Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 1, No. 3, pp. 625–630, September 2005

ON THE LEVEL-BOUNDEDNESS OF THE NATURAL RESIDUAL FUNCTION FOR VARIATIONAL INEQUALITY PROBLEMS*

Nobuo Yamashita and Masao Fukushima

Abstract: In this paper we consider the level-boundedness of the natural residual function for the variational inequality problem (VIP). We first introduce the concept of strong coercivity for vector-valued mappings. The strong coercivity is related to the weak coercivity of vector-valued mappings and is weaker than the strong monotonicity of mappings. We show that the natural residual function associated with VIP is level-bounded under the strong coercivity of the mapping involved in the VIP.

Key words: *variational inequality problem, level-boundedness, coercivity, natural residual*

Mathematics Subject Classification: *47J20, 65K10, 90C30*

1 Introduction

The purpose of this paper is to give a sufficient condition under which the natural residual function for the variational inequality problem is level-bounded. The variational inequality problem (VIP) [2] is to find a vector $x \in S$ such that

$$\langle F(x), y - x \rangle \geq 0 \quad \forall y \in S, \tag{1}$$

where $F : R^n \to R^n$, S is a nonempty closed convex subset of R^n, and $\langle \cdot, \cdot \rangle$ denotes the inner product. Throughout we assume that S is not a singleton. We note that the continuity of F is not assumed.

For solving VIP (1), reformulation approaches have drawn much attention in the last decade [3, 4, 8, 10]. A reformulation approach constructs a system of equations or a minimization problem equivalent to VIP, and solves the equivalent problem instead of the original VIP. The objective function of the equivalent minimization problem is called a merit function. In this approach, it is important to use a merit function which has some desirable properties such as differentiability, level-boundedness and error boundedness [2].

Until now, a number of merit functions for VIP have been proposed. Among them, the natural residual function and the D-gap function are particularly popular merit functions. The natural residual function $r : R^n \to R$ for VIP (1) is defined by

$$r(x) := \|x - [x - F(x)]_+\|,$$

*This work was supported in part by a Grant-in-Aid for Scientific Research from Japan Society for the Promotion of Science.

where $[x]_+$ denotes the projection of x onto S and $\|\cdot\|$ denotes the Euclidean norm. It is well-known that $r(x) = 0$ if and only if x is a solution of VIP. Therefore, we may obtain a solution of VIP by globally minimizing the function r on R^n. The D-gap function was proposed by Peng [7] and further studied by Yamashita et al. [10]. The D-gap function is differentiable everywhere, while the natural residual function is not.

Recently, various descent methods based on the natural residual function or the D-gap function have been proposed [8, 3, 4, 10]. Such methods have global convergence under the level-boundedness of a merit function. A function $f : R^n \to R$ is said to be level-bounded [9] if for every $\alpha \in R$ the set $\{x \in R^n \mid f(x) \le \alpha\}$ is bounded. The level-boundedness of f corresponds to the following property:

$$\lim_{\|x\|\to\infty} f(x) = \infty. \tag{2}$$

Therefore, we may also regard (2) as the definition of the level-boundedness. If f is level-bounded, then a sequence generated by an appropriate descent method for f has at least one accumulation point. Since the D-gap function and the natural residual function have the same growth property [7], the D-gap function is level-bounded if and only if the natural residual function is level-bounded. Therefore we will confine ourselves to investigating conditions under which the natural residual function is level-bounded.

A well-known sufficient condition for the natural residual function r to be level-bounded is the strong monotonicity of F [8]. In this paper, we give a weaker condition for the level-boundedness of r. The condition is related to the weak coercivity of the mapping F and is weaker than the strong monotonicity of F.

2 Strong Coercivity

In this section, we introduce the concept of strong coercivity of a vector-valued mapping and study its properties. First we recall the definition of the weak coercivity of a vector-valued mapping [5].

Definition 2.1. *A mapping $F : R^n \to R^n$ is said to be weakly coercive if there exists $y \in R^n$ such that*

$$\lim_{\|x\|\to\infty} \frac{\langle F(x), x-y\rangle}{\|x-y\|} = +\infty. \tag{3}$$

By definition, for the weak coercivity of the mapping F it is sufficient to have at least one vector y satisfying (3). The following example shows that the natural residual function r is not necessarily level-bounded even if F is weakly coercive. Let $F : R^2 \to R^2$ be defined by

$$F(x) = \begin{cases} \begin{pmatrix} x_2(x_1^2+x_2^2) \\ -x_1(x_1^2+x_2^2)+x_2 \end{pmatrix} & \text{if } x_1 \ge 0 \\ \begin{pmatrix} x_2(x_1^2+x_2^2)+x_1^3 \\ -x_1(x_1^2+x_2^2)-2x_1x_2+x_2 \end{pmatrix} & \text{if } x_1 < 0. \end{cases} \tag{4}$$

This mapping is weakly coercive since

$$\lim_{\|x\|\to\infty} \frac{\langle F(x), x-y\rangle}{\|x-y\|} = +\infty \tag{5}$$

holds with $y = (0,1)^T$. Now let $S = \{(x_1,0)^T \in R^2 \mid x_1 \ge 0\}$. Then the natural residual function r is not level-bounded. To see this, consider the sequence $x^k = (k,0)^T, k = 1,2,\cdots$.

Then we have $F(x^k) = (0, -k^3)^T$. Moreover, $x^k - F(x^k) = (k, k^3)^T$, and hence $[x^k - F(x^k)]_+ = (k, 0)^T$. Therefore $r(x^k) = \|x^k - [x^k - F(x^k)]_+\| = 0$ for all k, that is, r is not level-bounded.

To give a sufficient condition for the level-boundedness of r, we define the following stronger concept of coercivity.

Definition 2.2. *Mapping $F : R^n \to R^n$ is said to be strongly coercive if*

$$\lim_{\|x\|\to\infty} \frac{\langle F(x), x - y\rangle}{\|x - y\|} = +\infty$$

holds for all $y \in R^n$.

The strong coercivity implies the weak coercivity, but not vice versa. To see this, consider the weak coercive mapping $F : R^2 \to R^2$ defined by (4). Since

$$\langle F(x), x\rangle = 0$$

for all x such that $x_1 \geq 0$ and $x_2 = 0$, (5) fails to hold for $y = 0$, implying F is not strongly coercive.

Recall that a mapping $F : R^n \to R^n$ is said to be strongly monotone if there exists a constant $\mu > 0$ such that

$$\langle F(x) - F(y), x - y\rangle \geq \mu \|x - y\|^2$$

for all $x, y \in R^n$. It is easy to see that any strongly monotone mapping F is strongly coercive. The converse is not true in general. For example, the mapping $F : R \to R$ defined by

$$F(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{|x|} & \text{if } x < 0 \end{cases}$$

is strongly coercive. But this mapping is not strongly monotone, although it is monotone.

Next, we give a condition equivalent to the strong coercivity, which will be useful in the subsequent analysis.

Theorem 2.1 *The mapping $F : R^n \to R^n$ is strongly coercive if and only if, for any bounded set $B \subseteq R^n$, any sequence $\{y^k\} \subseteq B$, and any sequence $\{x^k\}$ such that $\|x^k\| \to \infty$, we have*

$$\lim_{k\to\infty} \frac{\langle F(x^k), x^k - y^k\rangle}{\|x^k - y^k\|} = +\infty. \tag{6}$$

Proof. The "if" part is evident. So we show the "only if" part. Suppose that F is strongly coercive. Let B be any bounded subset of R^n. First, we show

$$\lim_{k\to\infty} \frac{\langle F(x^k), x^k - y^k\rangle}{\|x^k\|} = +\infty. \tag{7}$$

Since B is bounded, there exists a bounded box $L := \{x \in R^n \mid l \leq x \leq u\}$ containing B, where $l, u \in R^n$ are vectors such that $l_i \leq u_i, i = 1, \ldots, n$. Since $B \subseteq L$, we have

$$\begin{aligned} \frac{\langle F(x^k), x^k - y^k\rangle}{\|x^k\|} &\geq \inf_{y\in B} \frac{\langle F(x^k), x^k - y\rangle}{\|x^k\|} \\ &\geq \inf_{y\in L} \frac{\langle F(x^k), x^k - y\rangle}{\|x^k\|}. \end{aligned} \tag{8}$$

Since L is a bounded box, the infimum on the right-hand side of (8) is attained at an extreme point of L. Let T be the set of extreme points of L. Then, by the strong coercivity of F,

$$\lim_{k\to\infty} \frac{\langle F(x^k), x^k - y\rangle}{\|x^k\|} = +\infty$$

holds for each $y \in T$. Since T is a finite set, we have

$$\begin{aligned}\lim_{k\to\infty} \min_{y\in L} \frac{\langle F(x^k), x^k - y\rangle}{\|x^k\|} &= \lim_{k\to\infty} \min_{y\in T} \frac{\langle F(x^k), x^k - y\rangle}{\|x^k\|} \\ &= +\infty.\end{aligned}$$

It then follows from (8) that (7) holds. Moreover, from the triangle inequality and the boundedness of $\{y^k\}$, we have

$$\begin{aligned}\lim_{k\to\infty} \frac{\langle F(x^k), x^k - y^k\rangle}{\|x^k - y^k\|} &\geq \lim_{k\to\infty} \frac{\langle F(x^k), x^k - y^k\rangle}{\|x^k\| + \|y^k\|} \\ &\geq \lim_{k\to\infty} \frac{\langle F(x^k), x^k - y^k\rangle}{2\|x^k\|}.\end{aligned}$$

This along with (7) shows (6). □

3 Level-Boundedness of the Natural Residual Function

We give the main result of the paper.

Theorem 3.1 *If F is strongly coercive, then the natural residual function r is level-bounded.*

Proof. Let y be any point belonging to S and let $\{x^k\}$ be any sequence such that $\|x^k\| \to \infty$. Suppose that there exists an $M > 0$ such that $r(x^k) \leq M$ for all k. Then $\{x^k - [x^k - F(x^k)]_+\}$ are bounded. Since $\{x^k\}$ is unbounded, $\{[x^k - F(x^k)]_+\}$ is also unbounded. Without loss of generality, we assume that there exists a positive constant τ such that $\|[x^k - F(x^k)]_+ - y\| > \tau$ for all k.

From the well-known property [1, Proposition B.11 (b)] of the projection mapping, we have

$$\langle x^k - F(x^k) - [x^k - F(x^k)]_+, y - [x^k - F(x^k)]_+\rangle \leq 0 \tag{9}$$

for all k. On the other hand, we have

$$\begin{aligned}&\langle x^k - [x^k - F(x^k)]_+ - F(x^k), y - [x^k - F(x^k)]_+\rangle \\ &= \langle x^k - [x^k - F(x^k)]_+, y - [x^k - F(x^k)]_+\rangle + \langle F(x^k), [x^k - F(x^k)]_+ - y\rangle \\ &\geq -\|x^k - [x^k - F(x^k)]_+\|\|[x^k - F(x^k)]_+ - y\| + \langle F(x^k), [x^k - F(x^k)]_+ - y\rangle \\ &\geq \|[x^k - F(x^k)]_+ - y\| \left(\frac{\langle F(x^k), [x^k - F(x^k)]_+ - y\rangle}{\|[x^k - F(x^k)]_+ - y\|} - M\right),\end{aligned} \tag{10}$$

where the first inequality follows from Cauchy-Schwarz inequality and the last inequality follows from $r(x^k) = \|x^k - [x^k - F(x^k)]_+\| \leq M$. Let $y^k = x^k - [x^k - F(x^k)]_+ + y$ for all k. Then $\{y^k\}$ is bounded. Moreover, we have

$$\lim_{k\to\infty} \frac{\langle F(x^k), [x^k - F(x^k)]_+ - y\rangle}{\|[x^k - F(x^k)]_+ - y\|} = \lim_{k\to\infty} \frac{\langle F(x^k), x^k - y^k\rangle}{\|x^k - y^k\|} = \infty,$$

where the last equality follows from the strong coercivity of F and Theorem 2.1. Hence the right-hand side of (10) eventually becomes positive as k tends to ∞. This contradicts (9). Consequently we have $r(x^k) \to \infty$. Since $\{x^k\}$ is arbitrary, r is level-bounded. □

Note that, since the D-gap function has the same growth property as the natural residual function, the D-gap function is also level-bounded if F is strongly coercive. Moreover, this theorem immediately yields the following local error bound result.

Corollary 3.1 *If F is strongly monotone and locally Lipschitz continuous, then for any $b_1 > 0$, there exists a $b_2 > 0$ such that*

$$r(x) \leq b_1 \quad \Rightarrow \quad b_2\|x - x^*\| \leq r(x),$$

where x^* is the unique solution of VIP.

Proof. Since the strong monotonicity of F implies the strong coercivity of F, the natural residual function r is level-bounded. So the level sets $\{x \mid r(x) \leq b_1\}$ are bounded for all b_1. Therefore, by the local Lipschitzian property of F, for any fixed $b_1 > 0$, there exists an $L > 0$ such that

$$\|F(y) - F(z)\| \leq L\|y - z\|$$

for all $y, z \in \{x \mid r(x) \leq b_1\}$. It then follows from [6, Theorem 3.1] that we have the desired property. □

4 Concluding Remarks

In this paper we have introduced the concept of strong coercivity for a vector-valued mapping. Moreover, we showed that the natural residual function associated with VIP is level-bounded under the strong coercivity of the mapping involved in the VIP.

The definition of strong coercivity does not depend on the feasible set S, whereas the level-boundedness of a merit function usually depends on S. Therefore it would be interesting to consider conditions that take into account the structure of the set S. For example, let S be defined as $S := \prod_{i=1}^{N} S_i$ with $S_i \subseteq R^{n_i}$ and $n = \sum_{i=1}^{N} n_i$. Then we can show that the natural residual function is level-bounded provided that, for all $y \in S$,

$$\lim_{\|x\|\to\infty} \max_{1\leq i\leq N} \frac{\langle F_i(x), x_i - y_i\rangle}{\|x - y\|} = +\infty, \tag{11}$$

where $F_i(x), x_i$ and y_i are n_i-dimensional vectors such that $F(x) = (F_1(x)^T, \cdots, F_N(x)^T)^T$, $x = (x_1^T, \cdots, x_N^T)^T$ and $y = (y_1^T, \cdots, y_N^T)^T$. Note that a uniform P-function satisfies (11) when S is the nonnegative orthant in R^n, that is, when VIP is a nonlinear complementarity problem.

Acknowledgement

We are grateful to Professor J.-S. Pang of Rensselaer Polytechnic Institute for his helpful comments.

References

[1] D.P. Bertsekas, *Nonlinear Programming*, Athena Scientific, Belmont, 1995.

[2] F. Facchinei and J.-S. Pang, *Finite-dimensional variational inequalities and complementarity problems*, Springer-Verlag, New York, 2003.

[3] H. Jiang, M. Fukushima, L. Qi and D. Sun, A trust region method for solving generalized complementarity problems, *SIAM Journal on Optimization* 8 (1998) 140–157.

[4] C. Kanzow and M. Fukushima, Solving box constrained variational inequalities by using the natural residual with D-gap function globalization, *Operations Research Letters* 23 (1998) 45–51.

[5] J.M. Ortega and W.C. Rheinboldt, *Iterative solution of nonlinear equations in several variables*, Academic Press, New York, 1970.

[6] J.-S. Pang, A posteriori error bounds for the linearly–constrained variational inequality problem, *Mathematics of Operations Research* 12 (1987) 474–484.

[7] J.-M. Peng, Equivalence of variational inequality problems to unconstrained minimization, *Math. Program.* 78 (1997) 347–355.

[8] J.M. Peng and M. Fukushima, A hybrid Newton method for solving the variational inequality problem via the D-gap function, *Math. Program.* 86 (1999) 367–386.

[9] R.T. Rockafellar and R.J-B. Wets, *Variational Analysis*, Springer-Verlag, Heidelberg, 1998.

[10] N. Yamashita, K. Taji and M. Fukushima, Unconstrained optimization reformulations of variational inequality problems, *Journal of Optimization Theory and Applications* 92 (1997) 439–456.

Manuscript received 23 May 2005
revised 6 July 2005
accepted for publication 6 July 2005

Nobuo Yamashita
Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan
E-mail address: nobuo@i.kyoto-u.ac.jp

Masao Fukushima
Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan
E-mail address: fuku@i.kyoto-u.ac.jp