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A REVISIT TO QUADRATIC PROGRAMMING WITH ONE INEQUALITY QUADRATIC CONSTRAINT VIA MATRIX PENCIL*

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Abstract: The quadratic programming over one inequality quadratic constraint (QP1QC) is a very special case of quadratically constrained quadratic programming (QCQP) and attracted much attention since early 1990's. It is now understood that, under the primal Slater condition, (QP1QC) has a tight SDP relaxation (P_{SDP}). The optimal solution to (QP1QC), if exists, can be obtained by a matrix rank one decomposition of the optimal matrix X^* to (P_{SDP}). In this paper, we pay a revisit to (QP1QC) by analyzing the associated matrix pencil of two symmetric real matrices A and B, the former matrix of which defines the quadratic term of the objective function whereas the latter for the constraint. We focus on the "undesired" (QP1QC) problems which are often ignored in typical literature: either there exists no Slater point, or (QP1QC) is unbounded below, or (QP1QC) is bounded below but unattainable. Our analysis is conducted with the help of the matrix pencil, not only for checking whether the undesired cases do happen, but also for an alternative way otherwise to compute the optimal solution in comparison with the usual SDP/rank-one-decomposition procedure.

Key words: *quadratically constrained quadratic program, matrix pencil, hidden convexity, Slater condition, unattainable SDP, simultaneously diagonalizable with congruence*

Mathematics Subject Classification: *90C20, 90C22, 90C26*

1 Introduction

Quadratically constrained quadratic programming (QCQP) is a classical nonlinear optimization problem which minimizes a quadratic function subject to a finite number of quadratic constraints. A general (QCQP) problem can not be solved exactly and is known to be NP-hard. (QCQP) problems with a single quadratic constraint, however, is polynomially solvable. It carries the following format

$$\text{(QP1QC)}: \quad \begin{array}{ll} \inf & F(x) = x^T Ax - 2f^T x \\ \text{s.t.} & G(x) = x^T Bx - 2g^T x \le \mu, \end{array} \tag{1.1}$$

where A, B are two $n \times n$ real symmetric matrices, μ is a real number and f, g are two $n \times 1$ vectors.

The problem (QP1QC) arises from many optimization algorithms, most importantly, the trust region methods. See, e.g., [9, 16] in which $B \succ 0$ and $g = 0$. Extensions to an indefinite

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B but still with $g = 0$ are considered in [2, 8, 14, 20, 24]. Direct applications of (QP1QC) having a nonhomogeneous quadratic constraint can be found in solving an inverse problem via regularization [6, 10] and in minimizing the double well potential function [5]. Solution methods for the general (QP1QC) can be found in [7, 15, 18].

As we can see from the above short list of review, it was not immediately clear until rather recent that the problem (QP1QC) indeed belongs to the class P. Two types of approaches have been most popularly adopted: the (primal) semidefinite programming (SDP) relaxation and the Lagrange dual method. Both require the (QP1QC) problem to satisfy either the primal Slater condition (e.g., [7, 15, 24, 25]):

Primal Slater Condition: There exists an x_0 such that $G(x_0) = x_0^T B x_0 - 2g^T x_0 < \mu$;

and/or the dual Slater condition (e.g., [2, 7, 15, 24, 25]):

Dual Slater Condition: There exists a $\sigma' \geq 0$ such that $A + \sigma' B \succ 0$.

Nowadays, with all the efforts and results in the literature, (QP1QC) is no longer a difficult problem and its structure can be understood easily with a standard SDP method, the conic duality, the S-lemma, and a rank-one decomposition procedure.

Under the primal Slater condition, the Lagrangian function of (QP1QC) is

$$d(\sigma) = \inf_{x \in \mathbb{R}^n} L(x, \sigma) := x^T(A + \sigma B)x - 2(f + \sigma g)^T x - \mu\sigma, \quad \sigma \geq 0, \tag{1.2}$$

with which the dual problem of (QP1QC) can be formulated as

$$\text{(D)} \qquad \sup_{\sigma \geq 0} d(\sigma).$$

Using Shor's relaxation scheme [19], the above Lagrange dual problem (D) has a semidefinite programming reformulation:

$$\begin{array}{lll} (\mathrm{D_{SDP}}): & \sup & s \\ & \text{s.t.} & \begin{bmatrix} A + \sigma B & -f - \sigma g \\ -f^T - \sigma g^T & -\mu\sigma - s \end{bmatrix} \succeq 0, \\ & & \sigma \geq 0. \end{array} \tag{1.3}$$

The conic dual of $(\mathrm{D_{SDP}})$ turns out to be the SDP relaxation of (QP1QC):

$$\begin{array}{lll} (\mathrm{P_{SDP}}): & \inf & \begin{bmatrix} A & -f \\ -f^T & 0 \end{bmatrix} \bullet Y \\ & \text{s.t.} & \begin{bmatrix} B & -g \\ -g^T & -\mu \end{bmatrix} \bullet Y \leq 0, \\ & & Y_{n+1,n+1} = 1 \\ & & Y \succeq 0, \end{array} \tag{1.4}$$

where $X \bullet Y := \mathrm{trace}(X^T Y)$ is the usual matrix inner product. Let $v(\cdot)$ denote the optimal value of problem $(\cdot)$. By the weak duality, we have

$$v(\text{QP1QC}) \geq v(\mathrm{P_{SDP}}) \geq v(\mathrm{D_{SDP}}). \tag{1.5}$$

To obtain the strong duality, observe that

$$\begin{aligned} v(\text{QP1QC}) &= \inf_{x \in \mathbb{R}} \left\{ F(x) \middle| G(x) \leq \mu \right\} \\ &= \sup \left\{ s \in \mathbb{R} \middle| \{x \in \mathbb{R}^n | F(x) - s < 0, G(x) - \mu \leq 0\} = \emptyset \right\}. \end{aligned} \tag{1.6}$$

From the definition of infimum, we have

$$\{x \in \mathbb{R}^n | F(x) - v(\text{QP1QC}) < 0, G(x) - \mu \le 0\} = \emptyset$$

and also for any $\epsilon > 0$,

$$\{x \in \mathbb{R}^n | F(x) - (v(\text{QP1QC}) + \epsilon) < 0, G(x) - \mu \le 0\} \neq \emptyset.$$

In other words, $v(\text{QP1QC})$ is the largest s such that $\{x \in \mathbb{R}^n | F(x) - s < 0, G(x) - \mu \le 0\} = \emptyset$, which justifies the validity of (1.6).

Under the primal Slater condition and by the S-lemma [18], we get

$$\begin{aligned}
&\sup\left\{ s \in \mathbb{R} \,\middle|\, \{x \in \mathbb{R}^n | F(x) - s < 0, G(x) - \mu \le 0\} = \emptyset \right\} \\
&= \sup\left\{ s \in \mathbb{R} \,\middle|\, \exists \sigma \ge 0 \text{ such that } F(x) - s + \sigma G(x) - \sigma\mu \ge 0,\ \forall x \in \mathbb{R}^n \right\} \\
&= \left\{ \begin{array}{ll} \sup & s \\ \text{s.t.} & \begin{bmatrix} A + \sigma B & -f - \sigma g \\ -f^T - \sigma g^T & -\mu\sigma - s \end{bmatrix} \bullet \begin{bmatrix} xx^T & x \\ x^T & 1 \end{bmatrix} \ge 0,\ \forall x \in \mathbb{R}^n \\ & \sigma \ge 0 \end{array} \right. \qquad (1.7) \\
&= \left\{ \begin{array}{ll} \sup & s \\ \text{s.t.} & \begin{bmatrix} A + \sigma B & -f - \sigma g \\ -f^T - \sigma g^T & -\mu\sigma - s \end{bmatrix} \succeq 0 \\ & \sigma \ge 0, \end{array} \right.
\end{aligned}$$

where the last equality holds because

$$\begin{bmatrix} A + \sigma B & -f - \sigma g \\ -f^T - \sigma g^T & -\mu\sigma - s \end{bmatrix} \bullet \begin{bmatrix} xx^T & x \\ x^T & 1 \end{bmatrix} \ge 0,\ \forall x \in \mathbb{R}^n \Leftrightarrow \begin{bmatrix} A + \sigma B & -f - \sigma g \\ -f^T - \sigma g^T & -\mu\sigma - s \end{bmatrix} \succeq 0.$$

Combining (1.3), (1.5), (1.6) and (1.7) together, we conclude that

$$v(\text{QP1QC}) = v(\text{P}_{\text{SDP}}) = v(\text{D}_{\text{SDP}}), \qquad (1.8)$$

which says that the SDP relaxation of (QP1QC) is tight and, when $v(\text{QP1QC}) > -\infty$, ($\text{D}_{\text{SDP}}$) is always attainable. The strong duality result also appeared, e.g., in [4, Appendix B].

Suppose $v(\text{QP1QC}) > -\infty$ and X^* is a positive semidefinite optimal matrix obtained from solving (P_{SDP}) and the rank of X^* is r. To obtain a rank-one optimal solution of (P_{SDP}) having the format $X^* = \left[(x^*)^T, 1\right]^T \left[(x^*)^T, 1\right]$, we can first apply the following rank-one decomposition procedure of Sturm and Zhang [21] to represent X^* as

$$X^* = \sum_{i=1}^{r} p_i p_i^T \qquad (1.9)$$

such that

$$p_i^T M(G) p_i \le 0,\ i = 1, 2, \ldots, r. \qquad (1.10)$$

where

$$M(G) = \begin{bmatrix} B & -g \\ -g^T & -\mu \end{bmatrix}; \quad \text{and we similarly denote } M(F) = \begin{bmatrix} A & -f \\ -f^T & 0 \end{bmatrix}.$$

Let $I = \{i \in \{1,2,\ldots,r\} \mid (p_i)_{n+1} \neq 0\}$ and $J = \{1,2,\ldots,r\} \setminus I$. As $X^*_{n+1,n+1} = 1$, (1.9) can be written as

$$X^* = \sum_{i\in I} \xi_i \begin{bmatrix} \widetilde{p}_i \\ 1 \end{bmatrix} \begin{bmatrix} \widetilde{p}_i \\ 1 \end{bmatrix}^T + \sum_{j\in J} \begin{bmatrix} \widetilde{p}_j \\ 0 \end{bmatrix} \begin{bmatrix} \widetilde{p}_j \\ 0 \end{bmatrix}^T$$

where $I \neq \emptyset$; $\xi_i \geq 0$ and $\sum_{i\in I} \xi_i = 1$. It follows from (1.10) that $\widetilde{p}_i$ for each $i \in I$ is a feasible solution of (QP1QC) whereas $\widetilde{p}_j,\ j \in J$ may not. There are three cases that can happen:

(a) Suppose $J = \emptyset$. Then,

$$v(\mathrm{P_{SDP}}) = X^* \bullet M(F) = \sum_{i=1}^{r} \xi_i \begin{bmatrix} \widetilde{p}_i \\ 1 \end{bmatrix}^T M(F) \begin{bmatrix} \widetilde{p}_i \\ 1 \end{bmatrix}.$$

Since each $\widetilde{p}_i,\ i = 1,2,\ldots,r$ is feasible,

$$\begin{bmatrix} \widetilde{p}_i \\ 1 \end{bmatrix}^T M(F) \begin{bmatrix} \widetilde{p}_i \\ 1 \end{bmatrix} \geq v(\mathrm{P_{SDP}}),\ i = 1,2,\ldots,r.$$

Because $\xi_i \geq 0$, it forces each $\widetilde{p}_i,\ i = 1,2,\ldots,r$ to be an optimal solution of (QP1QC).

(b) Suppose it holds that

$$\begin{bmatrix} \widetilde{p}_i \\ 1 \end{bmatrix}^T M(G) \begin{bmatrix} \widetilde{p}_i \\ 1 \end{bmatrix} = 0,\ \forall i \in I.$$

Let (σ, s) be any feasible solution of $(\mathrm{D_{SDP}})$ in (1.3). Then

$$\begin{aligned} v(\mathrm{P_{SDP}}) &= X^* \bullet M(F) \\ &\geq s + X^* \bullet \left(M(F) + \sigma M(G) + s \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \right) && (1.11) \\ &\geq s + \sum_{i\in I} \xi_i \begin{bmatrix} \widetilde{p}_i \\ 1 \end{bmatrix}^T \left(M(F) + \sigma M(G) + s \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \right) \begin{bmatrix} \widetilde{p}_i \\ 1 \end{bmatrix} && (1.12) \\ &= \sum_{i\in I} \xi_i \begin{bmatrix} \widetilde{p}_i \\ 1 \end{bmatrix}^T M(F) \begin{bmatrix} \widetilde{p}_i \\ 1 \end{bmatrix} \\ &\geq \sum_{i=1}^{r} (\xi_i v(\mathrm{P_{SDP}})) = v(\mathrm{P_{SDP}}), \end{aligned}$$

where (1.11) is true since $X^* \bullet M(G) \leq 0$ and $\sigma \geq 0$, while (1.12) holds since

$$M(F) + \sigma M(G) + s \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \succeq 0 \ \text{ and } X^* \succeq \sum_{i\in I} \xi_i \begin{bmatrix} \widetilde{p}_i \\ 1 \end{bmatrix} \begin{bmatrix} \widetilde{p}_i \\ 1 \end{bmatrix}^T.$$

Consequently,

$$v(\mathrm{P_{SDP}}) = \sum_{i\in I} \xi_i \begin{bmatrix} \widetilde{p}_i \\ 1 \end{bmatrix}^T M(F) \begin{bmatrix} \widetilde{p}_i \\ 1 \end{bmatrix}$$

and each $\widetilde{p}_i,\ i \in I$ is an optimal solution of (QP1QC).

(c) Suppose $J \neq \emptyset$ and there is an index $k \in I$ such that

$$\begin{bmatrix} \widetilde{p}_k \\ 1 \end{bmatrix}^T M(G) \begin{bmatrix} \widetilde{p}_k \\ 1 \end{bmatrix} < 0.$$

Let $j \in J$. We first notice that $p_k p_k^T + p_j p_j^T$ can be written as the sum of two different rank one matrices as follows:

$$p_k p_k^T + p_j p_j^T = \frac{1}{1+\epsilon^2}(p_k + \epsilon p_j)(p_k + \epsilon p_j)^T + \frac{1}{1+\epsilon^2}(\epsilon p_k - p_j)(\epsilon p_k - p_j)^T$$

where $\epsilon \neq 0$. Since $p_k^T M(G) p_k < 0$ and $p_j^T M(G) p_j \leq 0$, we can choose $\epsilon(p_k^T M(G) p_j) \geq 0$ and $|\epsilon|$ sufficiently small such that

$$(p_k + \epsilon p_j)^T M(G)(p_k + \epsilon p_j) = p_k^T M(G) p_k + 2\epsilon(p_k^T M(G) p_j) + \epsilon^2 p_j^T M(G) p_j < 0,$$
$$(\epsilon p_k - p_j)^T M(G)(\epsilon p_k - p_j) = \epsilon^2 p_k^T M(G) p_k - 2\epsilon(p_k^T M(G) p_j) + p_j^T M(G) p_j < 0.$$

Moreover, $(p_k + \epsilon p_j)_{n+1} \neq 0$, $(\epsilon p_k - p_j)_{n+1} \neq 0$. In other words, we can replace p_k by $\frac{1}{\sqrt{1+\epsilon^2}}(p_k + \epsilon p_j)$ and p_j by $\frac{1}{\sqrt{1+\epsilon^2}}(\epsilon p_k - p_j)$ to eliminate the index j from the set J. Repeat the process until $J = \emptyset$ and we eventually return to case (a) where any $\widetilde{p}_i$, $i = 1, 2, \ldots, r$ is an optimal solution of (QP1QC).

In summary, by the SDP relaxation, first one lifts the (QP1QC) problem to a linear matrix inequality in a (much) higher dimensional space where a conic (convex) optimization problem ($\mathrm{P_{SDP}}$) is solved. Then, by incorporating a matrix rank-one decomposition procedure we can run it back to a solution of (QP1QC). The idea is neat, but it suffers from a huge computational task should a large scale (QP1QC) be encountered.

In contrast, the Lagrange dual approach seems to be more natural but the analysis is often subject to serious assumptions. We can see that the Lagrange function (1.2) makes a valid lower bound for the (QP1QC) problem only when $A + \sigma B \succeq 0$. Furthermore, the early analysis, such as those in [2, 7, 15, 24, 25], all assumed the dual Slater condition in order to secure the strong duality result. Actually, the dual Slater condition is a very restrictive assumption since it implies that, see [13], the two matrices A and B can be simultaneously diagonalizable via congruence (SDC), which itself is already very limited:

Simultaneously Diagonalizable via Congruence (SDC): A and B are simultaneously diagonalizable via congruence (SDC) if there exists a nonsingular matrix C such that both $C^T AC$ and $C^T BC$ are diagonal.

Nevertheless, solving (QP1QC) via the Lagrange dual enjoys a computational advantage over the SDP relaxation. It has been studied thoroughly in [7] that the optimal solution x^* to (QP1QC) problems, while satisfying the dual Slater condition, is either $\bar{x} = \lim_{\sigma \to \sigma^*} (A + \sigma B)^{-1}(f + \sigma g)$, or obtained from $\bar{x} + \alpha_0 \tilde{x}$ where σ^* is the dual optimal solution, $\tilde{x}$ is a vector in the null space of $A + \sigma^* B$ and α_0 is some constant from solving a quadratic equation. For (QP1QC) problems under the SDC condition while violating the dual Slater condition, they are either unbounded below or can be transformed equivalently to an unconstraint quadratic problem. Since the dual problem is a single-variable concave programming, it is obvious that the computational cost for x^* using the dual method is much cheaper than solving a semidefinite programming plus an additional run-down procedure.

Our paper pays a revisit to (QP1QC) trying to solve it by continuing and extending the study beyond the dual Slater condition $A + \sigma B \succ 0$ (or SDC) into the hard case that $A + \sigma B \succeq 0$. Our analysis was motivated by an early result by J. J. Moré [15, Theorem 3.4] in 1993, which states:

Under the primal Slater condition and assuming that $B \neq 0$, a vector x^* is a global minimizer of (QP1QC) if and only if there exists a pair $(x^*, \sigma^*) \in R^n \times R$ such that x^* is primal feasible satisfying $x^{*T}Bx^* - 2g^Tx^* \leq \mu$; $\sigma^* \geq 0$ is dual feasible; the pair satisfies the complementarity $\sigma^*(x^{*T}Bx^* - 2g^Tx^* - \mu) = 0$; and a second order condition $A + \sigma^*B \succeq 0$ holds.

Notice that the speciality of the statement is to obtain a superior version of the second order necessary condition $A + \sigma^*B \succeq 0$. Normally in the context of a general nonlinear programming, one can only conclude that $A + \sigma^*B$ is positive semidefinite on *the tangent subspace* of $G(x^*) = 0$ when x^* lies on the boundary. The original proof was lengthy, but it can now be understood from the theory of semidefinite programming without much difficulty.

To this end, we first solve (QP1QC) separately without the primal Slater condition in Section 2. In Section 3, we establish new results for the positive semideifinite matrix pencil

$$I_{\succeq}(A, B) = \{\sigma \in \mathbb{R} \mid \ A + \sigma B \succeq 0\}$$

with which, in Section 4, we characterize necessary and sufficient conditions for two unsolvable (QP1QC) cases: either (QP1QC) is unbounded below, or bounded below but unattainable. The conditions are not only polynomially checkable, but provide a solution procedure to obtain an optimal solution x^* of (QP1QC) when the problem is solvable. In Section 5, we use numerical examples to show why (QP1QC), while failing the (SDC) condition, become the hard case. Conclusion remarks are made in section 6.

2 (QP1QC) Violating Primal Slater Condition

In this section, we show that (QP1QC) with no Slater point can be separately solved without any condition. Notice that, if $\mu > 0$, $x = 0$ satisfies the Slater condition. Consequently, if the primal Slater condition is violated, we must have $\mu \leq 0$, and

$$\min_{x \in \mathbb{R}^n} G(x) = x^TBx - 2g^Tx \geq \mu, \tag{2.1}$$

which implies that $G(x)$ is bounded from below and thus

$$B \succeq 0; \ g \in R(B); \ G(B^+g) = -g^TB^+g \geq \mu$$

where $R(B)$ is the range space of B, and B^+ is the Moore-Penrose generalized inverse of B. Conversely, if $B \succeq 0$, then $G(x)$ is convex. Moreover, if $g \in R(B)$, the set of critical points $\{x \big| Bx = g\} \neq \emptyset$, and every point in this set assumes the global minimum $-g^TB^+g$. Suppose $G(B^+g) = -g^TB^+g \geq \mu$, it implies that $\{x \big| x^TBx - 2g^Tx < \mu\} = \emptyset$. In other words,

$$\text{(QP1QC) violates the primal Slater condition if and only if } \left\{ \begin{array}{l} \mu \leq 0; \\ B \succeq 0; \\ g \in R(B); \\ -g^TB^+g \geq \mu. \end{array} \right.$$

In this case, the constraint set is reduced to

$$\{x \big| G(x) \leq \mu\} = \left\{ \begin{array}{ll} \emptyset, & \text{if } -g^TB^+g > \mu; \\ \{x \big| Bx = g\}, & \text{if } -g^TB^+g = \mu. \end{array} \right.$$

Namely, (QP1QC) is either infeasible (when $-g^T B^+ g > \mu$) or reduced to a quadratic programming with one linear equality constraint. In the latter case, the following problem is to be dealt with:

$$\begin{array}{ll} \inf\limits_{x\in\mathbb{R}^n} & x^T Ax - 2f^T x \\ \text{s.t.} & Bx = g. \end{array} \tag{2.2}$$

All the feasible solutions of (2.2) can be expressed as $x = B^+g + Vy,\ \forall y \in \mathbb{R}^{\dim N(B)}$, where $N(B)$ denotes the null space of B and V is the matrix basis of $N(B)$. The objective function becomes

$$\begin{aligned} x^T Ax - 2f^T x &= (B^+g + Vy)^T A (B^+g + Vy) - 2f^T (B^+g + Vy) \\ &= y^T V^T AVy + 2\left(V^T AB^+ g - V^T f\right)^T y + g^T B^+ AB^+ g - 2f^T B^+ g \\ &= z^T \hat{D} z + 2\Lambda^T z + \alpha, \end{aligned}$$

where $z = Q^T y$, Q is an orthonormal matrix such that $V^T AV = Q\hat{D}Q^T$; $\hat{D} = diag\left(\hat{d}_1, \hat{d}_2, \dots, \hat{d}_n\right)$; $\Lambda = Q^T\left(V^T AB^+ g - V^T f\right)$ and $\alpha = g^T B^+ AB^+ g - 2f^T B^+ g$. Let

$$h(z) = \sum_{i\in I}\left(\hat{d}_i z_i^2 + 2\Lambda_i z_i\right) + \alpha,\ I = \{1, 2, \dots, n\}.$$

Then, a feasible (QP1QC) violating the primal Slater condition can be summarized as follows:

- If $\min\limits_{i\in I}\left\{\hat{d}_i\right\} < 0$ or there is an index $i_0 \in I$ such that $\hat{d}_{i_0} = 0$ with $\Lambda_{i_0} \neq 0$, then (QP1QC) is unbounded from below.
- Otherwise, $h(z)$ is convex, $v(QP1QC) = \alpha - \sum_{i\in J}\frac{\Lambda_i^2}{\hat{d}_i}$ where $J = \{i \in I | \hat{d}_i > 0\}$. Moreover, $x^* = B^+g + VQz$ with

$$z_i = \begin{cases} -\frac{\Lambda_i}{\hat{d}_i}, & \text{if } i \in J; \\ 0, & \text{if } i \in I \setminus J. \end{cases}$$

  is one of the optimal solutions to (QP1QC).

3 New Results on Matrix Pencil and SDC

As we have mentioned before, we will use matrix pencils, that is, one-parameter families of matrices of the form $A + \sigma B$ as a main tool to study (QP1QC). An excellent survey for matrix pencils can be found in [22]. In this section, we first summarize important results about matrix pencils followed by new findings of the paper.

For any symmetric matrices A, B, define

$$\begin{aligned} I_{\succ}(A,B) &:= \{\sigma \in \mathbb{R} \mid\ A + \sigma B \succ 0\}, && (3.1) \\ I_{\succeq}(A,B) &:= \{\sigma \in \mathbb{R} \mid\ A + \sigma B \succeq 0\}, && (3.2) \\ II_{\succ}(A,B) &:= \{(\mu,\sigma) \in \mathbb{R}^2 \mid\ \mu A + \sigma B \succ 0\}, && (3.3) \\ II_{\succeq}(A,B) &:= \{(\mu,\sigma) \in \mathbb{R}^2 \mid\ \mu A + \sigma B \succeq 0\}, && (3.4) \\ Q(A) &:= \{v \mid\ v^T Av = 0\}, && (3.5) \\ N(A) &:= \{v \mid\ Av = 0\}. && (3.6) \end{aligned}$$

The first result characterizes when $I_{\succ}(A,B) \neq \emptyset$ and when $II_{\succ}(A,B) \neq \emptyset$.

Theorem 3.1 ([22]). *If $A, B \in \mathbb{R}^{n\times n}$ are symmetric matrices, then $I_{\succ}(A,B) \neq \emptyset$ if and only if*

$$w^T Aw > 0, \ \forall w \in Q(B), \ w \neq 0. \tag{3.7}$$

Furthermore, $II_{\succ}(A,B) \neq \emptyset$ implies that

$$Q(A) \cap Q(B) = \{0\}, \tag{3.8}$$

which is also sufficient if $n \geq 3$.

The second result below extends the discussion in Theorem 3.1 to the positive semidefinite case $I_{\succeq}(A,B) \neq \emptyset$ assuming that B is indefinite. This result was also used by Moré to solve (QP1QC) in [15]. An example shown in [15] indicates that the assumption about the indefiniteness of B is critical.

Theorem 3.2 ([15]). *If $A, B \in R^{n\times n}$ are symmetric matrices and B is indefinite, then $I_{\succeq}(A,B) \neq \emptyset$ if and only if*

$$w^T Aw \geq 0, \ \forall w \in Q(B). \tag{3.9}$$

Other known miscellaneous results related to our discussion for (QP1QC) are also mentioned below. In particular, the proof of Lemma 3.4 is constructive.

Lemma 3.3 ([15]). *Both $I_{\succ}(A,B)$ and $I_{\succeq}(A,B)$ are intervals.*

Lemma 3.4 ([12, 13]). *$II_{\succ}(A,B) = \{(\lambda,\nu) \in R^2 \mid \lambda A + \nu B \succ 0\} \neq \emptyset$ implies that A and B are simultaneously diagonalizable via congruence(SDC). When $n \geq 3$, $II_{\succ}(A,B) \neq \emptyset$ is also necessary for A and B to be SDC.*

In the following we discuss necessary conditions for $I_{\succeq}(A,B) \neq \emptyset$ (Theorem 3.5) and sufficient conditions for $II_{\succeq}(A,B) \neq \emptyset$ (Theorem 3.6) without the indefiniteness assumption of B.

Theorem 3.5. *Let $A, B \in R^{n\times n}$ be symmetric matrices and suppose $I_{\succeq}(A,B) \neq \emptyset$. If $I_{\succeq}(A,B)$ is a single-point set, then B is indefinite and $I_{\succ}(A,B) = \emptyset$. Otherwise, $I_{\succeq}(A,B)$ is an interval and*

(a) *$Q(A) \cap Q(B) = N(A) \cap N(B)$.*

(b) *$N(A+\sigma B) = N(A) \cap N(B)$, for any σ in the interior of $I_{\succeq}(A,B)$.*

(c) *A, B are simultaneously diagonalizable via congruence.*

Proof. Suppose $I_{\succeq}(A,B) = \{\sigma\}$ is a single-point set. Then for any $\sigma_1 < \sigma < \sigma_2$, neither $A + \sigma_1 B$ nor $A + \sigma_2 B$ is positive semidefinite. There are vectors v_1 and v_2 such that

$$\begin{aligned} 0 > v_1^T(A+\sigma_1 B)v_1 = v_1^T(A+\sigma B)v_1 + (\sigma_1 - \sigma)v_1^T Bv_1 \geq (\sigma_1-\sigma)v_1^T Bv_1; \\ 0 > v_2^T(A+\sigma_2 B)v_2 = v_2^T(A+\sigma B)v_2 + (\sigma_2 - \sigma)v_2^T Bv_2 \geq (\sigma_2-\sigma)v_2^T Bv_2. \end{aligned}$$

It follows that

$$v_1^T Bv_1 > 0 \text{ and } v_2^T Bv_2 < 0,$$

so B is indefinite. Furthermore, suppose there is a $\hat{\sigma} \in I_{\succ}(A,B)$ such that $A + \hat{\sigma}B \succ 0$. Then, $A + \sigma B \succ 0$ for $\sigma \in (\hat{\sigma}-\delta, \hat{\sigma}+\delta)$ and some $\delta > 0$ sufficiently small, which contradicts to the fact that $I_{\succeq}(A,B)$ is a singleton.

Now assume that $I_{\succeq}(A,B)$ is neither empty nor a single-point set. According to Lemma 3.3, $I_{\succeq}(A,B)$ is an interval. To prove (a), choose $\sigma_1,\sigma_2 \in I_{\succeq}(A,B)$ with $\sigma_1 \neq \sigma_2$. If $x \in Q(A) \cap Q(B)$, we have $x^T(A+\sigma_1 B)x = 0$. Since $A+\sigma_1 B \succeq 0$, $x^T(A+\sigma_1 B)x = 0$ implies that $(A+\sigma_1 B)x = 0$. Similarly, we have $(A+\sigma_2 B)x = 0$. Therefore,

$$0 = (A+\sigma_1 B)x - (A+\sigma_2 B)x = (\sigma_1 - \sigma_2)Bx,$$

indicating that $Bx = 0$. It follows also that $Ax = 0$. In other words, $Q(A) \cap Q(B) \subseteq N(A) \cap N(B)$. Since it is trivial to see that $N(A) \cap N(B) \subseteq Q(A) \cap Q(B)$, we have the statement (a).

To prove (b), choose $\sigma_1, \sigma, \sigma_2 \in int\,(I_{\succeq}(A,B))$ such that $\sigma_1 < \sigma < \sigma_2$. Then, for all $x \in N(A+\sigma B)$,

$$\begin{aligned} 0 \le x^T(A+\sigma_1 B)x = x^T(A+\sigma B)x + (\sigma_1 - \sigma)x^T Bx, \\ 0 \le x^T(A+\sigma_2 B)x = x^T(A+\sigma B)x + (\sigma_2 - \sigma)x^T Bx, \end{aligned}$$

which implies that

$$0 \le x^T Bx \le 0,$$

and hence $x^T Bx = 0$. Moreover, if $x \in N(A+\sigma B)$, we have not only $(A+\sigma B)x = 0$, but $x^T(A+\sigma B)x = 0$. Since $x^T Bx = 0$, there also is $x^T Ax = 0$. By the statement (a), we have proved that

$$N(A+\sigma B) \subseteq Q(A) \cap Q(B) = N(A) \cap N(B).$$

With the fact that $N(A) \cap N(B) \subseteq N(A+\sigma B)$ for $\sigma \in int\,(I_{\succeq}(A,B))$, we find that $N(A+\sigma B)$ is invariant for any $\sigma \in int\,(I_{\succeq}(A,B))$, and the statement (b) follows.

For the statement (c), let $V \in R^{n\times r}$ be the basis matrix of $N(A+\sigma B)$ for some $\sigma \in int\,(I_{\succeq}(A,B))$, where $r = \dim(N(A+\sigma B))$. Extend V to a nonsingular matrix $[U\ V] \in R^{n\times n}$ such that $U^T U$ is nonsingular and $U^T V = 0$. Then,

$$\begin{bmatrix} U^T \\ V^T \end{bmatrix} (A+\sigma B)\,[U\ V] = \begin{bmatrix} U^T(A+\sigma B)U & U^T(A+\sigma B)V \\ V^T(A+\sigma B)U & V^T(A+\sigma B)V \end{bmatrix} = \begin{bmatrix} U^T AU + \sigma U^T BU & 0 \\ 0 & 0 \end{bmatrix}.$$

Consequently,

$$U^T AU + \sigma U^T BU \succ 0, \tag{3.10}$$

which can be verified by applying u^T and u, $u \in R^{n-r}$, to $U^T AU + \sigma U^T BU$. Then, $(u^T U^T)(A+\sigma B)(Uu) = 0$ if and only if $(A+\sigma B)(Uu) = 0$. Since U is the orthogonal complement of V, it must be $u = 0$.

By Lemma 3.4, $U^T AU$ and $U^T BU$ are simultaneously diagonalizable via congruence, i.e., there exists nonsingular square matrix U_1 such that both $U_1^T U^T AUU_1$ and $U_1^T U^T BUU_1$ are diagonal. Let W be such that

$$\begin{bmatrix} U_1 & 0 \\ 0 & W \end{bmatrix}$$

is nonsingular. By the statement (b), V also spans $N(A) \cap N(B)$ and thus $AV = BV = 0$. It indicates that

$$\begin{aligned} \begin{bmatrix} U_1^T & 0 \\ 0 & W^T \end{bmatrix} \begin{bmatrix} U^T \\ V^T \end{bmatrix} A \begin{bmatrix} U & V \end{bmatrix} \begin{bmatrix} U_1 & 0 \\ 0 & W \end{bmatrix} &= \begin{bmatrix} U_1^T U^T AUU_1 & 0 \\ 0 & 0 \end{bmatrix}, \\ \begin{bmatrix} U_1^T & 0 \\ 0 & W^T \end{bmatrix} \begin{bmatrix} U^T \\ V^T \end{bmatrix} B \begin{bmatrix} U & V \end{bmatrix} \begin{bmatrix} U_1 & 0 \\ 0 & W \end{bmatrix} &= \begin{bmatrix} U_1^T U^T BUU_1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

That is, A and B are simultaneously diagonalizable via congruence. The statement (c) is proved. □

Theorem 3.6. *Let $A, B \in R^{n\times n}$ be symmetric matrices. Suppose $Q(A) \cap Q(B) = N(A) \cap N(B)$ and*

$$\dim\{N(A) \cap N(B)\} \leq n - 3. \tag{3.11}$$

Then,

(a') *$II_{\succ}(U^TAU, U^TBU) \neq \emptyset$ where U is the basis matrix spanning the orthogonal complement of $N(A) \cap N(B)$;*

(b') *$II_{\succeq}(A, B) \neq \emptyset$; and*

(c') *A, B are simultaneously diagonalizable via congruence.*

Proof. Following the same notation in the proof of Theorem 3.5, we let $V \in R^{n\times r}$ be the basis matrix of $N(A) \cap N(B)$, $r = \dim(N(A) \cap N(B))$, and then extend V to a nonsingular matrix $[U\ V] \in R^{n\times n}$ such that U^TU is nonsingular and $U^TV = 0$. For any $x \in Q(U^TAU) \cap Q(U^TBU)$, we have $x^TU^TAUx = x^TU^TBUx = 0$ and thus $Ux \in Q(A) \cap Q(B)$. By assumption, $Q(A) \cap Q(B) = N(A) \cap N(B)$, there must exist $z \in R^r$ such that

$$Ux = Vz.$$

Therefore,

$$x = (U^TU)^{-1}U^TVz = 0,$$

which shows that

$$Q(U^TAU) \cap Q(U^TBU) = \{0\}. \tag{3.12}$$

It follows from Assumption (3.11) that

$$\dim(U^TAU) = \dim(U^TBU) = n - r \geq 3.$$

According to Theorem 3.1, (3.12) implies that

$$II_{\succ}(U^TAU, U^TBU) \neq \emptyset. \tag{3.13}$$

Namely, there exists $(\mu, \sigma) \in \mathbb{R}^2$ such that $\mu(U^TAU) + \sigma(U^TBU) \succ 0$, which proved the statement (a'). Now, for $y \in R^n$, we can write $y = Uu + Vv$, where $u \in R^{n-r}, v \in R^r$. Then,

$$\begin{aligned}
& y^T(\mu A + \sigma B)y \\
= \ & (u^TU^T + v^TV^T)(\mu A + \sigma B)(Uu + Vv) \\
= \ & u^T(\mu U^TAU + \sigma U^TBU)u \geq 0,
\end{aligned}$$

which proves $II_{\succeq}(A, B) \neq \emptyset$. From (3.13) and Lemma 3.4, we again conclude that U^TAU and U^TBU are simultaneously diagonalizable via congruence. It follows from the same proof for the statement (c) in Theorem 3.5 that A, B are indeed simultaneously diagonalizable via congruence. The proof is complete. □

Remark 3.7. When $Q(A) \cap Q(B) = \{0\}$, the assumption of Theorem 3.6 becomes $N(A) \cap N(B) = \{0\}$. In this case, the dimension condition (3.11) is equivalent to $n \geq 3$ and the basis matrix U is indeed I_n, the identical matrix of dimension n. From the statement (a') in Theorem 3.6, we conclude that $II_{\succ}(A, B) \neq \emptyset$. In other words, we have generalized the part of conclusion in Theorem 3.1 that "$Q(A) \cap Q(B) = \{0\}$ and $n \geq 3$" implies "$II_{\succ}(A, B) \neq \emptyset$".

As an immediate corollary of Theorem 3.6, we obtain sufficient conditions to make A and B simultaneously diagonalizable via congruence.

Corollary 3.8. *Let $A, B \in R^{n\times n}$ be symmetric matrices. Suppose one of the following conditions is satisfied,*

(S1) *$I_{\succeq}(A,B)$ contains more than one point;*

(S2) *$I_{\succeq}(B,A)$ contains more than one point;*

(S3) *$Q(A)\cap Q(B) = N(A)\cap N(B)$, with $\dim\{N(A)\cap N(B)\} \neq n-2$,*

then A, B are simultaneously diagonalizable via congruence.

Proof. Conditions (S1) and (S2) directly implies that $I_{\succeq}(A,B)$ (and $I_{\succeq}(B,A)$ respectively) is non-empty and is not a single point set. The conclusion follows from the statement (c) of Theorem 3.5. Condition (S3) is a slight generalization of Theorem 3.6 where the situation for $\dim\{N(A)\cap N(B)\} \le n-3$ has already been proved. It remains to show that A, B are simultaneously diagonalizable via congruence when $\dim\{N(A)\cap N(B)\} = n \vee (n-1)$. Notice that $\dim\{N(A)\cap N(B)\} = n$ implies $A = B = 0$ (in which case the conclusion follows immediately), whereas $\dim\{N(A)\cap N(B)\} = n-1$ implies that matrices A and B must be expressed as $A = \alpha uu^T,\ B = \beta uu^T$ for some vector $u \in R^n$ and $\alpha^2+\beta^2 \neq 0$. Suppose $\beta \neq 0$, then $A = \frac{\alpha}{\beta}B$ so the two matrices are certainly simultaneously diagonalizable via congruence. $\square$

Example 3.9 below shows that the non-single-point assumption of $I_{\succeq}(A,B)$ is necessary to assure statement (a) and (c); and the dimensionality assumption in (S3) is also necessary.

Example 3.9. Consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.14}$$

Then there is only one $\sigma = 0$ such that $A + \sigma B \succeq 0$, but

$$\begin{aligned} Q(A)\cap Q(B) &= \left\{ \begin{bmatrix} 0 \\ * \end{bmatrix} \right\}; \\ N(A)\cap N(B) &= \{0\}; \\ \dim\{N(A)\cap N(B)\} &= n-2. \end{aligned}$$

It is not difficult to verify that here A and B cannot be simultaneously diagonalizable via congruence.

Remark 3.10. Corollary 3.8 extends Lemma 3.4 since $\{(\lambda,\nu)\in R^2 \mid \lambda A + \nu B \succ 0\} \neq \emptyset$ implies that either $I_{\succ}(A,B) \neq \emptyset$ or $I_{\succ}(B,A) \neq \emptyset$, belonging to (S1) and (S2), respectively.

Remark 3.11. Both (S1) and (S2) are polynomially checkable. A detailed proof is provided in Remark 4.2 of Section 4 below. In literature, there are necessary and sufficient conditions for A and B to be simultaneously diagonalizable via congruence, see [1] and references therein. But none is polynomially checkable.

4 Solving (QP1QC) via Matrix Pencils

In this section, we use matrix pencils developed in the previous section to establish necessary and sufficient conditions for the "undesired" (QP1QC) which are unbounded below, and which are bounded below but unattainable. All the discussions here are subject to the assumption of the primal Slater condition since the other type of (QP1QC) has been otherwise solved in Section 2 above. Interestingly, probing into the two undesired cases eventually leads to a complete characterization for the solvability of (QP1QC).

Theorem 4.1. *Under the primal Slater condition, (QP1QC) is unbounded below if and only if the system of σ:*

$$\begin{cases} A+\sigma B \succeq 0, \\ \sigma \geq 0, \\ f+\sigma g \in R(A+\sigma B), \end{cases} \tag{4.1}$$

has no solution.

Proof. Since the primal Slater condition is satisfied, if the infimum of (QP1QC) is finite, by the strong duality [11], $v(\mathrm{P_{SDP}}) = v(\mathrm{D_{SDP}})$ and the optimal value is attained for $(\mathrm{D_{SDP}})$. Then, the system (4.1) has a solution. Conversely, suppose the system (4.1) has a solution. Then $(\mathrm{D_{SDP}})$ is feasible which provides a finite lower bound for $(\mathrm{P_{SDP}})$. As the primal Slater condition is satisfied, (QP1QC) is feasible and must be bounded from below. □

Remark 4.2. Checking whether the system (4.1) has a solution can be done in polynomial time. To this end, denote

$$\sigma_l^* := \min_{A+\sigma B\succeq 0} \sigma, \tag{4.2}$$

$$\sigma_u^* := \max_{A+\sigma B\succeq 0} \sigma, \tag{4.3}$$

and let $V \in R^{n\times r}$ be the basis matrix of $N(A)\cap N(B)$ and $U \in R^{n\times(n-r)}$ satisfy $[U\ V]$ is nonsingular and $U^TV = 0$. From

$$\begin{bmatrix} U^T \\ V^T \end{bmatrix}(A+\sigma B)\,[U\ V] = \begin{bmatrix} U^T(A+\sigma B)U & U^T(A+\sigma B)V \\ V^T(A+\sigma B)U & V^T(A+\sigma B)V \end{bmatrix} = \begin{bmatrix} U^TAU+\sigma U^TBU & 0 \\ 0 & 0 \end{bmatrix}, \tag{4.4}$$

it can be seen that, if $U^TBU = 0$ and $U^TAU \not\succeq 0$, then $I_{\succeq}(A,B) = \emptyset$ and the system (4.1) has no solution. It follows that (QP1QC) must be unbounded below in this case.

Assume that $I_{\succeq}(A,B) \neq \emptyset$ and $\bar\sigma \in I_{\succeq}(A,B)$. Then, we observe from

$$A+\sigma B = (A+\bar\sigma B)+(\sigma-\bar\sigma)B$$

that $\sigma_l^* = -\infty$ if and only if $B \preceq 0$. Similarly, $\sigma_u^* = \infty$ if and only if $B \succeq 0$.

Now suppose $-\infty < \sigma_l^* \leq \sigma_u^* < \infty$. From the proof for the statement (c) in Theorem 3.5 and (3.10), if $\bar\sigma \in int\,(I_{\succeq}(A,B))$, then $U^TAU+\bar\sigma U^TBU \succ 0$. Conversely, if $U^TAU + \bar\sigma U^TBU \succ 0$, there must be some $\epsilon > 0$ such that $U^TAU+(\bar\sigma\pm\epsilon)U^TBU \succ 0$ and thus $A+(\bar\sigma\pm\epsilon)B \succeq 0$. It holds that $\bar\sigma \in int\,(I_{\succeq}(A,B))$. Consequently, σ_l^* and σ_u^*, as end points of $I_{\succeq}(A,B)$, must be roots of the polynomial equation

$$\det(U^TAU+\sigma U^TBU) = 0. \tag{4.5}$$

We first find all the roots of the polynomial equation (4.5), denoted by $r_1, \ldots, r_{n-r}$, which can be done in polynomial time. Then,

$$\sigma_l^* = \min\{r_i : \ U^T AU + r_i U^T BU \succeq 0, \ i = 1, \ldots, n-r\},$$
$$\sigma_u^* = \max\{r_i : \ U^T AU + r_i U^T BU \succeq 0, \ i = 1, \ldots, n-r\}.$$

Case (a) If $\sigma_u^* < 0$, (4.1) has no solution.

Case (b) $\sigma_u^* = 0$ or $\sigma_l^* = \sigma_u^* > 0$. Checking the feasibility of (4.1) is equivalent to verify whether the linear equation

$$(A + \sigma B)x = f + \sigma g$$

has a solution for $\sigma = 0$ or $\sigma = \sigma_l^* = \sigma_u^*$.

Case (c) $\sigma_u^* > 0$ and $\sigma_l^* < \sigma_u^*$. According to Theorem 3.5, the basis matrix V spans $N(A + \sigma' B) = N(A) \cap N(B)$ for any $\sigma' \in (\sigma_l^*, \sigma_u^*)$.

- Subcase (c1) $\sigma_l^* < 0$. (4.1) has a solution if and only if either the linear equation

$$(A + \sigma_u^* B)x = f + \sigma_u^* g$$

has a solution or there is a $\sigma \in [0, \sigma_u^*)$ such that

$$(A + \sigma B)x = f + \sigma g \tag{4.6}$$

has a solution. According to (4.4), the equation (4.6) is equivalent to

$$\begin{bmatrix} U^T AU + \sigma U^T BU & 0 \\ 0 & 0 \end{bmatrix} [U \ V]^{-1} x = \begin{bmatrix} U^T \\ V^T \end{bmatrix} (f + \sigma g).$$

Since $U^T AU + \sigma U^T B \succ 0$ for $\sigma \in [0, \sigma_u^*)$, the equation (4.6) has a solution if and only if

$$V^T f + \sigma V^T g = 0. \tag{4.7}$$

- Subcase (c2) $\sigma_l^* \geq 0$. (4.1) has a solution if and only if at least one of the following three conditions holds:

$$V^T f + \sigma V^T g = 0, \text{ for some } \sigma \in (\sigma_l^*, \sigma_u^*); \tag{4.8}$$
$$(A + \sigma_l^* B)x = f + \sigma_l^* g \text{ has a solution;}$$
$$(A + \sigma_u^* B)x = f + \sigma_u^* g \text{ has a solution.}$$

We show either (4.7) or (4.8) is polynomially solvable. Since V, f, g are fixed, if $V^T f = V^T g = 0$, $V^T f + \sigma V^T g = 0$ holds for any σ. If $V^T f$ and $V^T g \neq 0$ are linear dependent, there is a unique σ such that $V^T f + \sigma V^T g = 0$. Otherwise, $V^T f + \sigma V^T g = 0$ has no solution.

From Theorem 4.1, it is clear that, if (QP1QC) is bounded from below, the positive semidefinite matrix pencil $I_{\succeq}(A, B)$ must be non-empty. In solving (QP1QC), we therefore divide into two cases: $I_{\succeq}(A, B)$ is an interval or $I_{\succeq}(A, B)$ is a singleton.

Theorem 4.3. *Under the primal Slater condition, if the optimal value of (QP1QC) is finite and $I_{\succeq}(A, B)$ is an interval, the infimum of (QP1QC) is always attainable.*

Proof. Let $[U\ V] \in R^{n\times n}$ be a nonsingular matrix such that V spans $N(A) \cap N(B)$. By Theorem 3.5, for any $\sigma' \in int\,(I_{\succeq}(A,B))$, V also spans $N(A+\sigma' B)$. It follows from (3.10) that

$$U^T AU + \sigma' U^T BU \succ 0,\ \forall \sigma' \in int\,(I_{\succeq}(A,B))\,. \tag{4.9}$$

Since $v(QP1QC) > -\infty$, there is a solution σ to the system (4.1) such that $\sigma \in I_{\succeq}(A,B) \cap [0,+\infty)$ and $f + \sigma g \in R(A+\sigma B)$. Since V spans $N(A) \cap N(B)$, V must span a subspace contained in $N(A+\sigma B)$. It follows from $f+\sigma g \in R(A+\sigma B)$ that

$$V^T f + \sigma V^T g = 0. \tag{4.10}$$

Consider the nonsingular transformation $x = Uu + Vv$ which splits (QP1QC) into u-part and v-part:

$$\min \quad F(u,v) = u^T U^T AUu - 2f^T Uu - 2f^T Vv \tag{4.11}$$
$$\text{s.t.} \quad G(u,v) = u^T U^T BUu - 2g^T Uu - 2g^T Vv \le \mu. \tag{4.12}$$

Case (d) $V^T f = V^T g = 0$. Then, (QP1QC) by (4.11)-(4.12) is reduced to a smaller (QP1QC) having only the variable u. Moreover, the smaller (QP1QC) satisfies the dual Slater condition due to (4.9). It is hence solvable. See, for example, [15, 25, 7].

Case (e) $V^T f = 0$, $V^T g \neq 0$. By (4.10), we have $\sigma = 0$ which implies that $A \succeq 0$ and $f \in R(A)$. In this case, since (4.12) is always feasible for any u, (4.11)-(4.12) becomes an unconstrained convex quadratic program (4.11), which is always solvable due to $f \in R(A)$.

Case (f) $V^T f \neq 0$. By (4.10), $V^T g \neq 0$, σ is unique and nonzero. Thus, $\sigma > 0$. Then, the constraint (4.12) is equivalent to

$$\sigma u^T U^T BUu - 2\sigma g^T Uu - 2\sigma g^T Vv \le \sigma\mu,$$

or equivalently, by (4.10),

$$-2f^T Vv = 2\sigma g^T Vv \ge \sigma u^T U^T BUu - 2\sigma g^T Uu - \sigma\mu. \tag{4.13}$$

Therefore, (4.11)-(4.12) has a lower bounding quadratic problem with only the variable u such that

$$F(u,v) \ge u^T (U^T AU + \sigma U^T BU)u - 2(f^T U + \sigma g^T U)u - \sigma\mu. \tag{4.14}$$

Since $A + \sigma B \succeq 0$, the right hand side of (4.14) is a convex unconstrained problem. With $f + \sigma g \in R(A+\sigma B)$, if $f + \sigma g = (A+\sigma B)w$, we can write $w = Uy^* + Vz^*$ to make $f + \sigma g = (A+\sigma B)Uy^*$ and thus obtain $U^T(f+\sigma g) \in R(U^T AU + \sigma U^T BU)$ with y^* being optimal to (4.14). Substituting $u = y^*$ into (4.13), due to $V^T f \neq 0$, there is always some $v = z^*$ such that $(u,v) = (y^*, z^*)$ makes (4.13) hold as an equality. Then, $G(y^*, z^*) \le \mu$ and $F(y^*, z^*)$ attains its lower bound in (4.14). Therefore, $(u,v) = (y^*, z^*)$ solves (4.11)-(4.12).

Summarizing cases (d), (e) and (f), we have thus completed the proof of this theorem. □

What remains to discuss is when the positive semidefinite matrix pencil $I_{\succeq}(A,B)$ is a singleton $\{\sigma^*\}$. By Theorem 4.1, (QP1QC) is bounded from below if and only if we must have $\sigma^* \geq 0$ such that, with V being the basis matrix of $N(A+\sigma^* B)$ and $r = \dim N(A+\sigma^* B)$, the set

$$S = \{(A + \sigma^* B)^+(f + \sigma^* g) + Vy \mid y \in R^r\} \tag{4.15}$$

is non-empty. According to the result (quoted in Introduction) by Moré in [15], x^* solves (QP1QC) if and only if $x^* \in S$ is feasible and (x^*, σ^*) satisfy the complementarity. We therefore have the following characterization for a bounded below but unattainable (QP1QC).

Theorem 4.4. *Suppose the primal Slater condition holds and the optimal value of (QP1QC) is finite. The infimum of (QP1QC) is unattainable if and only if* $I_{\succeq}(A, B)$ *is a single-point set* $\{\sigma^*\}$ *with* $\sigma^* \geq 0$*, and the quadratic (in)equality*

$$\begin{cases} G\left((A + \sigma^* B)^+(f + \sigma^* g) + Vy\right) = \mu, & \text{if } \sigma^* > 0, \\ G\left(A^+ f + Vy\right) \leq \mu, & \text{if } \sigma^* = 0, \end{cases} \tag{4.16}$$

has no solution in y*, where* V *is the basis matrix of* $N(A + \sigma^* B)$.

Proof. Since $v(QP1QC)$ is assumed to be finite, by Theorem 4.1, the system (4.1) has at least one solution. By Theorem 4.3, the problem can be unattainable only when $I_{\succeq}(A, B)$ is a single-point set $\{\sigma^*\}$ with $\sigma^* \geq 0$. In addition, the set S in (4.15) can not have a feasible solution which, together with σ^*, satisfies the complementarity. In other words, the system (4.16) can not have any solution in y, which proves the necessity of the theorem. The sufficient part of the theorem which guarantees a bounded below but unattainable (QP1QC) is almost trivial. □

Remark 4.5. Under the condition that $I_{\succeq}(A, B)$ is a single-point set, that is, $\sigma_l^* = \sigma_u^*$ in (4.2)-(4.3), we show how (4.16) can be checked and how (QP1QC) can be solved in polynomial time.

Case(g) $\sigma_l^* = \sigma_u^* = 0$. The set S in (4.15) can be written as:

$$x^* = A^+ f + Vy. \tag{4.17}$$

The constraint $x^{*T} B x^* - 2g^T x^* \leq \mu$ on the set S is expressed as

$$y^T V^T B V y + 2[f^T A^+ B V - g^T V] y \leq \widetilde{\mu}, \tag{4.18}$$

where

$$\widetilde{\mu} = \mu - f^T A^+ B A^+ f + 2g^T A^+ f.$$

Then, (4.16) has a solution if and only if

$$\widetilde{\mu} \geq L^* = \min_y \; y^T V^T B V y + 2[f^T A^+ B V - g^T V] y. \tag{4.19}$$

Namely, if $\widetilde{\mu} < L^*$, the (QP1QC) is unattainable. Otherwise, we show how to get an optimal solution of (QP1QC).

- Subcase (g1) $L^* > -\infty$. It happens only when $V^T B V \succeq 0$ and $V^T B A^+ f - V^T g \in R(V^T B V)$. Therefore,

  $$y^* := -(V^T B V)^+(V^T B A^+ f - V^T g)$$

  is a solution to both (4.19) and (QP1QC).

- Subcase (g2) $L^* = -\infty$. It happens when either V^TBV has a negative eigenvalue with a corresponding eigenvector $\widetilde{y}$ or there is a vector $\widetilde{y} \in N(V^TBV)$ such that

$$\widetilde{y}^T(V^TBA^+f - V^Tg) < 0.$$

Then, for any sufficient large number k,

$$y^* = k\widetilde{y}$$

is feasible to (4.18) and thus solves the (QP1QC)

Case(h) $\sigma_l^* = \sigma_u^* > 0$. (QP1QC) is attainable if the constraint is satisfied as an equality by some point in S. Restrict the equality constraint $G(x) = \mu$ to the set S yields

$$y^TV^TBVy + 2[(f + \sigma_l^*g)^T(A + \sigma_l^*B)^+BV - g^TV]y = \widetilde{\mu}, \tag{4.20}$$

where

$$\widetilde{\mu} = \mu - (f + \sigma_l^*g)^T(A + \sigma_l^*B)^+B(A + \sigma_l^*B)^+(f + \sigma_l^*g) + 2g^T(A + \sigma_l^*B)^+(f + \sigma_l^*g).$$

Consider the unconstrained quadratic programming problems:

$$L^*/U^* = \inf / \sup \ y^TV^TBVy + 2[(f + \sigma_l^*g)^T(A + \sigma_l^*B)^+BV - g^TV]y. \tag{4.21}$$

We claim that (4.16) (as well as (QP1QC)) has a solution if and only if $\widetilde{\mu} \in [L^*, U^*]$. We only have to prove the "if" part, so $\widetilde{\mu} \in [L^*, U^*]$ is now assumed. Notice that $L^* > -\infty$ ($U^* < +\infty$) if and only if $V^TBV \succeq (\preceq) 0$ and $V^TB(A + \sigma_l^*B)^+(f + \sigma_l^*g) - V^Tg \in R(V^TBV)$.

- Subcase (h1) $L^* > -\infty$, $U^* < +\infty$. This is a trivial case as it happens if and only if $V^TBV = 0$, $V^TB(A + \sigma_l^*B)^+(f + \sigma_l^*g) - V^Tg = 0$. Consequently, $L^* = U^* = 0$ and any vector y is a solution to (4.16) and solves (QP1QC).
- Subcase (h2) $L^* > -\infty$, $U^* = +\infty$. The infimum of (4.21) is attained at

$$\widehat{y} = -(V^TBV)^+[V^TB(A + \sigma_l^*B)^+(f + \sigma_l^*g) - V^Tg].$$

Furthermore, either V^TBV has a positive eigenvalue with a corresponding eigenvector $\widetilde{y}$; or there is a vector $\widetilde{y} \in N(V^TBV)$ such that

$$\widetilde{y}^T[V^TB(A + \sigma_l^*B)^+(f + \sigma_l^*g) - V^Tg] > 0.$$

Starting from $\widehat{y}$ and moving along the direction $\widetilde{y}$, we find that

$$h(\alpha) := (\widehat{y} + \alpha\widetilde{y})^TV^TBV(\widehat{y} + \alpha\widetilde{y}) + 2[(f + \sigma_l^*g)^T(A + \sigma_l^*B)^+BV - g^TV](\widehat{y} + \alpha\widetilde{y})$$

is a convex quadratic function in the parameter $\alpha \in R$ with $h(0) = L^*$. The range of $h(\alpha)$ must cover all the values above L^*, particularly the value $\widetilde{\mu} \in [L^*, U^*]$. Then, the quadratic equation

$$h(\alpha) = \widetilde{\mu}$$

has a root at $\alpha^* \in (0, +\infty)$ which generates a solution $y^* = \widehat{y} + \alpha^*\widetilde{y}$ to (4.20).

- Subcase (h3) $L^* = -\infty$, $U^* < +\infty$. Multiplying both sides of the equation (4.20) by -1, we turn to Subcase (h2).

- Subcase (h4) $L^* = -\infty$, $U^* = +\infty$. Notice that, $L^* = -\infty$ implies that either V^TBV has a negative eigenvalue with an eigenvector $\widehat{y}$ or there is a vector $\widehat{y} \in N(V^TBV)$ such that

$$\widehat{y}^T[V^TB(A + \sigma_l^* B)^+(f + \sigma_l^* g) - V^T g] < 0.$$

Also, $U^* = +\infty$ implies that either V^TBV has a positive eigenvalue with an eigenvector $\widetilde{y}$ or there is a vector $\widetilde{y} \in N(V^TBV)$ such that

$$\widetilde{y}^T[V^TB(A + \sigma_l^* B)^+(f + \sigma_l^* g) - V^T g] > 0.$$

Define

$$h_1(\alpha) := \widehat{y}^T V^T BV \widehat{y}\alpha^2 + 2[(f + \sigma_l^* g)^T (A + \sigma_l^* B)^+ BV - g^T V]\widehat{y}\alpha, \quad (4.22)$$
$$h_2(\beta) := \widetilde{y}^T V^T BV \widetilde{y}\beta^2 + 2[(f + \sigma_l^* g)^T (A + \sigma_l^* B)^+ BV - g^T V]\widetilde{y}\beta. \quad (4.23)$$

where $h_1(\alpha)$ is concave quadratic whereas $h_2(\beta)$ convex quadratic. Since $h_1(0) = h_2(0) = 0$, the ranges of $h_1(\alpha)$ and $h_2(\beta)$, while they are taken in union, cover the entire R. Therefore, if $\widetilde{\mu} = 0$, $y^* = 0$ is a solution to (4.20). If $\widetilde{\mu} < 0$, $y^* = \alpha^*\widehat{y}$ with $h_1(\alpha^*) = \widetilde{\mu}$ is a solution to (4.20). If $\widetilde{\mu} > 0$, $y^* = \beta^*\widetilde{y}$ with $h_2(\beta^*) = \widetilde{\mu}$ is the desired solution.

5 (QP1QC) without SDC

Simultaneously diagonalizable via congruence (SDC) of a finite collection of symmetric matrices $A_1, A_2, \ldots, A_m$ is a very interesting property in optimization due to its tight connection with the convexity of the cone $\{(\langle A_1x, x\rangle, \langle A_2x, x\rangle, \ldots, \langle A_mx, x\rangle) | x \in R^n\}$. See [12] for the reference. In [7], Feng et al. concluded that (QP1QC) problems under the SDC condition is either unbounded below or has an attainable optimal solution whereas those having no SDC condition are said to be in the *hard case*. In the following, we construct three examples to illustrate why the hard case is complicate. The examples show that A and B cannot be simultaneously diagonalizable via congruence, while (QP1QC) could be unbounded below; could have an unattainable solution; or attain the optimal value.

Example 5.1. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; f = g = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; \mu = 0.$$

If A and B were simultaneously diagonalizable via congruence, then there would be a nonsingular matrix $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that P^TAP and P^TBP are diagonal matrices. That is,

$$P^TAP = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 - c^2 & ab - cd \\ ab - cd & b^2 - d^2 \end{bmatrix},$$

and

$$P^TBP = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2ac & ad + bc \\ ad + bc & 2bd \end{bmatrix}$$

are diagonal matrices. That is, $ab - cd = 0$, $ad + bc = 0$, and $ad - bc \neq 0$ since P is nonsingular. Since $bc = -ad$, we have $2ad \neq 0$, and hence a, b, c, and d are nonzeros. From $ab - cd = 0$, we have $a = \frac{cd}{b}$; and from $ad + bc = 0$, we have $a = \frac{-bc}{d}$. It implies that $\frac{cd}{b} + \frac{bc}{d} = 0$, which leads to a contradiction that $c(b^2 + d^2) = 0$ and $b = d = 0$. In other words, A and B cannot be simultaneously diagonalizable via congruence.

For these A and B, (QP1QC) becomes

$$\text{(QP1QC)}: \quad \begin{array}{ll} \inf & x_1^2 - x_2^2 \\ \text{s.t.} & 2x_1x_2 \leq 0. \end{array}$$

We can see that, for any $\sigma \geq 0$, $A + \sigma B = \begin{bmatrix} 1 & \sigma \\ \sigma & -1 \end{bmatrix}$. Since $(A + \sigma B)_{2,2} = -1$, $A + \sigma B$ cannot be positive semidefinite for any $\sigma \geq 0$. By Theorem 4.1, (QP1QC) is unbounded below.

Example 5.2. Let

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \ f = g = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \mu = -2.$$

Again, A and B can not be simultaneously diagonalizable via congruence. Suppose in the contrary there exists a nonsingular matrix $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that P^TAP and P^TBP are diagonal matrices:

$$P^TAP = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c^2 & cd \\ cd & d^2 \end{bmatrix},$$

and

$$P^TBP = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -2ac & -ad - bc \\ -ad - bc & -2bd. \end{bmatrix}$$

Then, we have $cd = 0$, $ad + bc = 0$, and $ad - bc \neq 0$ since P is nonsingular. If $c = 0$, then $ad = 0$, which contradicts to $ad - bc \neq 0$. If $d = 0$, then $bc = 0$, again contradicts to $ad - bc \neq 0$. Hence A and B cannot be SDC.

For these A and B, (QP1QC) becomes

$$\text{(QP1QC)}: \quad \begin{array}{ll} \inf & x_2^2 \\ \text{s.t.} & x_1x_2 \geq 1. \end{array}$$

We can easily see that the optimal solution of (QP1QC) is unattainable since x_2 can be asymptotically approaching 0, but can not be 0.

Example 5.3. Let

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ f = g = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \mu = 0.$$

Suppose first that there exists a nonsingular matrix $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that P^TAP and P^TBP are diagonal matrices. That is,

$$P^TAP = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c^2 & cd \\ cd & d^2 \end{bmatrix},$$

and

$$P^T BP = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2ac & ad+bc \\ ad+bc & 2bd \end{bmatrix}$$

are diagonal matrices. It follows that $cd = 0$, $ad + bc = 0$, and $ad - bc \neq 0$ since P is nonsingular. By $bc = -ad$, we have $2ad \neq 0$, and hence a, b, c, and d are nonzeros, which contradicts to $cd = 0$. Thus A and B cannot be simultaneously diagonalizable via congruence. In this example, (QP1QC) becomes

$$\begin{array}{lll} \text{(QP1QC)}: & \inf & x_2^2 \\ & \text{s.t.} & 2x_1x_2 \leq 0 \end{array}$$

which has the optimal solution set $\{(x, 0)|x \in \mathbb{R}\}$.

6 Conclusions

In this paper, we analyzed the positive semi-definite pencil $I_{\succeq}(A, B)$ for solving (QP1QC) without any primal Slater condition, dual Slater condition, or the SDC condition. Given any (QP1QC) problem, we are now able to check whether it is infeasible, or unbounded below, or bounded below but unattainable in polynomial time. If neither of the undesired cases happened, the solution of (QP1QC) can be obtained via different relatively simple subproblems in various subcases. In other words, once a given (QP1QC) problem is properly classified, its solution can be computed readily. Therefore, we believe that our analysis in this paper has the potential to become an efficient algorithm for solving (QP1QC) if carefully implemented.

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