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SOLVING CONIC QUADRATICALLY CONSTRAINED QUADRATIC PROGRAMMING PROBLEMS*

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Abstract: The conic quadratically constrained quadratic programming problem (CQCQP in short) is a generalization of the quadratically constrained quadratic programming problem (QCQP in short) with wider applications. Few research has been done on general CQCQP directly. Most known results focus on the local optimality conditions or the lower bounds of the conic nonlinear programming problems. In this paper, following Lu et al. ("KKT solution and conic relaxation for solving quadratically constrained quadratic programming problems, SIAM J. Optim., 21 (2011), 1475–1490."), we study the KKT conditions and linear conic reformulations of CQCQP. A global optimality condition is provided and based on the extended maximal property of the optimal Lagrangian vectors, a computational scheme is also proposed. An example is given to illustrate the effectiveness of the scheme.

Key words: *conic quadratically constrained quadratic programming; conic programming; global optimality condition; KKT condition*

Mathematics Subject Classification: *90C20, 90C26, 90C46*

1 Introduction

Let $\mathcal{S}^n$ be the set of $n \times n$ real symmetric matrices, $\mathcal{S}^n_+$ be the set of $n \times n$ positive semidefinite matrices, and $\mathcal{K} \subset \mathbb{R}^m$ be a proper cone. Here a proper cone means a pointed closed convex cone with a nonempty interior.

Consider the following conic quadratically constrained quadratic programming problem:

$$\begin{aligned} \min \quad & f(x) = x^T A_0 x + 2b_0^T x + c_0 \\ \text{s.t.} \quad & g(x) \in \mathcal{K} \subset \mathbb{R}^m, \end{aligned} \qquad \text{(CQCQP)} \qquad (1.1)$$

where $A_0 \in \mathcal{S}^n$, $b_0 \in \mathbb{R}^n$, $c_0 \in \mathbb{R}$ and $g(x) = [g_1(x), \ldots, g_m(x)]^T$ in the form

$$g_i(x) = x^T A_i x + 2b_i^T x + c_i$$

with $A_i \in \mathcal{S}^n$, $b_i \in \mathbb{R}^n$ and $c_i \in \mathbb{R}$, for $i = 1, \ldots, m$. (For simplicity, we do not differentiate the minimum and the infimum.)

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Notice that the quadratically constrained quadratic programming problem is a special case of CQCQP with $\mathcal{K} = \mathbb{R}^m_+$. The linear and bilinear matrix inequality problems, which are widely used in optimal control (e.g. Leibfritz [15]), are also special cases of CQCQP. The study of CQCQP problems is also motivated by the robust optimization (see Ben-Tal and Nemirovski [5], El-Ghaoui and Lebret [11], El-Ghaoni, Oustry and Lebret [12] and references therein). For example, given a quadratically constrained quadratic programming:

$$\begin{array}{ll} \min & x^T A_0 x + 2b_0^T x + c_0 \\ \text{s.t.} & x^T A_i x + 2b_i^T x + c_i \le 0, \quad i = 1, \dots, I, \end{array} \qquad \text{(QCQP)} \qquad (1.2)$$

consider its robust version RQCQP

$$\begin{array}{ll} \min & x^T A_0 x + 2b_0^T x + c_0 \\ \text{s.t.} & x^T A_i x + 2b_i^T x + c_i \le 0, \ \forall (A_i, b_i, c_i) \in \mathcal{U}_i, i = 1, \dots, I, \end{array} \qquad \text{(RQCQP)} \qquad (1.3)$$

where

$$\mathcal{U}_i = \left\{ (A, b, c) \in \mathcal{S}^n \times \mathbb{R}^n \times \mathbb{R} \;\middle|\; \begin{array}{l} (A, b, c) = (A_i^0, b_i^0, c_i^0) + \sum_{j=1}^{m_i} u_i^j (A_i^j, b_i^j, c_i^j), \\ \text{for some positive integer } m_i > 0 \\ \text{ with } u_i \in \mathbb{R}^{m_i}, \|u_i\|_2 \le 1, \\ (A_i^j, b_i^j, c_i^j) \in \mathcal{S}^n \times \mathbb{R}^n \times \mathbb{R}, j = 0, \dots, m_i. \end{array} \right\} \qquad (1.4)$$

$i = 1, \dots, I$. Notice that, for any $u_i \in \mathbb{R}^{m_i}$ and $\|u_i\|_2 \le 1$, we have

$$x^T A_i x + 2b_i^T x + c_i = (x^T A_i^0 x + 2(b_i^0)^T x + c_i^0) + \sum_{j=1}^{m_i} u_i^j (x^T A_i^j x + 2(b_i^j)^T x + c_i^j).$$

Then, for each i, $x^T A_i x + 2b_i^T x + c_i \le 0, \forall (A_i, b_i, c_i) \in \mathcal{U}_i$, is equivalent to

$$\left\| \begin{pmatrix} x^T A_i^1 x + 2(b_i^1)^T x + c_i^1 \\ \vdots \\ x^T A_i^{m_i} x + 2(b_i^{m_i})^T x + c_i^{m_i} \end{pmatrix} \right\|_2 \le -(x^T A_i^0 x + 2(b_i^0)^T x + c_i^0), \qquad (1.5)$$

which is a quadratic second order cone constraint. Hence, solving the RQCQP problem is equivalent to solving its corresponding CQCQP problem. Some of the polynomial optimization problems can also be formulated as the CQCQP problem. For example, given a double well function constraint

$$\frac{1}{2}(x^T A_1^T A_1 x + 2b_1^T x + c_1)^2 + x^T A_2 x + 2b_2^T x + c_2 \le 0 \qquad (1.6)$$

with $A_1 \in \mathbb{R}^{p \times n}$, $A_2 \in \mathcal{S}^n$, $b_1, b_2 \in \mathbb{R}^n$ and $c_1, c_2 \in \mathbb{R}$, it can be written as

$$\frac{1}{2}(x^T A_1^T A_1 x + 2b_1^T x + c_1)^2 + x^T A_3 x + 2b_3^T x + c_3 \le x^T (A_3 - A_2) x + 2(b_3 - b_2)^T x + c_3 - c_2 \quad (1.7)$$

for any $A_3 \in \mathcal{S}^n$, $b_3 \in \mathbb{R}^n$ and $c_3 \in \mathbb{R}$. When both $(x^T A_3 x + 2b_3^T x + c_3)$ and $(x^T (A_3 - A_2) x + 2(b_3 - b_2)^T x + c_3 - c_2)$ are nonnegative over $\mathbb{R}^n$, inequality (1.7) can be rewritten as a quadratic second order cone constraint.

One commonly used conic quadratic constraint is $X - xx^T \in \mathcal{S}^n_+$ while solving the relaxation of the QCQP problem. This special constraint can be reformulated as an equivalent conic linear constraint $\begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \in \mathcal{S}^{n+1}_+$ and thus be solved easily. For general conic quadratic constraints, in the literature, few research papers work on CQCQP problems directly. However, there are many results on the conic nonlinear programming problems. For example, when $f(x)$ is convex and $-g(x)$ is $\mathcal{K}$-convex (see Boyd [9]), CQCQP becomes a convex programming problem whose optimality conditions have been extensively studied. See e.g., Massam [17], Borwein and Wolkowicz [8], Berman [6], Ban and Song [3]. For conic nonlinear programming problems over polyhedral cones, the Fritz-John conditions are studied in Craven and Mond [10] and Alders and Sposito [2]. The first and second order optimality conditions for problems over positive semidefinite cone are studied in Shapiro [23]. Over second order cones, the Aubin property which is equivalent to the strong regular condition is studied in Outrata and Ramírez [18]. Over convex cones, the saddle-point conditions are studied in Sposito and David [24]. Based on the Farkas Lemma over cones developed in Sposito and David [25], regular conditions, KKT conditions and complementary slackness are studied in Sposito [26] and Alders and Sposito [1]. A more general regular condition can be seen in Robinson [19]. Strong regular condition and equivalent conditions which ensure the uniqueness of the Lagrangian multipliers are studied in Robinson [20] and Shapiro [22]. For more results regarding the optimality conditions for the conic nonlinear programming problems, one may refer to Bonnans and Shapiro [7]. In Kojima, Kim and Waki [13], a general framework for the convex relaxation of conic polynomial programming problems is introduced. The positive semidefinite (SDP in short) relaxation for the conic polynomial programming problem is studied in Kojima and Muramatsu [14].

Most results in the literature focus on the local optimality conditions or the lower bounds of the conic nonlinear programming problems. In [16], Lu et al. proposed a scheme to design algorithms solving QCQP problems globally. In their approach, based on the cone of nonnegative quadratic functions introduced in Sturm and Zhang [27], they reformulated the QCQP problem as a linear conic programming problem COP and its dual problem COD. They showed that when the QCQP problem has a finite optimal value, then these three problems have the same optimal objective value. They also proved that if a KKT solution (x^* and its corresponding Lagrangian multipliers) satisfies the "extended global optimality condition", then x^* is a global optimal solution to the QCQP problem. In order to obtain the optimal Lagrangian multipliers, they showed that the Lagrangian multipliers are a global optimal solution to the COD problem and, furthermore, if the linear independent constraint qualification (LICQ) is satisfied at x^*, then the Lagrangian multipliers have a "maximal property" (Lemma 4.5 in Lu et al. [16]) among all the optimal solutions of COD. However, solving COP and COD is as difficult as solving QCQP, and, therefore, a relaxation scheme is introduced to gain polynomial solvability. They also proved the extended global optimality condition for the relaxed problems.

Inspired by the work of Lu et al. [16], we propose a similar scheme for solving the CQCQP problem, extend the maximal property of the Lagrangian multipliers, and relax the condition required. In Section 2, we introduce the KKT condition for CQCQP and its corresponding linear conic reformulations. In Section 3, the copositiveness condition is introduced which is similar to the extended global optimality condition in [16]. We extend the maximal property of the Lagrangian vector and relax the condition for the maximal property. Our results can be applied to extend the results of Lu et al. [16] directly. In Section 4, an approximation scheme is proposed for the computation purpose and the copositiveness condition is used to test global optimality. We also use one example to illustrate the effectiveness of the algorithm. Some discussions and conclusions are given in Section 5.

2 KKT Conditions and Linear Conic Reformulation.

Let $\mathcal{K}^*$ be the dual cone of $\mathcal{K}$. Given $\lambda \in \mathcal{K}^*$, denote

$$\begin{aligned} A(\lambda) &= A_0 - \sum_{i=1}^m \lambda_i A_i, \\ b(\lambda) &= b_0 - \sum_{i=1}^m \lambda_i b_i, \\ c(\lambda) &= c_0 - \sum_{i=1}^m \lambda_i c_i. \end{aligned} \tag{2.1}$$

Then the Lagrangian of CQCQP becomes

$$L(x,\lambda) = f(x) - \sum_{i=1}^m \lambda_i g_i(x) = x^T A(\lambda) x + 2b^T(\lambda)x + c(\lambda). \tag{2.2}$$

Note that when $\mathcal{K}$ is a matrix cone, any matrix in $\mathcal{K}$ is treated as a vector in the above equation.

Given a local minimal solution x^* and its corresponding Lagrangian vector $\lambda^* \in \mathcal{K}^*$, the KKT condition can be denoted as

$$\begin{aligned} &\nabla_x L(x^*,\lambda^*) = \nabla f(x^*) - \sum_{i=1}^m \lambda_i^* \nabla g_i(x^*) = 0, \\ &\sum_{i=1}^m \lambda_i^* g_i(x^*) = 0, \\ &g(x^*) \in \mathcal{K} \text{ and } \lambda^* \in \mathcal{K}^*. \end{aligned} \tag{2.3}$$

Here $\nabla_x L$ is the partial derivative of the function L with respect to x, assuming f and g are differentiable.

In general, we could not guarantee the existence of the Lagrangian vector at x^*. In the literature, a commonly used regularity condition is the Robinson condition (refer to Bonnans and Shapiro [7], Eqn. 2.163), i.e.,

$$0 \in \mathrm{int}\{z \in \mathbb{R}^m \mid z = g(x^*) + \nabla g(x^*)h - k, \text{ for some } h \in \mathbb{R}^n, k \in \mathcal{K}\}, \tag{2.4}$$

where $\mathrm{int}(\cdot)$ means the interior of a set and $\nabla g(x^*) \in \mathbb{R}^{m\times n}$ is the Jacobean of g at x^*. When $\mathcal{K}$ has a nonempty interior, the condition can be equivalently written as

$$\exists h \in \mathbb{R}^n \text{ such that } g(x^*) + \nabla g(x^*)h \in \mathrm{int}(\mathcal{K}).$$

Under the Robinson condition, the Lagrangian vector exists at x^* and the set of Lagrangian vectors is a closed bounded set. (See Theorem 3.9 of [7].)

The KKT conditions are merely necessary for a local minimum solution of CQCQP. Given a KKT solution (x^*, λ^*), a commonly seen global optimality condition is given in the follows.

Lemma 2.1 (Positive semidefiniteness condition)**.** *Let (x^*, λ^*) be a KKT solution of CQCQP. If $A(\lambda^*)$ is positive semidefinite, then x^* is an optimal solution of CQCQP.*

A more general global optimality condition for QCQP is provided in Lu et al. [16]. To obtain a similar result for CQCQP, we need to introduce the corresponding linear conic programming problems.

Denote the feasible domain of CQCQP by

$$\mathrm{dom(CQCQP)} = \{x \in \mathbb{R}^n \mid g(x) \in \mathcal{K}\}.$$

For any given matrices M_1 and M_2, in a common practice, we treat each of them as a vector and define the inner product of $M_1 \cdot M_2$ as the ordinary inner product of two vectors. Then we define the following two cones:

$$\mathcal{D}_{n+1} = \left\{ U \in \mathcal{S}^{n+1} \;\middle|\; U \cdot \begin{bmatrix} 1 & x^T \\ x & xx^T \end{bmatrix} \geq 0, \forall x \in \mathrm{dom(CQCQP)} \right\}, \tag{2.5}$$

and

$$\mathcal{D}^*_{n+1} = \mathrm{cl\,cone} \left\{ X = \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}^T \;\middle|\; x \in \mathrm{dom(CQCQP)} \right\}. \tag{2.6}$$

where, cl cone$(\cdot)$ is the closure of the conic hull of a set.

Observe that cone $\mathcal{D}_{n+1}$ is the set of coefficients of all quadratic functions that are nonnegative over the feasible domain of CQCQP while $\mathcal{D}^*_{n+1}$ is the smallest closed convex cone that contains the feasible domain of CQCQP in the homogenous form. They are both closed and convex by definition. Corollary 1 of Sturm and Zhang [27] further shows that they are dual to each other.

Consider the following linear conic programming problem:

$$\begin{aligned} \min \quad & \begin{bmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{bmatrix} \cdot Y \\ \text{s.t.} \quad & Y_{11} = 1, \\ & Y \in \mathcal{D}^*_{n+1}. \end{aligned} \qquad \text{(COP0)} \tag{2.7}$$

Following the discussion made on the problem (MP) in Section 3 of Sturm and Zhang [27], we have the next result.

Theorem 2.2. *Problem COP0 and Problem CQCQP are equivalent in the sense that they have the same optimal objective value.*

Let $G_i = \begin{bmatrix} c_i & b_i^T \\ b_i & A_i \end{bmatrix}$ denote the matrix form of the coefficients of the ith constraint in CQCQP and $Y \in \mathcal{D}^*_{n+1}$. We can represent $g(x)$ by

$$G(Y) = \begin{bmatrix} G_1 \cdot Y \\ \vdots \\ G_m \cdot Y \end{bmatrix} \in \mathbb{R}^m. \tag{2.8}$$

Then we have the next result.

Lemma 2.3. *If $Y \in \mathcal{D}^*_{n+1}$ then $G(Y) \in \mathcal{K}$.*

Proof. Given $Y \in \mathcal{D}^*_{n+1}$, there exists a sequence of $Y_k \in \mathrm{cone}\{X = \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}^T \mid x \in \mathrm{dom(CQCQP)}\}$ such that

$$\lim_{k \to \infty} Y_k = Y.$$

By the definition of conic hull, we have

$$Y_k = \sum_{j=1}^{s_k} \mu_j \begin{bmatrix} 1 \\ x_j \end{bmatrix} \begin{bmatrix} 1 \\ x_j \end{bmatrix}^T \text{ with } x_j \in \mathrm{dom(CQCQP)} \text{ and } \mu_j \geq 0 \text{ for some positive integer } s_k.$$

Since $x_j \in \text{dom(CQCQP)}$ and $\mu_j \geq 0$, we have

$$G(\begin{bmatrix} 1 \\ x_j \end{bmatrix} \begin{bmatrix} 1 \\ x_j \end{bmatrix}^T) = g(x_j) \in \mathcal{K}.$$

Consequently,

$$G(Y_k) = G(\sum_{j=1}^{s_k} \mu_j \begin{bmatrix} 1 \\ x_j \end{bmatrix} \begin{bmatrix} 1 \\ x_j \end{bmatrix}^T) = \sum_{j=1}^{s_k} \mu_j G(\begin{bmatrix} 1 \\ x_j \end{bmatrix} \begin{bmatrix} 1 \\ x_j \end{bmatrix}^T) \in \mathcal{K}.$$

Since $\mathcal{K}$ is closed, we have $G(Y) = \lim_{k\to\infty} G(Y_k) \in \mathcal{K}$. □

Lemma 2.3 indicates that $G(Y) \in \mathcal{K}$ is valid for COP0. By adding this redundant constraints, we define the following linear conic programming problem:

$$\begin{array}{ll} \min & \begin{bmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{bmatrix} \cdot Y \\ \text{s.t.} & G(Y) \in \mathcal{K}, \\ & Y_{11} = 1, \\ & Y \in \mathcal{D}_{n+1}^*. \end{array} \qquad \text{(COP1)} \qquad (2.9)$$

Then COP1 and CQCQP must have the same optimal objective value. Moreover, the conic dual of COP1 becomes

$$\begin{array}{ll} \max & \sigma \\ \text{s.t.} & \begin{bmatrix} c(\lambda) - \sigma & b^T(\lambda) \\ b(\lambda) & A(\lambda) \end{bmatrix} \in \mathcal{D}_{n+1}, \\ & \lambda \in \mathcal{K}^*. \end{array} \qquad \text{(COD1)} \qquad (2.10)$$

Similar to the proof of Theorem 3.3 in [16], we have the next theorem.

Theorem 2.4. *If CQCQP has a finite optimal value, then the optimal objective values of problems CQCQP, COP1 and COD1 are equal.*

Let (x^*, λ^*) be a KKT solution of CQCQP and define a matrix

$$M(x^*, \lambda^*) = \begin{bmatrix} c(\lambda^*) - f(x^*) & b^T(\lambda^*) \\ b(\lambda^*) & A(\lambda^*) \end{bmatrix}. \qquad (2.11)$$

We can extend the positive semidefiniteness condition as follows.

Theorem 2.5 (Copositiveness condition)**.** *Let (x^*, λ^*) be a KKT solution of CQCQP. If $M(x^*, \lambda^*) \in \mathcal{D}_{n+1}$, then $(\sigma^*, \lambda^*) = (f(x^*), \lambda^*)$ is globally optimal to COD1 and x^* is globally optimal to CQCQP.*

Proof. From the condition

$$M(x^*, \lambda^*) = \begin{bmatrix} c(\lambda^*) - f(x^*) & b^T(\lambda^*) \\ b(\lambda^*) & A(\lambda^*) \end{bmatrix} \in \mathcal{D}_{n+1},$$

we know $(\sigma^*, \lambda^*) = (f(x^*), \lambda^*)$ is a feasible solution of COD1. Since x^* is feasible for CQCQP and $\sigma^* = f(x^*)$, we know x^* is optimal to CQCQP and (σ^*, λ^*) is optimal to COD1. □

Note that, given a KKT solution, if $M(x^*, \lambda^*) \in \mathcal{S}_+^{n+1} \subset \mathcal{D}_{n+1}$, then $A(\lambda^*) \in \mathcal{S}_+^n$ which means the positive semidefiniteness condition holds. On the other hand, if the positive semidefiniteness condition holds, then x^* is an optimal solution of CQCQP. Hence,

$$f(x^*) = L(x^*, \lambda^*) = c(\lambda^*) - (x^*)^T A(\lambda^*) x^*$$

and, from the KKT condition, we have

$$M(x^*, \lambda^*) = \begin{bmatrix} (x^*)^T A(\lambda^*) x^* & -(A(\lambda^*) x^*)^T \\ -A(\lambda^*) x^* & A(\lambda^*) \end{bmatrix} \in \mathcal{S}_+^{n+1}.$$

This shows that the positive semidefiniteness condition is equivalent to $M(x^*, \lambda^*) \in \mathcal{S}_+^{n+1} \subset \mathcal{D}_{n+1}$ and, therefore, is a special case of the copositiveness condition.

The copositiveness condition discussed here is similar to the extended global optimality condition for QCQP in Lu et al. [16].

3 Finding Global Optimal Solutions

We have already extend the global optimality condition proposed in Lu et al. [16] to the copositiveness condition for CQCQP. In this section, we will establish similar properties as Lu et al. [16] to solve CQCQP problems globally. Notice that, in Lu et al. [16], Algorithm 1 is based on the fact that the optimal Lagrangian vector λ^* has some maximal property among all the optimal solutions of COD, i.e., $\lambda^* \geq \bar{\lambda}$, for all $\bar{\lambda}$ being optimal in COD. In our problem, Robinson regularity condition cannot guarantee such maximal property.

Example 3.1. Consider the following problem.

$$\begin{array}{ll} \min & f(x) = x_1 + x_2 \\ \text{s.t.} & g_1(x) = (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 - \frac{1}{2} \leq 0 \\ & g_2(x) = (x_1 + x_2 - \frac{1}{2})^2 - \frac{1}{4} \leq 0 \end{array}$$

In this example, $\mathcal{K} = \mathcal{K}^* = \{x \in \mathbb{R}^2 | x_1 \leq 0, x_2 \leq 0\}$. One can easily verify that $x^* = (0,0)^T$ is the unique optimal solution. Robinson regularity condition is satisfied by choosing $h = (1,1)^T$. Any (x^*, λ) with $\lambda_1 \leq 0, \lambda_2 \leq 0$ and $\lambda_1 + \lambda_2 = -1$ satisfies both the KKT condition and the copositiveness condition. If we choose $\sigma^* = 0$, $\lambda^* = (-\frac{1}{2}, -\frac{1}{2})^T$ and $\bar{\lambda} = (0,-1)^T$, then both (σ^*, λ^*) and $(\sigma^*, \bar{\lambda})$ are optimal solutions of COD1. However, $\lambda^* - \bar{\lambda} = (-\frac{1}{2}, \frac{1}{2})^T \notin \mathcal{K}^*$, which means the maximal property does not hold.

Interestingly, although under Robinson regularity condition such maximal property does not hold, the idea for obtaining the optimal Lagrangian vector still works. In Lu et al. [16], the optimal Lagrangian vector is solved by maximizing an auxiliary linear conic programming problem in Theorem 4.6. The maximal property in their work guarantees the existence and uniqueness of the solution.

Under Robinson regularity condition, the set of Lagrangian multipliers corresponding to x^* is nonempty, convex, bounded and closed. (See Theorem 3.9 in [7]) If we can guarantee any optimal solution of the auxiliary programming lies in the set of Lagrangian multipliers corresponding to x^*, then the proposed solution scheme still works.

Suppose (x^*, λ^*) is a KKT solution satisfying the copositiveness condition. Then from Theorem 2.5, $(\sigma^*, \lambda^*) = (f(x^*), \lambda^*)$ is an optimal solution of COD1. Suppose $(\sigma^*, \bar{\lambda})$ is another optimal solution of COD1, then

$$L(x, \bar{\lambda}) - \sigma^* = \begin{bmatrix} 1 \\ x \end{bmatrix}^T \begin{bmatrix} c(\bar{\lambda}) - \sigma^* & b^T(\bar{\lambda}) \\ b(\bar{\lambda}) & A(\bar{\lambda}) \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0, \forall x \in \text{dom}(CQCQP).$$

Notice that

$$0 \geq -\bar{\lambda}^T g(x^*) = f(x^*) - \bar{\lambda}^T g(x^*) - \sigma^* = L(x^*, \bar{\lambda}) - \sigma^* \geq 0.$$

We have $\bar{\lambda}^T g(x^*) = 0$. Hence,

$$L(x, \bar{\lambda}) - L(x^*, \bar{\lambda}) = \begin{bmatrix} 1 \\ x \end{bmatrix}^T \begin{bmatrix} c(\bar{\lambda}) - \sigma^* & b^T(\bar{\lambda}) \\ b(\bar{\lambda}) & A(\bar{\lambda}) \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0, \forall x \in \mathrm{dom}(CQCQP).$$

This means that x^* is an optimal solution to the problem

$$\begin{array}{ll} \min & L(x, \bar{\lambda}) \\ \text{s.t.} & g(x) \in \mathcal{K}. \end{array} \tag{3.1}$$

From Lemma 3.7 in Bonnans and Shapiro [7], we have

$$[\nabla f(x^*) - \bar{\lambda}^T \nabla g(x^*)] d \geq 0, \ \forall d \in T_{\mathrm{dom}(CQCQP)}(x^*),$$

where $T_{\mathrm{dom}(CQCQP)}(x^*)$ is the tangent cone of $\mathrm{dom}(CQCQP)$ at x^*. Since

$$\nabla f(x^*) - (\lambda^*)^T \nabla g(x^*) = 0,$$

we have

$$(\lambda^* - \bar{\lambda}) \nabla g(x^*) d \geq 0, \ \forall d \in T_{\mathrm{dom}(CQCQP)}(x^*). \tag{3.2}$$

From Corollary 2.91 and Example 2.62 in [7], we have

$$\begin{aligned} T_{\mathrm{dom}(CQCQP)}(x^*) &= \{d \in \mathbb{R}^n | \nabla g(x^*) d \in T_{\mathcal{K}}(g(x^*))\} \\ &= \{d \in \mathbb{R}^n | \nabla g(x^*) d \in \mathrm{cl}([g(x^*)] + \mathcal{K})\} \end{aligned} \tag{3.3}$$

where $T_{\mathcal{K}}(g(x^*))$ is the tangent cone of $\mathcal{K}$ at $g(x^*)$ and $[g(x^*)]$ is the one dimensional subspace of $\mathbb{R}^m$ generated by $g(x^*)$.

Therefore,

$$\nabla g(x^*) T_{\mathrm{dom}(CQCQP)}(x^*) = \mathrm{cl}([g(x^*)] + \mathcal{K}) \cap \nabla g(x^*) \mathbb{R}^n \tag{3.4}$$

and its dual is

$$\begin{aligned} & \{\nabla g(x^*) T_{\mathrm{dom}(CQCQP)}(x^*)\}^* = \{\mathrm{cl}([g(x^*)] + \mathcal{K}) \cap \nabla g(x^*) \mathbb{R}^n\}^* \\ = \ & \{\mathrm{cl}([g(x^*)] + \mathcal{K})\}^* + \{\nabla g(x^*) \mathbb{R}^n\}^* \\ = \ & \{[g(x^*)] + \mathcal{K}\}^* + \{y \in \mathbb{R}^m | y^T \nabla g(x^*) = 0\} \\ = \ & \left\{[g(x^*)]^\perp \cap \mathcal{K}^*\right\} + \{y \in \mathbb{R}^m | y^T \nabla g(x^*) = 0\} \end{aligned}$$

in which the first equality holds by (3.4), the second and fourth equalities holds due to Corollary 16.4.2 in [21], and the third equality holds by the definition of dual operator. Here $[g(x^*)]^\perp$ is the orthogonal complement space of $[g(x^*)]$.

From (3.2) and $(\lambda^*)^T g(x^*) = \bar{\lambda}^T g(x^*) = 0$, we have

$$(\lambda^* - \bar{\lambda}) \in \left\{[g(x^*)]^\perp \cap \mathcal{K}^*\right\} + \{y \in \mathbb{R}^m | y^T \nabla g(x^*) = 0\}$$

and $(\lambda^* - \bar{\lambda})^T g(x^*) = 0$. This implies $(\lambda^* - \bar{\lambda}) = \lambda^1 + \lambda^2$ with $\lambda^1 \in \left\{[g(x^*)]^\perp \cap \mathcal{K}^*\right\}$ and $\lambda^2 \in \{y \in \mathbb{R}^m | y^T \nabla g(x^*) = 0\} \cap [g(x^*)]^\perp$. Define $\tilde{\lambda} \triangleq \lambda^* - \lambda^2 = \bar{\lambda} + \lambda^1$. One can easily check that $(x^*, \tilde{\lambda})$ is a KKT solution of CQCQP and $\tilde{\lambda} - \bar{\lambda} \in K^*$.

The above discussion reveals a fact that for any optimal solution $\bar{\lambda}$ of COD1, there exists another optimal solution $\tilde{\lambda}$ such that $(x^*, \tilde{\lambda})$ is a KKT solution of CQCQP and $\tilde{\lambda} - \bar{\lambda} \in \mathcal{K}^*$. Therefore, after solving COD1, we can use a similar auxiliary problem to find the optimal Lagrangian multipliers under Robinson regularity condition.

Theorem 3.2 (Extended maximal property)**.** *Let* (x^*, λ^*) *be a KKT solution satisfying the copositiveness condition. If Robinson regularity condition holds at* x^* *and let* σ^* *be the optimal value of COD1, then, for any* $h \in \text{int}(\mathcal{K})$*, any optimal solution* $\tilde{\lambda}$ *of*

$$\begin{aligned} \max \quad & h^T\lambda \\ \text{s.t.} \quad & \begin{bmatrix} c(\lambda) - \sigma^* & b^T(\lambda) \\ b(\lambda) & A(\lambda) \end{bmatrix} \in \mathcal{D}_{n+1}, \qquad \text{(COD2)} \\ & \lambda \in \mathcal{K}^*, \end{aligned} \tag{3.5}$$

forms a KKT solution $(x^*, \tilde{\lambda})$ *of CQCQP.*

Proof. Notice that any feasible solution of COD2 together with σ^* forms an optimal solution of COD1. From the above discussion, suppose $\bar{\lambda}$ is a feasible solution of COD2, then there is $\tilde{\lambda}$ such that $\tilde{\lambda} - \bar{\lambda} \in \mathcal{K}^*$ and $(x^*, \tilde{\lambda})$ is a KKT solution of CQCQP, which means $\tilde{\lambda}$ is also a feasible solution of COD2. Since $h \in \text{int}(\mathcal{K})$, then $(\tilde{\lambda} - \bar{\lambda})^T h > 0$. Therefore, any optimal solution of COD2 together with x^* forms a KKT solution of CQCQP. □

Notice that if $\{y \in \mathbb{R}^m | y^T \nabla g(x^*) = 0\} = \{0\}$, then we have the same maximal property as in [16]. One sufficient condition is that all the rows of $\nabla g(x^*)$ are linearly independent, which is the linear independence constraint qualification (LICQ).

Lemma 3.3. *Let* (x^*, λ^*) *be a KKT solution satisfying the copositiveness condition. If LICQ holds at* x^* *in CQCQP, then, for any* $(\sigma^*, \bar{\lambda})$ *being an optimal solution of COD1, we have* $\lambda^* - \bar{\lambda} \in \mathcal{K}^*$*.*

Proof. First, notice that LICQ implies the Robinson condition. Since $\nabla g(x^*)$ is of full row rank, we have $\{y \in \mathbb{R}^m | y^T \nabla g(x^*) = 0\} = \{0\}$. Therefore, from the above discussion, $\lambda^* - \bar{\lambda} \in \mathcal{K}^*$. □

Corollary 3.4 (Maximal property)**.** *Let* (x^*, λ^*) *be a KKT solution satisfying the copositiveness condition. If LICQ holds at* x^**, then, for any* $h \in \text{int}(\mathcal{K})$*,* λ^* *is the unique optimal solution of COD2.*

Proof. Note that if (σ^*, λ^*) and $(\sigma^*, \bar{\lambda})$ are two different optimal solutions of COD1, then they are both feasible solutions of COD2. Suppose λ^* is the optimal Lagrangian vector, then we have $\lambda^* - \bar{\lambda} \in \mathcal{K}^*$. Since $h \in \text{int}(\mathcal{K})$, then $h^T(\lambda^* - \bar{\lambda}) > 0$. Hence λ^* is always strictly better than any other solutions $\bar{\lambda}$ in COD2 which means λ^* is the unique optimal solution of COD2. □

After finding the Lagrangian vector λ^*, if we further know that $A(\lambda^*)$ is invertible, then x^* can be obtained by solving $\nabla_x L(x, \lambda^*) = 0$, i.e.,

$$x^* = -A^{-1}(\lambda^*) b(\lambda^*). \tag{3.6}$$

4 Computation Scheme and Example

Solving COD1 and COD2 may not be easy. In order to achieve polynomial solvability of CQCQP within an arbitrary precision, we have to make additional assumptions and use approximations of the corresponding linear conic reformulations. Here we assume that $\mathcal{K}$ is computable, i.e., any linear conic programming problems over $\mathcal{K}$ could be solved in polynomial time. Moreover, the approximating problems are obtained via replacing $\mathcal{D}^*_{n+1}$

by a computable $\mathcal{C}_{n+1}^*$ satisfying $\mathcal{D}_{n+1}^* \subset \mathcal{C}_{n+1}^* \subset \mathcal{S}_+^{n+1}$ in COP1, and $\mathcal{D}_{n+1}$ by a computable $\mathcal{C}_{n+1}$ satisfying $\mathcal{S}_+^{n+1} \subset \mathcal{C}_{n+1} \subset \mathcal{D}_{n+1}$ in COD1, where $\mathcal{C}_{n+1}^*$ and $\mathcal{C}_{n+1}$ are dual to each other.

In this way, we have the following two corresponding conic approximation problems that can be solved in polynomial time:

$$\begin{aligned} \max \quad & \sigma \\ \text{s.t.} \quad & \begin{bmatrix} c(\lambda) - \sigma & b^T(\lambda) \\ b(\lambda) & A(\lambda) \end{bmatrix} \in \mathcal{C}_{n+1}, \qquad \text{(COD1}'\text{)} \\ & \lambda \in \mathcal{K}^*. \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \max \quad & h^T \lambda \\ \text{s.t.} \quad & \begin{bmatrix} c(\lambda) - \sigma^* & b^T(\lambda) \\ b(\lambda) & A(\lambda) \end{bmatrix} \in \mathcal{C}_{n+1}, \qquad \text{(COD2}'\text{)} \\ & \lambda \in \mathcal{K}^*. \end{aligned} \tag{4.2}$$

where σ^* is the optimal objective value of COD1$'$ and $h \in \text{int}(\mathcal{K})$.

In this setting, we get a similar computation scheme:

Algorithm 4.1 (CQCQP algorithm)**.**

STEP 1: Given an instance of CQCQP, solve the corresponding COD1$'$ to find its optimal objective value σ^*. If failed, then stop and note that the CQCQP problem cannot be solved by using the current approximation.

STEP 2: Choose an $h \in \text{int}(\mathcal{K})$ and solve COD2$'$ to find the optimal λ^*.

STEP 3: Compute $x^* = -A^+(\lambda^*)b(\lambda^*)$, where $A^+(\lambda^*)$ is the Moore-Penrose inverse of $A(\lambda^*)$ (see Ben-Israel and Greville [4]). Particularly, $A^+(\lambda^*) = A^{-1}(\lambda^*)$ when $A(\lambda^*)$ is nonsingular.

STEP 4: If x^* is a feasible solution and $f(x^*) = \sigma^*$, then return x^* as a global optimal solution of CQCQP with the objective value $f(x^*) = \sigma^*$. Otherwise, return σ^* as a lower bound for CQCQP.

Remark. Since COD1$'$ is only an inner approximation of COD1, it may not be feasible and the algorithm could stop at STEP 1. If we let $\mathcal{C}_{n+1} = \mathcal{S}_+^{n+1}$, then the dual problem of COD1$'$ becomes the classic SDP relaxation of CQCQP.

The next theorem validates Algorithm 4.1.

Theorem 4.2. *If Algorithm 4.1 returns a solution x^*, then x^* is an optimal solution of CQCQP. If Algorithm 4.1 returns a value σ^* then it is a lower bound for CQCQP.*

Proof. If Algorithm 4.1 returns a feasible x^*, then $f(x^*) = \sigma^*$ and (σ^*, λ^*) is optimal to COD2$'$. Hence we know $M(x^*, \lambda^*) \in \mathcal{C}_{n+1} \subset \mathcal{D}_{n+1}$ and

$$0 \leq \begin{bmatrix} 1 \\ x^* \end{bmatrix}^T \begin{bmatrix} c(\lambda^*) - f(x^*) & b^T(\lambda^*) \\ b(\lambda^*) & A(\lambda^*) \end{bmatrix} \begin{bmatrix} 1 \\ x^* \end{bmatrix} = -\sum_{i=1}^m \lambda_i^* g_i(x^*) \leq 0.$$

This implies $\sum_{i=1}^{m} \lambda_i^* g_i(x^*) = 0$. Since $A(\lambda^*)x^* + b(\lambda^*) = 0$, (x^*, λ^*) satisfies the KKT conditions, i.e.,

$$\nabla_x L(x^*, \lambda^*) = 2A(\lambda^*)x^* + 2b(\lambda^*) = 0,$$

$$(\lambda^*)^T g(x^*) = 0,$$

$$g(x^*) \in \mathcal{K}, \lambda^* \in \mathcal{K}^*.$$

Consequently, (x^*, λ^*) satisfies the copositiveness condition. By Theorem 2.5, x^* must be an optimal solution of CQCQP with the objective value $f(x^*) = \sigma^*$.

If Algorithm 4.1 returns a value σ^*, since $\mathcal{C}_{n+1} \subset \mathcal{D}_{n+1}$, the feasible domain of COD1$'$ is contained in the feasible domain of COD1. Hence σ^* is a lower bound of CQCQP. □

Now let us first go back to Example 1. Although there are multiple optimal solutions of COD1 and the maximal property does not hold, choose $h = (-2, -1)^T$, after solving COD2, we can get $\lambda^* = (-1, 0)^T$. Hence, $x^* = -A(\lambda^*)^{-1}b(\lambda^*) = (0, 0)^T$ is an optimal solution.

Next, we use another simple example to illustrate the effectiveness of the proposed algorithm and to show that the proposed algorithm indeed improves the results obtained by using the positive semidefiniteness condition. This example is a modification of Example 2 in Lu et al. [16].

Example 4.3.

$$\begin{aligned} \min \quad & f(x) = x^T A_0 x + 2b_0^T x + c_0 \\ \text{s.t.} \quad & \begin{bmatrix} g_{11}(x) & g_{12}(x) \\ g_{12}(x) & g_{22}(x) \end{bmatrix} \in \mathcal{S}_+^2 \end{aligned}$$

where $g_{ij}(x) = x^T A_{ij} x + 2b_{ij}^T x + c_{ij}$ with

$$A_0 = \begin{bmatrix} -2 & 10 & 2 \\ 10 & 4 & 1 \\ 2 & 1 & -7 \end{bmatrix}, \quad b_0 = \begin{bmatrix} -12 \\ -6 \\ 56 \end{bmatrix}, c_0 = 0$$

$$A_{11} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad b_{11} = \begin{bmatrix} 1 \\ 0 \\ 8 \end{bmatrix}, c_{11} = -64$$

$$A_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -25 \end{bmatrix}, \quad b_{12} = \begin{bmatrix} 0 \\ 0 \\ 175 \end{bmatrix}, c_{12} = -1200$$

$$A_{22} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}, \quad b_{22} = \begin{bmatrix} 0 \\ 1 \\ 32 \end{bmatrix}, c_{22} = -256.$$

The optimal objective value of this problem is known to be 448 and the corresponding optimal solution is $x = [0, 0, 8]^T$.

The constraints $g_{11}(x) \geq 0$ and $g_{22}(x) \geq 0$ imply that the feasible domain is a subset of $\mathbb{R}_+^3$. (From $g_{22}(x) \geq 0$, we know $x_2 \leq -2$ or $x_2 \geq 0$. Together with $g_{11}(x) \geq 0$, the feasible domain is contained in $\mathbb{R}_+^3$.) We can easily check that $\mathcal{S}_+^4 \subset (\mathcal{S}_+^4 + \mathcal{N}^4) \subset \mathcal{D}_4$, where $\mathcal{N}^4$ is the set of 4×4 matrices with nonnegative elements. Therefore, we can choose $\mathcal{C}_4 = \mathcal{S}^4_+ + \mathcal{N}^4$ as the inner approximation of $\mathcal{D}_4$ in the approximations COD1′ and COD2′.

Solving COD1′ leads to an objective value of $\sigma^* = 448$. Taking this value into COD2′ for solution, we find the Lagrangian multipliers

$$\lambda^* = \begin{bmatrix} \lambda^*_{11} & \lambda^*_{12} \\ \lambda^*_{12} & \lambda^*_{22} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

and the matrices

$$M(x^*, \lambda^*) = \begin{bmatrix} 320 & -16 & -8 & -40 \\ -16 & 4 & 10 & 2 \\ -8 & 10 & 6 & 1 \\ -40 & 2 & 1 & 5 \end{bmatrix} \quad \text{and} \quad A(\lambda^*) = \begin{bmatrix} 4 & 10 & 2 \\ 10 & 6 & 1 \\ 2 & 1 & 5 \end{bmatrix}.$$

Since $A(\lambda^*)$ is invertible, we get $x^* = -A^{-1}(\lambda^*)b(\lambda^*) = [0, 0, 8]^T$ and $f(x^*) = 448$.

Notice that x^* is feasible and $f(x^*) = \sigma^*$, from Theorem 4.2, x^* is an optimal solution of CQCQP with the optimal value $f(x^*) = 448$.

Here, $M(x^*, \lambda^*)$ is not positive semidefinite, but it can be divided into the sum of a positive semidefinite matrix and a nonnegative matrix as following:

$$M(x^*, \lambda^*) = \begin{bmatrix} 320 & -16 & -8 & -40 \\ -16 & 4 & 0 & 2 \\ -8 & 0 & 6 & 1 \\ -40 & 2 & 1 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{C}_4.$$

It is worth mentioning that if we use the classic positive semidefinite relaxation problem, i.e., let $\mathcal{C}_4 = \mathcal{S}^4_+$, then we can only obtain a lower bound of 445.83.

5 Conclusion

In this paper, we have extended the study of QCQP in Lu et al. [16] to the CQCQP problem. We introduced linear conic reformulations of CQCQP using the cone of nonnegative quadratic functions and proved the copositiveness condition which is an extended global optimality condition. In order to extend the computation scheme proposed in Lu et al. [16], the extended maximal property of the optimal Lagrangian multipliers is proved. Since the first orthant is a special case of CQCQP, our results readily applies to Lu et al. [16] and improve their results. In that special case, the Robinson condition is $g(x^*) + \nabla g(x^*)h < 0$ for some $h \in \mathbb{R}^n$. Approximations of COD1 and COD2 are used in the computation scheme to provide computational efficiency.

Since the conic programming problem over the cone $\mathcal{D}_{n+1}$ may not be solved efficiently in general, a larger computable $\mathcal{C}_{n+1}$ in the approximation problem will lead to a better result in polynomial time. Also notice that although the problems COP0 and COP1 are equivalent, their conic dual problems are not. The property of the optimal Lagrangian multipliers can only be obtained from the conic dual of the problem COP1. This fact raises an issue on how to find and add valid conic form constraints so that the corresponding dual problem possesses desired properties.

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