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GENERAL REGULARIZED NONCONVEX VARIATIONAL INEQUALITIES AND IMPLICIT GENERAL NONCONVEX WIENER-HOPF EQUATIONS

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Abstract: In this paper, we investigate and analyze the general nonconvex variational inequalities introduced by Noor [M. A. Noor, An extragradient algorithm for solving general nonconvex variational inequalities, Appl. Math. Lett. 23 (2010) 917-921] and [M. A. Noor, Iterative methods for general nonconvex variational inequalities, Albanian J. Math. 3 (2009) 117-127] and point out that the basic results in the above mentioned papers have fatal errors, for example, the algorithms and results in these papers are not valid. To overcome with the problems in theses papers, two new classes of nonconvex variational inequalities, called general regularized nonconvex variational inequalities, and a new class of Wiener-Hopf equations, called general nonconvex Wiener-Hopf equations, are introduced and, further, by using the projection operator technique, the equivalence between the general regularized nonconvex variational inequalities and the fixed point problems as well as the general nonconvex Wiener-Hopf equations is proved. Also, by using these equivalence formulations, we discuss the existence and uniqueness of solutions of the general regularized nonconvex variational inequalities and construct some new projection iterative algorithms for approximating the solutions of the general regularized nonconvex variational inequalities. Finally, we also show that the approximate solutions obtained by our iterative algorithms converge to the solutions of the general regularized nonconvex variational inequalities and, as a consequence, we derive the improvement version of the algorithms and results presented in the above mentioned papers.

Key words: *general regularized nonconvex variational inequalities, iterative method, general nonconvex Wiener-Hopf equations, Prox-regularity, convergence analysis*

Mathematics Subject Classification: *47H05; 47J20*

1 Introduction

Variational inequalities, which were introduced and studied by Stampacchia [28] in the early sixties, can be considered as a natural and significant extension of the variational principles, the origin of which can be traced back to Fermat, Euler, Leibniz, Newton, Lagrange and Bernoulli brothers. The techniques and ideas of the variational inequalities are being applied in a variety of diverse areas of sciences and proved to be productive and innovative. These activities have motivated to generalize and extend the variational inequalities and related optimization problems in several directions using new and novel techniques. The projection method and its variant forms represent important tools for finding the approximate

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solution of various kinds of variational and quasi-variational inequalities and variational inclusions, the origin of which can be traced back to Lions and Stampacchia [20] (see, for example, [1, 2, 4, 11–15, 26, 29]). The projection type methods were developed in 1970's and 1980's. The main idea in this technique is to establish the equivalence between the variational inequalities and the fixed point problems using the concept of projection. This alternative formulation enables us to suggest some iterative methods for computing the approximate solution.

It is worth mentioning that most of the results regarding the existence and iterative approximation of solutions to variational inequality problems have been investigated and considered so far with the restriction to the case where the underlying set is a convex set. In recent years, the concept of convex set has been generalized in many directions, which has potential and important applications in various fields. It is well known that the uniformly prox-regular sets are nonconvex and include the convex sets as special cases (for more details, see, for example, [3, 8, 17, 18, 25]).

Recently, Noor [22] introduced and considered a class of general nonconvex variational inequalities and asserted that this class is equivalent to the fixed point problems as well as the Wiener-Hopf equations. By using this equivalence formulation, he studied the existence and uniqueness of solution of the general nonconvex variational inequalities and suggested some projection iterative algorithms for solving the general nonconvex variational inequalities. He also discussed the convergence analysis of the suggested iterative algorithms under some certain conditions.

Very recently, Noor [21] introduced and considered a class of general nonconvex variational inequalities on the uniformly prox-regular sets and claimed that the general nonconvex variational inequalities are equivalent to the fixed point problems. He used this equivalence formulation to suggest and analyze an extragradient method for solving the general nonconvex variational inequalities. He also studied the convergence analysis of the suggested iterative method under some certain conditions.

In this paper, we point out that the equivalence formulations used by Noor [21, 22] are not correct. Hence the algorithms and results in [21, 22] are not valid. To overcome with the problems in [21, 22], we introduce two new classes of nonconvex variational inequalities, called general regularized nonconvex variational inequalities, and a new class of Wiener-Hopf equations, called general nonconvex Wiener-Hopf equations, and then, by the projection operator technique, we establish the equivalence between the general regularized nonconvex variational inequalities and the fixed point problems as well as the general nonconvex Wiener-Hopf equations. By using these equivalence formulations, the existence and uniqueness of solutions of the general regularized nonconvex variational inequalities is discussed and some new projection iterative algorithms for approximating the solutions of the general regularized nonconvex variational inequalities are constructed. Finally, we also verify that the approximate solutions obtained by our iterative algorithms converge to the solutions of the general regularized nonconvex variational inequalities and, as a consequence, we derive the improvement of the algorithms and results presented in [22].

2 Preliminaries and Basic Facts

Throughout this paper, let $\mathcal{H}$ be a real Hilbert space which is equipped with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$ and K be a nonempty closed subset of $\mathcal{H}$. We denote by $d_K(\cdot)$ or $d(\cdot, K)$ the usual distance function to the subset K, i.e., $d_K(u) = \inf_{v \in K} \|u - v\|$. Let us recall the following well-known definitions and some auxiliary results of nonlinear convex analysis and nonsmooth analysis [16–18, 25].

Definition 2.1. Let $u \in \mathcal{H}$ be a point not lying in K. A point $v \in K$ is called *a closest point or a projection of* u *onto* K if, $d_K(u) = \|u - v\|$. The set of all such closest points is denoted by $P_K(u)$, i.e.,

$$P_K(u) := \{v \in K : d_K(u) = \|u - v\|\}.$$

Definition 2.2. The *proximal normal cone* of K at a point $u \in K$ is given by

$$N_K^P(u) := \{\xi \in \mathcal{H} : \exists \alpha > 0 : u \in P_K(u + \alpha\xi)\}.$$

Clarke et al. [17], in Proposition 1.1.5, give a characterization of $N_K^P(u)$ as follows:

Lemma 2.3. *Let K be a nonempty closed subset in $\mathcal{H}$. Then $\xi \in N_K^P(u)$ if and only if there exists a constant $\alpha = \alpha(\xi, u) > 0$ such that $\langle \xi, v - u\rangle \leq \alpha\|v - u\|^2$ for all $v \in K$.*

The above inequality is called the *proximal normal inequality.* The special case in which K is closed and convex is an important one. In Proposition 1.1.10 of [17], the authors give the following characterization of the proximal normal cone for the closed and convex subset $K \subset \mathcal{H}$:

Lemma 2.4. *Let K be a nonempty, closed and convex subset in $\mathcal{H}$. Then $\xi \in N_K^P(u)$ if and only if $\langle \xi, v - u\rangle \leq 0$ for all $v \in K$.*

Definition 2.5. Let X be a real Banach space and $f : X \to \mathbb{R}$ be Lipschitz with constant τ near a point $x \in X$, that is, for some $\varepsilon > 0$, we have $|f(y) - f(z)| \leq \tau\|y - z\|$ for all $y, z \in B(x; \varepsilon)$, where $B(x; \varepsilon)$ denotes the open ball of radius $\varepsilon > 0$ and centered at x. The *generalized directional derivative* of f at x in the direction v, denoted as $f^\circ(x; v)$, is defined as follows:

$$f^\circ(x; v) = \limsup_{y \to x, t \downarrow 0} \frac{f(y + tv) - f(y)}{t},$$

where $y \in X$ and t is a positive number.

The generalized directional derivative can be used to develop a notion of tangency that does not require K to be smooth or convex.

Definition 2.6. The *tangent cone* $T_K(x)$ to K at a point x in K is defined as follows:

$$T_K(x) := \{v \in \mathcal{H} : d_K^\circ(x; v) = 0\}.$$

Similarly, we define the *normal cone* of K at x by polarity with respect to $T_K(x)$ as follows:

$$N_K(x) := \{\xi : \langle \xi, v\rangle \leq 0, \quad \forall v \in T_K(x)\}.$$

The *Clarke normal cone*, denoted by $N_K^C(x)$, is given by $N_K^C(x) = \overline{co}[N_K^P(x)]$, where $\overline{co}[S]$ means the closure of the convex hull of S. It is clear that $N_K^P(x) \subseteq N_K^C(x)$. The converse is not true in general. Note that $N_K^C(x)$ is always closed and convex cone, whereas $N_K^P(x)$ is always convex, but may not be closed (see [16, 17, 25]).

In 1995, Clarke et al. [18] introduced and studied a new class of nonconvex sets, called *proximally smooth sets*, and, subsequently Poliquin et. al in [25] investigated the aforementioned sets, under uniformly prox-regular sets. These have been successfully used in many nonconvex applications in areas such as optimizations, economic models, dynamical systems, differential inclusions, etc. For such as applications, see [5–7, 9]. This class seems particularly well suited to overcome the difficulties which arise due to the nonconvexity assumptions on K. We take the following characterization proved in [18] as a definition of this class. We point out that the original definition was given in terms of the differentiability of the distance function (see [18]).

Definition 2.7. For any $r \in (0, +\infty]$, a subset K_r of $\mathcal{H}$ is called *normalized uniformly prox-regular*(or *uniformly r-prox-regular* [18]) if every nonzero proximal normal to K_r can be realized by an r-ball. This means that, for all $\bar{x} \in K_r$ and $0 \neq \xi \in N_{K_r}^P(\bar{x})$,

$$\langle \frac{\xi}{\|\xi\|}, x - \bar{x} \rangle \leq \frac{1}{2r}\|x - \bar{x}\|^2, \quad \forall x \in K_r,$$

or, equivalently, for all $\bar{x} \in K_r$ and $0 \neq \xi \in N_{K_r}^P(\bar{x})$ with $\|\xi\| = 1$, one has

$$\langle \xi, x - \bar{x} \rangle \leq \frac{1}{2r}\|x - \bar{x}\|^2, \quad \forall x \in K_r.$$

Obviously, the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of $\mathcal{H}$, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets, see [10, 18].

Lemma 2.8 ([18])**.** *A closed set $K \subseteq \mathcal{H}$ is convex if and only if it is proximally smooth of radius r for all $r > 0$.*

If $r = +\infty$, then in view of Definition 2.7 and Lemma 2.8, the uniform r-prox-regularity of K_r is equivalent to the convexity of K_r, which makes this class of great importance. For the case of that $r = +\infty$, we set $K_r = K$.

The following proposition summarizes some important consequences of the uniform prox-regularity needed in the sequel. The proof of this results can be found in [18, 25].

Proposition 2.9. *Let $r > 0$ and K_r be a nonempty closed and uniformly r-prox-regular subset of $\mathcal{H}$. Set $U(r) = \{u \in \mathcal{H} : d_{K_r}(u) < r\}$. Then the following statements hold:*

(a) For all $x \in U(r)$, one has $P_{K_r}(x) \neq \emptyset$.

(b) For all $r' \in (0, r)$, P_{K_r} is Lipschitz continuous with constant $\frac{r}{r-r'}$ on $U(r') = \{u \in \mathcal{H} : d_{K_r}(u) < r'\}$.

(c) The proximal normal cone is closed as a set-valued mapping.

As a direct consequent of part (c) of Proposition 2.9, for any uniformly r-prox-regular subset K_r of $\mathcal{H}$, we have $N_{K_r}^C(x) = N_{K_r}^P(x)$. Therefore, we define $N_{K_r}(x) := N_{K_r}^C(x) = N_{K_r}^P(x)$ for such a class of sets.

In order to make clear the concept of r-prox-regular sets, we state the following concrete example: The union of two disjoint intervals $[a, b]$ and $[c, d]$ is r-prox-regular with $r = \frac{c-b}{2}$ (see [8, 17, 25]). The finite union of disjoint intervals is also r-prox-regular and r depends on the distances between the intervals.

3 Formulations and Some Basic Comments

Let K_r be an uniformly r-prox-regular set and suppose further that $T, g, h : \mathcal{H} \to \mathcal{H}$ are three different nonlinear operators. Recently, Noor [22] introduced and considered the problem of finding $u \in K_r$ such that

$$\langle \rho Tu + u - h(u), h(v) - u \rangle \geq 0, \quad \forall v \in \mathcal{H} : h(v) \in K_r, \tag{3.1}$$

where $\rho > 0$ is a given constant and is called the *general nonconvex variational inequality*. He considered several special cases of the problem (3.1). Noor [22] has also mentioned that the problem (3.1) is equivalent to that of finding $u \in K_r$ such that

$$0 \in \rho Tu + u - h(u) + \rho N^P_{K_r}(u), \tag{3.2}$$

where $N^P_{K_r}(u)$ denotes the normal cone of K_r at u in the sense of nonconvex analysis.

In Section 3 of [22], the author has claimed that the problem (3.1) is equivalent to the fixed point problem.

Lemma 3.1 ([22])**.** *$u \in K_r$ is a solution of the problem* (3.1) *if and only if*

$$u = P_{K_r}[h(u) - \rho Tu], \tag{3.3}$$

where P_{K_r} is the projection of $\mathcal{H}$ onto the uniformly r-prox-regular set K_r.

By a careful reading the proof of Lemma 3.1 in [22], we found that there are two fatal errors in the proof of Lemma 3.1. Firstly, in view of Proposition 2.9, it should be pointed that for any $r' \in (0, r)$, the projection of points in $U(r') = \{u \in \mathcal{H} : d_{K_r}(u) < r'\}$ onto the set K_r exists and is unique, that is, for any $x \in U(r')$, the set $P_{K_r}(x)$ is nonempty and singleton. Equation (3.3) and Proposition 2.9 imply that the point $h(u) - \rho Tu$ should be belonged to $U(r')$ for some $r' \in (0, r)$. Unfortunately, it is not necessarily true. Indeed, the equation (3.3) is not necessarily well-defined. If $h(u) \in K_r$ and $\rho < \frac{r'}{1+\|Tu\|}$ for some $r' \in (0, r)$, then we have

$$\begin{aligned} d_{K_r}(h(u) - \rho Tu) &= \inf_{v \in K_r} \|h(u) - \rho Tu - v\| \\ &\leq \|h(u) - \rho Tu - h(u)\| = \rho\|Tu\| \\ &< \frac{r'\|Tu\|}{1 + \|Tu\|} < r'. \end{aligned}$$

Therefore, $h(u) - \rho Tu \in U(r')$, that is, the set $P_{K_r}[h(u) - \rho Tu]$ is nonempty and singleton. Hence, in the statement of Lemma 3.1, the constant ρ should be satisfied $\rho < \frac{r'}{1+\|Tu\|}$ for some $r' \in (0, r)$.

Secondly, we note that the author [22] used the general nonconvex variational inclusion (3.2) as an equivalence formulation of the general nonconvex variational inequality (3.1). Since $N^P_{K_r}(u)$ is a cone, the problem (3.2) is equivalent to the following problem:

$$0 \in \rho Tu + u - h(u) + N^P_{K_r}(u). \tag{3.4}$$

Therefore, according to the proof of Lemma 3.1, the problem (3.1) is equivalent to the problem (3.4). Unfortunately, it is not necessarily true.

Remark 3.2. Every solution of the problem (3.1) is a solution of the problem (3.4), but the converse is not necessarily true.

Proof. Let $u \in K_r$ be a solution of the problem (3.1). Then we have

$$\langle \rho Tu + u - h(u), h(v) - u \rangle \geq 0, \quad \forall v \in \mathcal{H} : h(v) \in K_r. \tag{3.5}$$

The inequality (3.5) implies that, for all $\alpha > 0$,

$$\langle \rho Tu + u - h(u), h(v) - u \rangle + \alpha \|h(v) - u\|^2 \geq 0, \quad \forall v \in \mathcal{H} : h(v) \in K_r. \tag{3.6}$$

It follows from the inequality (3.6) and Lemma 2.3 that

$$-(\rho Tu + u - h(u)) \in N^P_{K_r}(u),$$

which leads to

$$0 \in \rho Tu + u - h(u) + N^P_{K_r}(u). \tag{3.7}$$

The converse of the above statement is not true in general. Indeed, suppose that the inclusion (3.7) holds for some $u \in K_r$. Then, Lemma 2.3 implies that the inequality (3.6) holds for some $\alpha > 0$. However, by using the inequality (3.6), we cannot deduce the inequality (3.5). □

The following example illustrates that the problem (3.6) does not imply the problem (3.5).

Example 3.3. Let $\mathcal{H} = \mathbb{R}$ and K_r be the union of two disjoint intervals $[0, \beta]$ and $[\gamma, \delta]$ where $0 < \beta < \gamma < \delta$. Then $K_r = [0, \beta] \cup [\gamma, \delta]$ is a r-prox-regular set with $r = \frac{\gamma - \beta}{2}$. Define $T, h : \mathcal{H} \to \mathcal{H}$ by

$$Tx = \theta e^{sx}, \quad h(x) = kx^l, \quad \forall x \in \mathcal{H},$$

where $s, \theta, l \in \mathbb{R}$, $\theta < 0$, $l > 0$ and $k > \beta^{1-l}$ are arbitrary but fixed. Take $u = \beta$ and let $\rho > 0$ and $\alpha \geq -\frac{\rho\theta e^{s\beta} + \beta(1 - k\beta^{l-1})}{\gamma - \beta}$ be arbitrary and fixed. Then, we have

$$\begin{aligned}
&\langle \rho T(u) + u - h(u), h(v) - u \rangle + \alpha \|h(v) - u\|^2 \\
&= (\rho\theta e^{s\beta} + \beta - k\beta^l)(kv^l - \beta) + \alpha(kv^l - \beta)^2 \\
&= (kv^l - \beta)(\alpha(kv^l - \beta) + \rho\theta e^{s\beta} + \beta(1 - k\beta^{l-1})), \quad \forall v \in \mathcal{H}.
\end{aligned} \tag{3.8}$$

If $v \in \left[0, \sqrt[l]{\frac{\beta}{k}}\right]$, then $-\beta \leq kv^l - \beta \leq 0$ and

$$\begin{aligned}
-\alpha\beta + \rho\theta e^{s\beta} + \beta(1 - k\beta^{l-1}) &\leq \alpha(kv^l - \beta) + \rho\theta e^{s\beta} + \beta(1 - k\beta^{l-1}) \\
&\leq \rho\theta e^{s\beta} + \beta(1 - k\beta^{l-1}).
\end{aligned}$$

To the case where $v \in \left[\sqrt[l]{\frac{\gamma}{k}}, \sqrt[l]{\frac{\delta}{k}}\right]$, we have $\gamma - \beta \leq kv^l - \beta \leq \delta - \beta$ and

$$\begin{aligned}
\alpha(\gamma - \beta) + \rho\theta e^{s\beta} + \beta(1 - k\beta^{l-1}) &\leq \alpha(kv^l - \beta) + \rho\theta e^{s\beta} + \beta(1 - k\beta^{l-1}) \\
&\leq \alpha(\delta - \beta) + \rho\theta e^{s\beta} + \beta(1 - k\beta^{l-1}).
\end{aligned}$$

The above facts guarantee that

$$(kv^l - \beta)(\alpha(kv^l - \beta) + \rho\theta e^{s\beta} + \beta(1 - k\beta^{l-1})) \geq 0, \quad \forall v \in \left[0, \sqrt[l]{\frac{\beta}{k}}\right] \cup \left[\sqrt[l]{\frac{\gamma}{k}}, \sqrt[l]{\frac{\delta}{k}}\right],$$

which deduces that

$$(kv^l - \beta)(\alpha(kv^l - \beta) + \rho\theta e^{s\beta} + \beta(1 - k\beta^{l-1})) \geq 0, \quad \forall v \in \mathcal{H} : kv^l \in [0, \beta] \cup [\gamma, \delta]. \tag{3.9}$$

Now, applying (3.8) and (3.9), it follows that

$$\langle \rho Tu + u - h(u), h(v) - u \rangle + \alpha \|h(v) - u\|^2 \geq 0, \quad \forall v \in \mathcal{H} : h(v) \in K_r.$$

However, it is obvious that $(\rho\theta e^{s\beta} + \beta(1 - k\beta^{l-1}))(kv^l - \beta) < 0$ for all $v \in \mathcal{H}$ with $kv^l \in [\gamma, \delta]$, that is,

$$\langle \rho Tu + u - h(u), h(v) - u \rangle < 0, \quad \forall v \in \mathcal{H} : h(v) \in [\gamma, \delta].$$

Hence the inequality

$$\langle \rho Tu + u - h(u), h(v) - u \rangle \geq 0, \quad \forall v \in \mathcal{H} : h(v) \in [\gamma, \delta]$$

can not hold for all $v \in \mathcal{H}$ with $h(v) \in K_r$.

Very recently, Noor [21] introduced and considered the problem of finding $u \in \mathcal{H}$ such that $g(u) \in K_r$ and

$$\langle Tu, g(v) - g(u) \rangle \geq 0, \quad \forall v \in \mathcal{H} : g(v) \in K_r, \tag{3.10}$$

which is called also the *general nonconvex variational inequality.*

Noor [21] has also mentioned that the problem (3.10) is equivalent to that of finding $u \in \mathcal{H}$ such that $g(u) \in K_r$ and

$$0 \in Tu + N^P_{K_r}(g(u)) \tag{3.11}$$

(the problem (5) of [21]), where $N^P_{K_r}(g(u))$ denotes the normal cone of K_r at $g(u)$ in the sense of nonconvex analysis.

In Section 3 of [21], the author has claimed that the problem (3.10) is equivalent to the fixed point problem.

Lemma 3.4 ([21])**.** *$u \in \mathcal{H} : g(u) \in K_r$ is a solution of the problem* (3.10) *if and only if*

$$g(u) = P_{K_r}[g(u) - \rho Tu], \tag{3.12}$$

where P_{K_r} is the same as in Lemma 3.1.

Similar to the mentioned argument for Lemma 3.1, we note that there are two fatal errors in the proof of Lemma 3.4. Firstly, equation (3.12) is not necessarily well-defined. Secondly, the author [21] used the general nonconvex variational inclusion (3.11) (the problem (5) in [21]) as an equivalence formulation of the general nonconvex variational inequality (3.10). But we show that unlike a claim of the author, the problems (3.10) and (3.11) are not equivalent.

Remark 3.5. Every solution of the problem (3.10) is a solution of the problem (3.11), but the converse is not necessarily true.

Proof. Let $u \in K_r$ be a solution of the problem (3.10). Then we have

$$\langle Tu, g(v) - g(u) \rangle \geq 0, \quad \forall v \in \mathcal{H} : g(v) \in K_r. \tag{3.13}$$

From (3.13) it follows that, for all $\alpha > 0$,

$$\langle Tu, g(v) - g(u)\rangle + \alpha\|g(v) - g(u)\|^2 \geq 0, \quad \forall v \in \mathcal{H} : g(v) \in K_r. \tag{3.14}$$

By using the inequality (3.14) and Lemma 2.3, it follows that

$$-Tu \in N^P_{K_r}(g(u))$$

and so

$$0 \in Tu + N^P_{K_r}(g(u)). \tag{3.15}$$

The converse of the above statement is not true in general. To see this fact, assume that the inclusion (3.15) holds for some $u \in \mathcal{H}$ with $g(u) \in K_r$. Then, from Lemma 2.3, we conclude that the inequality (3.14) holds for some $\alpha > 0$. However, by using the inequality (3.14), we cannot deduce the inequality (3.13). $\square$

The following example shows that cannot be concluded the inequality (3.13) from the inequality (3.14).

Example 3.6. Let $\mathcal{H}$ and K_r be the same as in Example 3.3. Define two mappings $T, g : \mathcal{H} \to \mathcal{H}$ by

$$Tx = \theta e^{sx}, \quad g(x) = kx^l, \quad \forall x \in \mathcal{H},$$

where $s, \theta, l \in \mathbb{R}$, $\theta < 0$, $l > 0$ and $k \in [\beta^{1-l}, \frac{\gamma}{\beta^l})$ are arbitrary but fixed. Take $u = \beta$ and let $\alpha \geq -\frac{\theta e^{s\beta}}{\gamma - k\beta^l}$ be arbitrary and fixed. Then we have

$$\begin{aligned}\langle Tu, g(v) - g(u)\rangle + \alpha\|g(v) - g(u)\|^2 &= \theta e^{s\beta}(kv^l - k\beta^l) + \alpha(kv^l - k\beta^l)^2 \\ &= k(v^l - \beta^l)(\alpha k(v^l - \beta^l) + \theta e^{s\beta}), \quad \forall v \in \mathcal{H}.\end{aligned} \tag{3.16}$$

If $v \in \left[0, \sqrt[l]{\frac{\beta}{k}}\right]$, then $-k\beta^l \leq k(v^l - \beta^l) \leq \beta - k\beta^l = \beta(1 - k\beta^{l-1})$ and

$$-\alpha\beta^l k + \theta e^{s\beta} \leq \alpha k(v^l - \beta^l) + \theta e^{s\beta} \leq \alpha\beta(1 - k\beta^{l-1}) + \theta e^{s\beta}.$$

For any $v \in \left[\sqrt[l]{\frac{\gamma}{k}}, \sqrt[l]{\frac{\delta}{k}}\right]$, we have $\gamma - k\beta^l \leq k(v^l - \beta^l) \leq \delta - k\beta^l$ and

$$\alpha(\gamma - k\beta^l) + \theta e^{s\beta} \leq \alpha k(v^l - \beta^l) + \theta e^{s\beta} \leq \alpha(\delta - k\beta^l) + \theta e^{s\beta}$$

and so

$$k(v^l - \beta^l)(\alpha k(v^l - \beta^l) + \theta e^{s\beta}) \geq 0, \quad \forall v \in \left[0, \sqrt[l]{\frac{\beta}{k}}\right] \cup \left[\sqrt[l]{\frac{\gamma}{k}}, \sqrt[l]{\frac{\delta}{k}}\right],$$

which deduces that

$$k(v^l - \beta^l)(\alpha k(v^l - \beta^l) + \theta e^{s\beta}) \geq 0, \quad \forall v \in \mathcal{H} : kv^l \in [0, \beta] \cup [\gamma, \delta]. \tag{3.17}$$

Now, from (3.16) and (3.17), it follows that

$$\langle Tu, g(v) - g(u)\rangle + \alpha\|g(v) - g(u)\|^2 \geq 0, \quad \forall v \in \mathcal{H} : g(v) \in K_r.$$

However, it is obvious that $k\theta e^{s\beta}(v^l - \beta^l) < 0$ for all $v \in \mathcal{H}$ with $kv^l \in [\gamma, \delta]$, that is,

$$\langle Tu, g(v) - g(u)\rangle < 0, \quad \forall v \in \mathcal{H} : g(v) \in K_r.$$

Hence the inequality $\langle Tu, g(v) - g(u)\rangle \geq 0$ cannot hold for all $v \in \mathcal{H}$ with $g(v) \in K_r$.

In view of the above arguments, the statements of Lemmas 3.1 and 3.4 are incorrect. The aforesaid lemmas are the main tools to construct Algorithms 3.1–3.5 and 4.1–4.3 in [22] and Algorithms 3.1–3.4 in [21]. Also, Lemmas 3.1 and 3.4 play key roles to derive the results in [22] and [21], respectively. Therefore, the results in [21, 22] are not valid.

4 General Regularized Nonconvex Variational Inequalities

Instead of the problems (3.1) and (3.10), in this section, we consider two new classes of general nonconvex variational inequalities and prove the equivalence between these problems and the general nonconvex variational inclusions (3.4) and (3.11) as well as the fixed point problems (3.3) and (3.12).

For given nonlinear operators $T, h : \mathcal{H} \to \mathcal{H}$ and constant $\rho > 0$, we consider the problem of finding $u \in K_r$ such that

$$\langle \rho Tu + u - h(u), h(v) - u\rangle + \frac{\|\rho Tu + u - h(u)\|}{2r}\|h(v) - u\|^2 \geq 0, \quad \forall v \in \mathcal{H}. \tag{4.1}$$

The problem (4.1) is called the *general regularized nonconvex variational inequality.*

If $r = \infty$, that is, $K_r = K$, the convex set in $\mathcal{H}$, then the problem (4.1) reduces to the problem of finding $u \in K$ such that

$$\langle \rho Tu + u - h(u), h(v) - u\rangle \geq 0, \quad \forall v \in \mathcal{H}, \tag{4.2}$$

which is called the *general variational inequality*, introduced and studied by Noor [24].

If $h \equiv I$ (: the identity operator), then the problem (4.2) reduces to the classical variational inequality.

In the next proposition, the equivalence between the general regularized nonconvex variational inequality (4.1) and the general nonconvex variational inclusion (3.4) is demonstrated.

Proposition 4.1. *Let T, h and ρ be the same as in the problem* (4.1) *such that $h(\mathcal{H}) = K_r$. Then the problem* (4.1) *is equivalent to the problem* (3.4).

Proof. Let $u \in K_r$ be a solution of the problem (4.1). If $\rho Tu + u - h(u) = 0$, then $0 \in \rho Tu + u - h(u) + N^P_{K_r}(u)$ because the zero vector always belongs to any normal cone. If $\rho Tu + u - h(u) \neq 0$, because $h(\mathcal{H}) = K_r$, then for all $v \in \mathcal{H}$, we have

$$\langle -(\rho Tu + u - h(u)), h(v) - u\rangle \leq \frac{\|\rho Tu + u - h(u)\|}{2r}\|h(v) - u\|^2.$$

Now, Lemma 2.3 implies that $-(\rho Tu + u - h(u)) \in N^P_{K_r}(u)$ and so

$$0 \in \rho Tu + u - h(u) + N^P_{K_r}(u),$$

that is, $u \in K_r$ is a solution of the problem (3.4).

Conversely, if $u \in K_r$ is a solution of the problem (3.4), then Definition 2.7 guarantees that $u \in K_r$ is a solution of the problem (4.1). This completes the proof. □

By using the projection operator technique and Proposition 4.1, we establish the equivalence between the problem (4.1) and the fixed point problem (3.3).

Lemma 4.2. *Let T, h and ρ be the same as in the problem* (4.1) *such that $h(\mathcal{H}) = K_r$. Then $u \in K_r$ is a solution of the problem* (4.1) *if and only if u satisfies the problem* (3.3), *provided that $\rho \leq \frac{r'}{1+\|Tu\|}$ for some $r' \in (0, r)$.*

Proof. Let $u \in K_r$ be a solution of the problem (4.1). Since $h(\mathcal{H}) = K_r$ and $\rho \leq \frac{r'}{1+\|Tu\|}$, for some $r' \in (0, r)$, it follows that equation (3.3) is well-defined. Then, by using Proposition 4.1, we have

$$\begin{aligned} 0 \in \rho Tu + u - h(u) + N^P_{K_r}(u) &\iff h(u) - \rho Tu \in u + N^P_{K_r}(u) \\ &\iff h(u) - \rho Tu \in (I + N^P_{K_r})(u) \\ &\iff u = P_{K_r}[h(u) - \rho Tu], \end{aligned}$$

where I is the identity operator and we have used the well-known fact that $P_{K_r} = (I + N^P_{K_r})^{-1}$. This completes the proof. $\square$

Instead of the problem (3.10), for given nonlinear operators $T, g : \mathcal{H} \to \mathcal{H}$, we now consider the problem of finding $u \in \mathcal{H}$ such that $g(u) \in K_r$ and

$$\langle Tu, g(v) - g(u)\rangle + \frac{\|Tu\|}{2r}\|g(v) - g(u)\|^2 \geq 0, \quad \forall v \in \mathcal{H}. \tag{4.3}$$

The problem (4.3) is called also the *general regularized nonconvex variational inequality*.

If $r = \infty$, then the problem (4.3) is equivalent to the problem of finding $u \in \mathcal{H}$ such that $g(u) \in K$ and

$$\langle Tu, g(v) - g(u)\rangle \geq 0, \quad \forall v \in \mathcal{H}, \tag{4.4}$$

which is called also the *general nonconvex variational inequality*, introduced and studied by Noor [23] in 1988.

If $g \equiv I$, then the problem (4.4) collapses to the classical variational inequality.

As in the proofs of Proposition 4.1 and Lemma 4.2, one can prove the equivalence between the general regularized nonconvex variational inequality (4.3) and the general nonconvex variational inclusion (3.11) as well as the fixed point problem (3.12) and we omit the proofs.

Proposition 4.3. *Let T and g be the same as in the problem* (4.3) *such that $g(\mathcal{H}) = K_r$. Then the problem* (4.3) *is equivalent to the problem* (3.11).

Lemma 4.4. *Let T and g be the same as in the problem* (4.3) *such that $g(\mathcal{H}) = K_r$. Then $u \in \mathcal{H}$ with $g(u) \in K_r$ is a solution of the problem* (4.3) *if and only if u satisfies equation* (3.12) *provided that $\rho \leq \frac{r'}{1+\|Tu\|}$ for some $r' \in (0, r)$.*

Definition 4.5. A nonlinear single-valued operator $T : \mathcal{H} \to \mathcal{H}$ is said to be:

(a) *monotone* if

$$\langle Tu - Tv, u - v\rangle \geq 0, \quad \forall u, v \in \mathcal{H};$$

(b) *κ-strongly monotone* if there exists a constant $\kappa > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq \kappa\|u - v\|^2, \quad \forall u, v \in \mathcal{H};$$

(c) *γ-Lipschitz continuous* if there exists a constant $\gamma > 0$ such that

$$\|Tu - Tv\| \leq \gamma\|u - v\|, \quad \forall u, v \in \mathcal{H}.$$

Now, we establish the existence and uniqueness of solution for the problem of general regularized nonconvex variational inequality (4.1).

Theorem 4.6. *Let T, h and ρ be the same as in the problem* (4.1) *such that $h(\mathcal{H}) = K_r$. Suppose further that T is α-strongly monotone and β-Lipschitz continuous and h is σ-strongly monotone and ς-Lipschitz continuous. If the constant ρ satisfies the following conditions: there exists $r' in(0, Y)$ such that*

$$\rho < \frac{r'}{1 + \|Tu\|}, \quad \forall u \in \mathcal{H} \tag{4.5}$$

and

$$\begin{cases} \left|\rho - \frac{\alpha}{\beta^2}\right| < \frac{\sqrt{\delta^2\alpha^2 - \beta^2(\delta^2 - (1-\delta\kappa)^2)}}{\delta\beta^2}, \\ \delta\alpha > \beta\sqrt{\delta^2 - (1-\delta\kappa)^2}, \quad \delta\kappa < 1, \\ \kappa = \sqrt{1 - 2\sigma + \varsigma^2}, \quad 2\sigma < 1 + \varsigma^2, \end{cases} \tag{4.6}$$

where $\delta = \frac{r}{r - r'}$, then the problem (4.1) *admits a unique solution.*

Proof. Define a mapping $F : K_r \to K_r$ by

$$F(u) = P_{K_r}[h(u) - \rho Tu], \quad \forall u \in K_r. \tag{4.7}$$

Since $h(\mathcal{H}) = K_r$ and $\rho < \frac{r'}{1+\|Tu\|}$ for some $r' \in (0, r)$ and for all $u \in \mathcal{H}$, it follows that $h(u) - \rho Tu \in U(r')$ for some $r' \in (0, r)$ and for all $u \in \mathcal{H}$. Hence the mapping F is well-defined.

Now, we prove that F is a contraction mapping. For this end, let $u, v \in K_r$ be given. By using (4.7) and Proposition 2.9, we obtain

$$\begin{aligned} \|F(u) - F(v)\| &= \|P_{K_r}[h(u) - \rho Tu] - P_{K_r}[h(v) - \rho Tv]\| \\ &\leq \delta\|h(u) - h(v) - \rho(Tu - Tv)\| \\ &\leq \delta(\|u - v - (h(u) - h(v))\| + \|u - v - \rho(Tu - Tv)\|), \end{aligned} \tag{4.8}$$

where $\delta = \frac{r}{r-r'}$. Since the operator h is σ-strongly monotone and ς-Lipschitz continuous, we have

$$\begin{aligned} \|u - v - (h(u) - h(v))\|^2 &= \|u - v\|^2 - 2\langle h(u) - h(v), u - v\rangle + \|h(u) - h(v)\|^2 \\ &\leq (1 - 2\sigma + \varsigma^2)\|u - v\|^2. \end{aligned} \tag{4.9}$$

From the α-strongly monotonicity and β-Lipschitz continuity of the operator T, it follows that

$$\begin{aligned} \|u - v - \rho(Tu - Tv)\|^2 &= \|u - v\|^2 - 2\rho\langle Tu - Tv, u - v\rangle + \rho^2\|Tu - Tv\|^2 \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|u - v\|^2. \end{aligned} \tag{4.10}$$

Substituting (4.9) and (4.10) in (4.8), we have

$$\|F(u) - F(v)\| \leq \theta\|u - v\|, \tag{4.11}$$

where

$$\theta = \delta(\kappa + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}), \tag{4.12}$$

and κ is the same as in (4.6). The condition (4.6) implies that $\theta < 1$. From the inequality (4.11), we deduce that F is a contraction mapping. According to Banach's fixed point theorem, there exists a unique point $u^* \in K_r$ such that $F(u^*) = u^*$. It follows from (4.7) that $u^* = P_{K_r}[h(u^*) - \rho Tu^*]$. Now, Lemma 4.2 guarantees that $u^* \in K_r$ is a solution of the problem (4.1). This completes the proof. □

As in the proof of Theorem 4.6, we can prove the existence and uniqueness of solution for the problem of general regularized nonconvex variational inequality (4.3) and we omit the proof.

Theorem 4.7. *Suppose that T and g are the same as in the problem* (4.3) *such that $g(\mathcal{H}) = K_r$. Let T be α-strongly monotone and β-Lipschitz continuous and g be σ-strongly monotone and ς-Lipschitz continuous. If there exists a constant $\rho > 0$ satisfying the condition* (4.5) *and the following condition*

$$\begin{cases} \left|\rho - \frac{\alpha}{\beta^2}\right| < \frac{\sqrt{\delta^2\alpha^2-\beta^2(\delta^2-(1-(1+\delta)\kappa)^2)}}{\delta\beta^2}, \\ \delta\alpha > \beta\sqrt{\delta^2-(1-(1+\delta)\kappa)^2}, \quad (1+\delta)\kappa < 1, \\ \kappa = \sqrt{1-2\sigma+\varsigma^2}, \quad 2\sigma < 1+\varsigma^2, \end{cases} \tag{4.13}$$

where δ is the same in Theorem 4.6*, then the problem* (4.3) *admits a unique solution.*

By utilizing Lemma 4.2, we suggest and analyze the following explicit projection iterative algorithms for solving the problem of general regularized nonconvex variational inequality (4.1).

Algorithm 4.8. Let T, h and ρ be the same as in the problem (4.1) such that $h(\mathcal{H}) = K_r$ and ρ be satisfied the condition (4.5). For an arbitrary chosen initial point $u_0 \in \mathcal{H}$, compute the sequence $\{u_n\}$ in $\mathcal{H}$ by the following iterative process:

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n P_{K_r}[h(u_n) - \rho T u_n], \quad \forall n \geq 0, \tag{4.14}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ with $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$.

If, for all $n \geq 0$, we have $\alpha_n = 1$, then Algorithm 4.8 reduces to the following iterative algorithm for solving the problem (4.1).

Algorithm 4.9. Suppose that T, h and ρ are the same as in Algorithm 4.8. For an arbitrary chosen initial point $u_0 \in \mathcal{H}$, compute the sequence $\{u_n\}$ in $\mathcal{H}$ by the following iterative process:

$$u_{n+1} = P_{K_r}[h(u_n) - \rho T u_n], \quad \forall n \geq 0,$$

Now, we suggest and analyze the following two-step iterative method for solving the problem (4.1).

Algorithm 4.10. Assume that T, h and ρ are the same as in Algorithm 4.8. For an arbitrary chosen initial point $u_0 \in \mathcal{H}$, compute the sequence $\{u_n\}$ in $\mathcal{H}$ in the following way:

$$\begin{aligned} u_{n+1} &= (1-\alpha_n)u_n + \alpha_n P_{K_r}[h(v_n) - \rho T v_n], \\ v_n &= (1-\beta_n)u_n + \beta_n P_{K_r}[h(u_n) - \rho T u_n], \quad \forall n \geq 0, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$.

If, for all $n \geq 0$, we have $\alpha_n = \beta_n = 1$, then Algorithm 4.10 collapses to the following two-step algorithm for solving the problem (4.1).

Algorithm 4.11. Let T, h and ρ be the same as in Algorithm 4.8. For an arbitrary chosen initial point $u_0 \in \mathcal{H}$, compute the sequence $\{u_n\}$ in $\mathcal{H}$ in the following way:

$$\begin{aligned} u_{n+1} &= P_{K_r}[h(v_n) - \rho T v_n], \\ v_n &= P_{K_r}[h(u_n) - \rho T u_n], \quad \forall n \geq 0. \end{aligned}$$

If $r = \infty$, that is, $K_r = K$, then Algorithms 4.8–4.11 collapse to iterative algorithms for solving the problem (4.2).

Now, we establish the strong convergence of the sequence generated by Algorithm 4.8 to the unique solution of the problem (4.1).

Theorem 4.12. *Let T, h and ρ be the same as in* Theorem 4.6*, and let all the conditions of* Theorem 4.6 *hold. If the constant ρ satisfies the conditions* (4.5) *and* (4.6)*, then the iterative sequence $\{u_n\}$ generated by* Algorithm 4.8 *converges strongly to the unique solution u^* of the problem* (4.1).

Proof. According to Theorem 4.6, the problem (4.1) has a unique solution $u \in K_r$. Since $h(\mathcal{H}) = K_r$ and $\rho < \frac{r'}{1+\|Tu\|}$ for some $r' \in (0, r)$, it follows from Lemma 4.2 that $u = P_{K_r}[h(u) - \rho Tu]$. Then, for each $n \geq 0$, we can write

$$u = (1 - \alpha_n)u + \alpha_n P_{K_r}[h(u) - \rho Tu], \tag{4.15}$$

where the sequence $\{\alpha_n\}$ is the same as in Algorithm 4.8. By using (4.14) and (4.15) and the assumptions, we have

$$\begin{aligned}\|u_{n+1} - u\| \leq &(1 - \alpha_n)\|u_n - u\| + \alpha_n\|P_{K_r}[h(u_n) - \rho Tu_n] - P_{K_r}[h(u) - \rho Tu]\| \\ \leq &(1 - \alpha_n)\|u_n - u\| + \alpha_n\delta(\|u_n - u - (h(u_n) - h(u))\| \\ &+ \|u_n - u - \rho(T(u_n) - T(u))\|) \\ \leq &(1 - \alpha_n)\|u_n - u\| + \alpha_n\delta(\kappa + \sqrt{1 - 2\alpha\rho + \rho^2\beta^2})\|u_n - u\| \\ = &(1 - (1 - \theta)\alpha_n)\|u_n - u\| \\ \leq &\prod_{i=0}^{n}[1 - (1 - \theta)\alpha_i]\|u_0 - u\|,\end{aligned} \tag{4.16}$$

where κ and θ are the same as in (4.6) and (4.12), respectively. Since $\sum_{n=0}^{\infty}\alpha_n = \infty$ and $\theta < 1$, it follows that

$$\lim_{n\to\infty}\prod_{i=0}^{n}[1 - (1 - \theta)\alpha_i] = 0. \tag{4.17}$$

Now, (4.16) and (4.17) imply that $u_n \to u$, as $n \to \infty$. This completes the proof. □

Similarly, one can discuss the convergence analysis of iterative Algorithms 4.9–4.11.

By using Lemma 4.4, we suggest and analyze the following explicit projection iterative algorithms for solving the problem of general regularized nonconvex variational inequality (4.3).

Algorithm 4.13. Let T and g be the same as in the problem (4.3) such that $g(\mathcal{H}) = K_r$. For an arbitrary chosen initial point $u_0 \in \mathcal{H}$, compute the sequence $\{u_n\}$ in $\mathcal{H}$ by the following iterative process:

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n\{u_n - g(u_n) + P_{K_r}[g(u_n) - \rho Tu_n]\}, \quad \forall n \geq 0,$$

where $\rho > 0$ is a constant satisfying the condition (4.5) and the sequence $\{\alpha_n\}$ is the same as in Algorithm 4.8.

If, for all $n \geq 0$, we have $\alpha_n = 1$, then Algorithm 4.13 reduces to the following iterative algorithm for solving the problem (4.3).

Algorithm 4.14. Assume that T and g are the same as in Algorithm 4.13. For an arbitrary chosen initial point $u_0 \in \mathcal{H}$, compute the sequence $\{u_n\}$ in $\mathcal{H}$ by the following iterative process:

$$u_{n+1} = u_n - g(u_n) + P_{K_r}[g(u_n) - \rho T u_n], \quad \forall n \geq 0,$$

where $\rho > 0$ is a constant satisfying the condition (4.5).

Now, we suggest and analyze the following two-step iterative method for solving the problem (4.3).

Algorithm 4.15. Suppose that T and g are the same as in Algorithm 4.13. For an arbitrary chosen initial point $u_0 \in \mathcal{H}$, compute the sequence $\{u_n\}$ in $\mathcal{H}$ in the following way:

$$\begin{cases} u_{n+1} = (1-\alpha_n)u_n + \alpha_n\{v_n - g(v_n) + P_{K_r}[g(v_n) - \rho T v_n]\}, \\ v_n = (1-\beta_n)u_n + \beta_n\{u_n - g(u_n) + P_{K_r}[g(u_n) - \rho T u_n]\}, \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$ and $\rho > 0$ is a constant satisfying the condition (4.5).

If, for all $n \geq 0$, we have $\alpha_n = \beta_n = 1$, then Algorithm 4.10 collapses to the following two-step iterative algorithm for solving the problem (4.3).

Algorithm 4.16. Let T and g be the same as in Algorithm 4.13. For an arbitrary chosen initial point $u_0 \in \mathcal{H}$, compute the sequence $\{u_n\}$ in $\mathcal{H}$ in the following way:

$$\begin{cases} u_{n+1} = v_n - g(v_n) + P_{K_r}[g(v_n) - \rho T v_n], \\ v_n = u_n - g(u_n) + P_{K_r}[g(u_n) - \rho T u_n], \quad \forall n \geq 0. \end{cases}$$

If $r = \infty$, that is, $K_r = K$, then Algorithms 4.13–4.16 collapse to the iterative algorithms for solving the problem (4.4).

In a similar way to that of proof of Theorem 4.7, one can establish the strong convergence of the sequence generated by Algorithm 4.13 to the unique solution of the problem (4.3) and we omit the proof.

Theorem 4.17. *Let T and g be the same as in Theorem 4.7 such that $g(\mathcal{H}) = K_r$, and let all the conditions of Theorem 4.7 hold. If there exists a constant $\rho > 0$ satisfying the conditions (4.5) and (4.13), then the iterative sequence $\{u_n\}$ generated by Algorithm 4.13 converges strongly to the unique solution u^* of the problem (4.3).*

In a similar way, one can discuss the convergence analysis of iterative Algorithms 4.14–4.16.

5 Wiener-Hopf Equations Technique

In this section, we introduce a new class of general nonconvex Wiener-Hopf equations which includes the Wiener-Hopf equations as special cases. By using Lemma 4.2 and the projection method, we prove the equivalence between the general noconvex Wiener-Hopf equations and the general regularized nonconvex variational inequalities of type (4.1).

Let T, h and ρ be the same as in the problem (4.1). Related to the problem (4.1), we consider the problem of finding $z \in \mathcal{H}$ such that

$$TP_{K_r}z + \rho^{-1}Q_{K_r}z = 0, \tag{5.1}$$

where $Q_{K_r} = I - hP_{K_r}$ and I is the identity operator. The problem (5.1) is called the *general nonconvex Winer-Hopf equation.*

If $r = \infty$ and $h \equiv I$, then the problem (5.1) collapses to the following problem:

Find $z \in \mathcal{H}$ such that

$$TP_K z + \rho^{-1}Q_K z = 0, \tag{5.2}$$

where $Q_K = I - P_K$. The equation (5.2) is original Wiener-Hopf equation mainly due to Shi [27].

Remark 5.1. It has been shown that the Wiener-Hopf equations have played an important and significant role in developing several numerical techniques for solving variational inequalities and related optimizations problems (see, for example, [19, 20, 27] and the references therein).

In the next lemma, the equivalence between the general regularized nonconvex variational inequality (4.1) and the general nonconvex Wiener-Hopf equation (5.1) is verified.

Lemma 5.2. *Let T, h and ρ be the same as in the problem* (4.1) *such that $h(\mathcal{H}) = K_r$ and ρ be satisfied the condition* (4.5) *for some $r' \in (0, r)$. Then $u \in K_r$ is a solution of the problem* (4.1) *if and only if the general nonconvex Wiener-Hopf equation* (5.1) *has a solution $z \in \mathcal{H}$ satisfying*

$$u = P_{K_r}z, \quad z = h(u) - \rho Tu. \tag{5.3}$$

Proof. Let $u \in K_r$ be a solution of the problem (4.1). Since $h(\mathcal{H}) = K_r$ and $\rho < \frac{r'}{1+\|Tu\|}$ for some $r' \in (0, r)$, it follows from Lemma 4.2 that

$$u = P_{K_r}[h(u) - \rho Tu]. \tag{5.4}$$

Taking $z = h(u) - \rho Tu$ in (5.4), we have

$$u = P_{K_r}z. \tag{5.5}$$

By using (5.5) and the fact that $z = h(u) - \rho Tu$, we have

$$z = hP_{K_r}z - \rho TP_{K_r}z.$$

It is clear that the above equation is equivalent to

$$TP_{K_r}z + \rho^{-1}Q_{K_r}z = 0,$$

where $Q_{K_r} = I - hP_{K_r}$, that is, $z \in \mathcal{H}$ is a solution of the general nonconvex Wiener-Hopf equation (5.1).

Conversely, if $z \in \mathcal{H}$ is a solution of the problem (5.1) satisfying

$$u = P_{K_r}z, \quad z = h(u) - \rho Tu,$$

then Lemma 4.2 implies that $u \in K_r$ is a solution of the problem (4.1). This completes the proof. □

By using the general nonconvex Wiener-Hopf equation (5.1) and Lemma 5.2, we get some fixed point formulations to construct a number of new projection iterative algorithms for solving the general regularized nonconvex variational inequality (4.1).

(I) By using (5.1) and (5.3), we have

$$\begin{aligned} TP_{K_r}z+\rho^{-1}Q_{K_r}z=0 &\Longleftrightarrow Q_{K_r}z=-\rho TP_{K_r}z\\ &\Longleftrightarrow z=hP_{K_r}z-\rho TP_{K_r}z\\ &\Longleftrightarrow z=h(u)-\rho Tu. \end{aligned}$$

This fixed point formulation enables us to construct the following iterative algorithm for solving the problem (4.1).

Algorithm 5.3. Let T, h and ρ be the same as in the problem (4.1) such that $h(\mathcal{H})=K_r$. Suppose further that the constant ρ is satisfied the condition (4.5), for some $r'\in(0,r)$. For a given $z_0\in U(r')$, compute the iterative sequence $\{z_n\}$ in $U(r')$ in the following way:

$$\begin{cases} u_n=P_{K_r}z_n,\\ z_{n+1}=h(u_n)-\rho Tu_n, \quad \forall n\geq 0. \end{cases} \tag{5.6}$$

(II) Applying (5.1) and (5.3), we obtain

$$\begin{aligned} TP_{K_r}z+\rho^{-1}Q_{K_r}z=0 &\Longleftrightarrow Q_{K_r}z=Q_{K_r}z-TP_{K_r}z-\rho^{-1}Q_{K_r}z\\ &\Longleftrightarrow Q_{K_r}z=-TP_{K_r}z+(1-\rho^{-1})Q_{K_r}z\\ &\Longleftrightarrow z=hP_{K_r}z-TP_{K_r}z+(1-\rho^{-1})Q_{K_r}z\\ &\Longleftrightarrow z=h(u)-Tu+(1-\rho^{-1})Q_{K_r}z. \end{aligned}$$

By this fixed point formulation, we can define the following iterative algorithm for solving the problem (4.1).

Algorithm 5.4. Let T, h and ρ be the same as in the problem (4.1) such that $h(\mathcal{H})=K_r$. Suppose further that the constant ρ is satisfied the condition (4.5), for some $r'\in(0,r)$. For any $z_0\in U(r')$, compute the iterative sequence $\{z_n\}$ in $U(r')$ in the following way:

$$\begin{cases} u_n=P_{K_r}z_n,\\ z_{n+1}=h(u_n)-Tu_n+(1-\rho^{-1})Q_{K_r}z_n, \quad \forall n\geq 0. \end{cases}$$

.

We now apply Lemma 5.2 and study the convergence analysis of Algorithm 5.3. In similar way, one can discuss the convergence analysis of Algorithm 5.4.

Theorem 5.5. *Let T, h and ρ be the same as in* Theorem 4.6 *and suppose further that all the conditions of* Theorem 4.6 *hold. Further, assume that the constant $\rho>0$ satisfies the conditions* (4.5) *and* (4.6). *Then there exists $z\in\mathcal{H}$ such that z is a solution of the problem* (5.1) *and the sequence $\{z_n\}$ generated by* Algorithm 5.3 *converges strongly to z.*

Proof. Theorem 4.6 guarantees that the problem (4.1) admits a unique solution $u\in K_r$. Hence Lemma 5.2 implies the existence of a unique point $z\in U(r')$ satisfying (5.3). By using (5.3), (5.6) and the assumptions, we have

$$\begin{aligned} \|z_{n+1}-z\| &= \|h(u_n)-h(u)-\rho(Tu_n-Tu)\|\\ &\leq \|u_n-u-(h(u_n)-h(u))\|+\|u_n-u-\rho(Tu_n-Tu)\|\\ &\leq (\kappa+\sqrt{1-2\rho\alpha+\rho^2\beta^2})\|u_n-u\|, \quad \forall n\geq 0, \end{aligned} \tag{5.7}$$

where κ is the same as in (4.6). From (5.3) and (5.6) and Proposition 2.9, it follows that

$$\|u_n - u\| = \|P_{K_r} z_n - P_{K_r} z\| \leq \delta \|z_n - z\|, \tag{5.8}$$

where δ is the same as in (4.6). Substituting (5.8) in (5.7), we have

$$\|z_{n+1} - z\| \leq \theta \|z_n - z\|, \tag{5.9}$$

where θ is the same as in (4.12). Applying (5.9) deduce that

$$\|z_{n+1} - z\| \leq \theta \|z_n - z\| \leq \theta^2 \|z_{n-1} - z\| \leq \cdots \leq \theta^{n+1} \|z_0 - z\|. \tag{5.10}$$

The condition (4.6) implies that $0 < \theta < 1$. Since $\theta \in (0, 1)$, it follows that the right side of the above inequality tends to zero, as $n \to \infty$, which implies that the sequence $\{z_n\}$ generated by Algorithm 5.3 converges strongly to z. This completes the proof. □

6 Conclusion

In this paper, we have investigated and analyzed the general nonconvex variational inequalities (1) from [21] and [22]. We have verified that the problem (1) in [21] is not equivalent to the fixed point problem (5) from [21], and also the problem (1) in [22] is not equivalent to the fixed point problem (8) from [22]. That is, Lemma 3.1 in [21, 22] is incorrect. Lemma 3.1 in [21, 22] is the main key to suggest the algorithms and to establish the strong convergence of the sequences generated by the proposed algorithms. Since Lemma 3.1 in [21, 22] is not valid, the algorithms and results in [21, 22] are not also valid.

To overcome with the problems in [21, 22], we have introduced two new classes of variational inequalities, named general regularized nonconvex variational inequalities and a new class of Wiener-Hopf equations, named general nonconvex Wiener-Hopf equations and by the projection operator technique, we have established the equivalence between the general regularized nonconvex variational inequalities and the fixed point problems as well as the general nonconvex Wiener-Hopf equations. By using the obtained equivalence formulations, the existence and uniqueness of solution for the general regularized nonconvex variational inequalities is discussed and some new projection iterative algorithms for approximating the solutions of the general regularized nonconvex variational inequalities are constructed.

We have also verified that the approximate solutions obtained by our iterative algorithms converge to the solutions of the general regularized nonconvex variational inequalities. As a consequence, we have derived the improvement of the algorithms and results presented in [22]. Meanwhile, the existence and uniqueness of solution for the general nonconvex variational inequalities (3.1) and (3.10) (the problem (1) in [21] and [22]) up to now has not been verified and remain still an open problem.

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