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OPTIMAL BOUNDARY CONTROL FOR A SCHRÖDİNGER EQUATION

Murat Subaşı* and Sidika Şule Şener

Abstract: In this paper we look for an answer to the question: To which extent can the solution of the Schrödinger equation be perturbed by the action of the Neumann boundary controls at a given final time in order to reach a given final target? Showing ill-posedness of the problem and after regularization, we get the Frechet differentiability and find the gradient of the new cost functional in terms of the solution of the adjoint problem. We constitute the minimizing sequence by the method of the projection of the gradient and prove its convergence to the optimal solution.

Key words: *optimal boundary control, Schrödinger equation, Ill-posedness and regularization*

Mathematics Subject Classification: *49J20, 35Q41, 65F22*

1 Introduction and Statement of the Problem

Optimal control theory governed by Schrödinger equations has begun to improve in 1980s. The cases of being the control in the coefficient of the equation were seriously investigated by [3,4]. Exact controllability studies on Schrödinger equations were dealt with on the other hand [13]. Machtyngier studied the exact controllability of Schrödinger equation in bounded domains with Dirichlet boundary condition using Hilbert Uniqueness Method [6]. Avdonin et al. recovered the spectral data from the Neumann-Dirichlet map of the Schrödinger equation [1]. Guo and Shao studied the basic properties of the Schrödinger equation defined on a bounded domain of $\mathbb{R}^n, n \geq 2$ with partial Dirichlet control [2]. Triggiani investigated the role of an $L_2(\Omega)$energy estimate in the theories of uniform stabilization and exact controllability for Schrödinger equations with Neumann boundary control [10]. The case that the control is in initial condition was investigated in [11].

The investigations in this paper are remarkable in some aspects: The controls in our problems are in the both Neumann boundary conditions. We give an example for non-uniqueness and ill-conditionedness for solution to given problem. After regularization, we use strong convexity of the functional to prove the well-posedness of the regularized problem. We constitute the minimizing sequence by the method of the projection of the gradient and indicate the convergence according to this method. This result is compatible with literature on gradient projection methods in optimal control problems [7] and references therein.

*Corresponding author

We consider the following problem;

$$J(v)=\int_0^l |\psi(x,T;v)-y(x)|^2\,dx\to\min \tag{1.1}$$

where the function $\psi(x,t;v)$ is the solution of the following 1-D Schrödinger boundary value problem;

$$i\frac{\partial\psi}{\partial t}+a_0\frac{\partial^2\psi}{\partial x^2}-a(x)\psi=f(x,t),(x,t)\in\Omega_T \tag{1.2}$$

$$\psi(x,0)=\phi(x),x\in(0,l) \tag{1.3}$$

$$\frac{\partial\psi}{\partial x}(0,t)=v_1(t),\frac{\partial\psi}{\partial x}(l,t)=v_2(t),t\in(0,T) \tag{1.4}$$

corresponding the controls $v(t)=\{v_1(t),v_2(t)\}$. Boundary conditions including controls can serve as a phenomenological description of apparatuses measuring quantum mechanical particles on the boundary of the domain. Here $a_0>0$, $\Omega_T=(0,l)\times(0,T)$,

$$0<\mu_0\le a(x)\le\mu_1 \text{ for } \overset{\circ}{\forall} x\in(0,l) \tag{1.5}$$

and

$$y(x),\phi(x)\in L_2(0,l),\quad f(x,t)\in L_2(\Omega_T). \tag{1.6}$$

The controls are defined on the bounded, closed and convex set

$$V:=\left\{v=v(t)=\{v_1(t),v_2(t)\}\in L_2^{(2)}(0,T):\ v_k\in L_2(0,T),|v_k(t)|\le\tilde{v}_k,k=1,2\right\} \tag{1.7}$$

The generalized solution of the problem (1.2)-(1.4) is the function $\psi\in W_2^{1,0}(\Omega_T)$ satisfying the following integral identity for $\forall\eta\in W_2^{1,1}(\Omega_T)$ with $\eta(x,T)=0$;

$$\begin{aligned}&\iint_{\Omega_T}(-i\psi\eta_t-a_0\psi_x\eta_x-a(x)\psi\eta)dxdt=\iint_{\Omega_T}f\eta dxdt\\&+i\int_0^l\phi(x)\eta(x,0)\,dx-a_0\int_0^T v_2(t)\eta(l,t)\,dt+a_0\int_0^T v_1(t)\eta(0,t)\,dt.\end{aligned} \tag{1.8}$$

So, we can state the following theorem whose proof can be done by Galerkin method used in [5].

Theorem 1.1. *Suppose that the conditions (1.5)-(1.6) hold. Then, the problem (1.2)-(1.4) has a unique solution in the sense (1.8) and this solution satisfies the following estimate;*

$$\|\psi\|^2_{W_2^{1,0}(\Omega_T)}\le c_0\left(\|\phi\|^2_{L_2(0,l)}+\|f\|^2_{L_2(\Omega_T)}\right). \tag{1.9}$$

2 Existence of the Optimal Solution

In this section we prove that the problem

$$\inf_{v\in V}J(v)=J_* \tag{2.1}$$

has at least one solution for the functional (1.1).

Firstly, we show that the cost functional (1.1) is continuous on V. Suppose that the function $\delta v = \{\delta v_1(t), \delta v_2(t)\} \in L_\infty^{(2)}(0,T) := L_\infty(0,T) \times L_\infty(0,T)$ is the increment to $v \in V$ such that $v+\delta v \in V$. Then the function $\psi_\delta = \psi_\delta(x,t) \equiv \psi(x,t;v+\delta v)$ is the solution of (1.2)-(1.4) corresponding to this element. So the solution $\psi = \psi(x,t) \equiv \psi(x,t;v)$ of the problem (1.2)-(1.4) will have the difference $\delta\psi = \delta\psi(x,t) \equiv \psi(x,t;v+\delta v) - \psi(x,t;v) = \psi_\delta - \psi$. Then we can say that the function $\delta\psi = \delta\psi(x,t)$ is the solution of the following problem;

$$i\frac{\partial \delta\psi}{\partial t} + a_0\frac{\partial^2 \delta\psi}{\partial x^2} - a(x)\delta\psi = 0, (x,t) \in \Omega \tag{2.2}$$

$$\delta\psi(x,0) = 0, x \in (0,l) \tag{2.3}$$

$$\frac{\partial \delta\psi}{\partial x}(0,t) = \delta v_1(t), \frac{\partial \delta\psi}{\partial x}(l,t) = \delta v_2(t), t \in (0,T). \tag{2.4}$$

The difference of the cost functional corresponding to the increment $v + \delta v \in V$ is such as

$$\begin{aligned} \delta J(v) &= J(v+\delta v) - J(v) \\ &= \int_0^l |\psi(x,T;v+\delta v) - y(x)|^2 dx - \int_0^l |\psi(x,T;v) - y(x)|^2 dx \end{aligned}$$

and

$$\begin{aligned} &\delta J(v) = J(v+\delta v) - J(v) \\ &= 2\int_0^l Re\left[\psi(x,T;v) - y(x)\right] \delta\overline{\psi}(x,T)\, dx + \|\delta\psi(x,T)\|^2_{L_2(0,l)}. \end{aligned} \tag{2.5}$$

Lemma 2.1. *Let $\delta\psi$ be a solution of the problem (2.2)-(2.4). Then the following inequality is valid;*

$$\|\delta\psi(x,T)\|^2_{L_2(0,l)} \leq c_1 \|\delta v\|^2_{L_\infty^{(2)}(0,T)} \tag{2.6}$$

where $\|\delta v\|_{L_\infty^{(2)}(0,T)} = \|\delta v_1\|_{L_\infty(0,T)} + \|\delta v_2\|_{L_\infty(0,T)}$.

Proof. If we multiply (2.2) by $\delta\overline{\psi}$ and integrate over Ω_t, we have

$$\begin{aligned} &\iint_{\Omega_t} \left(i\delta\psi_t\delta\overline{\psi} - a_0|\delta\psi_x|^2 - a(x)|\delta\psi|^2\right) dxdt = a_0\int_0^t \overline{\delta\psi}(0,t)\,\delta v_1 dt \\ &-a_0\int_0^t \overline{\delta\psi}(l,t)\,\delta v_2 dt. \end{aligned} \tag{2.7}$$

Subtracting complex conjugate of (2.7) from itself, we get the following for $\forall t \in [0,T]$;

$$i\iint_{\Omega_t} \frac{d}{dt}|\delta\psi|^2 dxdt = 2ia_0\int_0^t \delta v_2 Im\overline{\delta\psi}(l,t) - 2ia_0\int_0^t \delta v_1 Im\overline{\delta\psi}(0,t)$$

and

$$\begin{aligned} &\int_0^l |\delta\psi(x,t)|^2 dx \leq 2a_0\left(\int_0^t |\delta v_2||\delta\psi(l,t)| + \int_0^t |\delta v_1||\delta\psi(0,t)|\right) \\ &\leq 2a_0\left(\|\delta v_2\|_{L_\infty(0,t)}\|\delta\psi(l,t)\|_{L_2(0,t)} + \|\delta v_1\|_{L_\infty(0,t)}\|\delta\psi(0,t)\|_{L_2(0,t)}\right) \\ &\leq a_0\left(\|\delta v_2\|^2_{L_\infty(0,t)} + \|\delta v_1\|^2_{L_\infty(0,t)} + \|\delta\psi(l,t)\|^2_{L_2(0,t)} + \|\delta\psi(0,t)\|^2_{L_2(0,t)}\right). \end{aligned}$$

Further

$$\exists\sigma_1 > 0, \|\delta\psi\,(l,t)\|_{L_2(0,t)} \leq \sigma_1\sqrt{T}\,\|\delta\psi\|_{L_2(\Omega_t)} \tag{2.8}$$

and

$$\exists\sigma_2 > 0, \|\delta\psi\,(0,t)\|_{L_2(0,t)} \leq \sigma_2\sqrt{T}\,\|\delta\psi\|_{L_2(\Omega_t)}\,. \tag{2.9}$$

So, we write for $\forall t \in [0,T]$;

$$\|\delta\psi\,(x,t)\|^2_{L_2(0,l)} \leq a_0\,\|\delta v\|^2_{L^{(2)}_\infty(0,t)} + a_0T\left(\sigma_1^2+\sigma_1^2\right)\int\limits_0^t \|\delta\psi\|^2_{L_2(0,l)}\,d\tau$$

Applying Gronwall inequality, we get

$$\|\delta\psi\,(x,t)\|^2_{L_2(0,l)} \leq \exp\left(ta_0T\left(\sigma_1^2+\sigma_1^2\right)\right)\|\delta v\|^2_{L^{(2)}_\infty(0,t)}$$

and for $t = T$, we have

$$\|\delta\psi\,(x,T)\|^2_{L_2(0,l)} \leq c_1\,\|\delta v\|^2_{L^{(2)}_\infty(0,T)}$$

where $c_1 = \exp\left(a_0T^2\left(\sigma_1^2+\sigma_1^2\right)\right)$. Hence lemma has been proved. □

Using this lemma and the estimate (1.9), since $y\,(x) \in L_2\,(0,l)$we can write from (2.5) by Cauchy-Bunyakovski inequality that

$$|\delta J\,(v)| \leq c_2\,\|\delta v\|^2_{L^{(2)}_\infty(0,T)}\,. \tag{2.10}$$

Therefore $\delta J\,(v) \to 0$ when $\|\delta v\|^2_{L^{(2)}_\infty(0,T)} \to 0$, namely the functional (1.1) is continuous on V.

Now we continue to prove the existence of optimal solution. Let us take a minimizing sequence $\{v^m\} \subset V$ such that

$$\lim_{m\to\infty} J\,(v^m) = J_* = \inf_{v\in V} J\,(v)$$

and $\psi_m = \psi_m\,(x,t) \equiv \psi\,(x,t;v^m)$ be solution of problem (1.2)-(1.4) corresponding $\{v^m\}$ for $m = 1,2,....$ We can write from (1.9) that

$$\|\psi_m\|^2_{W_2^{1,0}(\Omega_T)} \leq c_0\left(\|\phi\|^2_{L_2(0,l)} + \|f\|^2_{L_2(\Omega_T)}\right) = c_3, m = 1,2,... \tag{2.11}$$

Here, c_3 is a number independent of the $m = 1,2,....$

Since the set V is a closed, bounded and convex subset of $L^{(2)}_\infty\,(0,T)$, one can select a subsequence $\{v^{m_k}\}$ of $\{v^m\}$ converging weakly-star to $v \in L^{(2)}_\infty\,(0,T)$. Let us show this subsequence by $\{v^m\}$ again for simplicity. Since the set V is weak-star closed, we say that$v \in V$. The sequence$\{\psi_m\,(x,t)\}$ is defined from a closed sphere to a closed sphere in $W_2^{1,0}\,(\Omega_T)$. So, it can be chosen a subsequence $\{\psi_{m_k}\,(x,t)\}$ of $\{\psi_m\,(x,t)\}$ weakly converging to $\psi\,(x,t)$ in $W_2^{1,0}\,(\Omega_T)$. Let us show this subsequence by $\{\psi_m\,(x,t)\}$ again for simplicity. This limit function $\psi\,(x,t)$ is also a solution to the problem (1.2)-(1.4) in sense of (1.8). It can be written from embedding theorem that

$$\psi_m\,(.,T) \longrightarrow \psi\,(.,T)\,, \text{weakly in } L_2\,(0,l)$$

for $m \to \infty$. Using this convergence, weak convergence of $\{v^m\}$ in $L_2^{(2)}(0,T)$ and continuity of functional (1.1), we get

$$J_* \le J(v) \le \lim_{m\to\infty} J(v^m) = J_*.$$

Consequently we get (2.1). Namely $v \in V$ is a solution of problem (1.1)-(1.4).

Although the solution of this optimal control problem is exist, the solution is not unique and not dependent continuously on data. In the following section we give an example for this situation.

3 Example of Non-Uniqueness and III-Conditionedness

Let us assume in the problem (1.2) that $a(x) = 1$. Denote by $\psi(x,t;v)$, $\psi(x,t;v+\delta v)$ the solutions of this problem, corresponding to the pairs $v = \{v_1, v_2\}$, $v + \Delta v = \{v_1 + \Delta v_1, v_2 + \Delta v_2\}$. Then the function $\delta\psi(x,t;v) = \delta(x,t;v+\delta v) - \psi(x,t;v)$ will be the solution of the problem (2.2)-(2.4). In the particular case, if we select

$$\begin{cases} \delta_{v_1}(t) = \int_0^t g(\tau)\, d\tau, \ t \in (0,T) \\ \delta_{v_2}(t) = \int_0^t h(\tau)\, d\tau, \ t \in (0,T) \end{cases} \tag{3.1}$$

then the solution of problem (2.2)-(2.4), with $a(x) = 1$, has the form

$$\delta\psi(x,t;v) = \begin{cases} 0 & (x,t) \in (0,l) \times [0,T) \\ \frac{1}{2l}\left[x^2 \delta_{v_2}(t) - (x-l)^2 \delta_{v_1}(t)\right] & (x,t) \in \{0,l\} \times (0,T] \end{cases} \tag{3.2}$$

which can be easily verified. For final time we find from (3.2) that

$$\delta\psi(x,T;v) = \frac{1}{2l}\left[x^2 \delta v_2(T) - (x-l)^2 \delta v_1(T)\right], x \in \{0,l\}. \tag{3.3}$$

To analyze the non-uniqueness of the solution, we need to show whether or not the condition

$$\delta\psi(x,T;v) = 0, \forall x \in [0,l], \tag{3.4}$$

holds for arbitrary functions $\delta v_1(t)$ and $\delta v_2(t)$. The situation is obvious for $x \in (0,l)$ from (3.2). The condition (3.4) is equivalent to the condition

$$\frac{1}{2l}\left[x^2 \int_0^T h(t)\, dt - (x-l)^2 \int_0^T g(t)\, dt\right] = 0, \tag{3.5}$$

for $x \in \{0,l\}$. If we consider the class of polynomial functions

$$\begin{cases} h(t) = at^n + bT^n, n \in \mathbb{N}_+ \ \text{ with } \ b = -\frac{a}{n+1}, a \in \mathbb{R} \\ g(t) = ct^m + dT^m, m \in \mathbb{N}_+ \ \text{with } d = -\frac{c}{n+1}, c \in \mathbb{R} \end{cases} \tag{3.6}$$

then we get $\int_0^T h(t)\, dt = 0$ and $\int_0^T g(t)\, dt = 0$. So condition (3.4) holds. Therefore if $\delta v_1(t)$ and $\delta v_2(t)$ are given by the functions given in (3.1), then any admissible data $\{v_1(t), v_2(t)\}$, $\{v_1(t) + \delta v_1(t), v_2(t) + \delta v_2(t)\}$ cannot be distinguished by the final state $\psi(x,T;v)$.

Now we will show that two close enough final states $\mu_1(x) := \psi(x,T;v)$, $\mu_2(x) := \psi(x,T;v+\delta v)$ may correspond to the different functions $v = \{v_1, v_2\}, v + \delta v = \{v_1+\delta v_1, v_2+\delta v_2\} \in V$.

Let us assume that the output data $\delta\mu(x) = \mu_2(x) - \mu_1(x) = \delta\psi(x,T;v)$ are given by the formula (3.2) and corresponding input data $\delta v = \{\delta v_1, \delta v_2\}$ given by (3.1). Let us choose the functions $g(t)$ and $h(t)$ in (3.1) such as

$$\begin{cases} h_\varepsilon(t) = at^n + b_\varepsilon T^n, n \in \mathbb{N}_+ \text{ with } b = -\frac{a}{n+1} + \varepsilon, a \in \mathbb{R} \\ g_\varepsilon(t) = ct^m + d_\varepsilon T^m, m \in \mathbb{N}_+ \text{ with } d = -\frac{c}{n+1} + \varepsilon, c \in \mathbb{R} \end{cases}$$

where $\varepsilon > 0$ is a small parameter. If we consider the $\delta\mu_\varepsilon(x)$ functions, obtained by using the functions $h_\varepsilon(t)$ and $g_\varepsilon(t)$, from (3.3) then we find $\|\delta\mu_\varepsilon(x)\|_{L_2[0,l]} = 0$ since $\delta\mu_\varepsilon(x)$ is zero almost everywhere in $[0,l]$. Corresponding input data norms are

$$\begin{aligned} \|\delta v_1(t)\|^2_{L_2[0,T]} &= \frac{2c^2m^2T^{2n+3}}{3(2m^4+13m^3+29m^2+27m+9)} \\ \|\delta v_2(t)\|^2_{L_2[0,T]} &= \frac{2a^2n^2T^{2n+3}}{3(2n^4+13n^3+29n^2+27n+9)} \end{aligned}$$

while $\varepsilon \to 0$. This shows ill-conditionedness of the problem.

4 Regularization of Optimal Control Problem

Since the problem is ill-posed, we use the parameter $\alpha > 0$ as the regularization parameter and write the functional;

$$J_\alpha(v) = \int_0^l |\psi(x,T;v) - y(x)|^2 dx + \alpha \|v\|^2_{L_2^{(2)}(0,T)} \to \min \tag{4.1}$$

where $\|v\|^2_{L_2^{(2)}(0,T)} = \left(\int_0^T v_0^2 dt + \int_0^T v_1^2 dt\right)$.

The parameter α can be found by regularization methods such as Tikhonov regularization, for numerical investigations [8, 9].

The functional (4.1) is continuous on V, as it is sum of a continuous functional (1.1) and a norm in the space $L_2^{(2)}(0,T)$, norm in this space is weakly lover semi-continuous.

Now, we show that the functional (4.1) is strongly convex with the constant $\chi > 0$, namely

$$J_\alpha(\beta v + (1-\beta)w) \le \beta J_\alpha(v) + (1-\beta) J_\alpha(w) - \chi\beta(1-\beta)\|v-w\|^2_{L_2^{(2)}(0,T)}$$

for $\forall v, w \in V$ and for $\forall \beta \in [0,1]$.

Since the problem (1.2)-(1.4) is linear and has a unique solution, we can write that

$$\psi(x,T;\beta v + (1-\beta)w) = \beta\psi(x,T;v) + (1-\beta)\psi(x,T;w).$$

On the other hand, we easily see that

$$\begin{aligned}
J_\alpha(\beta v+(1-\beta)w) &= \int_0^l |\psi(x,T;\beta v+(1-\beta)w)-y(x)|^2\,dx+\alpha\,\|\beta v+(1-\beta)w\|^2_{L_2^{(2)}(0,T)} \\
&\le \beta\left\{\int_0^l |\psi(x,T;v)-y(x)|^2\,dx+\alpha\,\|v-w\|^2_{L_2^{(2)}(0,T)}\right\} \\
&+(1-\beta)\left\{\int_0^l |\psi(x,T;w)-y(x)|^2\,dx+\alpha\,\|v-w\|^2_{L_2^{(2)}(0,T)}\right\} \\
&-\alpha\beta(1-\beta)\,\|v-w\|^2_{L_2^{(2)}(0,T)}
\end{aligned}$$

So, the functional (4.1) is strongly convex with $\chi=\alpha$.

We proved that the cost functional is continuous and strongly convex(so strictly convex) for strong convexity constant $\alpha>0$ on the weakly compact set V. So according to the generalized Weierstrass theorem [12], the optimal solution $v_*\in V$ to the problem

$$J_\alpha(v_*)=\inf_{v\in V} J_\alpha(v)$$

is exist and unique.

5 Frechet Differentiability and Lipschitz Continuity of the Gradient for Regularized Problem

Let us consider the adjoint boundary value problem for given optimal control problem;

$$i\frac{\partial\eta}{\partial t}+a_0\frac{\partial^2\eta}{\partial x^2}-a(x)\eta=0,\,(x,t)\in\Omega \tag{5.1}$$

$$\eta(x,T)=-2i\left[\psi(x,T)-y(x)\right],x\in(0,l) \tag{5.2}$$

$$\frac{\partial\eta}{\partial x}(0,t)=0,\,\frac{\partial\eta}{\partial x}(l,t)=0,t\in(0,T) \tag{5.3}$$

Lemma 5.1. *Let $\delta\psi$ be a solution of the problem (2.2)-(2.4) and η be a solution of the problem (5.1)-(5.3), then the following equality is valid;*

$$\begin{aligned}
2\int_0^l Re\left[\psi(x,T;v)-y(x)\right]\delta\overline{\psi}(x,T)\,dx &= a_0\int_0^T Re\eta(l,t)\,\overline{\delta v_2}(t)\,dt \\
&-a_0\int_0^T Re\eta(0,t)\,\overline{\delta v_1}(t)\,dt.
\end{aligned} \tag{5.4}$$

Proof. Let us multiply (5.1) by $\delta\overline{\psi}$ and after partial integration if we take complex conjugate we have

$$\begin{aligned}
\int_\Omega (i\delta\psi_t+a_0\delta\psi_{xx}-a(x)\delta\psi)\,\overline{\eta}dxdt &= 2\int_0^l \left[\overline{\psi}(x,T)-\overline{y}(x)\right]\delta\psi(x,T)\,dx \\
&+a_0\int_0^T \overline{\eta}(0,t)\,\delta v_1 dt-a_0\int_0^T \overline{\eta}(l,t)\,\delta v_2 dt
\end{aligned}$$

Multiplying (2.2) by $\overline{\eta}$ and then subtracting from above equation, we write

$$2\int_0^l \left[\overline{\psi}(x,T)-\overline{y}(x)\right]\delta\psi(x,T)\,dx = a_0\int_0^T \overline{\eta}(l,t)\,\delta v_2 dt - a_0\int_0^T \overline{\eta}(0,t)\,\delta v_1 dt.$$

Adding complex conjugate to itself of above equation, we get (5.4). □

Theorem 5.2. *The functional $J_\alpha(v)$ is Frechet differentiable and its gradient is given by the formula;*

$$J'_\alpha(v) = \{\, -a_0 Re\eta(0,t)+2\alpha v_1, a_0 Re\eta(l,t)+2\alpha v_2\}\,. \tag{5.5}$$

The increment of the functional (4.1) is

$$\begin{aligned}
\delta J_\alpha(v) &= J_\alpha(v+\delta v)-J_\alpha(v)\\
&= 2\int_0^l Re\left[\psi(x,T;v)-y(x)\right]\delta\overline{\psi}(x,T)\,dx + \|\delta\psi(x,T)\|^2_{L_2(0,l)}\\
&+2\alpha\int_0^T \left(v_1\overline{\delta v_1}+v_2\overline{\delta v_2}\right)dt+\alpha\|\delta v\|^2_{L_2^{(2)}(0,T)}
\end{aligned}$$

The following can be written using (5.4);

$$\begin{aligned}
\delta J_\alpha(v) &= J_\alpha(v+\delta v)-J_\alpha(v)\\
&= a_0\int_0^T Re\eta(l,t)\,\overline{\delta v_2}dt - a_0\int_0^T Re\eta(0,t)\,\overline{\delta v_1}dt + \|\delta\psi(x,T)\|^2_{L_2(0,l)}\\
&+2\alpha\int_0^T \left(v_1\overline{\delta v_1}+v_2\overline{\delta v_2}\right)dt+\alpha\|\delta v\|^2_{L_2^{(2)}(0,T)}\\
&= \int_0^T \left[a_0 Re\eta(l,t)+2\alpha v_2\right]\overline{\delta v_2}(t)\,dt - \int_0^T \left[a_0 Re\eta(0,t)+2\alpha v_1\right]\overline{\delta v_1}(t)\,dt\\
&+\|\delta\psi(x,T)\|^2_{L_2(0,l)}+\alpha\|\delta v\|^2_{L_2^{(2)}(0,T)}
\end{aligned}$$

By the definition of Frechet differential at $v\in V$,

$$J_\alpha(v+\delta v)-J_\alpha(v) = \left\langle J'_\alpha(v),\delta v\right\rangle_{L_2^{(2)}(0,T)} + o\left(\|\delta v\|^2_{L_2^{(2)}(0,T)}\right)$$

we get the formula (5.5) using (2.6).

Theorem 5.3. *The following estimate is valid for the gradient of the functional (4.1);*

$$\|J'_\alpha(v+\delta v)-J'_\alpha(v)\|_{L_2^{(2)}(0,T)} \le L\,\|\delta v\|_{L_2^{(2)}(0,T)}\,. \tag{5.6}$$

where $L=\left(\max\left\{a_0^2 T c_4\left(\sigma_3^2+\sigma_4^2\right),8\alpha^2\right\}\right)^{1/2}$.

Proof. We can evaluate the increment of the gradient as;

$$\|J'_\alpha(v+\delta v)-J'_\alpha(v)\|^2_{L_2^{(2)}(0,T)} = \int_0^T \left[\left(a_0 Re\delta\eta(0,t)+2\alpha\delta v_1\right)^2+\left(a_0 Re\delta\eta(l,t)+2\alpha\delta v_2\right)^2\right]dt \tag{5.7}$$

where $\delta\eta$ is the solution of the following problem

$$i\frac{\partial\delta\eta}{\partial t}+a_0\frac{\partial^2\delta\eta}{\partial x^2}-a(x)\delta\eta=0,(x,t)\in\Omega \tag{5.8}$$

$$\delta\eta\left(x,T\right)=-2i\delta\psi\left(x,T\right),x\in(0,l) \tag{5.9}$$

$$\frac{\partial\delta\eta}{\partial x}\left(0,t\right)=0,\frac{\partial\delta\eta}{\partial x}\left(l,t\right)=0,t\in\left(0,T\right). \tag{5.10}$$

Similar to (2.6), we can write the following estimate for the solution of the problem (5.8)-(5.10);

$$\left\|\delta\eta\right\|^2_{L_2(\Omega_T)}\le c_4\left\|\delta v\right\|^2_{L_2^{(2)}(0,T)} \tag{5.11}$$

Since $(a+b)^2\le 2\left(a^2+b^2\right)$, we get from (5.7) that

$$\left\|J'_\alpha\left(v+\delta v\right)-J'_\alpha\left(v\right)\right\|^2_{L_2^{(2)}(0,T)}=\int\limits_0^T\left[\left(a_0Re\delta\eta\left(0,t\right)+2\alpha\delta v_1\right)^2+\left(a_0Re\delta\eta\left(l,t\right)+2\alpha\delta v_2\right)^2\right]dt$$
$$\le 2\int\limits_0^T\left(a_0^2\left|\delta\eta\left(0,t\right)\right|^2+4\alpha^2\delta v_1^2+a_0^2\left|\delta\eta\left(l,t\right)\right|^2+4\alpha^2\delta v_2^2\right)dt$$

Using here similar inequalities to (2.8)-(2.9) for $\delta\eta\left(l,t\right)$ and $\delta\eta\left(0,t\right)$, we have

$$\begin{aligned}\left\|J'_\alpha\left(v+\delta v\right)-J'_\alpha\left(v\right)\right\|^2_{L_2^{(2)}(0,T)}&\le 2a_0^2T\left(\sigma_3^2+\sigma_4^2\right)\left\|\delta\eta\right\|^2_{L_2(\Omega_T)}+8\alpha^2\left\|\delta v\right\|^2_{L_2^{(2)}(0,T)}\\&\le 2a_0^2T\left(\sigma_3^2+\sigma_4^2\right)c_4\left\|\delta v\right\|^2_{L_2^{(2)}(0,T)}+8\alpha^2\left\|\delta v\right\|^2_{L_2^{(2)}(0,T)}\\&\le L^2\left\|\delta v\right\|^2_{L_2^{(2)}(0,T)}\end{aligned}$$

and so using (5.11) we get (5.6), where $L^2=\max\left\{a_0^2Tc_4\left(\sigma_3^2+\sigma_4^2\right),8\alpha^2\right\}$. □

6 Continuous Dependence of Solutions for Regularized Problem

Let $v^0\left(t\right)=\left\{v_1^0\left(t\right),v_2^0\left(t\right)\right\}\in V$ be the initial point. Then we set the minimizing sequence according to the method of projection of the gradient [7,12] by

$$v^{m+1}=P_V\left(v^m-\beta_kJ'_\alpha\left(v^m\right)\right),m=0,1,2,... \tag{6.1}$$

where $P_V\left(v^m-\beta_kJ'_\alpha\left(v^m\right)\right)$ is the projection of the element $v^m-\beta_kJ'_\alpha\left(v^m\right)$ on the set V.

Projection on the set $V$is established in this way;

$$v^{m+1}=\begin{cases}v^m-\beta_kJ'_\alpha\left(v^m\right) & ,\left\|v^m-\beta_kJ'_\alpha\left(v^m\right)\right\|_{L_\infty^{(2)}(0,T)}\le\widetilde{v}_k\\ \frac{\widetilde{v}_k\left(v^m-\beta_kJ'_\alpha\left(v^m\right)\right)}{\left\|v^m-\beta_kJ'_\alpha\left(v^m\right)\right\|_{L_\infty^{(2)}(0,T)}} & ,\left\|v^m-\beta_kJ'_\alpha\left(v^m\right)\right\|_{L_\infty^{(2)}(0,T)}>\widetilde{v}_k\end{cases}.$$

Since the set V is closed and convex this projection is exist and unique. According the definition of differentiability we have;

$$J_\alpha\left(v^{m+1}\right)-J_\alpha\left(v^m\right)=\beta_k\left[-\left\|J'_\alpha\left(v^m\right)\right\|^2_{L_2^{(2)}(0,T)}+\frac{O\left(\beta_k\right)}{\beta_k}\right]<0$$

for small enough β_k values. Appropriate values for β_k will be given in the following theorem. In any step if one gets $v^{m+1}=v^m$ then iteration is stopped. Due to the convexity of functional (4.1) and conditions on the set V, this element v^m will be the solution v_* of the problem.

Theorem 6.1. *Let $v^0 \in V$ be the initial point. The sequence $\{v^m\}$ defined by (6.1) converges to the unique minimum element v_* of the functional $J_\alpha(v)$ for $\beta_k = \beta \in \left(0, 4\alpha L^{-2}\right)$. Moreover the following inequality holds;*

$$\|v^m - v_*\|_{L_2^{(2)}(0,T)} \leq \left\|v^0 - v_*\right\|_{L_2^{(2)}(0,T)} .q^m, m = 0, 1, ...$$

where $q = \left(1 - 4\alpha\beta + \beta^2 L^2\right)^{\frac{1}{2}} \in (0,1)$.

Proof. We define the mapping $A : V \to V$ with

$$Av = P_V\left(v - \beta J'_\alpha(v)\right).$$

It can easily be shown that the mapping P_V is a contraction mapping, namely;

$$\|P_V(u) - P_V(v)\|_{L_2^{(2)}(0,T)} \leq \|u - v\|_{L_2^{(2)}(0,T)} \, \forall u, v \in V.$$

We must show that the mapping A also holds this property when $0 < \beta < 4\alpha L^{-2}$. We know that

$$\left\langle J'_\alpha(u) - J'_\alpha(v), u - v\right\rangle \geq 2\alpha \|u - v\|^2_{L_2^{(2)}(0,T)} \, \forall u, v \in V. \tag{6.2}$$

So, we write the following using (6.2) and Lipschitz continuity of the functional;

$$\begin{aligned}
\|Au - Av\|^2_{L_2^{(2)}(0,T)} &= \|P_V(u - \beta J'_\alpha(u)) - P_V(v - \beta J'_\alpha(v))\|^2_{L_2^{(2)}(0,T)} \\
&\leq \|(u - v) - \beta\left(J'_\alpha(u) - J'_\alpha(v)\right)\|^2_{L_2^{(2)}(0,T)} \\
&= \|u - v\|^2_{L_2^{(2)}(0,T)} - 2\beta\left\langle J'_\alpha(u) - J'_\alpha(v), u - v\right\rangle \\
&\quad + \beta^2 \|J'_\alpha(u) - J'_\alpha(v)\|^2_{L_2^{(2)}(0,T)} \\
&\leq \left(1 + \beta^2 L^2 - 4\beta\alpha\right) \|u - v\|^2_{L_2^{(2)}(0,T)}
\end{aligned}$$

Then, we have

$$\|Au - Av\|_{L_2^{(2)}(0,T)} \leq q \|u - v\|_{L_2^{(2)}(0,T)}$$

with $q = \left(1 + \beta^2 L^2 - 4\beta\alpha\right)^{\frac{1}{2}}$. The condition $\beta < 4\alpha L^{-2}$ gives $q \in (0,1)$.

The statement (6.1) can be written as $v^{m+1} = Av^m$. Then by the contraction mapping principle, the sequence $\{v^m\}$ converges to the fixed point $v_* = Av_*$ of the operator A with the factor q. We know that the minimum element v_* is unique. So we get followings;

$$\begin{aligned}
\|v^m - v_*\|_{L_2^{(2)}(0,T)} &= \left\|Av^{m-1} - Av_*\right\|_{L_2^{(2)}(0,T)} \\
&\leq q\left\|v^{m-1} - v_*\right\|_{L_2^{(2)}(0,T)} \\
&\leq q^2\left\|v^{m-2} - v_*\right\|_{L_2^{(2)}(0,T)} \\
&\;\;\vdots \\
&\leq q^m\left\|v^0 - v_*\right\|_{L_2^{(2)}(0,T)}
\end{aligned}$$

Hence, the theorem 6.1 has been proven. □

Conclusions

In this paper we have found out that using the sequence (6.1) the solution of the Schrödinger equation can be perturbed by the action of the Neumann boundary controls at a given final time in order to reach a given final target. To do this the gradient (5.5) is used. The rate of convergence, in which the relations of α, β, L can be seen, is given in Theorem 6.1. Furthermore in this problem the regularization parameter α is, at the same time, the strong convexity constant for the cost functional (4.1).

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Murat Subaşı
Atatürk University, Science Faculty, Department of Mathematics, 25240, Erzurum, Turkey
E-mail address: msubasi@atauni.edu.tr

Sidika Şule Şener
Atatürk University, Science Faculty, Department of Mathematics, 25240, Erzurum, Turkey
E-mail address: senersule@atauni.edu.tr