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STRONG CONVERGENCE OF PROJECTION METHODS FOR BREGMAN ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS AND EQUILIBRIUM PROBLEMS IN BANACH SPACES

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Abstract: In this paper, using Bregman functions, we prove strong convergence theorems by the hybrid projection method for finding a common element of the set of fixed points of a Bregman asymptotically quasi-nonexpansive mapping and the set of solutions of an equilibrium problem in the framework of Banach spaces. The method is computationally described, and application to the equilibrium problem is demonstrated through an illustrative example. Our results improve and generalize many known results in the current literature.

Key words: *Bregman asymptotically quasi-nonexpansive mapping, Bregman function, uniformly convex function, uniformly smooth function, fixed point, strong convergence*

Mathematics Subject Classification: *47H10, 37C25*

1 Introduction

Throughout this paper, we assume that E is a real Banach space with the norm $\|.\|$ and the dual space E^* and C is a nonempty and closed convex subset of E. Let E be a smooth, strictly convex and reflexive Banach space and J the normalized duality mapping of E. The generalized projection Π_C from E onto C is denoted by

$$\Pi_C(x) = \operatorname{argmin}_{y\in C}\phi(y, x)$$

for all $x \in E$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$. Let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point $p \in C$ is said to be an asymptotic fixed point of T if there exists a sequence $\{x_n\}$ in C which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$. Following Matsushita and Takahashi [8, 9], a mapping $T : C \to C$ is said to be relatively nonexpansive if the following conditions are satisfied:
(1) $F(T)$ is nonempty;

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(2) $\phi(u, Tx) \leq \phi(u, x)$, $\forall u \in F(T),\ x \in C$;
(3) $\hat{F}(T) = F(T)$.

In this paper, we also denote the set of real numbers and the set of positive integers by $\mathbb{R}$ and $\mathbb{N}$, respectively. Let $f : C \times C \to \mathbb{R}$ be a bifunction. Consider the following equilibrium problem: Find $p \in C$ such that

$$f(p, y) \geq 0, \qquad \forall y \in C. \tag{1.1}$$

For solving the equilibrium problem, let us assume that $f : C \times C \to \mathbb{R}$ satisfies the following conditions:
(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, *i.e.*, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $y \in C$, the function $x \longmapsto f(x, y)$ is upper semicontinuous;
(A4) for each $x \in C$, the function $y \longmapsto f(x, y)$ is convex and lower semicontinuous.
The set of solutions of problem (1.1) is denoted by $EP(f)$.

Recently, Takahashi and Zembayashi [19] proved the following strong convergence theorem for relatively nonexpansive mappings in a Banach space.

Theorem 1.1. *Let E be a uniformly smooth and strictly convex Banach space. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and T a relatively nonexpansive mapping from C into itself such that $F(T) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTx_n), \\ u_n \in C \quad \text{such that} \qquad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \qquad \forall y \in C, \\ H_n = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x \end{cases}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the normalized duality mapping on E, $\{\alpha_n\} \subset [0, 1]$ satisfies $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap EP(f)}x$, where $\Pi_{F(T)\cap EP(f)}$ is the generalized projection of E onto $F(T) \cap EP(f)$.

Let $g : E \to \mathbb{R}$ be a convex function. Then the *directional derivative* $d^+g(x)(y)$ of g at $x \in E$ with the direction $y \in E$ is defined by

$$d^+g(x)(y) = \lim_{t\downarrow 0} \frac{g(x + ty) - g(x)}{t}. \tag{1.2}$$

The function g is said to be *Gâteaux differentiable* at x if $\lim_{t\to 0} \frac{g(x+ty)-g(x)}{t}$ exists for any $y \in E$ (see, for example, [2, p. 12] or [7, p. 508]). In this case, we denote $d^+g(x)$ by $\nabla g(x)$. The function $g : E \to \mathbb{R}$ is said to be *Gâteaux differentiable* if it is Gâteaux differentiable everywhere. If $g : E \to \mathbb{R}$ is a Gâteaux differentiable function, then the *Bregman distance* [1, 3] corresponding to g is the function $D : E \times E \to \mathbb{R}$ defined by

$$D(x, y) = g(x) - g(y) - \langle x - y, \nabla g(y)\rangle, \quad \forall x, y \in E. \tag{1.3}$$

It is clear that $D(x, y) \geq 0$ for all $x, y \in E$. In the case when E is a smooth Banach space, setting $g(x) = \|x\|^2$ for all $x \in E$, we have that $\nabla g(x) = 2Jx$ for all $x \in E$ and hence

$$\begin{aligned} D(x,y) &= \|x\|^2 - \|y\|^2 - \langle x-y, \nabla g(y)\rangle \\ &= \|x\|^2 - \|y\|^2 - \langle x-y, 2Jy\rangle \\ &= \|x\|^2 - \|y\|^2 - \langle x, 2Jy\rangle + 2\|y\|^2 \\ &= \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \\ &= \phi(x,y) \end{aligned}$$

for all $x, y \in E$.

In this paper, using Bregman functions, we prove strong convergence theorems by the hybrid projection method for finding a common element of the set of fixed points of a Bregman asymptotically quasi-nonexpansive mapping and the set of solutions of an equilibrium problem in the framework of Banach spaces. The method is computationally described, and application to the equilibrium problem is demonstrated through an illustrative example. Our results improve and generalize many known results in the current literature; see, for example, [8, 9, 4, 6, 11, 12].

2 Preliminaries

For any $x \in E$, we denote the value of $x^* \in E^*$ at x by $\langle x, x^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. For any sequence $\{x_n^*\}$ in E^*, we denote the strong convergence of $\{x_n^*\}$ to $x^* \in E^*$ by $x_n^* \to x^*$ and the weak convergence by $x_n^* \rightharpoonup x^*$. The modulus δ of convexity of E is denoted by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. Let $S = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in S$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called *smooth.* If the limit (2.1) is attained uniformly in $x, y \in S$, then E is called *uniformly smooth.* The Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S$ and $x \ne y$. It is well-known that E is uniformly convex if and only if E^* is uniformly smooth. It is also known that if E is reflexive, then E is strictly convex if and only if E^* is smooth; for more details, see [17].

Let $A : E \to 2^{E^*}$ be a set-valued mapping. We define the domain and range of A by $D(A) = \{x \in E : Ax \ne \emptyset\}$ and $R(A) = \cup_{x\in E} Ax$, respectively. The graph of A is denoted by $G(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}$. The mapping $A \subset E \times E^*$ is said to be *monotone* [16] if $\langle x-y, x^*-y^*\rangle \ge 0$ whenever $(x,x^*), (y,y^*) \in A$. It is also said to be *maximal monotone* [15] if its graph is not contained in the graph of any other monotone operator on E. If $A \subset E \times E^*$ is maximal monotone, then we can show that the set $A^{-1}0 = \{z \in E : 0 \in Az\}$ is closed and convex. A function $f : E \to (-\infty, +\infty]$ is said to be *proper* if the domain $D(f) = \{x \in E : f(x) < \infty\}$ is nonempty. It is also called *lower semicontinuous* if $\{x \in E : f(x) \le r\}$ is closed for all $r \in \mathbb{R}$. We say that f is *upper semicontinuous* if $\{x \in E : f(x) \ge r\}$ is closed for all $r \in \mathbb{R}$. The function f is said to be *convex* if

$$f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha) f(y) \tag{2.2}$$

for all $x, y \in E$ and $\alpha \in (0,1)$. It is also said to be *strictly convex* if the strict inequality holds in (2.2) for all $x, y \in D(f)$ with $x \neq y$ and $\alpha \in (0,1)$. For a proper lower semicontinuous convex function $f : E \to (-\infty, +\infty]$, the *subdifferential* ∂f of f is defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle y - x, x^* \rangle \leq f(y), \quad \forall y \in E\} \tag{2.3}$$

for all $x \in E$. It is well-known that $\partial f \subset E \times E^*$ is maximal monotone [13, 14]. For any proper lower semicontinuous convex function $f : E \to (-\infty, +\infty]$, the *conjugate function* f^* of f is defined by

$$f^*(x^*) = \sup_{x \in E}\{\langle x, x^* \rangle - f(x)\}$$

for all $x^* \in E^*$. It is well-known that $f(x) + f^*(x^*) \geq \langle x, x^* \rangle$ for all $(x, x^*) \in E \times E^*$. It is also known that $(x, x^*) \in \partial f$ is equivalent to

$$f(x) + f^*(x^*) = \langle x, x^* \rangle. \tag{2.4}$$

We also know that if $f : E \to (-\infty, +\infty]$ is a proper lower semicontinuous function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper weak* lower semicontinuous convex function; see [18] for more details on convex analysis. The convex function $g : E \to \mathbb{R}$ is also said to be *Fréchet differentiable* at $x \in E$ (see, for example, [2, p. 13] or [7, p. 508]) if for all $\epsilon > 0$, there exists $\delta > 0$ such that $\|y - x\| \leq \delta$ implies that

$$| g(y) - g(x) - \langle y - x, \nabla g(x) \rangle | \leq \epsilon \|y - x\|.$$

The function $g : E \to \mathbb{R}$ is said to be *Fréchet differentiable* if it is Fréchet differentiable everywhere. It is well-known that if a continuous convex function $g : E \to \mathbb{R}$ is Gâteaux differentiable, then ∇g is norm-to-weak* continuous (see, for example, [2, Proposition 1.1.10]). Also, it is known that if g is Fréchet differentiable, then ∇g is norm-to-norm continuous (see, [7, p. 508]). The mapping ∇g is said to be weakly sequentially continuous if $x_n \rightharpoonup x$ implies that $\nabla g(x_n) \rightharpoonup^* \nabla g(x)$ (for more details, see [2, Theorem 3.2.4] or [7, p. 508]). The function g is said to be *strongly coercive* if

$$\|x_n\| \to \infty \Longrightarrow \frac{g(x_n)}{\|x_n\|} \to \infty.$$

It is also said to be *bounded on bounded sets* if $g(U)$ is bounded for each bounded subset U of E.

The following definition is slightly different from that in Butnariu and Iusem [2].

Definition 2.1 ([7])**.** The function $g : E \to \mathbb{R}$ is said to be a Bregman function if the following conditions are satisfied:

(1) g is continuous, strictly convex and Gâteaux differentiable;

(2) the set $\{y \in E : D(x, y) \leq r\}$ is bounded for all $x \in E$ and $r > 0$.

The following lemma follows from Butnariu and Iusem [2] and Zălinscu [20].

Lemma 2.2. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a strongly coercive Bregman function. Then*

(1) $\nabla g : E \to E^*$ *is one-to-one, onto and norm-to-weak* continuous;*

(2) $\langle x - y, \nabla g(x) - \nabla g(y) \rangle = 0$ *if and only if* $x = y$;

(3) $\{x \in E : D(x,y) \leq r\}$ *is bounded for all* $y \in E$ *and* $r > 0$;

(4) $D(g^*) = E^*$, g^* *is Gâteaux differentiable and* $\nabla g^* = (\nabla g)^{-1}$.

Let E be a reflexive Banach space, $g : E \to \mathbb{R}$ a strongly coercive Bregman function and $D : E \times E \to \mathbb{R}$ the Bregman distance corresponding to g. Then, $g^* : E^* \to \mathbb{R}$ is convex and Gâteaux differentiable [20]. Let $D_* : E^* \times E^* \to \mathbb{R}$ be the function defined by

$$D_*(x^*, y^*) = g^*(x^*) - g^*(y^*) - \langle \nabla g^*(y^*), x^* - y^* \rangle \tag{2.5}$$

for $x^*, y^* \in E^*$, where ∇g^* is the directional derivative of g^*. It follows from (2.2)-(2.5) and Lemma 2.2 (4) that

$$\begin{aligned} D_*(\nabla g(x), \nabla g(y)) &= g^*(\nabla g(x)) - g^*(\nabla g(y)) - \langle \nabla g^*(\nabla g(y)), \nabla g(x) - \nabla g(y) \rangle \\ &= g^*(\nabla g(x)) - g^*(\nabla g(y)) - \langle y, \nabla g(x) - \nabla g(y) \rangle \\ &= [\langle x, \nabla g(x) \rangle - g(x)] - [\langle y, \nabla g(y) \rangle - g(y)] - \langle y, \nabla g(x) - \nabla g(y) \rangle \\ &= \langle x, \nabla g(x) \rangle - g(x) - \langle y, \nabla g(y) \rangle + g(y) - \langle y, \nabla g(x) \rangle + \langle y, \nabla g(y) \rangle \\ &= g(y) - g(x) - \langle y - x, \nabla g(x) \rangle \\ &= D(y, x) \end{aligned} \tag{2.6}$$

for all $x, y \in E$. On the other hand, we know from [10] that for $x \in E$ and $x_0 \in C$, $D(x_0, x) = \min_{y \in C} D(y, x)$ if and only if

$$\langle y - x_0, \nabla g(x) - \nabla g(x_0) \rangle \leq 0, \qquad \forall y \in C. \tag{2.7}$$

Furthermore, there exists a unique $x_0 \in C$ such that

$$D(x_0, x) = \min_{y \in C} D(y, x).$$

The *Bregman projection* P_C from E onto C is defined by $P_C(x) = x_0$ for all $x \in E$. It is also well-known that P_C has the following property:

$$D(y, P_C x) + D(P_C x, x) \leq D(y, x) \tag{2.8}$$

for all $y \in C$ and $x \in E$ (see [2] for more details). Let B be the unit ball of the Banach space E. Let $rB := \{z \in E : \|z\| \leq r\}$ for all $r > 0$. Then a function $g : E \to \mathbb{R}$ is said to be *uniformly convex on bounded sets* ([20, pp. 203, 221]) if $\rho_r(t) > 0$ for all $r, t > 0$, where $\rho_r : [0, +\infty) \to [0, \infty]$ is defined by

$$\rho_r(t) = \inf_{x, y \in rB, \|x - y\| = t, \alpha \in (0,1)} \frac{\alpha g(x) + (1-\alpha) g(y) - g(\alpha x + (1-\alpha) y)}{\alpha(1-\alpha)}$$

for all $t \geq 0$. The function ρ_r is called the gauge of uniform convexity of g. The function g is also said to be *uniformly smooth on bounded sets* ([20, pp. 207, 221]) if $\lim_{t \downarrow 0} \frac{\sigma_r(t)}{t} = 0$ for all $r > 0$, where $\sigma_r : [0, +\infty) \to [0, \infty]$ is defined by

$$\sigma_r(t) = \sup_{x \in rB, y \in S, \alpha \in (0,1)} \frac{\alpha g(x + (1-\alpha) t y) + (1-\alpha) g(x - \alpha t y) - g(x)}{\alpha(1-\alpha)}$$

for all $t \geq 0$.

Remark 2.3. Let $r > 0$ be a constant and $g : E \to \mathbb{R}$ a convex function which is uniformly convex on bounded sets. Then

$$g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y) - \alpha(1-\alpha)\rho_r(\|x-y\|)$$

for all $x, y \in rB$ and $\alpha \in (0,1)$, where ρ_r is the gauge of uniform convexity of g.

We know the following two results; see [20, Proposition 3.6.4].

Theorem 2.4. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a convex function which is bounded on bounded sets. Then the following assertions are equivalent:*
(1) *g is strongly coercive and uniformly convex on bounded sets;*
(2) *$D(g^*) = E^*$, g^* is bounded on bounded sets and uniformly smooth on bounded sets;*
(3) *$D(g^*) = E^*$, g^* is Fréchet differentiable and ∇g^* is uniformly norm-to-norm continuous on bounded sets.*

Theorem 2.5. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a continuous convex function which is strongly coercive. Then the following assertions are equivalent:*
(1) *g is bounded on bounded sets and uniformly smooth on bounded sets;*
(2) *g^* is Fréchet differentiable and ∇g^* is uniformly norm-to-norm continuous on bounded sets;*
(3) *$D(g^*) = E^*$, g^* is strongly coercive and uniformly convex on bounded sets.*

The following two lemmas have been proved in [7].

Lemma 2.6. *Let $g : E \to \mathbb{R}$ be a Gâteaux differentiable function which is uniformly convex on bounded sets. If $\{x_n\}$ and $\{y_n\}$ are bounded sequences in E such that $\lim_{n\to\infty} D(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.7. *Let E be a reflexive Banach space, $g : E \to \mathbb{R}$ a strongly coercive Bregman function and V the function defined by*

$$V(x, x^*) = g(x) - \langle x, x^* \rangle + g^*(x^*)$$

for all $x \in E$ and $x^* \in E^*$. Then

$$D(x, \nabla g^*(x^*)) = V(x, x^*).$$

The following result has been proved in [20].

Theorem 2.8. *Let $g : E \to (-\infty, +\infty]$ be a function. Then the following assertions are equivalent:*
(1) *g is convex and lower semicontinuous;*
(2) *g is convex and weakly lower semicontinuous;*
(3) *$epi(g)$ is convex and closed;*
(4) *$epi(g)$ is convex and weakly closed,*
where $epi(g) = \{(x,t) \in E \times \mathbb{R} : g(x) \leq t\}$ denotes the epigraph of g.

3 Main results

Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a function. We say that g is *strongly admissible*, if it is convex, continuous, strongly coercive and Gâteaux differentiable. Then the Bregman distance [1, 3] satisfies that

$$D(x, z) = D(x, y) + D(y, z) + \langle x - y, \nabla g(y) - \nabla g(z) \rangle, \quad \forall x, y, z \in E. \tag{3.1}$$

A mapping $T : C \to C$ is said to be *closed*, if for any sequence $\{x_n\} \subset C$ with $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then we have $Tx_0 = y_0$.
A mapping $T : C \to C$ is said to be *Bregman asymptotically quasi-nonexpansive*, if $F(T) \neq \emptyset$ and there exists a real sequence $\{k_n\}_{n=1}^{\infty} \subset [1, \infty)$ with $k_n \to 1$ as $k \to \infty$ such that

$$D(p, T^n x) \leq k_n D(p, x), \quad \forall x \in C,\ p \in F(T),\ n \in \mathbb{N}.$$

A mapping $T : C \to C$ is said to be *Bregman quasi-nonexpansive*, if $F(T) \neq \emptyset$ and

$$D(p, Tx) \leq D(p, x), \quad \forall x \in C,\ p \in F(T).$$

A mapping $T : C \to C$ is said to be *Bregman relatively nonexpansive* if the following conditions are satisfied:
(1) $F(T)$ is nonempty;
(2) $D(p, Tv) \leq D(p, v)$, $\forall p \in F(T),\ v \in C$;
(3) $\hat{F}(T) = F(T)$,
where $\hat{F}(T)$ is the set of asymptotic fixed points of T.
A mapping $T : C \to C$ is said to be λ*-uniformly continuous* if there exists a positive real number λ such that

$$\|T^n x - T^n y\| \leq \lambda \|x - y\|, \quad \forall x, y \in C,\ n \in \mathbb{N}.$$

In this section, we prove the strong convergence theorems for Bregman asymptotically quasi-nonexpansive mappings in reflexive Banach spaces. We first prove that the set of fixed points of a closed Bregman asymptotically quasi-nonexpansive mapping is closed and convex.

Lemma 3.1. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a strongly admissible function which is bounded on bounded sets and uniformly convex on bounded sets. Let $T : C \to C$ be a closed Bregman asymptotically quasi-nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$. Then $F(T)$ is closed and convex.*

Proof. We first show that $F(T)$ is closed. Let $\{p_n\}$ be a sequence in $F(T)$ such that $p_n \to p$ as $n \to \infty$. Then we have $\{p_n\}$ is a bounded sequence in E. We claim that $p \in F(T)$. Since g is continuous, we conclude that $g(p_n) \to g(p)$ as $n \to \infty$. This implies that

$$\begin{aligned} D(p_n, p) &= g(p_n) - g(p) - \langle p_n - p, \nabla g(p) \rangle \\ &\leq |g(p_n) - g(p)| + \|p_n - p\| \|\nabla g(p)\| \to 0, \quad (n \to \infty). \end{aligned}$$

In view of the definition of T, we obtain

$$D(p_n, Tp) \leq k_1 D(p_n, p) \to 0, \quad (n \to \infty).$$

This implies that

$$0 = \lim_{n\to\infty} D(p_n, Tp) = \lim_{n\to\infty} [g(p_n) - g(Tp) - \langle p_n - Tp, \nabla g(Tp) \rangle] = D(p, Tp).$$

It follows from Lemma 2.6 that $Tp = p$. Thus we have $p \in F(T)$.
Let us show that $F(T)$ is convex. For any $p, q \in F(T),\ t \in (0, 1)$, we set $x = tp + (1 - t)q$.

We prove that $x \in F(T)$. By the definition of Bregman distance, we get

$$
\begin{aligned}
D(x, T^n x) &= g(x) - g(T^n x) - \langle x - T^n x, \nabla g(T^n x)\rangle \\
&= g(x) - g(T^n x) - \langle tp + (1-t)q - T^n x, \nabla g(T^n x)\rangle \\
&= g(x) - g(T^n x) - t\langle p - T^n x, \nabla g(T^n x)\rangle - (1-t)\langle q - T^n x, \nabla g(T^n x)\rangle \\
&\quad + tg(p) + (1-t)g(q) - [tg(p) + (1-t)g(q)] \\
&= g(x) + t[g(p) - g(T^n x) - \langle p - T^n x, \nabla g(T^n x)\rangle] \\
&\quad + (1-t)[g(q) - g(T^n x) - \langle q - T^n x, \nabla g(T^n x)\rangle] \\
&= g(x) + tD(p, T^n x) + (1-t)D(q, T^n x) - [tg(p) + (1-t)g(q)] \\
&\leq g(x) + tk_n D(p, x) + (1-t)k_n D(q, x) - tg(p) - (1-t)g(q) \\
&= g(x) + k_n[t(g(p) - g(x) - \langle p - x, \nabla g(x)\rangle) \\
&\quad + (1-t)(g(q) - g(x) - \langle q - x, \nabla g(x)\rangle)] - tg(p) - (1-t)g(q) \\
&= g(x) + k_n[-g(x) - \langle t(p - x), \nabla g(x)\rangle - \langle (1-t)(q - x), \nabla g(x)\rangle] \\
&\quad + k_n[tg(p) + (1-t)g(q)] - tg(p) - (1-t)g(q) \\
&= g(x) - k_n g(x) - k_n[\langle tp + (1-t)q - x, \nabla g(x)\rangle] \\
&\quad + k_n[tg(p) + (1-t)g(q)] - tg(p) - (1-t)g(q) \\
&= (k_n - 1)(-g(x)) + (k_n - 1)tg(p) + (k_n - 1)(1-t)g(q) \\
&= (k_n - 1)[-g(x) + tg(p) + (1-t)g(q)].
\end{aligned}
$$

This implies that $\lim_{n\to\infty} D(x, T^n x) = 0$. Thus for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$D(x, T^n x) \leq \epsilon, \quad \forall n \geq n_0.$$

This means that the sequence $\{D(x, T^n x)\}$ is bounded. In view of Definition 2.1, we conclude that the sequence $\{T^n x\}$ is bounded. Then, by Lemma 2.6, we obtain $\lim_{n\to\infty} \|x - T^n x\| = 0$. Thus we have $T^{n+1}x \to x$, that is $T(T^n x) \to x$. Since T is closed, we deduce that $Tx = x$, which completes the proof. □

In order to get the strong convergence theorems for Bregman asymptotically quasi-nonexpansive mappings in reflexive Banach spaces, we need to prove the following key Lemma. Using ideas in [19], we can prove the following result.

Lemma 3.2. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a strongly admissible function which is bounded on bounded sets and uniformly convex on bounded sets. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). For $r > 0$ and $x \in E$, we define a mapping $T_r : E \to C$ as follows:*

$$T_r(x) = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, \nabla g(z) - \nabla g(x)\rangle \geq 0 \quad for \ all \ y \in C\}$$

for all $x \in E$. Then, the following statements hold:
(1) T_r is single-valued;
(2) T_r is a Bregman firmly nonexpansive-type mapping [5], i.e., for all $x,\ y \in E$,

$$\langle T_r x - T_r y, \nabla g(T_r x) - \nabla g(T_r y)\rangle \leq \langle T_r x - T_r y, \nabla g(x) - \nabla g(y)\rangle;$$

(3) $F(T_r) = EP(f)$;
(4) $EP(f)$ is closed and convex;
(5) T_r is a Bregman asymptotically quasi-nonexpansive mapping;
(6) T_r is a closed mapping;
(7) $D(q, T_r x) + D(T_r x, x) \leq D(q, x),\ \forall q \in F(T_r)$.

Proof. Using ideas in [19, Lemma 2.7], we have that $T_r(x) \neq \emptyset$ for all $x \in E$. We claim that T_r is single-valued. For any $x \in E$ and $r > 0$, let $z_1, z_2 \in T_r x$. Then, we have

$$f(z_1, z_2) + \frac{1}{r}\langle z_2 - z_1, \nabla g(z_1) - \nabla g(x)\rangle \geq 0$$

and

$$f(z_2, z_1) + \frac{1}{r}\langle z_1 - z_2, \nabla g(z_2) - \nabla g(x)\rangle \geq 0.$$

Adding the two inequalities, we obtain

$$\langle z_2 - z_1, \nabla g(z_1) - \nabla g(z_2)\rangle \geq 0.$$

Since ∇g is strictly monotone, we conclude that $z_1 = z_2$.

Next, we show that T_r is a Bregman firmly nonexpansive-type mapping. Let $x, y \in E$ be arbitrary. Then, we have

$$f(T_r x, T_r y) + \frac{1}{r}\langle T_r y - T_r x, \nabla g(T_r x) - \nabla g(x)\rangle \geq 0$$

and

$$f(T_r y, T_r x) + \frac{1}{r}\langle T_r x - T_r y, \nabla g(T_r y) - \nabla g(y)\rangle \geq 0.$$

Adding the two inequalities, we obtain

$$f(T_r x, T_r y) + f(T_r y, T_r x) + \frac{1}{r}\langle T_r y - T_r x, \nabla g(T_r x) - \nabla g(T_r y) - \nabla g(x) + \nabla g(y)\rangle \geq 0.$$

In view of (A2) and $r > 0$, we get

$$\langle T_r y - T_r x, \nabla g(T_r x) - \nabla g(T_r y) - \nabla g(x) + \nabla g(y)\rangle \geq 0.$$

Thus, we have

$$\langle T_r x - T_r y, \nabla g(T_r x) - \nabla g(T_r y)\rangle \leq \langle T_r x - T_r y, \nabla g(x) - \nabla g(y)\rangle.$$

We prove that $F(T_r) = EP(f)$. It is obvious that

$$\begin{aligned} u \in F(T_r) &\iff u = T_r u \\ &\iff f(u, y) + \tfrac{1}{r}\langle y - u, \nabla g(u) - \nabla g(u)\rangle \geq 0, \quad \forall y \in C \\ &\iff f(u, y) \geq 0, \quad \forall y \in C \\ &\iff u \in EP(f). \end{aligned}$$

Now, we show that $EP(f)$ is closed and convex. In view of (3), we have $EP(f) = F(T_r)$. For any $x,\ y \in E$, it follows from (2) that

$$\langle T_r x - T_r y, \nabla g(T_r x) - \nabla g(T_r y)\rangle \leq \langle T_r x - T_r y, \nabla g(x) - \nabla g(y)\rangle.$$

Further, we have

$$\begin{aligned} D(T_r x, T_r y) + D(T_r y, T_r x) &= g(T_r x) - g(T_r y) - \langle T_r x - T_r y, \nabla g(T_r y)\rangle \\ &\quad + g(T_r y) - g(T_r x) - \langle T_r y - T_r x, \nabla g(T_r x)\rangle \\ &= \langle T_r x - T_r y, \nabla g(T_r x) - \nabla g(T_r y)\rangle, \end{aligned}$$

and

$$\begin{aligned}
&D(T_rx,y)+D(T_ry,x)-D(T_rx,x)-D(T_ry,y)\\
&=g(T_rx)-g(y)-\langle T_rx-y,\nabla g(y)\rangle+g(T_ry)-g(x)-\langle T_ry-x,\nabla g(x)\rangle\\
&\quad-[g(T_rx)-g(x)-\langle T_rx-x,\nabla g(x)\rangle]-[g(T_ry)-g(y)-\langle T_ry-y,\nabla g(y)\rangle]\\
&=\langle T_rx,\nabla g(x)-\nabla g(y)\rangle-\langle T_ry,\nabla g(x)-\nabla g(y)\rangle\\
&=\langle T_rx-T_ry,\nabla g(x)-\nabla g(y)\rangle.
\end{aligned}$$

Thus we have

$$D(T_rx,T_ry)+D(T_ry,T_rx)\le D(T_rx,y)+D(T_ry,x)-D(T_rx,x)-D(T_ry,y). \tag{3.2}$$

In view of (3.2), for all $x,\ y\in E$, we obtain

$$D(T_rx,T_ry)+D(T_ry,T_rx)\le D(T_rx,y)+D(T_ry,x).$$

Letting $y=u\in F(T_r)$ in the above inequality, we deduce that

$$D(u,T_rx)\le D(u,x).$$

This shows that T_r is a Bregman asymptotically quasi-nonexpansive mapping.
Next, we prove that T_r is a closed mapping. Let the sequences $\{x_n\}\subset E$ and $\{z_n\}\subset C$ be such that $x_n\to x$ and $z_n\to z$ with $z_n=T_rx_n$. It is enough to show that $z=T_rx$. Since $z_n=T_rx_n$, we have that for any $y\in C$,

$$0\le f(z_n,y)+\frac{1}{r}\langle y-z_n,\nabla g(z_n)-\nabla g(x_n)\rangle.$$

In view of (A3), we conclude that

$$0\le f(z,y)+\frac{1}{r}\langle y-z,\nabla g(z)-\nabla g(x)\rangle.$$

Thus we have $z=T_rx$ and hence T_r is a closed mapping. Employing Lemma 3.1, we conclude that $F(T_r)=EP(f)$ is closed and convex.
Finally, we have from (2) and (3.2) that

$$D(T_rx,T_ry)+D(T_ry,T_rx)\le D(T_rx,y)+D(T_ry,x)-D(T_rx,x)-D(T_ry,y).$$

Letting $y=q\in F(T_r)$ in the above inequality, we obtain

$$D(q,T_rx)+D(T_rx,x)\le D(q,x).$$

This completes the proof. □

Now, we are ready to prove the strong convergence theorems for Bregman asymptotically quasi-nonexpansive mappings in a reflexive Banach space.

Theorem 3.3. *Let E be a reflexive Banach space and $g:E\to\mathbb{R}$ a strongly coercive Bregman function which is bounded on bounded sets, and uniformly convex and uniformly smooth on bounded sets. Let f be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $T:C\to C$ be a closed Bregman asymptotically quasi-nonexpansive mapping with a sequence $\{k_n\}\subset[1,\infty)$, $k_n\to 1$ which is λ-uniformly continuous for some $\lambda>0$. Suppose that*

$F := F(T) \cap EP(f)$ is a nonempty and bounded subset of C, where $EP(f)$ is the set of solutions to the equilibrium problem (1.1). Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ C_0 = C, \\ y_n = \nabla g^*[\alpha_n \nabla g(x_n) + (1-\alpha_n)\nabla g(T^n x_n)], \\ u_n \in C \quad \text{such that} \qquad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, \nabla g(u_n) - \nabla g(y_n)\rangle \geq 0, \qquad \forall y \in C, \\ C_{n+1} = \{z \in C_n : D(z, u_n) \leq D(z, x_n) + (k_n - 1)\theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x \text{ and } n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{3.3}$$

where ∇g is the directional derivative of g and $\theta_n = \sup\{D(z, x_n) : z \in F\}$ for all $n \in \mathbb{N} \cup \{0\}$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then the sequence $\{x_n\}$ defined in (3.3) converges strongly to $P_F x$, where P_F is the Bregman projection from E onto F.

Proof. We divide the proof into several steps.

Step 1. We show that C_n is closed and convex for each $n \in \mathbb{N} \cup \{0\}$.
It is clear that $C_0 = C$ is closed and convex. Let C_m be closed and convex for some $m \in \mathbb{N}$. From the definition of C_{m+1}, we have that

$$D(z, u_m) \leq D(z, x_m) + (k_m - 1)\theta_m$$

which is equivalent to

$$\langle z, \nabla g(x_m) - \nabla g(u_m)\rangle \leq g(u_m) - g(x_m) + \langle x_m, \nabla g(x_m)\rangle - \langle u_m, \nabla g(u_m)\rangle + (k_m - 1)\theta_m.$$

An easy argument shows that C_{m+1} is closed and convex. Thus we have C_n is closed and convex for each $n \in \mathbb{N} \cup \{0\}$.

Step 2. We claim that $F \subset C_n$ for all $n \geq 0$.
It is obvious that $F \subset C_0 = C$. Assume now that $F \subset C_m$ for some $m \in \mathbb{N}$. Employing Lemma 2.7, for any $w \in F \subset C_m$, we obtain

$$\begin{aligned} D(w, u_m) &= D(w, T_{r_m} y_m) \\ &\leq D(w, y_m) \\ &= D(w, \nabla g^*[\alpha_m \nabla g(x_m) + (1-\alpha_m)\nabla g(T^m x_m)]) \\ &= V(w, \alpha_m \nabla g(x_m) + (1-\alpha_m)\nabla g(T^m x_m)) \\ &\leq \alpha_m V(w, \nabla g(x_m)) + (1-\alpha_m) V(w, \nabla g(T^m x_m)) \\ &= \alpha_m D(w, x_m) + (1-\alpha_m) D(w, T^m x_m) \\ &\leq \alpha_m D(w, x_m) + (1-\alpha_m) k_m D(w, x_m) \\ &\leq \alpha_m D(w, x_m) + k_m D(w, x_m) - \alpha_m D(w, x_m) \\ &= k_m D(w, x_m) \\ &\leq D(w, x_m) + (k_m - 1)\theta_m. \end{aligned} \tag{3.4}$$

This shows that $w \in C_{m+1}$. Thus, we have $F \subset C_n$ for each $n \in \mathbb{N} \cup \{0\}$.

Step 3. We prove that $\{x_n\}$ and $\{T^n x_n\}$ are bounded sequences in C.
In view of (2.8), we conclude that

$$D(x_n, x) = D(P_{C_n} x, x) \leq D(w, x) - D(w, x_n) \leq D(w, x)$$

for all $w \in F \subset C_n$ and for each $n \in \mathbb{N} \cup \{0\}$. This implies that the sequence $\{D(x_n, x)\}$ is bounded and hence there exists $M > 0$ such that

$$D(x_n, x) \leq M, \qquad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.5}$$

We claim that the sequence $\{x_n\}$ is bounded. Assume on the contrary that $\|x_n\| \to \infty$ as $n \to \infty$. From the definition of Bregman distance and (3.5), it follows that

$$\begin{aligned} M \geq D(x_n, x) &= g(x_n) - g(x) - \langle x_n - x, \nabla g(x)\rangle \\ &= g(x_n) - g(x) - \langle x_n, \nabla g(x)\rangle + \langle x, \nabla g(x)\rangle \\ &\geq g(x_n) - g(x) - \|x_n\|\|\nabla g(x)\| + \langle x, \nabla g(x)\rangle, \quad \forall n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{3.6}$$

Without loss of generality, we may assume that $\|x_n\| \neq 0$ for each $n \in \mathbb{N}$. This, together with (3.6), implies that

$$\tfrac{M}{\|x_n\|} \geq \tfrac{g(x_n)}{\|x_n\|} - \tfrac{g(x)}{\|x_n\|} - \|\nabla g(x)\| + \tfrac{\langle x, \nabla g(x)\rangle}{\|x_n\|}, \qquad \forall n \in \mathbb{N}. \tag{3.7}$$

Since g is strongly coercive, by letting $n \to \infty$ in (3.7), we conclude that $0 \geq \infty$, which is a contradiction. Therefore, $\{x_n\}$ is bounded. Since T is a Bregman asymptotically quasi-nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$, we have for any $q \in F(T)$ that

$$D(q, T^n x_m) \leq k_1 D(q, x_m), \quad \forall n, m \in \mathbb{N}.$$

This, together with Definition 2.1 and the boundedness of $\{x_n\}$, implies that $\{T^n x_n\}$ is bounded.

Step 4. We show that $x_n \to p$ for some $p \in F(T) \cap EP(f)$, where $p = P_{F(T)\cap EP(f)}x$. By the Step 3, we have that $\{x_n\}$ is bounded. Since E is reflexive, without loss of generality, we may assume that $x_n \rightharpoonup p$ for some $p \in E$. Since C_n is closed and convex, we conclude that $p \in C_n$ for each $n \in \mathbb{N}$. Then, we have

$$D(x_n, x) \leq D(p, x), \qquad \forall n \in \mathbb{N}. \tag{3.8}$$

Since g is continuous, by Theorem 2.8 we conclude that the Bregman distance is weakly lower-semicontinuous in the first argument. This, together with (3.8) implies, that

$$\begin{aligned} D(p, x) &= g(p) - g(x) - \langle p - x, \nabla g(x)\rangle \\ &\leq \liminf_{n\to\infty}[g(x_n) - g(x) - \langle x_n - x, \nabla g(x)\rangle] \\ &\leq \limsup_{n\to\infty}[g(x_n) - g(x) - \langle x_n - x, \nabla g(x)\rangle] \\ &= \limsup_{n\to\infty} D(x_n, x) \\ &\leq D(p, x). \end{aligned}$$

Thus, we have

$$\lim_{n\to\infty} D(x_n, x) = D(p, x). \tag{3.9}$$

In view of (3.9), we obtain

$$\begin{aligned} \lim_{n\to\infty}[g(x_n) - g(p) - \langle x_n - p, \nabla g(x)\rangle] &= \lim_{n\to\infty}[g(x_n) - g(x) - \langle x_n - x, \nabla g(x)\rangle \\ &\quad - g(p) + g(x) + \langle p - x, \nabla g(x)\rangle] \\ &= \lim_{n\to\infty}[D(x_n, x) - D(p, x)] \\ &= 0. \end{aligned} \tag{3.10}$$

Since $x_n \rightharpoonup p$, it follows from (3.10) that

$$\begin{aligned} \lim_{n\to\infty} D(x_n, p) &= \lim_{n\to\infty}[g(x_n) - g(p) - \langle x_n - p, \nabla g(p)\rangle] \\ &= \lim_{n\to\infty}[g(x_n) - g(p) - \langle x_n - p, \nabla g(x)\rangle \\ &\quad - \langle x_n - p, \nabla g(p) - \nabla g(x)\rangle] \\ &= 0. \end{aligned} \tag{3.11}$$

On the other hand, since $\{x_n\}$ is bounded, we have from Lemma 2.6 and (3.11) that

$$\lim_{n\to\infty} \|x_n - p\| = 0. \tag{3.12}$$

Now, we show that $p \in F(T)$. By the construction of C_n, we have that $C_{n+1} \subset C_n$ and $x_{n+1} = P_{C_{n+1}} x \in C_n$. It follows that

$$\begin{aligned} D(x_{n+1}, x_n) &= D(x_{n+1}, P_{C_n} x) \\ &\le D(x_{n+1}, x) - D(P_{C_n} x, x) \\ &= D(x_{n+1}, x) - D(x_n, x). \end{aligned} \tag{3.13}$$

In view of (3.13), we obtain

$$\lim_{n\to\infty} D(x_{n+1}, x_n) = 0. \tag{3.14}$$

Since $x_{n+1} \in C_{n+1}$, we conclude that

$$D(x_{n+1}, u_n) \le D(x_{n+1}, x_n) + (k_n - 1)\theta_n.$$

This, together with (3.14), implies that

$$\lim_{n\to\infty} D(x_{n+1}, u_n) = 0. \tag{3.15}$$

Employing Lemma 2.6 and (3.14)-(3.15), we deduce that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \tag{3.16}$$

In view of (3.12) and (3.16), we get

$$\lim_{n\to\infty} \|u_n - p\| = 0. \tag{3.17}$$

Thus, we have $\{u_n\}$ is a bounded sequence.
From (3.16), it follows that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0.$$

Since ∇g is uniformly norm-to-norm continuous on any bounded subset of E, we obtain

$$\lim_{n\to\infty} \|\nabla g(x_n) - \nabla g(u_n)\| = 0. \tag{3.18}$$

The function g is bounded on bounded subsets of E and therefore ∇g is also bounded on bounded subsets of E^* (see, for example, [2, Proposition 1.1.11] for more details). This, together with Step 3, implies that the sequences $\{\nabla g(x_n)\}$, $\{\nabla g(u_n)\}$ and $\{\nabla g(T^n x_n)\}$ are bounded in E^*.
Let $s = \sup\{\|\nabla g(x_n)\|, \|\nabla g(T^n x_n)\| : n \in \mathbb{N}\}$ and let $\rho_s^* : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function g^*. We prove that for any $w \in F(T)$

$$D(w, y_n) \le D(w, x_n) + (k_n - 1)\theta_n - \alpha_n(1 - \alpha_n)\rho_s^*(\|\nabla g(x_n) - \nabla g(T^n x_n)\|). \tag{3.19}$$

Let us show (3.19). For any given $w \in F(T)$, in view of the definition of Bregman distance, Remark 2.3 and (2.4), we obtain

$$
\begin{aligned}
D(w,y_n) &= g(w) - g(y_n) - \langle w - y_n, \nabla g(y_n)\rangle \\
&= g(w) + g^*(y_n) - \langle y_n, \nabla g(y_n)\rangle - \langle w, \nabla g(y_n)\rangle + \langle y_n, \nabla g(y_n)\rangle \\
&= g(w) + g^*(\alpha_n \nabla g(x_n) + (1-\alpha_n)\nabla g(T^n x_n)) \\
&\quad - \langle w, \alpha_n \nabla g(x_n) + (1-\alpha_n)\nabla g(T^n x_n))\rangle \\
&\le \alpha_n g(w) + (1-\alpha_n) g(w) + \alpha_n g^*(\nabla g(x_n)) + (1-\alpha_n) g^*(\nabla g(T^n x_n)) \\
&\quad - \alpha_n(1-\alpha_n)\rho_s^*(\|\nabla g(x_n) - \nabla g(T^n x_n)\|) \\
&\quad - \alpha_n \langle w, \nabla g(x_n)\rangle - (1-\alpha_n)\langle w, \nabla g(T^n x_n)\rangle \\
&= \alpha_n[g(w) + g^*(\nabla g(x_n)) - \langle w, \nabla g(x_n)\rangle] \\
&\quad + (1-\alpha_n)[g(w) + g^*(\nabla g(T^n x_n)) - \langle w, \nabla g(T^n x_n)\rangle] \\
&\quad - \alpha_n(1-\alpha_n)\rho_s^*(\|\nabla g(x_n) - \nabla g(T^n x_n)\|) \\
&= \alpha_n[g(w) - g(x_n) + \langle x_n, \nabla g(x_n)\rangle - \langle w, \nabla g(x_n)\rangle] \\
&\quad + (1-\alpha_n)[g(w) - g(T^n x_n) + \langle T^n x_n, \nabla g(T^n x_n)\rangle - \langle w, \nabla g(T^n x_n)\rangle] \\
&\quad - \alpha_n(1-\alpha_n)\rho_s^*(\|\nabla g(x_n) - \nabla g(T^n x_n)\|) \\
&= \alpha_n D(w,x_n) + (1-\alpha_n) D(w, T^n x_n) \\
&\quad - \alpha_n(1-\alpha_n)\rho_s^*(\|\nabla g(x_n) - \nabla g(T^n x_n)\|) \\
&\le \alpha_n D(w,x_n) + (1-\alpha_n) k_n D(w, x_n) \\
&\quad - \alpha_n(1-\alpha_n)\rho_s^*(\|\nabla g(x_n) - \nabla g(T^n x_n)\|) \\
&\le D(w,x_n) + (k_n - 1) D(w, x_n) \\
&\quad - \alpha_n(1-\alpha_n)\rho_s^*(\|\nabla g(x_n) - \nabla g(T^n x_n)\|) \\
&\le D(w,x_n) + (k_n - 1)\theta_n \\
&\quad - \alpha_n(1-\alpha_n)\rho_s^*(\|\nabla g(x_n) - \nabla g(T^n x_n)\|).
\end{aligned}
$$

In view of (3.19), we obtain

$$
\begin{aligned}
D(w,u_n) &= D(w, T_{r_n} y_n) \\
&\le D(w, y_n) \\
&\le D(w,x_n) + (k_n - 1)\theta_n - \alpha_n(1-\alpha_n)\rho_s^*(\|\nabla g(x_n) - \nabla g(T^n x_n)\|).
\end{aligned}
\tag{3.20}
$$

It follows from (3.14)-(3.16) that

$$
\begin{aligned}
D(x_n,u_n) &= D(x_n, x_{n+1}) + D(x_{n+1}, u_n) - \langle x_n - x_{n+1}, \nabla g(x_{n+1}) - \nabla g(u_n)\rangle \\
&\le D(x_n, x_{n+1}) + D(x_{n+1}, u_n) + \|x_n - x_{n+1}\|\|\nabla g(x_{n+1}) - \nabla g(u_n)\| \\
&\to 0
\end{aligned}
\tag{3.21}
$$

as $n \to \infty$. This, together with Lemma 2.6, implies that

$$
\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.22}
$$

By the definition of Bregman distance and (3.22), we obtain

$$
\begin{aligned}
|g(x_n) - g(u_n)| &= |D(x_n, u_n) + \langle x_n - u_n, \nabla g(u_n)\rangle| \\
&\le D(x_n, u_n) + \|x_n - u_n\|\|\nabla g(u_n)\| \\
&\to 0
\end{aligned}
\tag{3.23}
$$

as $n \to \infty$. From (3.18) and (3.22)-(3.23), we conclude that

$$\begin{aligned}|D(w,x_n) - D(w,u_n)| &= |g(w) - g(x_n) - \langle w - x_n, \nabla g(x_n)\rangle \\ &\quad - [g(w) - g(u_n) - \langle w - u_n, \nabla g(u_n)\rangle]| \\ &= |g(u_n) - g(x_n) + \langle x_n - w, \nabla g(x_n)\rangle + \langle w - u_n, \nabla g(u_n)\rangle| \\ &= |g(u_n) - g(x_n) + \langle x_n - w, \nabla g(x_n)\rangle - \langle x_n - w, \nabla g(u_n)\rangle \\ &\quad + \langle x_n - w, \nabla g(u_n)\rangle + \langle w - u_n, \nabla g(u_n)\rangle| \\ &= |g(u_n) - g(x_n) + \langle x_n - w, \nabla g(x_n) - \nabla g(u_n)\rangle + \langle x_n - u_n, \nabla g(u_n)\rangle| \\ &\le |g(u_n) - g(x_n)| + \|x_n - w\|\|\nabla g(x_n) - \nabla g(u_n)\| + \|x_n - u_n\|\|\nabla g(u_n)\| \\ &\to 0\end{aligned} \tag{3.24}$$

as $n \to \infty$. In view of (3.24) and taking into account $k_n \to 1$ as $n \to \infty$, we obtain

$$D(w,x_n) - D(w,u_n) + (k_n - 1)\theta_n \to 0 \quad \text{as } n \to \infty. \tag{3.25}$$

In view of (3.20) and (3.25), we conclude that

$$\begin{aligned}\alpha_n(1-\alpha_n) \quad & \rho_s^*(\|\nabla g(x_n) - \nabla g(T^n x_n)\|) \\ & \le D(w,x_n) - D(w,u_n) + (k_n - 1)\theta_n \\ & \to 0\end{aligned}$$

as $n \to \infty$. From the assumption $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, we have

$$\lim_{n\to\infty} \rho_s^*(\|\nabla g(x_n) - \nabla g(T^n x_n)\|) = 0.$$

Therefore, from the property of ρ_s^* we deduce that

$$\lim_{n\to\infty} \|\nabla g(x_n) - \nabla g(T^n x_n)\| = 0. \tag{3.26}$$

Since ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we arrive at

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0. \tag{3.27}$$

Since ∇g is uniformly norm-to-norm continuous on bounded subsets of E, from (3.12) we infer that

$$\lim_{n\to\infty} \|\nabla g(x_n) - \nabla g(p)\| = 0. \tag{3.28}$$

It follows from (3.26) and (3.28) that

$$\lim_{n\to\infty} \|\nabla g(T^n x_n) - \nabla g(p)\| = 0. \tag{3.29}$$

Since ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we arrive at

$$\lim_{n\to\infty} \|T^n x_n - p\| = 0. \tag{3.30}$$

Furthermore, by the assumption that T is λ-uniformly continuous, we have that

$$\begin{aligned}\|T^{n+1}x_n - T^n x_n\| &\le \|T^{n+1}x_n - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| + \|x_n - T^n x_n\| \\ &\le (\lambda+1)\|x_{n+1} - x_n\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + \|x_n - T^n x_n\|.\end{aligned}$$

This, together with (3.16) and (3.27), implies that

$$\lim_{n\to\infty} \|T^{n+1}x_n - T^n x_n\| = 0.$$

Hence, from (3.30), we have $\lim_{n\to\infty}\|TT^n x_n - p\| = 0$. In view of (3.30) and the closedness of T, we conclude that $Tp = p$. Thus, we have $p \in F(T)$.

Next, we show that $p \in EP(f)$. It follows from (3.20) that

$$D(w, y_n) \le D(w, x_n) + (k_n - 1)\theta_n. \tag{3.31}$$

From Lemma 3.2 (7), (3.31) and $u_n = T_{r_n} y_n$, we conclude that

$$\begin{aligned} D(u_n, y_n) &= D(T_{r_n} y_n, y_n) \\ &\le D(w, y_n) - D(w, T_{r_n} y_n) \\ &\le D(w, x_n) - D(w, T_{r_n} y_n) + (k_n - 1)\theta_n \\ &= D(w, x_n) - D(w, u_n) + (k_n - 1)\theta_n \\ &\to 0 \end{aligned} \tag{3.32}$$

as $n \to \infty$. In view of (3.32) and Lemma 2.6, we obtain

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.33}$$

Since ∇g is uniformly norm-to-norm continuous on any bounded subset of E, it follows from (3.33) that

$$\lim_{n\to\infty} \|\nabla g(u_n) - \nabla g(y_n)\| = 0. \tag{3.34}$$

By the assumption $r_n \ge a$, we have

$$\lim_{n\to\infty} \frac{\|\nabla g(u_n) - \nabla g(y_n)\|}{r_n} = 0. \tag{3.35}$$

In view of $u_n = T_{r_n} y_n$, we obtain

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, \nabla g(u_n) - \nabla g(y_n)\rangle \ge 0, \qquad \forall y \in C.$$

From condition (A2), we deduce that

$$\begin{aligned} \|y - u_n\| \frac{\|\nabla g(u_n) - \nabla g(y_n)\|}{r_n} &\ge \frac{1}{r_n}\langle y - u_n, \nabla g(u_n) - \nabla g(y_n)\rangle \\ &\ge -f(u_n, y) \ge f(y, u_n) \ge 0, \ \forall y \in C. \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we have from (3.35) and (A4) that

$$f(y, p) \le 0, \qquad \forall y \in C.$$

For $t \in (0, 1]$ and $y \in C$, let $y_t = ty + (1 - t)p$. Then we have $y_t \in C$, which yields that $f(y_t, p) \le 0$. Thus, we have from (A1) that

$$0 = f(y_t, y_t) \le tf(y_t, y) + (1 - t)f(y_t, p) \le tf(y_t, y).$$

Dividing by t, we get

$$f(y_t, y) \ge 0, \qquad \forall y \in C.$$

Letting $t \downarrow 0$, from the condition (A3), we obtain that

$$f(p, y) \ge 0, \qquad \forall y \in C.$$

This means that $p \in EP(f)$. Therefore, $p \in F(T) \cap EP(f)$.
Finally, we show that $p = P_F x$. From $x_n = P_{C_n} x$, we conclude that

$$\langle z - x_n, \nabla g(x_n) - \nabla g(x) \rangle \geq 0, \quad \forall z \in C_n.$$

Since $F \subset C_n$ for each $n \in \mathbb{N}$, we obtain

$$\langle z - x_n, \nabla g(x_n) - \nabla g(x) \rangle \geq 0, \quad \forall z \in F. \tag{3.36}$$

Letting $n \to \infty$ in (3.36), we deduce that

$$\langle z - p, \nabla g(p) - \nabla g(x) \rangle \geq 0, \quad \forall z \in F.$$

In view of (2.7), we have $p = P_F x$ which completes the proof. □

A similar argument as in the proof of Theorem 3.3 proves the following result.

Theorem 3.4. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a strongly coercive Bregman function which is bounded on bounded sets, and uniformly convex and uniformly smooth on bounded sets. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $T : C \to C$ be a closed Bregman quasi-nonexpansive mapping which is λ-uniformly continuous for some $\lambda > 0$. Suppose that $F := F(T) \cap EP(f)$ is a nonempty and bounded subset of C, where $EP(f)$ is the set of solutions to the equilibrium problem (1.1). Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in C \quad \textit{chosen arbitrarily}, \\ C_0 = C, \\ y_n = \nabla g^*[\alpha_n \nabla g(x_n) + (1 - \alpha_n)\nabla g(Tx_n)], \\ u_n \in C \quad \textit{such that} \qquad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, \nabla g(u_n) - \nabla g(y_n) \rangle \geq 0, \qquad \forall y \in C, \\ C_{n+1} = \{z \in C_n : D(z, u_n) \leq D(z, x_n)\}, \\ x_{n+1} = P_{C_{n+1}} x \ \textit{and} \ n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{3.37}$$

where ∇g is the directional derivative of g. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then the sequence $\{x_n\}$ defined in (3.37) converges strongly to $P_F x$, where P_F is the Bregman projection from E onto F.

Remark 3.5. Theorem 3.3 improves Theorem 1.1 in the following aspects.
(1) For the structure of Banach spaces, we extend the duality mapping to more general case, that is, a convex, continuous, strongly coercive Bregman function which is bounded on bounded sets, and uniformly convex and uniformly smooth on bounded sets.
(2) For the mappings, we extend the mapping from a relatively nonexpansive mapping to a Bregman asymptotically quasi-nonexpansive mapping. We extend the assumption $\hat{F}(T) = F(T)$ to a closed mapping, where $\hat{F}(T)$ is the set of asymptotic fixed points of the mapping T.
(3) For the algorithm, we remove the set W_n in Theorem 1.1.

At the end of this paper, we include a concrete example in support of Theorem 3.4.

Let E be a reflexive Banach space with the dual space E^* and let A be a maximal monotone operator from E to E^*. For any $r > 0$, let the mapping $J_r : E \to D(A)$ be defined by $J_r = (\nabla g + rA)^{-1}\nabla g$. The mapping J_r is called the *resolvent* of A. It is well-known that $A^{-1}(0) = F(J_r)$ for each $r > 0$ (for more details, see, for example [17]).

Using Theorem 3.3, we obtain the following strong convergence theorem for maximal monotone operators.

Theorem 3.6. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a strongly coercive Bregman function which is bounded on bounded sets, and uniformly convex and uniformly smooth on bounded sets. Let A be a maximal monotone operator from E to E^* such that $A^{-1}(0) \neq \emptyset$. Let $r > 0$ and $J_r = (\nabla g + rA)^{-1}\nabla g$ be the resolvent of A. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in E \quad \text{chosen arbitrarily}, \\ C_0 = E, \\ y_n = \nabla g^*[\alpha_n \nabla g(x_n) + (1-\alpha_n)\nabla g(J_r x_n)], \\ C_{n+1} = \{z \in C_n : D(z, u_n) \leq D(z, x_n)\}, \\ x_{n+1} = P_{C_{n+1}} x \text{ and } n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{3.38}$$

where ∇g is the directional derivative of g. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Then the sequence $\{x_n\}$ defined in (3.38) converges strongly to $P_{A^{-1}(0)}x$, where $P_{A^{-1}(0)}$ is the Bregman projection from E onto $A^{-1}(0)$.

Proof. Letting $f \equiv 0$ and $T = J_r$ in Theorem 3.4, from (3.37) we obtain (3.38). We need only to show that T satisfies all the conditions in Theorem 3.4. It is easy to see that T is a Bregman relatively nonexpansive mapping (for more details, see, for example [7]). Thus, we have

$$D(p, J_r v) \leq D(p, v), \qquad \forall v \in E,\ p \in F(J_r)$$

and

$$\hat{F}(J_r) = F(J_r) = A^{-1}(0),$$

where $\hat{F}(J_r)$ is the set of all asymptotic fixed points of J_r. Therefore, in view of Theorem 3.4 we have the conclusions of Theorem 3.6. This completes the proof. □

4 Numerical Example

In this section, in order to demonstrate the effectiveness, realization and convergence of algorithm of Theorem 3.2, we consider the following simple example.

Example 4.1. Let $T : [0,2] \to [0,2]$ be defined by

$$Tx = \begin{cases} 0 & \text{if } x \neq 2, \\ 1 & \text{if } x = 2. \end{cases}$$

Then T is a quasi-nonexpansive mapping. Indeed, for any $x \in [0,2)$, we have that $T^n x = 0$ for all $n \geq 1$. Thus,

$$|T^n x - 0|^2 = 0 \leq |x - 0|^2.$$

The other cases can be verified similarly. It is worth mentioning that T is neither nonexpansive nor continuous. Now, we define a function $f : [0,2] \times [0,2] \to \mathbb{R}$ by

$$f(x,y) = y^2 + xy - 2x^2, \quad x, y \in [0,2].$$

It is easy to that the function f satisfies (A1)-(A4). In view of Lemma 3.2, we have that T_r is single-valued. Next, we deduce a formula for T_r. For any $y \in [0,2]$, $r > 0$

$$\begin{aligned} f(z,y) + \tfrac{1}{r}\langle y - z, z - x\rangle \geq 0 &\iff y^2 + zy - 2z^2 + \tfrac{1}{r}(y-z)(z-x) \geq 0 \\ &\iff ry^2 + rzy - 2rz^2 + yz - yx - z^2 + zx \geq 0 \\ &\iff ry^2 + (rz + z - x)y + zx - z^2 - 2rz^2 \geq 0 \\ &\iff ry^2 + (rz + z - x)y + zx - (1+2r)z^2 \geq 0 \\ &\iff ry^2 + ((1+r)z - x)y + zx - (1+2r)z^2 \geq 0. \end{aligned}$$

Let $G(y) = ry^2 + ((1+r)z - x)y + zx - (1+2r)z^2$. Then $G(y)$ is a quadratic function of y with coefficients $a = r$, $b = (1+r)z - x$ and $c = zx - (1+2r)z^2$. Thus we have

$$\begin{aligned}
\Delta &= b^2 - 4ac \\
&= [(1+r)z - x]^2 - 4r[zx - (1+2r)z^2] \\
&= (r+1)^2 z^2 - 2x(r+1)z + x^2 - 4rzx + 4r(1+2r)z^2 \\
&= (r^2 + 2r + 1)z^2 - 2rzx - 2zx + x^2 - 4rzx + 4rz^2 + 8r^2 z^2 \\
&= r^2 z^2 + 2rz^2 + z^2 + 4rz^2 + 8r^2 z^2 + x^2 - 2(1+3r)zx \\
&= 9r^2 z^2 + 6rz^2 + z^2 + x^2 - 2(1+3r)zx \\
&= ((1+3r)z)^2 + x^2 - 2(1+3r)zx \\
&= ((1+3r)z - x)^2.
\end{aligned}$$

Then we have $G(y) \geq 0$ if and only if $\Delta \leq 0$. That is, $[(1+3r)z - x]^2 \leq 0$. Therefore, $z = \frac{x}{1+3r}$, which yields that $T_r(x) = \frac{x}{1+3r}$. Moreover, by Lemma 3.2 we have that $F(T_r) = EP(f) = \{0\}$. Let $\alpha_n = \frac{1}{2}$ and $r_n = 1$ for all $n \geq 1$. Under the above assumptions, the given algorithm (3.3) in Theorem 3.3 is simplified as follows:

$$\begin{cases}
x_0 = x \in [0,2] \quad \text{chosen arbitrarily}, \\
C_0 = [0,2], \\
y_n = \frac{1}{2}x_n + \frac{1}{2}T^n x_n, \\
u_n \in [0,2] \quad \text{such that} \quad y^2 + u_n y - 2u_n^2 + (y - u_n)(u_n - y_n) \geq 0, \qquad \forall y \in [0,2], \\
C_{n+1} = \{z \in C_n : |z - u_n| \leq |z - x_n|\}, \\
x_{n+1} = P_{C_{n+1}} x \quad \text{and} \quad n \in \mathbb{N} \cup \{0\}.
\end{cases} \tag{4.1}$$

We know that, in one dimensional case, the set C_{n+1} is a closed interval. If we set $[a_{n+1}, b_{n+1}] := C_{n+1}$, then the projection point x_{n+1} of $x \in C$ onto C_{n+1} can be expressed as:

$$x_{n+1} := P_{C_{n+1}} x = \begin{cases} x, & \text{if } x \in [a_{n+1}, b_{n+1}]; \\ b_{n+1}, & \text{if } x > b_{n+1}; \\ a_{n+1}, & \text{if } x < a_{n+1}. \end{cases}$$

Choose $x_0 = x = 1$. Then the iteration process (4.1) becomes

$$C_0 = [0,2], \ y_n = \frac{1}{2}x_n, \ \ u_n = \frac{1}{4}x_n, \ \ C_{n+1} = \Big[0, \frac{5}{8}x_n\Big], \ \ x_{n+1} = (\frac{5}{8})^n. \tag{4.2}$$

In this section, we give some numerical experiment results (based on Matlab) as follows:

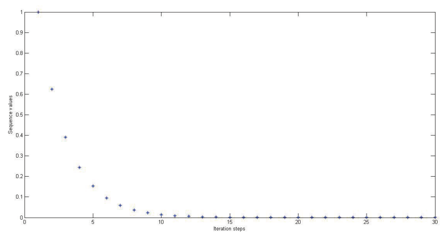

Figure 1: Iteration chart of the sequence $\{x_n\}$ in Example 4.1 with initial value $x_0 = 1$

n	x_n	y_n	u_n
1	1.000000000000000e+000	5.000000000000000e-001	2.500000000000000e-001
2	6.250000000000000e-001	3.125000000000000e-001	1.562500000000000e-001
3	3.906250000000000e-001	1.953125000000000e-001	9.765625000000000e-002
4	2.441406250000000e-001	1.220703125000000e-001	6.103515625000000e-002
5	1.525878906250000e-001	7.629394531250000e-002	3.814697265625000e-002
6	9.536743164062500e-002	4.768371582031250e-002	2.384185791015625e-002
7	5.960464477539063e-002	2.980232238769531e-002	1.490116119384766e-002
8	3.725290298461914e-002	1.862645149230957e-002	9.313225746154785e-003
9	2.328306436538696e-002	1.164153218269348e-002	5.820766091346741e-003
10	1.455191522836685e-002	7.275957614183426e-003	3.637978807091713e-003
11	9.094947017729282e-003	4.547473508864641e-003	2.273736754432321e-003
12	5.684341886080802e-003	2.842170943040401e-003	1.421085471520200e-003
13	3.552713678800501e-003	1.776356839400251e-003	8.881784197001252e-004
14	2.220446049250313e-003	1.110223024625157e-003	5.551115123125783e-004
15	1.387778780781446e-003	6.938893903907228e-004	3.469446951953614e-004
16	8.673617379884036e-004	4.336808689942018e-004	2.168404344971009e-004
17	5.421010862427522e-004	2.710505431213761e-004	1.355252715606881e-004
18	3.388131789017201e-004	1.694065894508601e-004	8.470329472543003e-005
19	2.117582368135751e-004	1.058791184067875e-004	5.293955920339377e-005
20	1.323488980084844e-004	6.617444900424221e-005	3.308722450212111e-005
21	8.271806125530277e-005	4.135903062765138e-005	2.067951531382569e-005
22	5.169878828456423e-005	2.584939414228212e-005	1.292469707114106e-005
23	3.231174267785264e-005	1.615587133892632e-005	8.077935669463161e-006
24	2.019483917365790e-005	1.009741958682895e-005	5.048709793414475e-006
25	1.262177448353619e-005	6.310887241768094e-006	3.155443620884047e-006
26	7.888609052210119e-006	3.944304526105059e-006	1.972152263052530e-006
27	4.930380657631324e-006	2.465190328815662e-006	1.232595164407831e-006
28	3.081487911019577e-006	1.540743955509789e-006	7.703719777548944e-007
29	1.925929944387236e-006	9.629649721936179e-007	4.814824860968089e-007
30	1.203706215242023e-006	6.018531076210113e-007	3.009265538105056e-007

Table 4.1. This table shows the values of sequence $\{x_n\}$ on 30th iteration steps (initial value $x_0 = 1$).

Conclusion

Table 4.1 and Figure 1 show that the sequence $\{x_n\}$ generated by (4.2) converges to 0 which solves the equilibrium problem.

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