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CONVERGENCE OF AN ALGORITHM FOR THE SPLIT COMMON FIXED-POINT OF ASYMPTOMATIC QUASI-NONEXPANSIVE OPERATORS*

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This paper is dedicated to Professor Shu-Cherng Fang in celebration of his 60th birthday.

Abstract: An iteration method for finding the split common fixed-point of asymptomatic quasi-nonexpansive operators is analyzed. The new iteration generated by the algorithm is a weighted average of three intermediate points. Weak convergence results are established in a uniformly convex Banach space.

Key words: *split common fixed-point problem, asymptomatic quasi-nonexpansive mapping, uniformly convex Banach space, weak convergence*

Mathematics Subject Classification: *37C25, 47H09, 90C25*

1 Introduction and Preliminaries

The fixed point problem is a classical problem in nonlinear analysis. It finds applications in a wide spectrum of fields such as economics, physics, and applied sciences. We are interested in the split common fixed-point problem (SCFP), which is a generalization of the split feasibility problem (SFP), and the latter is in turn a generalization of the convex feasibility problem (CFP), see [1, 2]. The CFP and SCFP have applications in many fields such as approximation theory [9], image reconstruction, radiation therapy [4, 12], and control [10].

Throughout this paper, we assume that E_1, E_2 are real Banach spaces, " $\rightharpoonup$ " denotes weak convergence. $\mathrm{Fix}\,(T)$ is the fixed point set of an operator T, i.e., $\mathrm{Fix}(T) := \{x \mid x = T(x)\}$, and I denotes the identity operator.

Let S be a nonempty closed convex subset of E_1 and $T : S \to S$ be a mapping. T is said to be nonexpansive if

$$\|T(x) - T(y)\| \leq \|x - y\|, \forall x, y \in S.$$

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T is said to be l_k-asymptotically nonexpansive if for all $x, y \in S$,

$$\|T^k(x) - T^k(y)\| \leq l_k \|x - y\|, \forall k \geq 1,$$

where $T^k = T(T(...(T)))$ (k times), $\{l_k\} \subset [1, \infty)$ is a sequence satisfying $l_k \to 1$ as $k \to \infty$.

A mapping T is called quasi-nonexpansive if for all $x \in S, z \in \text{Fix}(T)$

$$\|T(x) - z\| \leq \|x - z\|.$$

A mapping T is called l_k- quasi-asymptotically nonexpansive if for all $x \in S, z \in \text{Fix}(T)$,

$$\|T^k(x) - z\| \leq l_k \|x - z\|, \forall k \geq 1,$$

where $\{l_k\} \subset [1, \infty)$ is a sequence satisfying $l_k \to 1$ as $k \to \infty$.

A mapping T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that for all $x, y \in S$,

$$\|T^k(x) - T^k(y)\| \leq L\|x - y\|, \forall k \geq 1.$$

Clearly, a l_k-asymptotically nonexpansive mapping must be uniformly l_k- Lipschitzian as well as asymptotically quasi-nonexpansive but the converse may not hold.

The SFP in finite dimensional spaces was first introduced by Censor and Elfving [6] for modeling inverse problems. The SFP in an infinite-dimensional Hilbert space can be found in [8, 15, 16, 18]. The SCFP for a class of quasi-nonexpansive mappings in the setting of Hilbert space was first introduced and studied by Moudafi [13]. In this paper we discuss SCFP for asymptomatic quasi-nonexpansive mapping in uniformly convex Banach space. We propose a method for solving SCFP and prove its weak convergence. A notable feature of the method is that it generates an iterative sequence and each new iteration is a weighted average of three intermediate points generated from the last iteration.

The split common fixed point problem for asymptomatic quasi-nonexpansive operators is defined as follows.

$$\text{Find } \ x^* \in C \text{ such } \ \text{that } Ax^* \in Q, \tag{1.1}$$

where $A : E_1 \to E_2$ is a bounded linear operator, $U : E_1 \to E_1$ and $T : E_2 \to E_2$ are two operators with nonempty fixed-point sets $\text{Fix}(U) = C$ and $\text{Fix}(T) = Q$, respectively. We denote the solution set of the two-operator SCFP by

$$\Gamma = \{y \in C \mid Ay \in Q\}. \tag{1.2}$$

To prove the convergence theorem in the next section, we first recall some results. Let E be a Banach space and let S be a nonempty bounded convex subset of E. E is said to have Opial property [14], if for any sequence $\{x^k\}$ in E, $x^k \rightharpoonup x^*$, then

$$\liminf_{k\to\infty} \|x^k - x^*\| < \liminf_{k\to\infty} \|x^k - y\|, \forall y \in E \text{ with } y \neq x^*. \tag{1.3}$$

A mapping $T : S \to E$ is called demi-closed with respect to $y \in E$ if for each sequence $\{x^k\}$ in S and each $x \in E, x^k \rightharpoonup x$ and $T(x^k) \to y$ imply that $x \in S$ and $T(x) = y$. The reader is referred to [7, 11] for a detailed discussion on the notion of demi-closed mappings. In what follows, only the particular case of demi-closedness at zero will be used, which is the particular case when $y = 0$.

Lemma 1.1. *Let $T : S \to S$ be a l_k-quasi asymptotically nonexpansive mapping. Then, for any $z \in \mathrm{Fix}(T)$ and $x \in S$:*

(a). $\langle x - T^k(x), x - z\rangle \geq \frac{1}{2}\|x - T^k(x)\|^2 - \frac{l_k^2-1}{2}\|x - z\|^2$;

(b). $\langle x - T^k(x), z - T^k(x)\rangle \leq \frac{1}{2}\|x - T^k(x)\|^2 + \frac{l_k^2-1}{2}\|x - z\|^2$.

Proof. Part (a). By the classical equality there holds

$$\langle x, y\rangle = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x - y\|^2,$$

and, from the fact that T is l_k-quasi asymptotically nonexpansive mapping, it follows that

$$\begin{aligned}
\langle x - T^k(x), x - z\rangle &= \frac{1}{2}\|x - T^k(x)\|^2 + \|x - z\|^2 - \frac{1}{2}\|T^k(x) - z\|^2 \\
&\geq \frac{1}{2}\|x - T^k(x)\|^2 + \|x - z\|^2 - \frac{l_k^2}{2}\|x - z\|^2 \\
&= \frac{1}{2}\|x - T^k(x)\|^2 - \frac{l_k^2 - 1}{2}\|x - z\|^2.
\end{aligned}$$

Part (b). From Part (a), we have

$$\begin{aligned}
\langle x - T^k(x), z - T^k(x)\rangle &= \langle x - T^k(x), z - x + xT^k(x)\rangle \\
&= \langle x - T^k(x), z - x\rangle + \langle x - T^k(x), x - T^k(x)\rangle \\
&= -\langle x - T^k(x), z - x\rangle + \|x - T^k(x)\|^2 \\
&\leq \frac{1}{2}\|x - T^k(x)\|^2 + \frac{l_k^2 - 1}{2}\|x - z\|^2.
\end{aligned}$$

This completes the proof. □

Lemma 1.2 ([17])**.** *Let E be a uniformly convex Banach space. Let $\{a_k\}, \{b_k\}$ and $\{c_k\}$ be three sequences in $(0, 1)$ satisfying $a_k+b_k+c_k = 1$ and $0 < \liminf_{k\to\infty} a_k < \liminf_{k\to\infty}(a_k+ b_k) \leq \limsup_{k\to\infty}(a_k + b_k) < 1$. Assume that $\{x^k\}, \{y^k\}$ and $\{z^k\}$ are three sequences in E. Then the conditions: $\limsup_{k\to\infty} \|x^k\| \leq d$, $\limsup_{k\to\infty} \|y^k\| \leq d, \limsup_{k\to\infty} \|z^k\| \leq d$, and $\limsup_{k\to\infty} \|a_kx^k+b_ky^k+c_kz^k\| = d$ imply that $\lim_{k\to\infty} \|x^k-y^k\| = \lim_{k\to\infty} \|y^k-z^k\| = \lim_{k\to\infty} \|z^k - x^k\| = 0$, where $d > 0$ is some constant.*

2 The Algorithm and Its Asymptotic Convergence

2.1 The Algorithm

We now describe the algorithm and prove its weak convergence.

Algorithm 2.1.

Initialization: Let $x^0 \in E_1$ be arbitrary.
Iterative step: For $k \in N$, set $u^k = x^k + \gamma\ A^T(T^k - I)A(x^k)$, and let

$$x^{k+1} = a_kx^k + b_ku^k + c_kU^k(u^k), k \in N, \tag{2.1}$$

where $\{a_k\}, \{b_k\}$ and $\{c_k\}$ are three sequences in $(0, 1)$ satisfying $a_k + b_k + c_k = 1$ and $0 < a \leq a_k, b_k, c_k \leq b < 1$, $\gamma \in (0, \frac{1}{\lambda})$ with λ being the spectral radius of the operator A^TA.

To prove its weak convergence we need the following proposition.

Proposition 2.2. *Let E_2 be uniformly convex Banach space and let $T : E_2 \to E_2$ be a l_k-asymptotically quasi- nonexpansive mapping with $\{l_k\} \subset [1, \infty)$ such that $\sum_{k=1}^{\infty}(l_k - 1) < \infty$. Then, u^k is a $\sqrt{(1 + \gamma\lambda(l_k^2 - 1))}$- asymptotically quasi-nonexpansive mapping.*

Proof. By the definition of u^k, we have

$$\begin{aligned}\|u^k - z\|^2 &= \|x^k + \gamma A^T(T^k - I)(Ax^k) - z\|^2 \\ &= \|x^k - z\|^2 + \gamma^2\|A^T(T^k - I)(Ax^k)\|^2 \\ &\quad +2\gamma\langle x^k - z, A^T(T^k - I)(Ax^k)\rangle \\ &\leq \|x^k - z\|^2 + \lambda\gamma^2\|(T^k - I)(Ax^k)\|^2 \\ &\quad +2\gamma\langle Ax^k - Az, (T^k - I)(Ax^k)\rangle,\end{aligned}$$

that is

$$\|u^k - z\|^2 \leq \|x^k - z\|^2 + \lambda\gamma^2\|(T^k - I)(Ax^k)\|^2 + 2\gamma\langle Ax^k - Az, (T^k - I)(Ax^k)\rangle. \quad (2.2)$$

Now, by setting $\theta := 2\gamma\langle Ax^k - Az, (T^k - I)(Ax^k)\rangle$ and using Part (b) of Lemma 1.1, we obtain

$$\begin{aligned}\theta &= 2\gamma\langle Ax^k - Az, (T^k - I)(Ax^k)\rangle \\ &= 2\gamma\langle Ax^k - Az + (T^k - I)(Ax^k) - (T^k - I)(Ax^k), (T^k - I)(Ax^k)\rangle \\ &= 2\gamma(\langle T^k(Ax^k) - Az, (T^k - I)(Ax^k)\rangle - \|(T - I)(Ax^k)\|^2) \\ &\leq 2\gamma(\frac{1}{2}\|(T^k - I)(Ax^k)\|^2 + \frac{l_k^2 - 1}{2}\|Ax^k - Az\|^2 - \|(T^k - I)(Ax^k)\|^2) \\ &= -\gamma\|(T^k - I)(Ax^k)\|^2 + \gamma\lambda(l_k^2 - 1)\|x^k - z\|^2.\end{aligned}$$

Combined with (2.2), it yields

$$\|u^k - z\|^2 \leq (1 + \gamma\lambda(l_k^2 - 1))\|x^k - z\|^2 - \gamma(1 - \lambda\gamma)\|(T^k - I)(Ax^k)\|^2. \quad (2.3)$$

By $\gamma \in (0, \frac{1}{\lambda})$, (2.3) implies that

$$\|u^k - z\| \leq \sqrt{(1 + \gamma\lambda(l_k^2 - 1))}\|x^k - z\|. \quad (2.4)$$

Since $\{l_k\} \subset [1, \infty)$ and $\sum_{k=1}^{\infty}(l_k - 1) < \infty$, it is easy to deduce that $\{\sqrt{(1 + \gamma\lambda(l_k^2 - 1))}\} \subset [1, \infty)$ and $\sqrt{(1 + \gamma\lambda(l_k^2 - 1))} \to 1$. Therefore, u^k is a $\sqrt{(1 + \gamma\lambda(l_k^2 - 1))}$- asymptotically quasi-nonexpansive mapping. □

2.2 Convergence of the Algorithm

In this subsection, we establish the weak convergence of Algorithm 2.1.

We first prove a lemma which is an important part of the proof of weak convergence theorem.

Lemma 2.3. *Given a bounded linear operator $A : E_1 \to E_2$, E_1 and E_2 are two uniformly convex Banach spaces, let $U : E_1 \to E_1$ and $T : E_2 \to E_2$ be two uniformly L-Lipschitzian and l_k-asymptotically quasi-nonexpansive mappings with $\{l_k\} \subset [1, \infty)$ such that $l_k \to 1$ and $\sum_{k\to\infty}(\sqrt{1 + \gamma\lambda(l_k^2 - 1)} - 1) < \infty$, $\mathrm{Fix}(U) = C \neq \emptyset$ and $\mathrm{Fix}(T) = Q \neq \emptyset$. Let $\{x^k\}$ be any sequence generated by Algorithm 2.1. Then $\lim_{k\to\infty}\|x^k - z\|$ exists and $\{\|u^k - z\|\}$ is bounded for each $z \in \Gamma$ provided that $\Gamma \neq \emptyset$.*

Proof. From (2.1), by Proposition 2.2, we obtain

$$\begin{aligned}\|x^{k+1}-z\| &= \|a_k(x^k-z)+b_k(u^k-z)+c_k(U^k(u^k)-z)\| \\ &\le a_k\|x^k-z\|+b_k\|u^k-z\|+(l_kc_k)\|u^k-z\| \\ &= (a_k+(b_k+l_kc_k)\sqrt{(1+\gamma\lambda(l_k^2-1))}\|x^k-z\|.\end{aligned}$$

Setting $v_k = a_k+(b_k+l_kc_k)\sqrt{(1+\gamma\lambda(l_k^2-1)}$, the above inequality takes the shape

$$\|x^{k+1}-z\| \le v_k\|x^k-z\|, \text{ for all } k\in N.$$

By induction, we have

$$\|x^{k+m}-z\| \le (\prod_{i=k}^{k+m-1} v_i)\|x^k-z\| \text{ for all } k,m\in N.$$

Note that

$$v_k-1 = a_k+(b_k+l_kc_k)\sqrt{1+\gamma\lambda(l_k^2-1)}-a_k-b_k-c_k$$

$$=(\sqrt{(1+\gamma\lambda(l_k^2-1)}-1)b_k+(\sqrt{l_k(1+\gamma\lambda(l_k^2-1)}-1)c_k,$$

Since $\sum_{k\to\infty}(\sqrt{1+\gamma\lambda(l_k^2-1)}-1)<\infty$, it is easy to get

$$\sum_{k\to\infty} l_k(\sqrt{(1+\gamma\lambda(l_k^2-1)}-1)<\infty.$$

Hence,

$$\sum_{k=1}^{\infty}(v_k-1)<\infty,$$

furthermore

$$\lim_{k\to\infty}\prod_{i=k}^{\infty} v_i = 1.$$

Therefore, $\lim_{k\to\infty}\|x^k-z\|$ exists. From (2.4), we obtain that $\{u^k-z\}$ is bounded. □

Theorem 2.4. *Given a bounded linear operator $A: E_1\to E_2$, E_1 and E_2 are two uniformly convex Banach spaces, let $U: E_1\to E_1$ and $T: E_2\to E_2$ be two uniformly L-Lipschitzian and l_k-asymptotically quasi- nonexpansive mappings with $\{l_k\}\subset[1,\infty)$ such that $\sum_{k=1}^{\infty}(l_k-1)<\infty$, $\mathrm{Fix}(U)=C\neq\emptyset, \mathrm{Fix}(T)=Q\neq\emptyset$. Assume that $U-I$ and $T-I$ are demi-closed at 0 and $\Gamma\neq\emptyset$. Then, any sequence $\{x^k\}$ generated by Algorithm 2.1 converges weakly to a point $x^*\in\Gamma$.*

Proof. We first prove that $\lim_{k\to\infty}\|U^k(u^k)-u^k\| = \lim_{k\to\infty}\|T^k(Ax^k)-Ax^k\| = 0$. Let $\lim_{k\to\infty}\|x^k-z\| = c$ where $c\ge 0$ is a real number. If $c=0$, the result is obvious. Assume $c>0$,

$$c = \lim_{k\to\infty}\|a_k(x^k-z)+b_k(u^k-z)+c_k(U^k(u^k)-z)\|. \tag{2.5}$$

Since $\|U^k(u^k)-z)\|\le l_k\|u^k-z\|$, therefore

$$\limsup_{k\to\infty}\|U^k(u^k)-z)\|\le c. \tag{2.6}$$

Applying Lemma 1.2 on (2.5) and (2.6), we obtain

$$\lim_{k\to\infty} \|U^k(u^k) - u^k\| = 0, \tag{2.7}$$

$$\lim_{k\to\infty} \|U^k(u^k) - x^k\| = 0 \tag{2.8}$$

and

$$\lim_{k\to\infty} \|u^k - x^k\| = 0. \tag{2.9}$$

Let $M = \max\{\|x^k - z\|, \|u^k - z\|\}$. From (2.3), we have

$$\begin{aligned}
\gamma(1-\lambda\gamma)\|(T^k - I)(Ax^k)\|^2 &\le (1 + \gamma\lambda(l_k^2 - 1))\|x^k - z\|^2 - \|u^k - z\|^2 \\
&= \|x^k - z\|^2 - \|u^k - z\|^2 + \gamma\lambda(l_k^2 - 1))\|x^k - z\|^2 \\
&\le M\|x^k - u^k\| + \gamma\lambda(l_k^2 - 1))\|x^k - z\|^2 \to 0.
\end{aligned}$$

Hence

$$\lim_{k\to\infty} \|T^k(Ax^k) - Ax^k\| = 0. \tag{2.10}$$

Now, we prove that $\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0$ and $\lim_{k\to\infty} \|u^{k+1} - u^k\| = 0$. As a matter of fact, it follows from (2.1) that

$$\begin{aligned}
\|x^{k+1} - x^k\| &= \|a_k x^k + b_k u^k + c_k U(u^k) - (a_k + b_k + c_k)x^k\| \\
&\le b_k\|u^k - x^k\| + c_k\|U(u^k) - x^k\| \\
&\le b\|u^k - x^k\| + c\|U(u^k) - x^k\|.
\end{aligned}$$

In view of (2.8) and (2.9), we have that

$$\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0. \tag{2.11}$$

Similarly, it follows from (2.1), (2.10) and (2.11) that

$$\begin{aligned}
\|u^{k+1} - u^k\| &= \|(x^{k+1} + \gamma\ A^T(T^{k+1} - I)Ax^{k+1}) - (x^k + \gamma\ A^T(T^k - I)Ax^k)\| \\
&\le \|x^{k+1} - x^k\| + \gamma\|A^T(T^{k+1} - I)Ax^{k+1}\| + \gamma\|A^T(T^k - I)Ax^k\| \to 0, (k\to\infty).
\end{aligned} \tag{2.12}$$

Next, we prove that $\lim_{k\to\infty} \|Ax^k - T(Ax^k)\| = 0$ and $\lim_{k\to\infty} \|u^k - U(u^k)\| = 0$. Setting $\eta_k := \|u^k - U^k(u^k)\|$, since U is uniformly-L-Lipschitzian continuous, it follows from (2.4), (2.7) and (2.12) that

$$\begin{aligned}
\|u^k - U(u^k)\| &\le \|u^k - U^k(u^k)\| + \|U^k(u^k) - U(u^k)\| \\
&\le \eta_k + L\|U^{k-1}(u^k) - u^k\| \\
&\le \eta_k + L(\|U^{k-1}(u^k) - U^{k-1}(u^{k-1})\| + \|U^{k-1}(u^{k-1}) - u^k\|) \\
&\le \eta_k + L^2(\|u^k - u^{k-1}\| + L(\|U^{k-1}(u^{k-1}) - u^{k-1}\| + \|u^{k-1} - u^k\|) \\
&\le \eta_k + L(L+1)\|u^k - u^{k-1}\| + L\eta_{k-1} \to 0, (k\to\infty).
\end{aligned} \tag{2.13}$$

Similarly, we have

$$\lim_{k\to\infty} \|Ax^k - T(Ax^k)\| = 0. \tag{2.14}$$

Finally, we prove that $x^k \rightharpoonup x^*$ and $u^k \rightharpoonup x^*$, where $x^* \in \Gamma$. Since $\{u^k\}$ is bounded, let $v = 0, 1, 2\cdots$ be the sequence of indices, such that $w - \lim_v u^{k_v} = x^*$. From (2.13), we have $\lim_{v\to\infty} \|u^{k_v} - U(u^{k_v})\| = 0$. Since U is demi-closed at zero, we know that $x^* \in \mathrm{Fix}(U)$. Moreover, it follows from (2.1) and (2.10) that

$$x^{k_v} = u(x^{k_v}) - \gamma A^T(T^{k_v} - I)Ax^{k_v} \rightharpoonup x^*.$$

Since A is linear bounded operator, it gets $Ax^{k_v} \rightharpoonup Ax^*$. In view of (2.14) we have $\lim_{v\to\infty} \|Ax^{k_v} - T(Ax^{k_v})\| = 0$. Again since T is demi-closed at zero, we know that $Ax^* \in \mathrm{Fix}(T)$. This implies that $x^* \in \Gamma$.

Assume that there exists another subsequence $\{u^{k_w}\}$ of $\{u^k\}$ such that $\{u^{k_w}\}$ converges weakly to a point $y^* \in H$ with $y^* \neq x^*$. Using the same argument above, we know that $y^* \in \Gamma$. Since each uniformly convex Banach space possesses Opial property, we have

$$\begin{aligned}\liminf_{v\to\infty} \|u^{k_v} - x^*\| &< \liminf_{v\to\infty} \|u^{k_v} - y^*\| = \liminf_{k\to\infty} \|u^k - y^*\| \\ &= \liminf_{v\to\infty} \|u^{k_w} - y^*\| < \liminf_{w\to\infty} \|u^{k_w} - x^*\| \\ &= \liminf_{k\to\infty} \|u^k - x^*\| = \liminf_{v\to\infty} \|u^{k_v} - x^*\|,\end{aligned}$$

which is a contradiction. This implies that $\{u^k\}$ converges weakly to the point $x^* \in \Gamma$. Since $x^k = u(x^k) - \gamma A^T(T^k - I)Ax^k$, we know that $\{x^k\}$ converges weakly to $x^* \in \Gamma$. The proof is completed. □

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