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NONLINEAR METRIC REGULARITY AND SUBREGULARITY

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Abstract: In this paper, we provide some necessary and/or sufficient conditions of (linear or nonlinear) error bounds for proper functions (not necessarily lower semicontinuous). As applications, we present some linear and nonlinear metric regularity and subregularity results for the set-valued mappings with closed-graph.

Key words: *error bound, metric regularity, metric subregularity, strong slope, global slope*

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1 Introduction

Throughout the paper, let (X, d) be a complete metric space and (Y, d) be a metric space, $\sigma \in (0, +\infty)$. We recall that a set-valued mapping $F : X \rightrightarrows Y$ is a mapping which assigns to every $x \in X$ a subset (possibly empty) $F(x)$ of Y. As usual, we use the notation $\text{gph}F := \{(x; y) \in X \times Y : y \in F(x)\}$ for the graph of F and $F^{-1} : Y \rightrightarrows X$ for the inverse of F. This inverse (which always exists) is defined by $F^{-1}(y) := \{x \in X : y \in F(x)\}$. And, $(x, y) \in \text{gph}F$ if and only if $(y, x) \in \text{gph}F^{-1}$.

A set-valued mapping F is said to be nonlinear metrically regular, if there exists a nondecreasing function $\gamma : (0, +\infty) \to (0, +\infty)$ such that

$$\gamma(d(x, F^{-1}(y))) \leq d(y, F(x)), \quad \forall (x, y) \text{ with } d(x, F^{-1}(y)) > 0, \tag{1.1}$$

where $d(x, F^{-1}(y)) = \inf_{z \in F^{-1}(y)} d(x, z)$ and $d(x, F(x)) = \inf_{z \in F(x)} d(y, z)$. When $\gamma(s) := \tau s$ for some $\tau > 0$, we say that F is (linear) metrically regular (with modulus τ), which has been much discussed in the literature (see [3, 4, 14, 19, 20] and the references therein). Several authors have studied nonlinear metric regularity (see [5, 11, 15, 25]). Ioffe [15] presented a systematic study of nonlinear regularity theory in metric spaces.

A weaker property (than the metric regularity of F) is that of the metric subregularity concerning generalized equations of the form

$$b \in F(x), \tag{1.2}$$

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where and throughout we assume that $b \in Y$ is a given point. (1.2) is said to be nonlinear metrically subregular, if there exists a nondecreasing function $\gamma : (0, +\infty) \to (0, +\infty)$ such that

$$\gamma(d(x, F^{-1}(b))) \le d(b, F(x)), \quad \forall x \text{ with } d(x, F^{-1}(b)) > 0. \tag{1.3}$$

When $\gamma(s) := \tau s$ for some $\tau > 0$, we say that (1.2) is (linear) metrically subregular (with modulus τ). This property provides an estimate of how far a candidate x can be from the solution set $F^{-1}(b)$ of the generalized equation (1.2). There exists a wide literature on linear metric subregularity (see [12, 18, 22, 23] and the references therein).

The notion of error bound is closely related to metric regularity and subregularity. Let $f : X \to \mathbb{R} \bigcup \{+\infty\}$ be a proper function, and $S := \{x \in X | f(x) \le 0\}$. Recall (cf. [7]) that f is said to have a nonlinear error bound if there exists a nondecreasing function $\gamma : (0, +\infty) \to (0, +\infty)$ such that

$$\gamma(d(x, S)) \le f(x) \quad \forall x \text{ with } d(x, S) > 0. \tag{1.4}$$

When $\gamma(s) := \tau s$ for some $\tau > 0$, we say that f has a linear error bound. Linear error bounds were studied by many authors (e.g. [1, 2, 13, 10, 17, 24]). We refer the interested reader to the surveys by Azé [1], Fabian- Henrion-Kruger-Outrata [10],Lewis-Pang [17], and Pang [24].

Our main objective in this paper is to use the theory of error bounds to study the linear and nonlinear metric regularity and metric subregularity of set-valued mappings. Applications of the theory of error bounds to the investigation of metric regularity of set-valued mappings have been recently studied and developed by many authors, see for instance [3, 4, 19, 20]. Especially in the survey paper [4], it was shown that this approach is powerful to provide a unified theory of the metric regularity.

The idea of this paper is mainly motivated by Ngai and Théra [19], where they used the error bound of the lower semicontinuous hull of the function $x \to d(y, F(x))$ to get two criteria for linear metric regularity (see [19, Theorem 3.2]). Indeed, Ngai and Théra [19, Theorem 3.2] gave two error bounds criteria for the function $x \to d(y, F(x))$ (not necessarily lower semicontinuous). Theorem 3.2 in [19] and its proof indicate that one can formulate the error bounds criteria by employing the lower semicontinuous hull.

This paper goes in the same direction. In Section 2, we establish some necessary and/or sufficient conditions of linear and nonlinear error bounds for proper (not necessarily lower semicontinuous) functions, and extend some results on error bounds in [2, 6, 7] to proper functions. By using the ideas similar to [19] and the results obtained in Section 2, we derive characterizations of the linear and nonlinear metric regularity and subregularity in Section 3.

Next, we recall some notions and results. The lower semicontinuous hull of a function f is defined by $\mathrm{cl} f(x) := \min\{\liminf_{y \to x} f(y), f(x)\}$. As usual, $\mathrm{dom} f := \{x \in X | f(x) < +\infty\}$ denotes the effective domain of f, $f_+(x) := \max\{f(x), 0\}$. $B_\rho(x) := \{y \in X | d(x, y) < \rho\}$ and $\bar{B}_\rho(x) := \{y \in X | d(x, y) \le \rho\}$, where $\rho > 0$. Let $\frac{0}{0} := 0$, $0 * \infty := 0$. For $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R} \bigcup \{+\infty\}$ with $\alpha < \beta$, let $[f \le \alpha] := \{x \in X | f(x) \le \alpha\}$ and $[\alpha \le f < \beta] := \{x \in X | \alpha \le f(x) < \beta\}$.

In the rest of this section, unless otherwise specified, we assume that $f : X \to \mathbb{R} \bigcup \{+\infty\}$ is a (proper) lower semicontinuous function. Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R} \bigcup \{+\infty\}$ with $\alpha < \beta$.

Definition 1.1 ([8])**.** The strong slope of f at $x \in \mathrm{dom}\, f$ is defined by

$$|\nabla f|(x) = \begin{cases} 0, & if \;\; x \text{ is a local minimum of f}, \\ \limsup\limits_{y \to x} \frac{f(x) - f(y)}{d(x, y)}, & otherwise. \end{cases}$$

For $x \notin \mathrm{dom} f$, let $|\nabla f|(x) = +\infty$.

To the best of our knowledge, the following notion (called nonlocal slope in [21] and global slope in [6]) was first introduced by Ngai and Théra [19, Theorem 2.1].

Definition 1.2. For $x \in \mathrm{dom}\ f$, let

$$^\diamond|\nabla f|_\alpha(x) = \sup_y \frac{f(x) - \max\{f(y), \alpha\}}{d(x,y)}.$$

For $x \notin \mathrm{dom} f$, let $^\diamond|\nabla f|_\alpha(x) = +\infty$. The extended real number $^\diamond|\nabla f|_\alpha(x)$ is called the global slope of f at x about level α.

By the definitions of strong slope and global slope, one has the following proposition.

Proposition 1.3. *If $f(x) > \alpha$, then $^\diamond|\nabla f|_\alpha(x) \geq |\nabla f|(x)$.*

Let $\sigma_{\alpha,\beta}(f)$ denote the supremum of the $\tau \in [0, +\infty)$ such that

$$\tau d(x, [f \leq \alpha]) \leq f(x) - \alpha \ \text{ whenever } f(x) \in (\alpha, \beta).$$

Then

$$\sigma_{\alpha,\beta}(f) d(x, [f \leq \alpha]) \leq f(x) - \alpha \ \text{ whenever } f(x) \in (\alpha, \beta). \tag{1.5}$$

The following theorem gives two linear error bound criteria for lower semicontinuous functions.

Theorem 1.4 ([6])**.** (i) $\inf\limits_{x\in[\alpha<f<\beta]} |\nabla f|(x) \leq \sigma_{\alpha,\beta}(f)$. (ii) $\inf\limits_{x\in[\alpha<f<\beta]} {}^\diamond|\nabla f|_\alpha(x) = \sigma_{\alpha,\beta}(f)$.

Proof. We give a quick proof for completeness and for the reader's convenience. Since $|\nabla f|(x) \leq {}^\diamond|\nabla f|_\alpha(x)$ whenever $f(x) > \alpha$, one only needs to prove (ii).

Let $\tau < \sigma_{\alpha,\beta}(f)$. By the definition of $\sigma_{\alpha,\beta}(f)$, for any $x \in [\alpha < f < \beta]$, there exists $y_x \in [f \leq \alpha]$ such that $f(x) - \alpha > \tau d(x, y_x)$. Thus $^\diamond|\nabla f|_\alpha(x) \geq \frac{f(x)-\max\{f(y_x),\alpha\}}{d(x,y_x)} \geq \tau$. By the arbitrariness of x and τ, one has $\inf\limits_{x\in[\alpha<f<\beta]} {}^\diamond|\nabla f|_\alpha(x) \geq \sigma_{\alpha,\beta}(f)$.

On the other hand, for any $\tau > \sigma_{\alpha,\beta}(f)$, there exists $\bar{x} \in [\alpha < f < \beta]$ such that $f(\bar{x}) - \alpha < \tau d(\bar{x}, [f \leq \alpha])$. Let $r \in (0, d(\bar{x}, [f \leq \alpha]))$ be such that $f(\bar{x}) - \alpha \leq \tau r$. Let $g(x) := (f(x) - \alpha)_+ \geq 0$. One has

$$g(\bar{x}) = f(\bar{x}) - \alpha \leq \inf_X \ g + \tau r.$$

By virtue of the Ekeland variational principle [9], there exists $x \in \bar{B}_r(\bar{x})$ such that $g(x) \leq g(\bar{x})$ and $g(x) < g(y) + \tau d(x, y)$ for every $y \in X \setminus \{x\}$. Since $x \in \bar{B}_r(\bar{x})$ and $r < d(\bar{x}, [f \leq \alpha])$, one has $g(x) = f(x) - \alpha$ and

$$f(x) - \alpha < (f(y) - \alpha)_+ + \tau d(x, y), \ \forall y \in X \setminus \{x\},$$

$$f(x) < (f(y) - \alpha)_+ + \alpha + \tau d(x, y) = \max\{f(y), \alpha\} + \tau d(x, y), \ \forall y \in X \setminus \{x\}.$$

Thus $^\diamond|\nabla f|_\alpha(x) \leq \tau$. By the arbitrariness of x and τ, one has $\inf\limits_{x\in[\alpha<f<\beta]} {}^\diamond|\nabla f|_\alpha(x) \geq \sigma_{\alpha,\beta}(f)$. □

Through the above theorem and (1.5), one can easily get the following corollary.

Corollary 1.5. (i) *If* $\inf_{x\in[\alpha<f<\beta]} |\nabla f|(x) \geq \sigma$, *then* $f(x) - \alpha \geq \sigma d(x, [f \leq \alpha])$, $\forall x \in [\alpha < f < \beta]$.

(ii) $\inf_{x\in[\alpha<f<\beta]} {}^{\diamond}|\nabla f|_{\alpha}(x) \geq \sigma$ *if and only if* $f(x) - \alpha \geq \sigma d(x, [f \leq \alpha])$, $\forall x \in [\alpha < f < \beta]$.

The following theorem is needed for our purposes.

Theorem 1.6 ([13])**.** *Let* $f : X \to \mathbb{R}\bigcup\{+\infty\}$ *be a proper function (not necessarily lower semicontinuous). Then the following two statements are equivalent.*

(i) $\sigma d(x, [f \leq 0]) \leq f_+(x)$ *for all* $x \in X$.

(ii) $\mathrm{cl}[f \leq 0] = [\mathrm{cl}f \leq 0]$ *and* $\sigma d(x, [\mathrm{cl}f \leq 0]) \leq (\mathrm{cl}f)_+(x)$ *for all* $x \in X$.

2 Linear and Nonlinear Error Bounds for Proper Functions

In this section, we establish some necessary and/or sufficient conditions of linear and nonlinear error bounds for proper (not necessarily lower semicontinuous) function $f : X \to \mathbb{R}\bigcup\{+\infty\}$. By the definitions of f_+ and lower semicontinuous hull, one can easily get the following lemma.

Lemma 2.1. $\mathrm{cl}f_+(x) = (\mathrm{cl}f)_+(x)$.

To simplify the expression, we prove the following two propositions.

Proposition 2.2. *The following statements are equivalent.*

(i) *There exists a sequence* $\{u_n\}_{n\in\mathbb{N}}$ *converging to* x *such that* $\lim_{n\to+\infty} \frac{\mathrm{cl}f(x)-f(u_n)}{d(x,u_n)} \geq \sigma$.

(ii) $|\nabla \mathrm{cl}f|(x) \geq \sigma$.

Proof. (i)⇒(ii). Let $\{u_n\}_{n\in\mathbb{N}}$ be as in the statement (i). It follows from $\lim_{n\to+\infty} \frac{\mathrm{cl}f(x)-f(u_n)}{d(x,u_n)} \geq \sigma > 0$ that x is not a local minimum of $\mathrm{cl}f$. Since $\mathrm{cl}f \leq f$,

$$|\nabla \mathrm{cl}f|(x) \geq \lim_{n\to+\infty} \frac{\mathrm{cl}f(x) - \mathrm{cl}f(u_n)}{d(x,u_n)} \geq \lim_{n\to+\infty} \frac{\mathrm{cl}f(x) - f(u_n)}{d(x,u_n)} \geq \sigma.$$

(i)⇐(ii). If $|\nabla \mathrm{cl}f|(x) \geq \sigma$, then there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ converging to x such that

$$\frac{\mathrm{cl}f(x) - \mathrm{cl}f(z_n)}{d(x,z_n)} > \sigma - \frac{1}{n} \text{ for all } n \in \mathbb{N}. \tag{2.1}$$

For each fixed $n \in \mathbb{N}$, if $\mathrm{cl}f(z_n) = f(z_n)$, then we can set $u_n = z_n$. And if $\mathrm{cl}f(z_n) = \liminf_{y\to z_n} f(y) < f(z_n)$, then there exists a sequence $\{y_k\}_{k\in\mathbb{N}}$ converging to z_n such that $\mathrm{cl}f(z_n) = \lim_{k\to+\infty} f(y_k) < f(z_n)$. According to (2.1), there exists a natural number K such that

$$\frac{\mathrm{cl}f(x) - f(y_k)}{d(x,y_k)} > \sigma - \frac{1}{n} \text{ and } d(z_n, y_k) < \frac{1}{n},\ \forall k \geq K.$$

Choose $k_n > K$, and let $u_n = y_{k_n}$. Thus, the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges to x such that

$$\lim_{n\to+\infty} \frac{\mathrm{cl}f(x) - f(u_n)}{d(x,u_n)} \geq \sigma.$$

□

Proposition 2.3. *The following statements are equivalent.*

(i) *For each* $\varepsilon > 0$ *and each sequence* $\{x_n\}_{n\in\mathbb{N}}$ *converging to* x*, there exists a sequence* $\{u_n\}_{n\in\mathbb{N}}$ *with* $\lim_{n\to+\infty} d(x,u_n) > 0$ *such that*

$$\limsup_{n\to+\infty} \frac{f(x_n) - f_+(u_n)}{d(x,u_n)} > \sigma - \varepsilon.$$

(ii) *For each* $\varepsilon > 0$*, there exists a sequence* $\{u_n\}_{n\in\mathbb{N}}$ *with* $\lim_{n\to+\infty} d(x,u_n) > 0$ *such that*

$$\lim_{n\to+\infty} \frac{\mathrm{cl}f(x) - f_+(u_n)}{d(x,u_n)} > \sigma - \varepsilon.$$

(iii) ${}^\diamond|\nabla \mathrm{cl}f|_0(x) \geq \sigma$.

Proof. According to $\frac{\mathrm{cl}f(x)-f(u_n)}{d(x,u_n)} = \frac{\mathrm{cl}f(x)-f(x_n)}{d(x,u_n)} + \frac{f(x_n)-f(u_n)}{d(x,u_n)}$ and $\mathrm{cl}f(x) = \min\{\liminf_{y\to x} f(y), f(x)\}$, one can easily show that (i)⇔(ii).

(ii)⇒(iii). Let $\varepsilon > 0$ and $\{u_n\}_{n\in\mathbb{N}}$ be as in the statement (ii). According to Lemma 2.1 and $\mathrm{cl}(f_+) \leq f_+$, it follows that

$$\begin{aligned} {}^\diamond|\nabla \mathrm{cl}f|_0(x) &\geq \sup_{n\in\mathbb{N}, u_n\neq x} \frac{\mathrm{cl}f(x) - (\mathrm{cl}f)_+(u_n)}{d(x,u_n)} \\ &\geq \lim_{n\to+\infty} \frac{\mathrm{cl}f(x) - \mathrm{cl}(f_+)(u_n)}{d(x,u_n)} \geq \lim_{n\to+\infty} \frac{\mathrm{cl}f(x) - f_+(u_n)}{d(x,u_n)} > \sigma - \varepsilon. \end{aligned}$$

By the arbitrariness of $\varepsilon > 0$, one has ${}^\diamond|\nabla \mathrm{cl}f|_0(x) \geq \sigma$.

(ii)⇐(iii). If ${}^\diamond|\nabla \mathrm{cl}f|_0(x) \geq \sigma$ and $\varepsilon > 0$, then there exists $u \neq x$ such that

$$\frac{\mathrm{cl}f(x) - \mathrm{cl}(f_+)(u)}{d(x,u)} > \sigma - \varepsilon.$$

It follows from the definition of $\mathrm{cl}(f_+)$ that there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ converging to u such that $\lim_{n\to+\infty} f_+(u_n) = \mathrm{cl}(f_+)(u)$. Thus, the sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfies the following conditions

$$\lim_{n\to+\infty} d(x,u_n) > 0 \text{ and } \lim_{n\to+\infty} \frac{\mathrm{cl}f(x) - f_+(u_n)}{d(x,u_n)} > \sigma - \varepsilon.$$

□

Lemma 2.4. *If* ${}^\diamond|\nabla \mathrm{cl}f|_0(x) \geq \sigma$ *for every* $x \in X \setminus \mathrm{cl}[f \leq 0]$*, then* $\mathrm{cl}[f \leq 0] = [\mathrm{cl}f \leq 0]$.

Proof. $\mathrm{cl}[f \leq 0] \subseteq [\mathrm{cl}f \leq 0]$ is obvious.

On the other hand, if $x \notin \mathrm{cl}[f \leq 0]$, then ${}^\diamond|\nabla \mathrm{cl}f|_0(x) = \sup \frac{\mathrm{cl}f(x)-(\mathrm{cl}f)_+(y)}{d(x,y)} \geq \sigma > 0$. Thus $\mathrm{cl}f(x) > 0$. This implies that $x \notin [\mathrm{cl}f \leq 0]$. □

With the help of the preceding results, we obtain characterizations of the linear error bound for the proper function f.

Theorem 2.5. *Consider the following statements.*

(i) $\sigma d(x, [f \leq 0]) \leq f_+(x)$ *for all* $x \in X$.

(ii) ${}^\diamond|\nabla \mathrm{cl}f|_0(x) \geq \sigma$ for all $x \in X \setminus \mathrm{cl}[f \leq 0]$.

(iii) $|\nabla \mathrm{cl}f|(x) \geq \sigma$ for all $x \in X \setminus \mathrm{cl}[f \leq 0]$.

Then, one has (i) ⇔ (ii) ⇐ (iii).

Proof. According to Proposition 1.3, it follows that (ii) $\Leftarrow$ (iii).

According to Theorem 1.6, Lemma 2.4 and Corollary 1.5, it follows that (i)$\Leftrightarrow$(ii). □

The following theorem gives characterizations of nonlinear error bounds for the proper function f.

Theorem 2.6. *Let $\rho \in (0,+\infty)$, $C = \mathrm{cl}[f \le 0]$, and let $\theta : (0,+\infty) \to (0,+\infty)$ be a continuous and nondecreasing function. Consider the following statements.*

(i) *For all $x \in B_\rho(C) \setminus C$, one has $f(x) \ge \inf\limits_{B_{2\rho}(C)\setminus C} f + \int_0^{d(x,[f\le 0])} \theta(s)ds$.*

(ii) *For all $x \in B_{2\rho}(C) \setminus C$, one has $|\nabla \mathrm{cl} f|(x) \ge \theta(d(x,C))$.*

(iii) *Let $x \in B_{2\rho}(C)\setminus C$. Then there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ converging to x such that*

$$\lim_{n\to+\infty} \frac{\mathrm{cl} f(x) - f(u_n)}{d(x,u_n)} \ge \theta(d(x,C)).$$

Then, one has (i)$\Leftarrow$(ii)$\Leftrightarrow$(iii).

Proof. According to Proposition 2.2, it follows that (ii)$\Leftrightarrow$ (iii).

(i)$\Leftarrow$(ii): According to [7, Theorem 4.1], the statement (ii) implies that

$$\mathrm{cl} f(x) \ge \inf_{B_{2\rho}(C)\setminus C} \mathrm{cl} f + \int_0^{d(x,C)} \theta(s)ds, \ \textit{for every}\ x \in B_\rho(C)\setminus C.$$

Since $C = \mathrm{cl}[f \le 0]$ is a closed set, $d(x,C) = d(x,[f\le 0])$ and $B_{2\rho}(C)\setminus C$ is an open set. Thus $\inf\limits_{B_{2\rho}(C)\setminus C} \mathrm{cl} f = \inf\limits_{B_{2\rho}(C)\setminus C} f$. And, taking $\mathrm{cl} f \le f$ into account, one has

$$f(x) \ge \inf_{B_{2\rho}(C)\setminus C} f + \int_0^{d(x,[f\le 0])} \theta(s)ds, \ \textit{for every}\ x \in B_\rho(C)\setminus C.$$

Thus (i)$\Leftarrow$(ii)$\Leftrightarrow$(iii). □

3 Linear and Nonlinear Metric Regularity and Subregularity

In this section, we apply the results established in Section 2 to the investigation of metric regularity and subregularity. Throughout this section, we assume that $\mathrm{gph}F$ is closed. For $y \in Y$, we define the function $f^y(\cdot) := d(y, F(\cdot))$. For simplicity of presentation, we prove the following lemma.

Lemma 3.1. (i) $F^{-1}(y) = [f^y \le 0] = \mathrm{cl}[f^y \le 0] = [\mathrm{cl} f^y \le 0]$.

(ii) *If $^\diamond|\nabla \mathrm{cl} f^y|_0(x) \ge \sigma$ for every $x \in X$ with $y \notin F(x)$, then $F^{-1}(y) \ne \emptyset$.*

(iii) *If $|\nabla \mathrm{cl} f^y|(x) \ge \sigma$ for every $x \in X$ with $y \notin F(x)$, then $F^{-1}(y) \ne \emptyset$.*

Proof. (i) Since $[\mathrm{cl} f^y \le 0]$ is closed and $\mathrm{cl} f^y \le f^y$, $[f^y \le 0] \subseteq \mathrm{cl}[f^y \le 0] \subseteq [\mathrm{cl} f^y \le 0]$. If $x \in F^{-1}(y)$, then $f^y(x) = 0$. Thus $F^{-1}(y) \subseteq [f^y \le 0] \subseteq \mathrm{cl}[f^y \le 0] \subseteq [\mathrm{cl} f^y \le 0]$.

Conversely, suppose that $x \in [\mathrm{cl} f^y \le 0]$. Since $f^y(x) \ge 0$, $\mathrm{cl} f^y(x) = 0$. There exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converging to x with $\lim\limits_{n\to+\infty} d(y, F(x_n)) = 0$. Thus, one can find a sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq Y$ such that $z_n \in F(x_n)$ and $\lim\limits_{n\to+\infty} d((x,y),(x_n,z_n)) = \lim\limits_{n\to+\infty} d(y,z_n) = 0$. Since the graph of F is closed, $(x,y) \in \mathrm{gph}F$, i.e., $x \in F^{-1}(y)$. Thus $F^{-1}(y) \supseteq [\mathrm{cl} f^y \le 0]$.

(ii) According to (i), one only needs to show $[\mathrm{cl}f^y \le 0] \ne \emptyset$. Assume, for contradiction, that $[\mathrm{cl}f^y \le 0] = \emptyset$. For $\bar{x} \in [0 < \mathrm{cl}f^y < +\infty]$, there exists $\sigma' \in (0, \sigma)$ and $r \in (0, +\infty)$ such that

$$\mathrm{cl}f^y(\bar{x}) \le \inf_X \ \mathrm{cl}f^y + \sigma' r.$$

By virtue of the Ekeland variational principle [9], one can find $x \in \bar{B}_r(\bar{x})$ such that $(0 <)\mathrm{cl}f^y(x) \le \mathrm{cl}f^y(\bar{x})$ and $\mathrm{cl}f^y(x) < \mathrm{cl}f^y(z) + \sigma' d(x, z)$ for every $z \in X \setminus \{x\}$. It follows that

$$^{\diamond}|\nabla \mathrm{cl}f^y|_0(x) \le \sigma' < \sigma,$$

which contradicts $^{\diamond}|\nabla \mathrm{cl}f^y|_0(x) \ge \sigma$. Thus $F^{-1}(y) \ne \emptyset$.

(iii) By Proposition 1.3, one knows that the conclusion (iii) is correct. □

3.1 Metric Subregularity

In this subsection, we consider the metric subregularity of (1.2). According to Lemma 3.1 (i), one has

$$F^{-1}(b) = [f^b \le 0] = \mathrm{cl}[f^b \le 0] = [\mathrm{cl}f^b \le 0].$$

By Theorem 2.5, one can easily get the following theorem, which gives characterizations of the linear metric subregularity.

Theorem 3.2. *Consider the following statements.*

(i) *For all $x \in X$, one has $\sigma d(x, F^{-1}(b)) \le d(b, F(x))$.*

(ii) $^{\diamond}|\nabla \mathrm{cl}f^b|_0(x) \ge \sigma$ for all $x \in X \setminus F^{-1}(b)$.

(iii) $|\nabla \mathrm{cl}f^b|(x) \ge \sigma$ for all $x \in X \setminus F^{-1}(b)$.

Then, one has (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii).

The following theorem gives characterizations of the nonlinear metric subregularity.

Theorem 3.3. *Let $\theta : (0, +\infty) \to (0, +\infty)$ be a continuous and nondecreasing function, and $\rho > 0$. Consider the following statements.*

(i) *For all $x \in B_\rho(F^{-1}(b)) \setminus F^{-1}(b)$, one has*

$$\begin{aligned} d(b, F(x)) &\ge \inf_{B_{2\rho}(F^{-1}(b)) \setminus F^{-1}(b)} d(b, F(\cdot)) + \int_0^{d(x, F^{-1}(b))} \theta(s) ds \\ &\ge \int_0^{d(x, F^{-1}(b))} \theta(s) ds. \end{aligned}$$

(ii) *For all $x \in B_{2\rho}(F^{-1}(b)) \setminus F^{-1}(b)$, one has $|\nabla \mathrm{cl}f^b|(x) \ge \theta(d(x, F^{-1}(b)))$.*

(iii) *Let $x \in B_{2\rho}(F^{-1}(b)) \setminus F^{-1}(b)$. Then there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ converging to x such that*

$$\lim_{n\to+\infty} \frac{\mathrm{cl}f^b(x) - f^b(u_n)}{d(x, u_n)} \ge \theta(d(x, F^{-1}(b))).$$

Then, one has (i) $\Leftarrow$ (ii) $\Leftrightarrow$ (iii).

Proof. Since $d(b, F(\cdot)) \ge 0$, the second inequality in the statement (i) is always true. According to Lemma 3.1 (i), one knows that $F^{-1}(b)$ is a closed set. According to Theorem 2.6, one has (i) $\Leftarrow$ (ii) $\Leftrightarrow$ (iii). □

3.2 Metric Regularity

By Theorem 3.2, one can get the following theorem, which gives characterizations of the linear metric regularity.

Theorem 3.4. *Consider the following statements.*
(i) *For all* $(x, y) \in X \times Y$*, one has* $\sigma d(x, F^{-1}(y)) \leq d(y, F(x))$.
(ii) *Let* $(x, y) \in X \times Y$ *with* $y \notin F(x)$*. Then* ${}^{\diamond}|\nabla \mathrm{cl} f^y|_0(x) \geq \sigma$.
(iii) *Let* $(x, y) \in X \times Y$ *with* $y \notin F(x)$*. Then* $|\nabla \mathrm{cl} f^y|(x) \geq \sigma$.
Then, one has (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii).

By Theorem 3.3, one can get the following theorem, which gives characterizations of the nonlinear metric regularity.

Theorem 3.5. *Let* $\theta : (0, +\infty) \to (0, +\infty)$ *be a continuous and nondecreasing function, and* $\rho > 0$*. Consider the following statements.*
(i) *For all* $(x, y) \in X \times Y$ *with* $y \notin F(x)$*, one has*

$$\begin{aligned} d(y, F(x)) &\geq \inf_{B_{2\rho}(F^{-1}(y)) \setminus F^{-1}(y)} d(y, F(\cdot)) + \int_0^{d(x, F^{-1}(y))} \theta(s) ds \\ &\geq \int_0^{d(x, F^{-1}(y))} \theta(s) ds. \end{aligned}$$

(ii) *Let* $(x, y) \in X \times Y$ *with* $y \notin F(x)$*. Then* $|\nabla \mathrm{cl} f^y|(x) \geq \theta(d(x, F^{-1}(y)))$.
(iii) *Let* $(x, y) \in X \times Y$ *with* $y \notin F(x)$*. Then there exists a sequence* $\{u_n\}_{n \in \mathbb{N}} \subseteq X$ *converging to* x *such that*

$$\lim_{n \to +\infty} \frac{\mathrm{cl} f^y(x) - f^y(u_n)}{d(x, u_n)} \geq \theta(d(x, F^{-1}(y))).$$

Then, one has (i) $\Leftarrow$ (ii) $\Leftrightarrow$ (iii).

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