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SOLVABILITY OF A PERTURBED VARIATIONAL INEQUALITY*

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Abstract: This paper aims to establish solvability of a perturbed variational inequality defined by a set-valued mapping without assuming any kind of monotonicity. It is shown that if a coercivity condition holds, then the perturbed variational inequality has a solution. Two main results are obtained. The coercivity condition assumed in the first result implies the solution set of the variational inequality is nonempty and bounded, and that in the second one implies only the existence of solution. The first result extends some known results, while the second result is new even if the mapping is single-valued.

Key words: *variational inequality, set-valued mapping, upper semicontinuous mapping, variational inequality property, perturbation*

Mathematics Subject Classification: *90C31; 47J20.*

1 Introduction

This paper aims to consider variational inequality defined by a set-valued mapping (in short, VI(F, K)): To find $x \in K$ and $\xi \in F(x)$ such that

$$\langle \xi, y - x \rangle \geq 0, \quad \forall\, y \in K, \tag{1.1}$$

where $K \subset \mathbb{R}^n$ is a nonempty closed convex set and $F : K \rightrightarrows \mathbb{R}^n$ is a set-valued mapping with nonempty compact convex values. The stationary point of a minimization problem over K is a solution of the problem (1.1), especially when the objective function is not differentiable. In this case, the mapping F is a certain kind of subdifferential of the objective function. The monograph [5] gives a detailed introduction on the theory and algorithms of variational inequality VI(F, K) when the mapping F is single-valued.

The study of properties of variational inequality under data perturbations has been one of the main topics in mathematical programming; see [4, 6, 7, 9–12, 15–17, 20].

Many references discussed stability of a variational inequality. When a solution of variational inequality is locally unique, [7] discussed the stability of this isolated solution to nonlinear complementarity problem, which is a special case of variational inequality; see

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also [6]. Related results are summarized in Section 5.3 of [5]. In this case, the defining mapping in the VI(F, K) is single-valued, and a main tool is the theory of topological degree.

If the solution is not locally unique, [18] discussed the stability of the solution set of a monotone linear complementarity problem, [13] presented a comprehensive study of the stability of the solution set when the defining mapping F is set-valued and maximal monotone (see also [2]), [8] discussed the stability of solution set when the mapping F is set-valued and stably pseudomonotone, which is of a broader class than (maximal) monotone mappings. From these references, it can be seen when the solution is not locally unique, a certain kind of monotonicity is needed for studying the stability of the solution set. Though monotonicity is somewhat restrictive, a merit of of this kind results is that they can apply to variational inequality defined by a set-valued mapping.

When the solution is not locally unique and the mapping has no monotonicity, Chapter 5 of [5] and [14, Proposition 6.2] studied the solvability of the perturbed variational inequality defined by a single-valued mapping. In particular, Corollary 5.5.12 in [5] shows that if a coercivity condition holds, then a perturbed variational inequality VI$(F + q, K)$ has a solution. The assumed coercivity condition implies that the solution set is nonempty and bounded, and the perturbation mapping q is nonlinear. The theoretical tool is topological degree.

This paper presents two results on the solvability of a perturbed variational inequality. Theorem 3.1 extends the aforementioned result of [5] in two ways: the defining mapping is allowed to be set-valued, and the coercivity condition is relaxed. Moreover, we do not use the theory of topological degree. Theorem 3.3 of this paper shows that another coercivity condition can ensure the solvability of a perturbed variational inequality VI$(F-q, K)$, which is new even if the mapping F is single-valued. The assumed coercivity condition implies that VI(F, K) has a solution but the solution set is not necessarily bounded, and the perturbation term q is along a direction in the barrier cone of the set K.

2 Preliminary Results

For $r > 0$, let $K_r := \{x \in K : \|x\| \le r\}$, $\mathbb{B}(0, r) := \{x \in X : \|x\| < r\}$, and $\overline{\mathbb{B}}(0, r) := \{x \in X : \|x\| \le r\}$.

Definition 2.1. $F : K \rightrightarrows \mathbb{R}^n$ is said to be upper semicontinuous on K if for every open set $V \subset \mathbb{R}^n$, the set $\{x \in K : F(x) \subset V\}$ is open in K.

Definition 2.2. F is said to be upper hemicontinuous on K if the restriction of F to every line segment of K is upper semicontinuous.

Definition 2.3. F is said to have *variational inequality property on* K if for every nonempty bounded closed convex subset D of K, VI(F, D) has a solution.

Proposition 2.4 ([9, Proposition 3.2])**.** *The following classes of mappings have the variational inequality property.*

(i) *Every upper semicontinuous set-valued mapping with nonempty compact convex values.*

(ii) *Every upper hemicontinuous and quasimonotone set-valued mapping with nonempty compact convex values.*

Now let us list some coercivity conditions.

(A1) There exists $r > 0$ such that for every $x \in K \setminus K_r$, there is $y \in K$ with $\|y\| < \|x\|$ satisfying $\sup_{x^* \in F(x)} \langle x^*, y - x\rangle \le 0$.

(A2) There exists $r > 0$ such that for every $x \in K \setminus K_r$, there is $y \in K$ with $\|y\| < \|x\|$ satisfying $\sup_{x^* \in F(x)} \langle x^*, y - x \rangle < 0$.

(B1) There exists $r > 0$ such that for every $x \in K \setminus K_r$, there is $y \in K_r$ satisfying $\sup_{x^* \in F(x)} \langle x^*, y - x \rangle \leq 0$.

(B2) There exists $r > 0$ such that for every $x \in K \setminus K_r$, there exists some $y \in K_r$ such that $\sup_{x^* \in F(x)} \langle x^*, y - x \rangle < 0$.

(C1) There exists $y_0 \in K$ such that the set

$$\{x \in K : \sup_{x^* \in F(x)} \langle x^*, y_0 - x \rangle > 0\}$$

is bounded, if nonempty.

(C2) There exists $y_0 \in K$ such that the set

$$\{x \in K : \sup_{x^* \in F(x)} \langle x^*, y_0 - x \rangle \geq 0\}$$

is bounded.

The conditions (A1) and (B1) are proposed in [3]. Single-valued versions of (C1) and (C2) can be seen in [5, Proposition 2.2.3]. The following proposition is easy to see; see Proposition 3.4 in [9].

Proposition 2.5. *The following statements hold.*

(i) *(C2)$\Longrightarrow$(B2)$\Longrightarrow$(A2).*

(ii) *(C1)$\Longrightarrow$(B1)$\Longrightarrow$(A1).*

(iii) *(C2)$\Longrightarrow$(C1), (B2)$\Longrightarrow$(B1), and (A2)$\Longrightarrow$(A1).*

Proof. (iii) holds obviously.

If (C2) holds, take $r > \|y_0\|$, then (B2) holds. If (B2) holds, then (A2) holds obviously. Thus (i) is verified. In a same manner, one can verify (ii). □

So, if F has the variational inequality property on K, then each of (C2), (B2), and (A2) implies that the solution set of variational inequality is bounded (actually, if (A2) holds, then the solution set is contained in K_r), and each of (C1), (B1), and (A1) implies the existence of solution (see Theorem 3.8 in [9]).

3 Solvability of Perturbed Variational Inequality

3.1 Perturbed by Nonlinear Mapping

We use $\mathbb{B}(0; \varepsilon, K_m)$ to denote the set of continuous functions $q : K_m \to \mathbb{R}^n$ satisfying $\|q(x)\| < \varepsilon$ for all $x \in K_m$. $\mathrm{Sol}(F, K)$ is used to denote the solution set of $\mathrm{VI}(F, K)$.

Theorem 3.1. *Assume that F is upper semicontinuous with nonempty compact convex values and the coercivity condition (A2) holds. Then for every $m > r$ there exists $\varepsilon > 0$ such that*

$$\mathrm{Sol}(F + q, K) \cap \overline{\mathbb{B}}(0, m) \neq \emptyset, \quad \forall\, q \in \mathbb{B}(0; \varepsilon, K_m). \tag{3.1}$$

Proof. Let $m > r$. If the conclusion does not hold, then for every $\varepsilon > 0$, there exists $q_\varepsilon \in \mathbb{B}(0;\varepsilon, K_m)$ such that $\mathrm{Sol}(F+q_\varepsilon, K)\cap\overline{\mathbb{B}}(0,m)=\emptyset$. By the definition, K_m is a bounded closed convex set. Take $x_\varepsilon \in \mathrm{Sol}(F+q_\varepsilon, K_m)$. Then $\|x_\varepsilon\| \le m$.

If for some $\varepsilon > 0$, $\|x_\varepsilon\| < m$, then x_ε solves $\mathrm{VI}(F+q_\varepsilon, K)$. Therefore, $x_\varepsilon \in \mathrm{Sol}(F+q_\varepsilon, K)\cap\overline{\mathbb{B}}(0,m)\neq\emptyset$.

If for each $\varepsilon > 0$, $\|x_\varepsilon\| = m$. Without loss of generality, we assume that $\lim_{\varepsilon\to 0+} x_\varepsilon = d \in K$ where $\|d\| = m$. Then the coercivity condition implies the existence of $y_0 \in K$ such that $\|y_0\| < \|d\| = m$ and $\sup_{\xi\in F(d)} \langle \xi, y_0 - d\rangle < 0$.

Since $\sup_{x\in K_m} \|q_\varepsilon(x)\| < \varepsilon$, $\lim_{\varepsilon\to 0+} \langle q_\varepsilon(x_\varepsilon), y_0 - x_\varepsilon\rangle = 0$. Since F is upper semicontinuous with nonempty compact values, it follows that

$$\limsup_{\varepsilon\to 0+}\{\sup_{\xi\in F(x_\varepsilon)} \langle \xi, y_0 - x_\varepsilon\rangle + \langle q_\varepsilon(x_\varepsilon), y_0 - x_\varepsilon\rangle\} \le \sup_{\xi\in F(d)} \langle \xi, y_0 - d\rangle < 0.$$

So there exists $\delta > 0$ such that

$$\sup_{\xi\in F(x_\varepsilon)} \langle \xi, y_0 - x_\varepsilon\rangle + \langle q_\varepsilon(x_\varepsilon), y_0 - x_\varepsilon\rangle < 0, \quad \forall\, \varepsilon \in (0,\delta). \tag{3.2}$$

Since $\|y_0\| < m$, for any given $y \in K$, there is $t \in (0,1)$ such that $y_0 + t(y - y_0) \in K_m$. It follows that for $\varepsilon \in (0,\delta)$,

$$\begin{aligned} 0 &\le \sup_{\xi\in F(x_\varepsilon)} \langle \xi + q_\varepsilon(x_\varepsilon), y_0 + t(y-y_0) - x_\varepsilon\rangle \\ &\le t \sup_{\xi\in F(x_\varepsilon)} \langle \xi + q_\varepsilon(x_\varepsilon), y - x_\varepsilon\rangle + (1-t)\sup_{\xi\in F(x_\varepsilon)} \langle \xi, y_0 - x_\varepsilon\rangle + (1-t)\langle q_\varepsilon(x_\varepsilon), y_0 - x_\varepsilon\rangle \\ &< t \sup_{\xi\in F(x_\varepsilon)} \langle \xi + q_\varepsilon(x_\varepsilon), y - x_\varepsilon\rangle. \end{aligned}$$

This shows that $x_\varepsilon \in \mathrm{Sol}(F+q_\varepsilon, K)$ for sufficiently small $\varepsilon > 0$, and hence $x_\varepsilon \in \mathrm{Sol}(F+q_\varepsilon, K)\cap\overline{\mathbb{B}}(0,m)\neq\emptyset$.

In either case, a contradiction is obtained. □

Remark 3.2. Theorem 3.1 extends Corollary 5.5.12 in [5]. The latter assumes the mapping F is single-valued and the coercivity condition (C2), while Theorem 3.1 allows F to be set-valued and relaxes the coercivity condition.

3.2 Perturbed Along a Direction

In the following, we use K_∞ and $\mathrm{barr}(K)$ to denote the recession cone and the barrier cone of K, respectively. That is,

$$\begin{aligned} K_\infty &:= \{d \in \mathbb{R}^n : \exists t_n \downarrow 0 \text{ and } x_n \in K \text{ such that } t_n x_n \to d\}, \\ \mathrm{barr}(K) &:= \{\xi \in \mathbb{R}^n : \sup_{x\in K} \langle \xi, x\rangle < +\infty\}. \end{aligned}$$

It is known that for a closed convex set, $K_\infty = \{0\}$ if and only if K is bounded; $\mathrm{barr}(K)$ is a convex cone; $\mathrm{barr}(K) = \mathbb{R}^n$ if and only if K is bounded; the recession cone K_∞ is the negative polar cone of the barrier cone $\mathrm{barr}(K)$. One can refer to [19] and [1] for related results. We use $\mathrm{int}(K)$ to denote the interior of K.

Theorem 3.3. *Assume that F has the variational inequality property on K and the coercivity condition (B1) holds. Then for any $q \in \mathrm{int}(\mathrm{barr}(K))$, there is $\delta \in (0, 1/r)$ such that*

$$\mathrm{Sol}(F - \varepsilon\, q, K)\cap\overline{\mathbb{B}}(0, 1/\varepsilon) \neq \emptyset, \quad \forall\, \varepsilon \in (0,\delta).$$

Proof. If the conclusion is not true, then for any $m \in \mathbb{N}$ satisfying $m > r$, there exists $\varepsilon_m > 0$ such that $\varepsilon_m < 1/m$, and $\mathrm{Sol}(F - \varepsilon_m q, K) \cap \overline{\mathbb{B}}(0, \varepsilon_m^{-1}) = \emptyset$.

Let $E_m := \{x \in K : \|x\| \leq \varepsilon_m^{-1}\}$. Since E_m is bounded, $\mathrm{Sol}(F - \varepsilon_m q, E_m)$ is nonempty for each $m \in \mathbb{N}$. Let $x_m \in \mathrm{Sol}(F - \varepsilon_m q, E_m)$.

If there is m such that $\|x_m\| < \varepsilon_m^{-1}$, then $x_m \in \mathrm{Sol}(F - \varepsilon_m q, K)$.

If for each m, $\|x_m\| = \varepsilon_m^{-1}$. Since $\varepsilon_m^{-1} > r$, it follows that $x_m \notin K_r$. By the assumption (B1), there exists $y_m \in K_r$ such that

$$\sup_{\xi \in F(x_m)} \langle \xi, y_m - x_m \rangle \leq 0.$$

Without loss of generality, assume $\frac{x_m}{\|x_m\|} \to d$. Then $d \in K_\infty$ and $d \neq 0$. Since K_∞ is the negative polar cone of the barrier cone $\mathrm{barr}(K)$ and $q \in \mathrm{int}(\mathrm{barr}\, K)$, we have $\langle q, d \rangle < 0$.

For any given $y \in K \setminus K_r$, let $t \in (0,1)$ satisfy $t < \frac{r}{\|y\| - r}$. Then for each m,

$$\|y_m + t(y - y_m)\| \leq t\|y\| + (1-t)\|y_m\| < 2r < \varepsilon_m^{-1},$$

so $y_m + t(y - y_m) \in E_m$. Since $x_m \in \mathrm{Sol}(F - \varepsilon_m q, E_m)$, it follows that

$$\begin{aligned} 0 &\leq \sup_{\xi \in F(x_m)} \langle \xi - \varepsilon_m q, y_m + t(y - y_m) - x_m \rangle \\ &= t \sup_{\xi \in F(x_m)} \langle \xi - \varepsilon_m q, y - x_m \rangle + (1-t) \sup_{\xi \in F(x_m)} \langle \xi, y_m - x_m \rangle + \varepsilon_m (1-t)(\langle q, x_m \rangle - \langle q, y_m \rangle) \\ &\leq t \sup_{\xi \in F(x_m)} \langle \xi - \varepsilon_m q, y - x_m \rangle + \varepsilon_m (1-t)(\langle q, x_m \rangle - \langle q, y_m \rangle). \end{aligned}$$

Since $\|x_m\| = \varepsilon_m^{-1}$, $\varepsilon_m \langle q, x_m \rangle = \left\langle q, \frac{x_m}{\|x_m\|} \right\rangle \to \langle q, d \rangle < 0$. Since $\{y_m\} \subset K_r$, $\{y_m\}$ is bounded, and so $\varepsilon_m \langle q, y_m \rangle \to 0$. It follows that for large enough m,

$$\sup_{\xi \in F(x_m)} \langle \xi - \varepsilon_m q, y - x_m \rangle \geq 0.$$

Therefore, $x_m \in \mathrm{Sol}(F - \varepsilon_m q, K)$.

In either case, we obtain a contradiction to $\mathrm{Sol}(F - \varepsilon_m q, K) \cap \overline{\mathbb{B}}(0, \varepsilon_m^{-1}) = \emptyset$. □

Corollary 3.4. *Assume that F is single-valued and continuous on K. If for some $y_0 \in K$, the set $\{x \in K : \langle F(x), y_0 - x \rangle > 0\}$ if nonempty, is bounded, then for any $q \in \mathrm{int}(\mathrm{barr}(K))$, there is $\delta > 0$ such that*

$$\mathrm{Sol}(F - \varepsilon\, q, K) \cap \overline{\mathbb{B}}(0, 1/\varepsilon) \neq \emptyset, \quad \forall\, \varepsilon \in (0, \delta).$$

Remark 3.5. It can be seen that the coercivity condition assumed in Corollary 3.4 is a single-valued version of the condition (C1) listed in the last section. To the best of our knowledge, this result is the first one to deal with stability of variational inequality under such weak coercivity condition.

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