect-CS}}$ is a bounded, closed set and $\mathscr{D}_{\mathscr{H}}$ and $\mathscr{D}^*_{\mathscr{H}}$ have linear matrix inequality representations when $\mathscr{H}$ is an ellipsoid. Hence, we can choose an ellipsoid to cover $\mathscr{F}_{\text{detect-CS}}$. Because $\mathscr{F}_{\text{detect-CS}} = \left\{ \begin{bmatrix} 1 \\ x \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^n \middle| \|x\|_p \leq 1 \right\}$, the box

$$\mathscr{R}_0 = \left\{ \begin{bmatrix} 1 \\ x \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^n \middle| x \in [-1, 1]^n \right\}$$

covers $\mathscr{F}_{\text{detect-CS}}$.

For a general box $\mathscr{R} = \left\{ \begin{bmatrix} 1 \\ x \end{bmatrix} \middle| a \leq x \leq b \right\}$ with $a < b$, define an ellipsoid

$$\mathscr{H}_{\mathscr{R}} = \left\{ \begin{bmatrix} 1 \\ x \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^n \middle| \sqrt{\begin{bmatrix} 1 \\ x \end{bmatrix}^T \begin{bmatrix} c^T P c & -(Pc)^T \\ -Pc & P \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}} \leq 1 \right\}.$$

where $P = \frac{4([Diag(b-a)]^2)^{-1}}{n} \succeq 0$ and $c = \frac{a+b}{2}$. Because $\mathscr{H}_{\mathscr{R}}$ can also be written as $\mathscr{H}_{\mathscr{R}} = \left\{ \begin{bmatrix} 1 \\ x \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^n \middle| \sum_{i=1}^n \frac{(x_i - \frac{a_i + b_i}{2})^2}{(b_i - a_i)^2} - \frac{n}{4} \leq 0 \right\}$ and $\frac{(x_i - \frac{a_i+b_i}{2})^2}{(b_i - a_i)^2} - \frac{1}{4} \leq 0$ for $a_i \leq x_i \leq b_i$, $i = 1, \ldots, n$, $\mathscr{H}_{\mathscr{R}}$ covers $\mathscr{R}$. Then $\mathscr{H}_{\mathscr{R}_0}$ is just $\mathscr{H}_{\mathscr{R}}$ with $a = -1_n$ and $b = 1_n$. Define

$$\begin{aligned} \mathscr{G}_{\mathscr{R}_0} &= \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^n \middle| \sqrt{\begin{bmatrix} t \\ x \end{bmatrix}^T \begin{bmatrix} c^T P c & -(Pc)^T \\ -Pc & P \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix}} \leq t \right\} \\ &= \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^n \middle| \sqrt{(x - ct)^T P (x - ct)} \leq t \right\} \end{aligned}$$

with $P = \frac{4([Diag(b-a)]^2)^{-1}}{n} \succeq 0$, $c = \frac{a+b}{2}$, $a = -1_n$ and $b = 1_n$. The following theorem shows that $cone(\mathscr{H}_{\mathscr{R}_0}) = \mathscr{G}_{\mathscr{R}_0}$.

Theorem 4.8. *If*

$$\mathscr{H} = \left\{ \begin{bmatrix} 1 \\ x \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^n \middle| \begin{bmatrix} 1 \\ x \end{bmatrix}^T Q \begin{bmatrix} 1 \\ x \end{bmatrix} \leq 1,\ Q = \begin{bmatrix} c^T Pc & -(Pc)^T \\ -Pc & P \end{bmatrix} \right\}$$

and

$$\mathscr{G} = \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^n \middle| \sqrt{\begin{bmatrix} t \\ x \end{bmatrix}^T Q \begin{bmatrix} t \\ x \end{bmatrix}} \leq t,\ Q = \begin{bmatrix} c^T Pc & -(Pc)^T \\ -Pc & P \end{bmatrix} \right\},$$

with $P \succ 0$, then $Q \succeq 0$, $\mathscr{G}$ is a second order cone, and $\mathscr{G} = cone(\mathscr{H})$.

Proof. If $c = 0$, then $Pc = 0$, and $P \succ 0$ implies $Q \succeq 0$. If $c \neq 0$, then due to $P \succ 0$, we have $c^T Pc > 0$ and $c^T Pc - (Pc)^T P^{-1} Pc = 0$. Owing to the Schur-complementary theorem, it means $Q \succeq 0$. So $\mathscr{G}$ is a second order cone.

For any $y = \begin{bmatrix} t \\ x \end{bmatrix} \in \mathscr{G}$, if $t = 0$, then $P \succ 0$ implies $x = 0$. So if $\begin{bmatrix} t \\ x \end{bmatrix}$ is not a zero vector, then $t > 0$ and $\bar{y} = \begin{bmatrix} 1 \\ \frac{x}{t} \end{bmatrix} \in \mathscr{H}$ which implies $y \in \text{cone}(\mathscr{H})$. Hence, $\mathscr{G} \subseteq \text{cone}(\mathscr{H})$. Conversely,

$$\text{cone}(\mathscr{H}) = \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^n \middle| \begin{bmatrix} t \\ x \end{bmatrix} = \sum_{i=1}^{m} \lambda_i \begin{bmatrix} 1 \\ x^i \end{bmatrix},\ \lambda_i \geq 0,\ \begin{bmatrix} 1 \\ x^i \end{bmatrix} \in \mathscr{H},\ i = 1, \ldots, m \right\}$$

with $m \in \mathbb{Z}_+$. For any $\begin{bmatrix} t \\ x \end{bmatrix} \in \text{cone}(\mathscr{H})$,

$$\sqrt{\begin{bmatrix} t \\ x \end{bmatrix}^T Q \begin{bmatrix} t \\ x \end{bmatrix}} = \left\| Q^{\frac{1}{2}} \begin{bmatrix} t \\ x \end{bmatrix} \right\|_2 \leq \sum_{i=1}^{m} \lambda_i \left\| Q^{\frac{1}{2}} \begin{bmatrix} 1 \\ x^i \end{bmatrix} \right\|_2 \leq \sum_{i=1}^{m} \lambda_i = t.$$

Therefore, $\text{cone}(\mathscr{H}) \subseteq \mathscr{G}$. Consequently, $\mathscr{G} = \text{cone}(\mathscr{H})$. □

According to Theorem 4.8, $\mathscr{F}_{\text{detect-P}} = cone(\mathscr{F}_{\text{detect-CS}}) \subseteq cone(\mathscr{H}_{\mathscr{R}_0}) = \mathscr{G}_{\mathscr{R}_0}$, thus $\mathscr{D}^*_{\mathscr{F}_{\text{detect-P}}} \subseteq \mathscr{D}^*_{\mathscr{G}_{\mathscr{R}_0}}$ and we can set $\mathscr{G} = \mathscr{G}_{\mathscr{R}_0}$.

Next we will derive the linear matrix inequality representations of $\mathscr{D}^*_{\mathscr{H}}$ for $\mathscr{H}$ defined in Theorem 4.8. For any $M \in \mathbb{S}^{n+1}$, determing $M \in \mathscr{D}_{\mathscr{H}}$ is equivalent to determing whether the optimal value of the following problem

$$\begin{aligned} V_7 = \min \quad & \begin{bmatrix} 1 \\ x \end{bmatrix}^T M \begin{bmatrix} 1 \\ x \end{bmatrix} \\ s.t. \quad & \begin{bmatrix} 1 \\ x \end{bmatrix}^T \begin{bmatrix} c^T Pc & -(Pc)^T \\ -Pc & P \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} \leq 1, \end{aligned} \tag{4.1}$$

is no less than zero.

The semidefinite reformulation of problem (4.1) is

$$\begin{aligned} V_8 = \min \quad & M \cdot Y \\ s.t. \quad & Y_{11} = 1, \\ & \begin{bmatrix} c^T Pc - 1 & -(Pc)^T \\ -Pc & P \end{bmatrix} \cdot Y \le 0, \\ & Y \succeq 0. \end{aligned} \tag{4.2}$$

Its conic dual problem becomes

$$\begin{aligned} V_9 = \max \quad & \sigma \\ s.t. \quad & M + \begin{bmatrix} -\sigma + \lambda(c^T Pc - 1) & -\lambda(Pc)^T \\ -\lambda Pc & \lambda P \end{bmatrix} \succeq 0, \\ & \lambda \ge 0, \ \sigma \in \mathbb{R}. \end{aligned} \tag{4.3}$$

Theorem 4.9. *Let*

$$\mathscr{H} = \left\{ \begin{bmatrix} 1 \\ x \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^n \middle| \sqrt{\begin{bmatrix} 1 \\ x \end{bmatrix}^T Q \begin{bmatrix} 1 \\ x \end{bmatrix}} \le 1, \ Q = \begin{bmatrix} c^T Pc & -(Pc)^T \\ -Pc & P \end{bmatrix} \right\}$$

with $P \succ 0$. Then for any $M \in \mathbb{S}^{n+1}$, $M \in \mathscr{D}_{\mathscr{H}}$ if and only if it satisfies

$$\begin{aligned} & M + \lambda \begin{bmatrix} (c^T Pc - 1) & -(Pc)^T \\ -Pc & P \end{bmatrix} \succeq 0, \\ & \lambda \ge 0. \end{aligned} \tag{4.4}$$

For a given $Y \in \mathbb{S}^{n+1}$, $Y \in \mathscr{D}^*_{\mathscr{H}}$ if and only if

$$\begin{aligned} & \begin{bmatrix} c^T Pc - 1 & -(Pc)^T \\ -Pc & P \end{bmatrix} \cdot Y \le 0, \\ & Y \succeq 0. \end{aligned} \tag{4.5}$$

Proof. Since $P \succ 0$, we can find a sufficiently large λ and a sufficiently small σ such that $M + \begin{bmatrix} -\sigma + \lambda(c^T Pc - 1) & -\lambda(Pc)^T \\ -\lambda Pc & \lambda P \end{bmatrix} \succ 0$. Therefore, problem (4.3) is strongly feasible, and there is no duality gap between problem (4.2) and (4.3). Also the gap between problem (4.1) and (4.2) is zero [21], then $V_7 \ge 0$ if and only if $V_9 \ge 0$. Notice that $V_9 \ge 0$ if and only if (4.4) is satisfied. Hence, $M \in \mathscr{D}_{\mathscr{H}}$ if and only if (4.4) is satisfied.

$Y \in \mathscr{D}^*_{\mathscr{H}}$ if and only if $Y \cdot M \ge 0$, for any $M \in \mathscr{D}_{\mathscr{H}}$. Because $M \in \mathscr{D}_{\mathscr{H}}$ if and only if there exists a matrix $S \in \mathbb{S}^{n+1}_+$ and a $\lambda \ge 0$ such that $M = S - \lambda \begin{bmatrix} (c^T Pc - 1) & -(Pc)^T \\ -Pc & P \end{bmatrix}$, $Y \cdot M \ge 0$ for any $M \in \mathscr{D}_{\mathscr{H}}$ if and only if (4.5) is satisfied. □

Based on that $\mathscr{D}_{\mathscr{G}}$ and $\mathscr{D}^*_{\mathscr{G}}$ have the linear matrix inequality representations, a lower bound for problem (CR-P) can be obtained. Next we will provide two ways to improve the lower bound of problem (CR-P). One way is adding redundant constraints defined as follows.

Definition 4.10. Assumed that $\mathscr{F}$ is a nonempty set in $\mathbb{R}^n$, if a constraint $f(Y) \le 0$ holds for any $Y \in \mathscr{D}^*_{\mathscr{F}}$, then we call $f(Y) \le 0$ a redundant constraint for $\mathscr{D}^*_{\mathscr{F}}$.

Let $Y = \begin{bmatrix} Y_{11} & Y^{12} \\ Y^{21} & Y^{22} \end{bmatrix}$ with $Y_{11} \in \mathbb{R}$, $Y^{21} \in \mathbb{R}^n$ and $Y^{22} \in \mathbb{S}^n$, then we have the following theorem,

Theorem 4.11. $\|Y^{21}\|_p \leq Y_{11}$ *is a redundant constraint for* $\mathscr{D}^*_{\mathscr{F}_{\text{detect-P}}}$ *when* $p \geq 1$. *Especially,* $\|Diag(Y^{22})\|_{\frac{p}{2}} \leq Y_{11}$ *is also a redundant constraint for* $\mathscr{D}^*_{\mathscr{F}_{\text{detect-P}}}$ *when* $p > 2$.

Proof. Because $\mathscr{D}^*_{\mathscr{F}_{\text{detect-P}}} = \mathscr{D}^*_{\mathscr{F}_{\text{detect-CS}}}$ and $\mathscr{F}_{\text{detect-CS}}$ is bounded and closed, for any $Y \in \mathscr{D}^*_{\mathscr{F}_{\text{detect-CS}}}$, there exists a $\tau \in \mathbb{Z}_+$ such that

$$Y = \sum_{i=1}^{\tau} \theta_i \begin{bmatrix} 1 \\ x^i \end{bmatrix} \begin{bmatrix} 1 \\ x^i \end{bmatrix}^T = \begin{bmatrix} \sum_{i=1}^{\tau} \theta_i & \sum_{i=1}^{\tau} \theta_i (x^i)^T \\ \sum_{i=1}^{\tau} \theta_i x^i & \sum_{i=1}^{\tau} \theta_i x^i (x^i)^T \end{bmatrix}$$

where $\theta_i \geq 0$ and $\|x^i\|_p \leq 1$. Because $\|\theta_i x^i\|_p \leq \theta_i$ and $\|x\|_p$ is a convex function, $\| \sum_{i=1}^{\tau} \theta_i x^i\|_p \leq \sum_{i=1}^{\tau} \|\theta_i x^i\|_p \leq \sum_{i=1}^{\tau} \theta_i$, that is $\|Y^{21}\|_p \leq Y_{11}$.

$\|x\|_{\frac{p}{2}}$ is a convex function for $p > 2$. For any $\begin{bmatrix} 1 \\ x^i \end{bmatrix} \in \mathscr{F}_{\text{detect-CS}}$, we have $(|x_1^i|^p + \cdots + |x_n^i|^p)^{\frac{1}{p}} \leq 1$, so $(|(x_i^1)^2|^{\frac{p}{2}} + \cdots + |(x_n^i)^2|^{\frac{p}{2}})^{\frac{2}{p}} \leq 1$. That is $\|Diag(x^i(x^i)^T)\|_{\frac{p}{2}} \leq 1$. Hence, $\| \sum_{i=1}^{n} Diag(\theta_i x^i (x^i)^T)\|_{\frac{p}{2}} \leq \sum_{i=1}^{n} \|Diag(\theta_i x^i (x^i)^T)\|_{\frac{p}{2}} \leq \sum_{i=1}^{n} \theta_i$, which implies $\|Diag(Y^{22})\|_{\frac{p}{2}} \leq Y_{11}$. □

Therefore, we can add the redundant constraints proposed in Theorem 4.11 into problem (RLB) to improve the lower bound. The other way to improve the lower bound is to refine the cover $\mathscr{G}$ for $\mathscr{F}_{\text{detect-P}}$. The intuitive idea behind the refinement of the cover is to divide the box $\mathscr{R}_0$ into a union of boxes $\mathscr{R}_i$ and generate corresponding ellipsoids $\mathscr{H}_i$, $i = 1, ..., m$. Let $\mathscr{H} = \mathscr{H}_1 \cup \mathscr{H}_2 \cup \cdots \cup \mathscr{H}_m$. Lemma 4.6 implies $\mathscr{F}_{\text{detect-P}} \subseteq \mathscr{G}$ with $\mathscr{G} = cone(\mathscr{H}_1) \cup \cdots \cup cone(\mathscr{H}_m)$ being a union of second order cones. Then $\mathscr{D}^*_{\mathscr{F}_{\text{detect-P}}} \subseteq \mathscr{D}^*_{\mathscr{H}_1} + \cdots + \mathscr{D}^*_{\mathscr{H}_m}$. Define

$$\mathscr{H}_i = \left\{ \begin{bmatrix} 1 \\ x \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^n \,\middle|\, \sqrt{\begin{bmatrix} 1 \\ x \end{bmatrix}^T Q_i \begin{bmatrix} 1 \\ x \end{bmatrix}} \leq 1,\ Q_i = \begin{bmatrix} c_i^T P_i c_i & -(P_i c_i)^T \\ -P_i c_i & P_i \end{bmatrix} \right\}$$

with $P_i \succ 0$, $i = 1, 2, \ldots, m$, then we can improve (RLB) as

$$\begin{aligned} V_{\text{RLB2}} = \min \quad & M \cdot Y \\ s.t. \quad & Y = Y^1 + \cdots + Y^m, \\ & Y_{11} = 1, \\ & \begin{bmatrix} c_i^T P_i c_i - 1 & -(P_i c_i)^T \\ -P_i c_i & P_i \end{bmatrix} \cdot Y^i \leq 0, \quad i = 1, 2, \ldots, m, \\ & Y^i \succeq 0,\ i = 1, \ldots, m, \\ & f_j(Y) \leq 0,\ j = 1, \ldots, l, \\ & g_j(Y) \leq 0,\ j = 1, \ldots, t, \end{aligned} \tag{RLB2}$$

where $f_j(Y) \leq 0$, $j = 1, \ldots, l$, and $g_j(Y) \leq 0$, $j = 1, \ldots, t$, represent the redundant constraints. When $2 > p \geq 1$, $l = 1$ and $f_1(Y) = \|Y^{21}\|_p - Y_{11}$. When $p > 2$, $l = 2$, $f_1(Y) = \|Y^{21}\|_p - Y_{11}$ and $f_2(Y) = \|Diag(Y^{22})\|_{\frac{p}{2}} - Y_{11}$. Moreover, $g_j(Y) \leq 0$ is a linear inequality for $j = 1, \ldots, t$. $g_j(Y)$ will be introduced in the next section.

Problem (RLB2) can be solved in polynomial time [2]. Denote its optimal solution as Y^*. Next theorem states that there is a method to decompose Y^* into the original space in polynomial time.

Theorem 4.12. *Let* $Q_i = \begin{bmatrix} c_i^T P_i c_i - 1 & -(P_i c_i)^T \\ -P_i c_i & P_i \end{bmatrix}$ *with* $P_i \succ 0$, $i = 1, 2, \ldots, m$. *For an optimal solution* $Y^* = (Y^1)^* + \cdots + (Y^m)^*$ *of problem* (RLB2), there is a decomposition running in polynomial-time

$$Y^* = \sum_{i=1}^{m} \sum_{j=1}^{r_i} \xi_{ij} \begin{bmatrix} 1 \\ x^{ij} \end{bmatrix} \begin{bmatrix} 1 \\ x^{ij} \end{bmatrix}^T \tag{4.6}$$

satisfying $\begin{bmatrix} 1 \\ x^{ij} \end{bmatrix}^T Q_i \begin{bmatrix} 1 \\ x^{ij} \end{bmatrix} \leq 1$, $\xi_{ij} > 0$, $i = 1, \ldots, m$, $j = 1, \ldots, r_i$ and $\sum_{i=1}^{m} \sum_{j=1}^{r_i} \xi_{ij} = 1$, where $r_i = rank((Y^i)^*)$.

Proof. There is a decomposition running is polynomial-time via Procedure 1 in [21]

$$Y^* = \sum_{i=1}^{m} \sum_{j=1}^{r_i} \begin{bmatrix} t^{ij} \\ y^{ij} \end{bmatrix} \begin{bmatrix} t^{ij} \\ y^{ij} \end{bmatrix}^T$$

satisfying $\begin{bmatrix} t^{ij} \\ y^{ij} \end{bmatrix}^T Q_i \begin{bmatrix} t^{ij} \\ y^{ij} \end{bmatrix} \leq 0$, $i = 1, \ldots, m$, $j = 1, \ldots, r_i$. Because $P_i \succ 0$, if $\begin{bmatrix} t^{ij} \\ y^{ij} \end{bmatrix} \neq 0$, then $t^{ij} \neq 0$. Thus, $Y^* = \sum_{i=1}^{m} \sum_{j=1}^{r_i} (t^{ij})^2 \begin{bmatrix} 1 \\ y^{ij}/t^{ij} \end{bmatrix} \begin{bmatrix} 1 \\ y^{ij}/t^{ij} \end{bmatrix}^T$. Denote $\xi_{ij} = (t^{ij})^2$ and $x^{ij} = y^{ij}/t^{ij}$, we have the decomposition as in (4.6). □

5 Conic Approximation Algorithm

This section presents a specific partition method for the feasible set to improve the cover for $\mathscr{F}_{\text{detect-P}}$. At the end of this section, an adaptive conic approximation algorithm is proposed for solving problem (CR-P).

Before providing the partition process, we first give the following two definitions.

Definition 5.1. For a decomposition of the optimal solution of problem (RLB2) as in (4.6), the sensitive point is defined as

$$y^* = \begin{bmatrix} 1 \\ x^* \end{bmatrix} = arg \min_{\begin{bmatrix} 1 \\ x \end{bmatrix} \in \mathscr{J}} \begin{bmatrix} 1 \\ x \end{bmatrix}^T M \begin{bmatrix} 1 \\ x \end{bmatrix} \tag{5.1}$$

where $\mathscr{J} = \left\{ \begin{bmatrix} 1 \\ x^{ij} \end{bmatrix}, \ i = 1, \ldots, m, \ j = 1, \ldots, r_i \right\}$. If the sensitive point is decomposed from $(Y^k)^*$, $k \in \{1, 2, \ldots, m\}$, then it belongs to the k-th ellipsoid, and the corresponding box is called the sensitive box.

The sensitive point may be not unique. If there are multiple sensitive points, we choose the one with the smallest index $i \in \{1, \ldots, m\}$ and the smallest $j \in \{1, \ldots, r_i\}$ as the sensitive point.

Similar to the definition of ε-copositivity over $\mathbb{R}^n_+$ which is proposed in [4], we give the following definition.

Definition 5.2. For a given $\varepsilon > 0$, $M \in \mathbb{S}^n$ is called ε-copositivity over a p-th order cone if $V_{\text{CR-P}} \geq -\varepsilon$. The ε-copositivity set is denoted as ε-$\mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$.

Assuming a partition $\mathscr{R} = \mathscr{R}_1 \cup \cdots \cup \mathscr{R}_m$ is obtained, we consider how to obtain a better partition for a better approximation. By the continuity of the objective function of problem (detect-CS), for any given error $\varepsilon > 0$, there exists an η_ε $(1 \geq \eta_\varepsilon > 0)$ such that for any y^1, y^2 in the set $\{y \in \mathbb{R}^{n+1} | \|y - z\|_\infty \leq 1,\ \|z\|_p \leq 1\}$, if $\|y^1 - y^2\|_\infty \leq \eta_\varepsilon$, then $|(y^1)^T M y^1 - (y^2)^T M y^2| \leq \varepsilon$. If the length of the sensitive box's longest edge is greater than η_ε, cut this longest edge to obtain two boxes and check whether each box is intersected with $\mathscr{F}_{\text{detect-CS}}$. If not, drop that box. Generate the corresponding $\mathscr{H}$ which is the union of ellipsoids based on the remained box(es).

Whether a box $\mathscr{R}_k = [a, b]$ is intersected with $\mathscr{F}_{\text{detect-CS}}$ can be determined by the following polynomial-time slovable problem

$$\begin{aligned} V_{10} = \min \quad & \|x - z\|_2 \\ s.t. \quad & a \leq z \leq b, \\ & \|x\|_p \leq 1. \end{aligned} \tag{5.2}$$

If there exists a point belonging to $\mathscr{F}_{\text{detect-CS}} \cap \mathscr{R}_k$, then it is obvious $V_{10} = 0$ and we have the following theorem:

Theorem 5.3. *If $V_{10} > 0$, then $\mathscr{F}_{\text{detect-CS}} \cap \mathscr{R}_k = \emptyset$.*

For a sensitive point $y^* = \begin{bmatrix} 1 \\ x^* \end{bmatrix} \in \mathbb{R}^{n+1}$, it is expected to add one additional cut such that the new cover of the feasible domain of problem (detect-P) does not contain ty^* for any $t > 0$. Actually, the cut $u^T x \leq vt$ satisfying the above requirement, where

$$u = \nabla \| \frac{x^*}{\|x^*\|_p} \|_p \quad \text{and} \quad v = (\nabla \| \frac{x^*}{\|x^*\|_p} \|_p)^T \frac{x^*}{\|x^*\|_p}. \tag{5.3}$$

Theorem 5.4. *$u^T Y^{21} \leq v Y_{11}$ is a redundant constraint for $\mathscr{D}^*_{\mathscr{F}_{\text{detect-P}}}$, where u and v are defined by (5.3).*

Proof. Similar to the proof of Theorem 4.11, for a $Y \in \mathscr{D}^*_{\mathscr{F}_{\text{detect-P}}}$, there exists a $\tau \in \mathbb{Z}_+$ such that $Y = \begin{bmatrix} \sum_{i=1}^{\tau} \theta_i (t^i)^2 & \sum_{i=1}^{\tau} \theta_i t^i (x^i)^T \\ \sum_{i=1}^{\tau} \theta_i t^i x^i & \sum_{i=1}^{\tau} \theta_i x^i (x^i)^T \end{bmatrix}$ where $\theta_i \geq 0$, $\begin{bmatrix} t^i \\ x^i \end{bmatrix} \in \mathscr{F}_{\text{detect-P}}$ and $u^T x^i \leq v t_i$, $i = 1, \ldots, \tau$, then $u^T Y^{21} = \sum_{i=1}^{\tau} \theta_i t^i u^T x^i \leq v \sum_{i=1}^{\tau} \theta_i (t^i)^2 = v Y_{11}$. $\square$

Based on above analysis, we present a conic approximation algorithm as follows.

A conic approximation algorithm for detecting whether $M \in \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$ $(p \geq 1)$.

Initialization: Given an error ε, set $\eta_\varepsilon = -1 + \sqrt{1 + \frac{\varepsilon}{\sum_{i,j=1}^{n} |M_{ij}|}} > 0$, the sensitive box $\mathscr{R}_0 = \left\{ \begin{bmatrix} 1 \\ x \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^n \middle| x \in [-1, 1]^n \right\}$, and the corresponding ellipsoid $\mathscr{H}_{\mathscr{R}_0} = \left\{ \begin{bmatrix} 1 \\ x \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^n \middle| \sqrt{\frac{x^T x}{n}} \leq 1 \right\}$. Let $l = -\infty$ be initial the lower bound. Set $t = 0$ and the iteration number $k = 0$.

Step 1. Check whether M satisfying the conditions in Theorem 3.3, 3.4 or 3.5. If yes, then the problem is solved, stop. Otherwise, set $k = 1$, go to Step 2.

Step 2. Solve problem (RLB2) and obtain an optimal value V_{RLB2} and an optimal solution Y^*. If $0 > V_{\text{RLB2}} \geq -\varepsilon$, then $M \in \varepsilon\text{-}\mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$. Stop. If $V_{\text{RLB2}} \geq 0$, then $M \in \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$. Stop. Otherwise, decompose Y^* as in (4.6), and check whether there exists an $x^{ij} \in \mathscr{F}_{\text{detect-P}}$ satisfying $(x^{ij})^T M x^{ij} < 0$. If so, then $M \notin \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$. Stop. If not, set $l = \max\{l, V_{\text{RLB2}}\}$ and go to Step 3.

Step 3. Find the sensitive point y^* and sensitive box $\mathscr{R}_i$. If the length of the sensitive box's longest edge is smaller than $\frac{2\eta_\varepsilon}{\sqrt{n}}$, stop. If $(y^*)^T M y^* \geq 0$, then $M \in \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$. If $0 > (y^*)^T M y^* \geq -\varepsilon$, then $M \in \varepsilon\text{-}\mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$. If $(y^*)^T M y^* < -\varepsilon$, then $M \notin \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$. Otherwise, bisect $\mathscr{R}_i$ along the longest edge such that $\mathscr{R}_i = \mathscr{R}_{i1} \cup \mathscr{R}_{i2}$. Solve problem (5.2) to check whether $\mathscr{R}_{i1} \cap \mathscr{F}_{\text{detect-CS}} = \emptyset$ or $\mathscr{R}_{i2} \cap \mathscr{F}_{\text{detect-CS}} = \emptyset$. If $\mathscr{R}_{i1} \cap \mathscr{F}_{\text{detect-CS}} = \emptyset$, then $\mathscr{R} = \mathscr{R} \cup \mathscr{R}_{i2} \setminus \mathscr{R}_i$. If $\mathscr{R}_{i2} \cap \mathscr{F}_{\text{detect-CS}} = \emptyset$, then $\mathscr{R} = \mathscr{R} \cup \mathscr{R}_{i1} \setminus \mathscr{R}_i$. Otherwise, $\mathscr{R} = \mathscr{R} \cup \mathscr{R}_{i1} \cup \mathscr{R}_{i2} \setminus \mathscr{R}_i$. Generate corresponding $\mathscr{H}_{\mathscr{R}}$. Calculate u and v in (5.3) based on the sensitive point y^*, set $t = t+1$ and let $g_t(Y) = u^T Y^{21} - vY_{11}$ in problem (RLB2). Set $k = k+1$ and go to step 2.

We have following results about the convergency of the proposed algorithm.

Theorem 5.5. *If the proposed conic approximation algorithm doesn't stop at Step 1 or 2, then, for any given $\varepsilon > 0$, there exists a positive integer N_ε, such that $|V_{\text{RLB2}} - V_{\text{CR-P}}| \leq \varepsilon$ at the N_ε-th iteration. Hence, the algorithm stops at Step 3.*

Proof. Due to the continuity of the objective function of problem (CR-P), for any given error $\varepsilon > 0$, there exists $1 \geq \eta_\varepsilon > 0$, such that $|(y^1)^T M y^1 - (y^2)^T M y^2| \leq \varepsilon$ for any y^1, y^2 in the set $\{y \in \mathbb{R}^{n+1} | \|y - z\|_\infty \leq 1,\ \|z\|_p \leq 1\}$ and $\|y^1 - y^2\|_\infty \leq \eta_\varepsilon$. From the description of the proposed algorithm, there exists an integer $N_\varepsilon > 0$, a sensitive point y^* and a feasible point y^0 in the sensitive box such that the length of the longest edge of the sensitive box is shorter than $\frac{2\eta_\varepsilon}{\sqrt{n}}$. Following the structure of $\mathscr{H}_{\mathscr{R}}$, the length of its j-th axis of $\mathscr{H}_{\mathscr{R}}$ is equal to $\frac{\sqrt{n}}{2}$ times the length of its corresponding j-th edge of $\mathscr{R}$. Then $\|y^* - y^0\|_\infty \leq \eta_\varepsilon$ and $|V_{\text{RLB2}} - V_{\text{CR-P}}| \leq |M \cdot y^*(y^*)^T - M \cdot y^0(y^0)^T| \leq \varepsilon$. $\square$

Theorem 5.5 indicates at the N_ε-th iteration, if $V_{\text{RLB2}} \geq 0$, then $V_{\text{CR-P}} \geq V_{\text{RLB2}} \geq 0$ which means $M \in \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$. If $0 > V_{\text{RLB2}} \geq -\varepsilon$, then $V_{\text{CR-P}} \geq V_{\text{RLB2}} \geq -\varepsilon$ which means $M \in \varepsilon\text{-}\mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$. Otherwise, $V_{\text{CR-P}} \leq V_{\text{RLB2}} + \varepsilon < 0$ which means $M \notin \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$.

Remark 5.6. Because $\mathscr{F}_{\text{detect-CS}} = \left\{ \begin{bmatrix} 1 \\ x \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^n \middle| \|x\|_p \leq 1 \right\}$, we can choose

$$\mathscr{R}_0 = \left\{ \begin{bmatrix} 1 \\ x \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^n \middle| x \in [-1, 1]^n \right\}$$

as the initial box. From the partition, we know that the length of each new box's edge is one half of the sensitive box's longest edge length after each iteration. Therefore, the length of the longest edge of the sensitive box $\mathscr{R}_t$ is shorter than $\frac{2\eta_\varepsilon}{\sqrt{n}}$ after at most $N_\varepsilon = (\frac{\sqrt{n}}{2\eta_\varepsilon})^n$ iterations.

For any given $\varepsilon > 0$, we calculate η_ε as follows. If $\|y^0\|_p \leq 1$ and $\|y^* - y^0\|_\infty \leq \eta_\varepsilon$, we have $\max_{i=1,\ldots,n} y_i^0 \leq 1$, $\max_{i=1,\ldots,n} y_i^* \leq 1+\eta_\varepsilon$ and

$$\begin{aligned}|(y^*)^T My^* - (y^0)^T My^0| =& |(y^*)^T My^* - (y^0)^T My^* + (y^0)^T My^* - (y^0)^T My^0| \\ =& |(y^0+y^*)^T M(y^*-y^0)| \\ =& |\sum_{i,j=1}^n M_{ij}(y_i^0+y_i^*)(y_j^*-y_j^0)| \\ \leq& \sum_{i,j=1}^n |M_{ij}|\eta_\varepsilon(1+\eta_\varepsilon+1)\end{aligned}$$

So η_ε can be taken as $-1+\sqrt{1+\frac{\varepsilon}{\sum_{i,j=1}^n |M_{ij}|}}$ such that $|(y^*)^T My^* - (y^0)^T My^0| \leq \varepsilon$. Consequently, in the worst case, we have

$$N_\varepsilon \leq (\frac{\sqrt{n}}{2\eta_\varepsilon})^n \leq \left(\frac{\sqrt{n}(1+\sqrt{1+\frac{\varepsilon}{\sum_{i,j=1}^n |M_{ij}|}})}{\frac{2\varepsilon}{\sum_{i,j=1}^n |M_{ij}|}}\right)^n.$$

$\boxed{6}$ Numerical Examples and Experiments

In this section, some numerical examples are provided in order to verify the efficiency of our proposed algorithm. We used MATLAB 2012 on a computer with Intel Core 2 CPU, 2.26 Ghz and 3G memory to implement the proposed conic approximation algorithm. We solve problem (RLB2) by the Matlab software package CVX [7]. The tolerance is set to be 1e-3. The matrix $M \in \mathbb{S}^n$ is randomly generated as follows. We first generate $\frac{n(n+1)}{2}$ numbers with standard normal distribution, then multiply each number by 100 and round it to the nearest integer. These $\frac{n(n+1)}{2}$ integers are put into in the upper triangular of the symmetric matrix M row by row, and the lower triangular of M is filled symmetric to the upper triangular.

When $p > 2$, we provide the following instance to demonstrate the process of our conic approximation algorithm.

Example 1. Let $p = 3$, $n = 3$ and

$$M = \begin{bmatrix} 412 & -125 & 37 & 189 \\ -125 & -33 & 39 & -123 \\ 37 & 39 & 337 & 19 \\ 189 & -123 & 19 & -32 \end{bmatrix}.$$

M does not belong to the polynomial-time solvable subclasses, so we solve it with our conic approximation algorithm. In the first iteration, we use the second order cone

$$\mathscr{G}_{\mathscr{R}} = \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R}\times\mathbb{R}^n | \sqrt{\frac{x^Tx}{n}} \leq t \right\}$$

to cover $\mathscr{F}_{\text{detect-P}}$. Then we solve problem (RLB2) and obtain an optimal solution Y^*. After that, Y^* is decomposed according to (4.6) and two solutions $y^1 = [1.0000\ \ -0.7596\ \ 0.0614\ \ -1.5554]^T$ and $y^2 = [1.0000\ \ 0.4721\ \ -0.1162\ \ -0.7060]^T$ are obtained. y^1 is not a feasible solution for problem (detect-P), but y^2 is a feasible solution with $(y^2)^T My^2 = 80.5845 > 0$.

Hence, the algorithm enters Step 3 and we bisect the feasible set along the longest edge of the sensitive box. Go back to Step 2 and repeat this process. Until the 43rd iteration, we obtain a feasible solution $y = [1.0000 \;\; 0.2228 \;\; -0.1049 \;\; -0.9959]^T$ from decomposition of Y^* and $y^T My = -0.8498 < 0$ which indicates $M \notin \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$. The CPU time is 45.3223 seconds.

In order to demonstrate the effectiveness of redundant constraints, we compute the same example without the redundant constraints proposed in Theorem 4.11. It means that the constraints $f_j(Y) \leq 0$, $j = 1, \ldots, l$, in problem (RLB2) are removed. Then in the 189th iteration, we obtain a feasible solution $y' = [1.0000 \;\; 0.3664 \;\; -0.0973 \;\; -0.9830]^T$ satisfying $(y')^T My' < 0$ and the CPU time is 156.2727 seconds.

Set $p = 3$ and $n = 3$. Firstly, we use the proposed conic approximation algorithm to solve five randomly generated matrices not belonging to the polynomial-time subclasses. Then we solve the same five matrices without the constraints $f_j(Y) \leq 0$, $j = 1, \ldots, l$, in problem (RLB2). Iteration numbers and the CPU time compose the results in Table 1.

Table 1: Effectiveness of redundant constraints in Theorem 4.11

With the redundant constraints		Without the redundant constraints	
iterations	CPU time(sec)	iterations	CPU time(sec)
13	12.6308	145	110.5474
1	0.5818	18	13.4114
1	0.5538	7	5.0131
19	17.9298	126	96.2440
5	4.6162	51	37.8957

Table 1 shows that the redundant constraints in Theorem 4.11 can reduce the number of iterations and the CPU time significantly.

When $2 > p \geq 1$, we set the maximum iterations to be 1000. 5000 random matrices M were generated for different dimension n. Computational results are provided in Tables 2 and 3. PS means the instances belong to the polynomial-time solvable subclasses while NPS means they do not. ♯1 represents the number of problems which belong to the polynomial-time solvable subclasses, ♯2 and ♯3 represent the number of instances which are solved in one iteration and greater than one iteration, respectively, ♯4 represents the number of instances which cannot be solved within 1000 iterations. If M is in the polynomial-time solvable subclasses, then its iteration number is denoted as 0.

Table 2: Comparisons between different p when $p < 2$

p	PS		NPS					average CPU(sec) for all instances
	♯1	average CPU(sec)	♯2	♯3	♯4	average iterations	average CPU(sec)	
1	4613	0.1253	1	305	81	396.5943	288.6615	22.4580
1.2	4902	0.1268	3	67	28	391.0102	445.5849	8.8578
1.4	4984	0.1284	0	9	7	562.1875	639.5027	2.1744
1.6	4999	0.1367	0	1	0	246	242.4806	0.1851
1.8	5000	0.1290	0	0	0	0	0	0.1290

For $1 \leq p < 2$, Tables 2 and 3 show that a majority of the instances satisfy $M_{11} < 0$ or can be solved in polynomial time according to Theorem 3.5, especially when p approaches to 2. There is an explanation for this phenomenon. The smaller $2 - p$ is, the larger the feasible region of problem (3.6) is while the feasible region of problem (3.4) is unchanged.

Table 3: Comparisons between different n when $p = 1.2$

n	PS		NPS					average CPU(sec) for all instances
	$\sharp 1$	average CPU(sec)	$\sharp 2$	$\sharp 3$	$\sharp 4$	average iterations	average CPU(sec)	
10	4902	0.1268	3	67	28	391.0102	445.5849	8.8578
20	4956	0.1545	1	28	15	472.8182	837.4583	7.5228
30	4968	0.2008	0	11	21	793.0625	2961.6688	19.1542
40	4982	0.2868	0	2	16	920.8889	6851.1667	24.9500
50	4986	0.4123	0	2	12	930.0714	7981.7286	22.7600

Then the sign of V_6 may change from positive to negative when p increases to 2, and more matrices satisfy the conditions in Theorem 3.5. The increasing number $\sharp 1$ in column PS of Table 2 precisely illustrates this point.

$\boxed{7}$ Conclusions and Future Work

We conjecture that the complexity of detecting $M \in \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$ for $1 \leq p \neq 2$ is NP-hard although some polynomial-time solvable subclasses of the detection problem are identified for $1 \leq p \neq 2$. Based on the properties of the cone of nonnegative quadratic forms over a second order cone, we used a conic approximation scheme to detect whether $M \in \mathscr{D}_{\mathscr{F}_{\text{detect-P}}}$ for those instances not belonging to the proposed polynomial-time solvable subclasses. For the future study, the complexity of the detection problem deserves our attention.

Acknowledgements

The authors are grateful to the two anonymous referees for their valuable comments to improve the quality of this paper.

References

[1] E.D. Andersen, C. Roos and T. Terlaky, Notes on duality in second order and p-order cone optimization, *Optim.* 51 (2002) 627–643.

[2] A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization, Analysis, Algorithms and Engineering Applications*, 1^{st} edition, MPS/SIAM Series on Optimization, Philadelphia, 2001.

[3] S. Boyd and L. Vandenberghe, *Convex Optimization*, 6^{th} edition, The United Kingdom at the University Press, Cambridge, 2008.

[4] S. Bundfuss and M. Dür, Algorithmic copositivity detection by simplicial partition, *Linear Algebra Appl.* 428 (2008) 1511–1523.

[5] S. Burer and J. Chen, A p-cone sequential relaxation procedure for 0-1 integer programs, *Optim. Methods Softw.* 24 (2009) 523–548.

[6] Z. Deng, S.-C. Fang, Q. Jin and W. Xing, Detecting copositivity of a symmetric matrix by an adaptive ellipsoid-based approximation scheme, *European J. Oper. Res.* 229 (2013) 21–28.

[7] M. Grant and S. Boyd, CVX: Matlab software for disciplined convex programming, version 2.0(beta) `http://cvxr.com/cvx`, 2013.

[8] Y. Hsia, Complexity and nonlinear semidefinite programming reformulation of l_1-constrained nonconvex quadratic optimization, *Optim. Lett.* 8 (2014) 1433–1442.

[9] Q. Jin, *Quadratically Constrained Quadratic Programming Problems and Extensions*, Ph.D thesis, NC state University in Raleigh, 2011.

[10] S.G. Johnson, Notes on the equivalence of norms, `http://math.mit.edu/~stevenj/18.335/norm-equivalence.pdf`, 2012.

[11] S. Khot, On the power of unique 2-prover 1 round games, in *Proceedings of the Thirty-Fourth Annual ACM Symposium on Theory of Computing*, ACM., New York, 2002, pp. 767–775.

[12] S. Khot and R. O'Donnell, SDP gaps and UGC-hardness for MAX-CUT-GAIN, in *47th Annual Symposium on Foundations of Computer Science*, IEEE Computer Society, 2006, pp. 217–226.

[13] G. Kindler, A. Naor and G. Schechtman, The UGC hardness threshold of the L_p Grothendieck problem, *Math. Oper. Res.* 35 (2010) 267–283.

[14] P.A. Krokhmal and P. Soberanis, Risk optimization with p-order conic constraints: A linear programming approach, *European J. Oper. Res.* 201 (2010) 653–671.

[15] C. Lu, Q. Jin, S.-C. Fang, Z. Wang and W. Xing, Adaptive computable approximation to cones of nonnegative quadratic functions, *Optim.* 64 (2014) 955–980.

[16] M.W. Margaret, Advances in cone-based preference modeling for decision making with multiple criteria, *Decis. Mak. Manuf. Serv.* 1 (2007) 153–173.

[17] N. Meinshausen and P. Bühlmann, High-dimensional graphs and variable selection with the lasso, *Ann. Statist.* 34 (2006) 1436–1462.

[18] K.G. Murty and S.N. Kabadi, Some NP-complete problems in quadratic and nonlinear programming, *Math. Program.* 39 (1987) 117–129.

[19] Y. Nesterov, H. Wolkowicz and Y. Ye, Semidefinite programming relaxations of non-convex quadratic optimization, *Handbook of Semidefinite Programming*, Springer US, 2000. pp. 361–419.

[20] D.H. Sherali, A.I. Loughani and S. Subramanian, Global optimization procedures for the capacitated Euclidean and l_p distance multifacility location-allocation problems, *Oper. Res.* 50 (2002) 433–448.

[21] J.F. Sturm and S. Zhang, On cones of nonnegative quadratic functions, *Math. Oper. Res.* 28 (2003) 246–267.

[22] Y. Tian, S.-C. Fang, Z. Deng and W. Xing, Computable representation of the cone of nonnegative quadratic forms over a general second-order cone and its application to completely positive programming, *J. Ind. Manag. Optim.* 9 (2013) 701–719.

[23] A. Vinel and P.A. Krokhmal, Polyhedral approximations in p-order cone programming, *Optim. Methods Softw. Online first* (2014), `http://www.tandfonline.com/doi/pdf/10.1080/10556788.2013.877905`.

[24] J. Zhou, D. Chen, Z. Wang and W. Xing, A conic approximation method for the 0-1 quadratic knapsack problem, *J. Ind. Manag. Optim.* 9 (2013) 531–547.

Manuscript received 14 November 2013
revised 26 June 2014
accepted for publication 2 July 2014

J. Zhou
Department of Applied Mathematics, Zhejiang University of Technology
Hangzhou, Zhejiang, 310032, China
E-mail address: zhoujing_0224@163.com

Z. Deng
School of Management, University of Chinese Academy of Sciences
Beijing, 100190, China
E-mail address: zhibindeng@ucas.ac.cn

S.-C. Fang
Edward P. Fitts Department of Industrial and Systems Engineering
North Carolina State University, Raleigh, NC 27695-7906, USA
E-mail address: fang@ncsu.edu

W. Xing
Department of Mathematical Sciences, Tsinghua University
Beijing, 100084, China
E-mail address: wxing@math.tsinghua.edu.cn