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EXISTENCE OF PERIODIC MINIMIZERS FOR A CLASS OF INFINITE HORIZON VARIATIONAL PROBLEMS

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Abstract: In this paper we study infinite horizon variational problems arising in continuum mechanics and establish the existence of a periodic minimizer.

Key words: *good function, infinite horizon, periodic minimizer*

Mathematics Subject Classification: *49J99*

1 Introduction

The study of variational problems and optimal control problems defined on infinite intervals has recently been a rapidly growing area of research. These problems arise in engineering [1, 2], in models of economic growth [9, 10, 16, 17, 22, 23], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [3, 18] and in the theory of thermodynamical equilibrium for materials [5, 8, 12-15, 19-21].

In this paper we consider the following problem on the half line:

$$\inf \left\{ \liminf_{T\to\infty} T^{-1} \int_0^T f(w(t), w'(t), w''(t))dt : \ w \in A_x \right\}, \qquad (P_\infty)$$

where

$$w \in A_x = \{v \in W^{2,1}_{loc}[0,\infty) \colon (v(0), v'(0)) = x\}.$$

Here $W^{2,1}_{loc}([0,\infty)) \subset C^1$ denotes the Sobolev space of functions possessing a locally integrable second derivative and f belongs to a space of functions to be described below.

The interest in variational problems of the form (P_∞) stems from the theory of thermodynamical equilibrium for second-order materials developed in [5, 8, 11, 12].

Denote by $\mathfrak{A}$ the set of all continuous functions $f\colon R^3 \to R$ such that for each $N > 0$ the function $|f(x, y, z)| \to \infty$ as $|z| \to \infty$ uniformly on the set $\{(x, y) \in R^2 \colon |x|, |y| \le N\}$. For the set $\mathfrak{A}$ we consider the uniformity which is determined by the following base

$$\begin{aligned} E(N,\epsilon,\Gamma) = \{(f,g) \in \mathfrak{A}\times\mathfrak{A} : \\ |f(x_1,x_2,x_3) - g(x_1,x_2,x_3)| \le \epsilon \ (x_i \in R,\ |x_i| \le N,\ i=1,2,3), \\ (|f(x_1,x_2,x_3)|+1)(|g(x_1,x_2,x_3)|+1)^{-1} \in [\Gamma^{-1},\Gamma] \\ ((x_1,x_2,x_3)\in R^3,\ |x_1|,|x_2|\le N)\}, \end{aligned} \qquad (1.1)$$

where $N > 0, \epsilon > 0, \Gamma > 1$ [6]. Clearly, the uniform space $\mathfrak{A}$ is Hausdorff and has a countable base. Therefore $\mathfrak{A}$ is metrizable (by a metric ρ). It is easy to verify that the uniform space $\mathfrak{A}$ is complete.

Let $a = (a_1, a_2, a_3, a_4) \in R^4$, $a_i > 0$ $(i = 1, 2, 3, 4)$ and let α, β, γ be positive numbers such that $1 \leq \beta < \alpha$, $\beta \leq \gamma$, $\gamma > 1$. Denote by $\mathfrak{M}(\alpha, \beta, \gamma, a)$ the set of all functions $f \in \mathfrak{A}$ such that:

$$f(w, p, r) \geq a_1|w|^\alpha - a_2|p|^\beta + a_3|r|^\gamma - a_4,\ (w, p, r) \in R^3; \tag{1.2}$$

$$f,\ \partial f/\partial p \in C^2,\ \partial f/\partial r \in C^3,\ \partial^2 f/\partial r^2(w, p, r) > 0 \text{ for all } (w, p, r) \in R^3; \tag{1.3}$$

there is a monotone increasing function $M_f : [0, \infty) \to [0, \infty)$ such that for every $(w, p, r) \in R^3$

$$\begin{aligned} \sup\{f(w, p, r),\ |\partial f/\partial w(w, p, r)|,\ |\partial f/\partial p(w, p, r)|,\ |\partial f/\partial r(w, p, r)|\} \\ \leq M_f(|w| + |p|)(1 + |r|^\gamma). \end{aligned} \tag{1.4}$$

Denote by $\bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ the closure of $\mathfrak{M}(\alpha, \beta, \gamma, a)$ in $\mathfrak{A}$. Leizarowitz and Mizel [8] and Coleman, Marcus and Mizel [5] considered problems of type (P_∞) with integrands $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$ in order to study certain models in in the theory of thermodynamical equilibrium for materials. For these models integrands f have the minus signs in (1.2). A typical example is an integrand

$$f(w, p, r) = \psi(w) - bp^2 + cr^2,\ (w, p, r) \in R^3,$$

where b, c are positive constants and $\psi(\cdot)$ is a smooth function satisfying

$$\psi(w) \geq a|w|^\alpha - d,\ w \in R$$

for some $\alpha > 2$, $a, d > 0$ [8, Section 6].

Consider any $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$. Since (1.2) has negative terms the function f can be unbounded. Nevertheless, as it was shown in [14, Lemma 2.2] the corresponding integral functional is bounded from below on any bounded interval. Of special interest is the minimal long-run average cost growth rate

$$\mu(f) = \inf\left\{\liminf_{T\to+\infty} T^{-1} \int_0^T f(w(t), w'(t), w''(t))dt\colon w \in A_x\right\}. \tag{1.5}$$

Using the boundedness from below of the integral functional it is not difficult to verify that $\mu(f)$ is well defined and is independent of the initial vector x. A function $w \in W^{2,1}_{loc}[0, \infty)$ is called an (f)-good function if the function $\phi^f_w\colon T \to \int_0^T [f(w(t), w'(t), w''(t)) - \mu(f)]dt$, $T \in (0, \infty)$ is bounded. For every $w \in W^{2,1}_{loc}[0, \infty)$ the function ϕ^f_w is either bounded or diverges to $+\infty$ as $T \to +\infty$ and moreover, if ϕ^f_w is a bounded function, then $\sup\{|(w(t), w'(t))|\colon t \in [0, \infty)\} < \infty$ [20]. This fact is a continuous version of a result by Leizarowitz [7] established for discrete time control systems. Its proof is based on the result of Leizarowitz [7] applied to a function U^f_T which is defined below.

Leizarowitz and Mizel [8] established that for every $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$ satisfying $\mu(f) < \inf\{f(w, 0, s)\colon (w, s) \in R^2\}$ there exists a periodic (f)-good function. In Zaslavski [19] it was shown that this resut is valid for every $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$. In this paper we generalize the main result of [19] and establish the existence of a periodic (f)-good function for any $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$.

Let $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$. For each $T > 0$ define a function $U_T^f\colon R^2 \times R^2 \to R$ by

$$U_T^f(x,y) = \inf\left\{ \int_0^T f(w(t), w'(t), w''(t)dt \colon w \in A_{x,y}^T \right\}, \tag{1.6}$$

where

$$A_{x,y}^T = \{v \in W^{2,1}[0,T] \colon (v(0), v'(0)) = x,\ (v(T), v'(T)) = y\}. \tag{1.7}$$

In [8], analyzing the problem (P_∞) Leizarowitz and Mizel studied the function $U_T^f\colon R^2 \times R^2 \to R$, $T > 0$ and established the following representation formula

$$U_T^f(x,y) = T\mu(f) + \pi^f(x) - \pi^f(y) + \theta_T^f(x,y),\ x, y \in R^2,\ T > 0, \tag{1.8}$$

where $\pi^f\colon R^2 \to R$ and $(T, x, y) \to \theta_T^f(x,y),\ x, y \in R^2,\ T > 0$ are continuous functions,

$$\pi^f(x) = \inf\left\{ \liminf_{T\to\infty} \int_0^T [f(w(t), w'(t), w''(t)) - \mu(f)]dt \colon w \in A_x \right\},\ x \in R^2, \tag{1.9}$$

$\theta_T^f(x,y) \geq 0$ for each $T > 0$, and each $x, y \in R^2$, and for every $T > 0$, and every $x \in R^2$ there is $y \in R^2$ satisfying $\theta_T^f(x,y) = 0$.

Leizarowitz and Mizel established the representation formula for any integrand $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$, but their result also holds for every $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ without change in the proofs.

In this work we establish the following result.

Theorem 1.1. *Let $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$. Then there exist an (f)-good function $v_f \in W^{2,\gamma}[0,\infty)$ and $T_f > 0$ such that $v_f(t + T_f) = v_f(t)$ for each $t \geq 0$. Moreover, if*

$$\mu(f) < \inf\{f(t, 0, 0) :\ t \in R\},$$

then there is $T_{f0} \in (0, T_f)$ such that v_f is strictly increasing in $[0, T_{f0}]$ and strictly decreasing in $[T_{f0}, T_f]$.

The existence of a periodic (f)-good function is a a central problem in the theory of thermodynamical equilibrium for second-order materials developed in [5, 8, 11, 12]. Our main result, Theorem 1.1, establishes the existence of periodic (f)-good functions and describes their structure for integrands f belonging to the space $\bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ which is essentially larger than the space $\mathfrak{M}(\alpha, \beta, \gamma, a)$. The full description of integrands belonging to the space $\bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ was obtained in [24]. The space $\bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ contains integrands which are not smooth. It should be mentioned that in [8, 19] the proofs of the existence of periodic (f)-good functions for integrands $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$ are strongly based on the smoothness of the integrands. In [20] we established that there exists a set $\mathcal{F} \subset \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ which is a countable intersection of open everywhere dense subsets of $\bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ such that for each $f \in \mathcal{F}$ the following turnpike property holds:

There exists a nonempty compact set $H(f) \subset R^2$ such that for each (f)-good function v

$$H(f) = \{z \in R^2 :\ \text{there is a sequence } \{t_i\}_{i=1}^\infty \subset (0,\infty)$$

$$\text{which satisfies } \lim_{i\to\infty} t_i = \infty \text{ and } \lim_{i\to\infty} (v(t_i), v'(t_i)) = z\}.$$

Now it follows from Theorem 1.1 that for any $f \in \mathcal{F}$,

$$H(f) = \{(w(t), w'(t)) : \ t \in [0, \infty)\},$$

where w is a periodic (f)-good function.

2 Proof of Theorem 1.1

We preface the proof of our main result with the following explanation.

Assume that $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$. Then there exists a sequence

$$\{f_n\}_{n=1}^{\infty} \subset \mathfrak{M}(\alpha, \beta, \gamma, a)$$

such that

$$\lim_{n\to\infty} f_n = f \text{ in } \mathfrak{A}.$$

By the result of [19] for each natural number n there exists a periodic (f_n)-good function w_n with a minimal period τ_n. We will show that if the sequence of $\{\tau_n\}_{n=1}^{\infty}$ is bounded, then a subsequence of $\{w_n\}_{n=1}^{\infty}$ convreges to a periodic (f)-good function. If the sequence $\{\tau_n\}_{n=1}^{\infty}$ is unbounded, then a subsequence of $\{w_n\}_{n=1}^{\infty}$ will converges to a constant function which will be (f)-good.

We denote by $|\cdot|$ the Euclidean norm in R^n. For $\tau > 0$ and $v \in W^{2,1}[0, \tau]$ we define X_v: $[0, \tau] \to R^2$ as follows:

$$X_v(t) = (v(t), v'(t)), \ t \in [0, \tau]. \tag{2.1}$$

We also use this definition for $v \in W^{2,1}_{loc}[0, \infty)$.

We consider functionals of the form

$$I^f(T_1, T_2, w) = \int_{T_1}^{T_2} f(w(t), w'(t), w''(t))dt \tag{2.2}$$

where $-\infty < T_1 < T_2 < +\infty$, $w \in W^{2,1}[T_1, T_2]$ and $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$.

The following result was established in [21, Proposition 5.2].

Propostion 2.1. *The function $f \to \mu(f)$ is continuous for $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$.*

Corollary 2.1. *If $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ and*

$$\mu(f) < \inf\{f(s, 0, 0) : \ s \in R\},$$

then there exists a neighborhood $\mathcal{U}$ of f in $\bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ such that for each $h \in \mathcal{U}$,

$$\mu(h) < \inf\{h(s, 0, 0) : \ s \in R\}.$$

We set for simplicity

$$\mathfrak{M} = \mathfrak{M}(\alpha, \beta, \gamma, a), \ \bar{\mathfrak{M}} = \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a).$$

The following four results were establsihed in [21].

Propostion 2.2. *[21, Proposition 5.3.] Asuume that* $f \in \bar{\mathfrak{M}}$, $\{f_k\}_{k=1}^{\infty} \subset \bar{\mathfrak{M}}$, $\{T_k\}_{k=1}^{\infty} \subset (0,\infty)$, $T \geq 0$, $\{w_k\}_{k=1}^{\infty} \subset W_{loc}^{2,1}[0,\infty)$, $f_k \to f$ *in* $\bar{\mathfrak{M}}$, $T_k \to T$ *as* $k \to \infty$, $w_k(t+T_k) = w_k(t)$ *for all* $t \in [0,\infty)$, $k = 1,2,\dots,$

$$I^{f_k}(0,T_k,w_k) = \mu(f_k)T_k,\ k = 1,2,\dots.$$

Then there exist $\tau > 0$ *and* $w \in W_{loc}^{2,1}[0,\infty)$*:*

$$w(t+\tau) = w(t) \text{ for all } t \in [0,\tau);\ I^f(0,\tau,w) = \tau\mu(f);$$

if $T > 0$ *then* $\tau = T$*; if* $T = 0$*, then* $w(t) = w(0)$ *for all* $t \in [0,\infty)$*.*

Proposition 2.3. *[21, Proposition 5.1.] Let* $f \in \bar{\mathfrak{M}}$*. Then there exist a neighborhood* $\mathcal{U}$ *of* f *in* $\bar{\mathfrak{M}}$ *and a number* $S > 0$ *such that for every* $g \in \mathcal{U}$ *and every* (g)*-good function* v*,*

$$|X_v(t)| \leq S \text{ for all large enough } t.$$

Proposition 2.4. *[21, Proposition 3.2.] Let* $f \in \bar{\mathfrak{M}}$, $0 < c_1 < c_2 < \infty$, $c_3 > 0$, $\epsilon \in (0,1)$*. Then there exists a neighborhood* V *of* f *in* $\mathfrak{A}$ *such that for every* $g \in V \cap \bar{\mathfrak{M}}$*, every* $T \in [c_1,c_2]$*, and each* $x,y \in R^2$ *satisfying* $|x|,|y| \leq c_3$*, the inequality* $|U_T^f(x,y) - U_T^g(x,y)| \leq \epsilon$ *holds.*

Proposition 2.5. *[21, Proposition 3.1.] Let* $f \in \bar{\mathfrak{M}}$, $0 < c_1 < c_2 < \infty$, $\epsilon > 0$, $D > 0$*. Then there exists a neighborhood* V *of* f *in* $\mathfrak{A}$ *such that for every* $g \in V \cap \bar{\mathfrak{M}}$*, every* $T \in [c_1,c_2]$*, and every* $w \in W^{2,1}[0,T]$ *satisfying*

$$\min\{I^f(0,T,w),\ I^g(0,T,w)\} \leq D$$

the inequality

$$|I^f(0,T,w) - I^g(0,T,w)| \leq \epsilon$$

holds.

We also need the following result (see [14, Lemma 3.1]).

Proposition 2.6. *Let* $f \in \mathfrak{M}$*. Assume that* $w \in W_{loc}^{2,1}[0,\infty)$, $\tau > 0$*,*

$$w(t+\tau) = w(t),\ t \in [0,\infty),\ I^f(0,\tau,w) = \tau\mu(f),$$

$$w(0) = \inf\{w(t):\ t \in R\},\ \text{and } w'(t) \neq 0 \text{ for some } t \in R.$$

Then there exist $\tau_1 > 0$, $\tau_2 > \tau_1$ *such that the function* w *is strictly increasing in* $[0,\tau_1]$*,* w *is strictly decreasing in* $[\tau_1,\tau_2]$*, and*

$$w(\tau_1) = \sup\{w(t):\ t \in [0,\infty)\},\ w(t+\tau_2) = w(t),\ t \in R.$$

Proposition 2.7. *Let* $f \in \bar{\mathfrak{M}}$*. For each* $z \in R^2$ *there exists* $w \in W_{loc}^{2,1}[0,\infty)$ *such that* $(w,w')(0) = z$*, the set* $\{(w,w')(t):\ t \in [0,\infty)\}$ *is bounded and for each* $s > 0$*,*

$$I^f(0,s,w) = s\mu(f) + \pi^f(X_w(0)) - \pi^f(X_w(s)).$$

For the proof see Propositions 1.2, 1.3 in [15]. There the result was proved for $f \in \mathfrak{M}$ but it can be easily extended for $f \in \bar{\mathfrak{M}}$.

Propostion 2.8. *Let $f \in \bar{\mathfrak{M}}$, $w \in W^{2,1}_{loc}[0,\infty)$,*

$$U^f_T(X_w(0), X_w(T)) = I^f(0,T,w)$$

for each $T > 0$, the set $\{X_w(t) : t \in [0,\infty)\}$ is bounded. Then w is (f)-good.

Proposition 2.8 follows from Proposition 2.7 by a simple modofication of the proof of Lemma 2.6 of [14].

Lemma 2.1. *Let $f \in \bar{\mathfrak{M}}$. Assume that $u \in W^{2,1}_{loc}[0,\infty)$,*

$$U^f_t(X_u(0), X_u(t)) = I^f(0,t,u) \text{ for any } t > 0, \tag{2.3}$$

$$\sup\{|X_u(t)| : t \in [0,\infty)\} < \infty \tag{2.4}$$

and

$$\text{either } u'(t) \ge 0 \text{ for all } t \in [0,\infty) \text{ or } u'(t) \le 0 \text{ for all } t \ge 0. \tag{2.5}$$

Then $\mu(f) = \inf\{f(t,0,0) : t \in R^1\}$.

Proof. Clearly u is either increasing on $[0,\infty)$ or decreasing on $[0,\infty)$ and there is

$$d_0 = \lim_{t\to\infty} u(t) \tag{2.6}$$

which is finite by (2.4). For each integer $i \ge 1$ define $u_i \in W^{2,1}_{loc}[0,\infty)$ by

$$u_i(t) = u(t+i),\ t \ge 0. \tag{2.7}$$

It follows from (2.7), (2.4), (2.3) and the continuity of $U^f_T (T > 0)$, that for each natural number $n \ge 1$ the sequence $\{I^f(0,n,u_i)\}_{i=1}^\infty$ is bounded. Combined with (2.4) and (1.2) this implies that for each natural number n

$$\sup\{\int_0^n |u_i''(t)|^\gamma dt : i = 1,2,\dots\} < \infty. \tag{2.8}$$

(2.8) and (2.4) imply that there exist a subsequence $\{u_{i_k}\}_{k=1}^\infty$ and $u_* \in W^{2,\gamma}_{loc}[0,\infty)$ such that for any natural number n

$$(u_{i_k}(t), u'_{i_k}(t)) \to (u_*(t), u'_*(t)) \text{ as } n \to \infty \text{ uniformly in } [0,n], \tag{2.9}$$

$$u''_{i_k} \to u''_* \text{ as } k \to \infty \text{ weakly in } L^\gamma[0,n]. \tag{2.10}$$

(2.9), (2.6) and (2.7) imply that

$$u_*(t) = d_0,\ t \in [0,\infty). \tag{2.11}$$

By the lower semicontinuity of integral functionals [4], (2.7), (2.3), (2.9), (2.10) and the continuity of U_n^f for any natural number n

$$I^f(0,n,u_*) \le \liminf_{k\to\infty} I^f(0,n,u_{i_k}) = \liminf_{k\to\infty} U_n^f(X_{u_{i_k}}(0), X_{u_{i_k}}(n))$$
$$= U_n^f(X_{u_*}(0), X_{u_*}(n))$$

and

$$I^f(0,n,u_*) = U_n^f(X_{u_*}(0), X_{u_*}(n)). \tag{2.12}$$

Proposition 2.8, (2.12) and (2.11) imply that $f(d_0,0,0)=\mu(f)$. The lemma is proved.

Lemma 2.2. *Let* $f \in \bar{\mathfrak{M}}$,

$$\mu(f) < \inf\{f(s,0,0):\ s \in R\} \tag{2.13}$$

$$\{f_n\}_{n=1}^{\infty} \subset \mathfrak{M},\ f_n \to f \ \textit{as}\ n\to\infty\ \textit{in}\ \bar{\mathfrak{M}},$$

and

$$\mu(f_n) < \inf\{f_n(s,0,0):\ s\in R^1\},\ n=1,2,\dots. \tag{2.14}$$

Assume that for each natural number $n \ge 1$,

$$w_n \in W^{2,1}_{loc}[0,\infty),\ \tau_n>0,\ 0<\tau_{n0}<\tau_n, \tag{2.15}$$

$$w_n(t+\tau_n) = w_n(t),\ t\in[0,\infty),\ I^f(0,\tau_n,w_n) = \mu(f_n)\tau_n, \tag{2.16}$$

w_n *is increasing in* $[0,\tau_{n0}]$ *and decreasing in* $[\tau_{n0},\tau_n]$.

Then

$$\sup\{\tau_n:\ n=1,2,\dots\} < \infty$$

and there exist a strictly increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$, *number* $0<\tau_0<\tau$ *and* $w \in W^{2,1}_{loc}[0,\infty)$ *such that*

$$\tau = \lim_{k\to\infty} \tau_{n_k},\ \tau_0 = \lim_{k\to\infty} \tau_{n_k 0},$$

$$w(t+\tau) = w(t),\ t\in[0,\infty),\ I^f(0,\tau,w) = \mu(f)\tau,$$

w *is strictly increasing in* $[0,\tau_0]$ *and strictly decreasing in* $[\tau_0,\tau]$ *and for any integer* $j\ge 0$,

$$w'_{n_k}(t)\to w'(t),\ w_{n_k}(t)\to w(t)\ \textit{as}\ k\to\infty\ \textit{uniformly in}\ [j,j+1],$$

$$w''_{n_k} \to w''\ \textit{weakly in}\ L^{\gamma}[j,j+1]\ \textit{as}\ k\to\infty.$$

Proof. By Proposition 2.3, (2.16) and (2.13)

$$\sup\{|(w_n(t), w'_n(t))|:\ t\in[0,\infty),\ n=1,2\dots\} < \infty. \tag{2.17}$$

(2.17) implies that for each $T>0$

$$\sup\{|U_T^f(X_{w_n}(s), X_{w_n}(s+T))|:\ s\ge 0,\ n=1,2,\dots\} < \infty. \tag{2.18}$$

(2.17), the relation $\lim_{n\to\infty} f_n = f$ and Proposition 2.4 imply that for any $T > 0$

$$|U_T^f(X_{w_n}(s), X_{w_n}(s+T)) - U_T^{f_n}(X_{w_n}(s), X_{w_n}(s+T))| \to 0 \text{ as } n \to \infty \tag{2.19}$$

uniformly in $s \in [0, \infty)$. It follows from (2.16), (2.19), (2.18) and (2.17) that for each $T > 0$ the set

$$\{I^{f_n}(s, s+T, w_n) : s \in [0, \infty), n = 1, 2, \dots\} =$$

$$\{U_T^{f_n}(X_{w_n}(s), X_{w_n}(s+T)) : s \in [0, \infty), n = 1, 2, \dots\}$$

is bounded. Combining with the relation $\lim_{n\to\infty} f_n = f$ and Proposition 2.5 this fact implies that for any $T > 0$

$$|I^f(s, s+T, w_n) - I^{f_n}(s, s+T, w_n)| \to 0 \text{ as } n \to \infty \tag{2.20}$$

uniformly in $s \in [0, \infty)$ and

$$\{I^f(s, s+T, w_n) : s \in [0, \infty), n = 1, 2, \dots\} < \infty. \tag{2.21}$$

We willl show that

$$\sup\{\tau_n : n = 1, 2, \dots\} < \infty.$$

Assume the contrary. Then

$$\sup\{\tau_n : n = 1, 2, \dots\} = \infty.$$

Then one of the following cases holds:

a) $\sup\{\tau_{n0} : n = 1, 2, \dots\} = \infty$, b) $\sup\{\tau_n - \tau_{n0} : n = 1, 2, \dots\} = \infty$. Consider the case a). By the lemma assumptions,

$$w_n'(t) \geq 0, \ t \in [0, \tau_{n_0}], \ n = 1, 2, \dots \tag{2.22}$$

We may assume without loss of generality that

$$\lim_{n\to\infty} \tau_{n0} = \infty. \tag{2.23}$$

It follows from (2.21), (2.17) and the growth condition (1.2) that for each $T > 0$

$$\sup\{\int_s^{s+T} |w_n''(t)|^\gamma dt : s \in [0, \infty), n = 1, 2, \dots\} < \infty.$$

By this inequality and (2.17) there are a subsequence $\{w_{n_k}\}_{k=1}^\infty$ and $w \in W^{2,\gamma}[0, \infty)$ such that for any $T > 0$

$$(w_{n_k}(t), w_{n_k}'(t)) \to (w(t), w'(t)) \text{ as } k \to \infty \tag{2.24}$$

uniformly on $t \in [0, T]$ and

$$w_{n_k}'' \to w'' \text{ weakly in } L^\gamma[0, T] \text{ as } n \to \infty. \tag{2.25}$$

By (2.24), (2.22) and (2.23)

$$w'(t) \geq 0 \text{ for all } t \geq 0. \tag{2.26}$$

By (2.24), (2.25) and the lower semicontinuity of the integral functionals, (2.20), (2.16), (2.19) for each $\tau_1 \ge 0$, $\tau_2 > \tau_1$,

$$I^f(\tau_1,\tau_2,w) \le \liminf_{k\to\infty} I^f(\tau_1,\tau_2,w_{n_k})$$

$$= \liminf_{k\to\infty} I^{f_{n_k}}(\tau_1,\tau_2,w_{n_k}) = \liminf_{k\to\infty} U^{f_{n_k}}_{\tau_2-\tau_1}((w_{n_k}(\tau_1),w'_{n_k}(\tau_1)),(w_{n_k}(\tau_2),w'_{n_k}(\tau_2)))$$

$$= \liminf_{k\to\infty} U^{f}_{\tau_2-\tau_1}((w_{n_k}(\tau_1),w'_{n_k}(\tau_1)),(w_{n_k}(\tau_2),w'_{n_k}(\tau_2))).$$

By these relations, (2.24) and continuity of U^f_T, $T>0$ for each $\tau_1 \ge 0, \tau_2 > \tau_1$

$$I^f(\tau_1,\tau_2,w) \le \liminf_{k\to\infty} U^{f}_{\tau_2-\tau_1}((w_{n_k}(\tau_1),w'_{n_k}(\tau_1)),(w_{n_k}(\tau_2),w'_{n_k}(\tau_2)))$$

$$= U^f_{\tau_2-\tau_1}((w(\tau_1),w'(\tau_1)),(w(\tau_2),w'(\tau_2))).$$

Thus

$$I^f(\tau_1,\tau_2,w) = U^f_{\tau_2-\tau_1}((w(\tau_1),w'(\tau_1)),(w(\tau_2),w'(\tau_2))) \tag{2.27}$$

for each $\tau_1 \ge 0$, $\tau_2 > \tau_1$. By (2.24), (2.17),

$$\sup\{|X_w(t)| :\ t \in [0,\infty)\} < \infty. \tag{2.28}$$

(2.27), (2.28), (2.26) and Lemma 2.1 imply that

$$\mu(f) = \inf\{f(t,0,0) :\ t \in R^1\},$$

a contradiction. Therefore the case a) does not hold and

$$\sup\{\tau_{n0} :\ n = 1,2,\dots\} < \infty.$$

Analogously we can show that the case b) does not hold and

$$\sup\{\tau_n - \tau_{n0} :\ n = 1,2,\dots\} < \infty.$$

Thus

$$\sup\{\tau_n :\ n = 1,2,\dots\} < \infty. \tag{2.29}$$

Extracting a subsequence and re-indexing we may assume without loss of generality that there exists

$$\tau = \lim_{n\to\infty} \tau_n. \tag{2.30}$$

We will show that $\tau > 0$. Indeed assume that $\tau = 0$. By (2.17) there is a constant $c_0 > 0$ such that

$$|w_n(t_2) - w_n(t_1)| \le c_0|t_2 - t_1|$$

for each $t_1, t_2 \in R$ and each natural number n. Combined with (2.16) this implies that for each natural number n

$$\max\{w_n(t) :\ t \in [0,\infty)\} - \min\{w_n(t) :\ t \in [0,\infty)\} \le c_0\tau_n \to 0 \text{ as } n \to \infty.$$

Together with (2.24) this implies that w is a constant and $\mu(f) = \inf\{f(t,0,0):\ t \in R\}$, a contradiction. Therefore $\tau > 0$. By (2.21), (2.17) and the growth condition (1.2), for each $T > 0$

$$\sup\{\int_s^{s+T} |w_n''(t)|^\gamma dt:\ s \in [0,\infty),\ n = 1,2,\dots\} < \infty.$$

By this inequality and (2.17) there are a subsequence $\{w_{n_k}\}_{k=1}^\infty$ and $w \in W^{2,\gamma}[0,\infty)$ such that for any $T > 0$

$$(w_{n_k}(t), w_{n_k}'(t)) \to (w(t), w'(t)) \text{ as } k \to \infty \tag{2.31}$$

uniformly on $t \in [0,T]$ and

$$w_{n_k}'' \to w'' \text{ weakly in } L^\gamma[0,T] \text{ as } n \to \infty. \tag{2.32}$$

By (2.31), (2.32) and the lower semicontinuity of the integral functionals [4], (2.20), (2.16), (2.19) for each $\tau_1 \geq 0$, $\tau_2 > \tau_1$,

$$\begin{aligned} I^f(\tau_1,\tau_2,w) &\leq \liminf_{k\to\infty} I^f(\tau_1,\tau_2,w_{n_k}) = \liminf_{k\to\infty} I^{f_{n_k}}(\tau_1,\tau_2,w_{n_k}) \\ &= \liminf_{k\to\infty} U_{\tau_2-\tau_1}^{f_{n_k}}((w_{n_k}(\tau_1), w_{n_k}'(\tau_1)), (w_{n_k}(\tau_2), w_{n_k}'(\tau_2))) \\ &= \liminf_{k\to\infty} U_{\tau_2-\tau_1}^{f}((w_{n_k}(\tau_1), w_{n_k}'(\tau_1)), (w_{n_k}(\tau_2), w_{n_k}'(\tau_2))). \end{aligned}$$

By these relations, (2.31) and continuity of U_T^f, $T > 0$ for each $\tau_1 \geq 0, \tau_2 > \tau_1$

$$\begin{aligned} I^f(\tau_1,\tau_2,w) &\leq \liminf_{k\to\infty} U_{\tau_2-\tau_1}^{f}((w_{n_k}(\tau_1), w_{n_k}'(\tau_1)), (w_{n_k}(\tau_2), w_{n_k}'(\tau_2))) \\ &= U_{\tau_2-\tau_1}^{f}((w(\tau_1), w'(\tau_1)), (w(\tau_2), w'(\tau_2))). \end{aligned}$$

Thus

$$I^f(\tau_1,\tau_2,w) = U_{\tau_2-\tau_1}^{f}((w(\tau_1), w'(\tau_1)), (w(\tau_2), w'(\tau_2))) \tag{2.33}$$

for each $\tau_1 \geq 0$, $\tau_2 > \tau_1$. By (2.31), (2.17),

$$\sup\{|X_w(t)|:\ t \in [0,\infty)\} < \infty. \tag{2.34}$$

(2.30), (2.31), (2.16) imply that for each $t \in [0,\infty)$

$$w(t+\tau) = \lim_{k\to\infty} w(t+\tau_{n_k}) = \lim_{k\to\infty} w_{n_k}(t+\tau_{n_k}) = \lim_{k\to\infty} w_{n_k}(t) = w(t)$$

and

$$w(t+\tau) = w(t) \text{ for all } t \in [0,\infty). \tag{2.35}$$

(2.35), (2.33) imply that w is an (f)-good function and

$$I^f(0,\tau,w) = \tau\mu(f). \tag{2.36}$$

We may suppose extracting a subsequence and re-indexing, if necessary, that there exists

$$\tau_0 = \lim_{k\to\infty} \tau_{n_k 0}.$$

By (2.31) and the lemma assumptions,

$$w'(t) \geq 0 \text{ for any } t \in [0, \tau_0),\ w'(t) \leq 0 \text{ for any } t \in (\tau_0, \tau).$$

Since $\mu(f) < \inf\{f(t,0,0) :\ t \in R\}$ we conclude that $\tau_0 \in (0,\tau)$, w is strictly increasing in $[0,\tau_0]$ and strictly decreasing in $[\tau_0,\tau]$. This completes the proof of the lemma.

Completion of the proof of Theorem 1.1. If $\mu(f) = \inf\{f(s,0,0) :\ s \in R\}$, then the assertion of the theorem is valild. Therefore we may assume without loss of generality that

$$\mu(f) < \inf\{f(s,0,0) :\ s \in R\}. \tag{2.37}$$

There exists a sequence $\{f_n\}_{n=1}^{\infty} \subset \mathfrak{M}$ such that

$$\lim_{n\to\infty} f_n = f \text{ in } \mathfrak{A}.$$

By Corollary 2.1 we may assume without loss of generality that for each natural number n

$$\mu(f_n) < \inf\{f_n(s,0,0) :\ s \in R^1\}. \tag{2.38}$$

Let $n \geq 1$ be a natural number. By the main result of [19] there exist a periodic (f_n)-good function $w_n \in W^{2,1}_{loc}[0,\infty)$ and $\tau > 0$ such that

$$w_n(t+\tau) = w_n(t),\ t \in [0,\infty),\ I^f(0,\tau,w_n) = \mu(f_n)\tau.$$

We may assume without loss of generality that

$$w_n(0) = \inf\{w_n(0) :\ t \in [0,\infty\}.$$

(Otherwise we consider a translation of w). By (2.38) w_n is not a constant. Now we can see that all the assumptions made in Proposition 2.6 hold with the integrand f_n and the periodic function w_n. Therefore, by Proposition 2.6 there exist

$$\tau_n > 0,\ 0 < \tau_{n0} < \tau_n$$

such that

$$w_n(t+\tau_n) = w_n(t),\ t \in [0,\infty),\ I^f(0,\tau_n,w_n) = \mu(f_n)\tau_n,$$

w_n is increasing in $[0,\tau_{n0}]$ and decreasing in $[\tau_{n0},\tau_n]$. Now we can easy to see that all the assumptions made in Lemma 2.2 hold. Therefore, by Lemma 2.2 there exist numbers $0 < \tau_0 < \tau$ and $w \in W^{2,1}_{loc}[0,\infty)$ such that

$$w(t+\tau) = w(t),\ t \in [0,\infty),\ I^f(0,\tau,w) = \mu(f)\tau,$$

w is strictly increasing in $[0,\tau_0]$ and strictly decreasing in $[\tau_0,\tau]$. This completes the proof of the theorem.

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