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CUT AND SPLIT METHOD FOR THE MINIMUM MAXIMAL FLOW PROBLEM *

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Dedicated to Professor Hiroshi Konno on his 65th birthday.

Abstract: This paper is concerned with the minimum maximal flow problem, i.e., a problem of minimizing the flow value attained by a maximal flow for a given network. The optimal value indicates how inefficiently the network can be utilized under restricted controllability. We discuss the extension of the gap function defining the set of all maximal flows and then formulate the problem as a D.C. optimization problem. Based on this formulation, we propose the cut and split method combined with a local search technique.

Key words: *minimum maximal flow, optimization over the efficient set, D.C. optimization, cut and split, global optimization*

Mathematics Subject Classification: *90B50, 90C35, 90C57, 90C59*

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1 Introduction

We consider a network $(V, s, t, E, \partial^+, \partial^-, \boldsymbol{c})$, where V is the set of $m+2$ nodes containing the source node s and the sink node t, E is the set of n arcs, ∂^+ and ∂^- are incidence functions, and $\boldsymbol{c}$ is the vector of dimension n whose hth element c_h is the capacity of arc h. We assume that each element of $\boldsymbol{c}$ is a positive integer throughout this paper. A vector $\boldsymbol{x}$ of dimension n is said to be a *feasible flow* if it satisfies the inequality $\boldsymbol{0} \leqq \boldsymbol{x} \leqq \boldsymbol{c}$, called the *capacity constraint*, and the equality $A\boldsymbol{x} = \boldsymbol{0}$, called the *conservation constraint*, where the $m \times n$ matrix A is the incidence matrix whose (v, h) element a_{vh} is given by

$$a_{vh} = \begin{cases} +1 & \text{if } \partial^+ h = v \\ -1 & \text{if } \partial^- h = v \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

Let X denote the set of all feasible flows, i.e.,

$$X = \{\, \boldsymbol{x} \in \mathrm{R}^n \mid A\boldsymbol{x} = \boldsymbol{0},\ \boldsymbol{0} \leqq \boldsymbol{x} \leqq \boldsymbol{c} \,\}, \tag{1.2}$$

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and X_V denote the set of all vertices of X. Note that each vertex $\boldsymbol{v} \in X_V$ is an integer vector by the total unimodularity of A and the integrality of the capacity vector $\boldsymbol{c}$. For a feasible flow $\boldsymbol{x}$, the *flow value of* $\boldsymbol{x}$, denoted by $\phi(\boldsymbol{x})$, is given by

$$\phi(\boldsymbol{x}) = \sum_{\partial^+ h = s} x_h - \sum_{\partial^- h = s} x_h. \tag{1.3}$$

Using the row vector $\boldsymbol{d}$ of dimension n whose element d_h is

$$d_h = \begin{cases} +1 & \text{if } \partial^+ h = s \\ -1 & \text{if } \partial^- h = s \\ 0 & \text{otherwise,} \end{cases} \tag{1.4}$$

we can simply write $\phi(\boldsymbol{x}) = \boldsymbol{dx}$. We assume that a given network has no t-s-path, which ensures that $\boldsymbol{dx} \geqq 0$ for all $\boldsymbol{x} \in X$. A feasible flow $\boldsymbol{x}$ is said to be a *maximal flow* if there is no feasible flow $\boldsymbol{y}$ such that $\boldsymbol{y} \geqq \boldsymbol{x}$ and $\boldsymbol{y} \neq \boldsymbol{x}$. We denote the set of all maximal flows by X_M, i.e.,

$$X_M = \{\, \boldsymbol{x} \in X \mid \nexists \boldsymbol{y} \in X \;:\; \boldsymbol{y} \geqq \boldsymbol{x},\ \boldsymbol{y} \neq \boldsymbol{x} \,\}. \tag{1.5}$$

We assume that X_M is nonempty. The problem of minimizing the flow value among maximal flows, called the *minimum maximal flow problem* and abbreviated by (mmF), is written as

$$(mmF) \qquad \left| \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \phi(\boldsymbol{x}) = \boldsymbol{dx} \\ \text{s.t.} & \boldsymbol{x} \in X_M. \end{array} \right.$$

The difficulty of the problem comes from the nonconvexity of X_M, and hence sophisticated algorithms are required. Note that (mmF) embraces the minimum maximal matching problem, which is an NP-hard problem (see e.g. Garay-Johnson [14]).

In the field of network flow theory such as maximum flow problem and minimum cost flow problem, we usually take it for granted that we can control each arc flow, namely we can freely increase and decrease each arc flow as long as the feasibility is met. However, when we are not allowed to decrease arc flows, the solution to be obtained depends on a given initial flow. Under the condition of such uncontrollability, it is meaningful to know the minimum flow value among the maximal flows since this value indicates how inefficiently a network can be utilized. The concept of uncontrollable flow was first raised by Iri [18, 19], which is closely related to but different from the maximal flow. Shi-Yamamoto first raised (mmF) and proposed an algorithm in [29]. After this several algorithms for (mmF) combining local search and global optimization technique have been proposed in e.g. Gotoh-Thoai-Yamamoto [15] and Shigeno-Takahashi-Yamamoto [30]. As will be seen in the next section, (mmF) is a special case of linear optimization problems over the efficient set. The algorithms for (mmF) mentioned above are mainly based on the algorithms for the linear optimization problem over the efficient set. However, we have neither theoretical evidence that these algorithms are efficient, nor comparative study from the viewpoint of computational time.

The purpose of this paper is to propose an algorithm for (mmF) within the framework of D.C. optimization. The algorithm is based on local search and global optimization technique.

In Section 2 we briefly review some fundamental theorems of the linear optimization problem over the efficient set, and then we reformulate the problem by the gap function. We also mention the connectedness of the efficient set. Section 3 is devoted to explaining the cut and split method. In Section 4 we extend the gap function and apply the cut and split method to (mmF). Finally, some conclusion and further works will be discussed in the last section.

Throughout this paper we use the following notations: R^n denotes the set of all real column vectors of dimension n. Let $\mathrm{R}^n_+ = \{\, \boldsymbol{x} \in \mathrm{R}^n \mid \boldsymbol{x} \geqq \mathbf{0} \,\}$ and $\mathrm{R}^n_{++} = \{\, \boldsymbol{x} \in \mathrm{R}^n \mid \boldsymbol{x} > \mathbf{0} \,\}$. Let R_n denote the set of all real row vectors of dimension n, R_{n+} and R_{n++} are defined in the similar way. We use $\boldsymbol{e}$ to denote the row vector of ones, $\mathbf{1}$ to denote the column vector of ones, and $\boldsymbol{e}_i$ to denote the ith unit row or column vector of an appropriate dimension. Let I denote the identity matrix of an appropriate size. We use $\boldsymbol{a}^\top$ and $A^\top$ to denote the transposed vector of $\boldsymbol{a}$ and the transposed matrix of A, respectively. For a set S, we denote the interior of S by $\operatorname{int} S$, the closure of S by $\operatorname{cl} S$, and the relative boundary of S by ∂S. We use P_V to denote the set of all vertices of a polyhedron P. For two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ of dimension n, let $[\boldsymbol{v}, \boldsymbol{w}]$ denote the line segment with endpoints $\boldsymbol{v}$ and $\boldsymbol{w}$.

2 Linear Optimization Problem Over the Efficient Set

Given a polyhedron $D = \{\, \boldsymbol{x} \in \mathrm{R}^n \mid B\boldsymbol{x} \leqq \boldsymbol{z},\ \boldsymbol{x} \geqq \mathbf{0} \,\}$ with $B \in \mathrm{R}^{m \times n}$ and $\boldsymbol{z} \in \mathrm{R}^m$, and a criterion matrix $C \in \mathrm{R}^{p \times n}$ with $p \geqq 2$, the *linear multicriteria problem* is

$$(MC) \qquad \left|\ \begin{array}{ll} \max\limits_{\boldsymbol{x}} & C\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in D. \end{array}\right.$$

The point $\boldsymbol{x} \in D$ is said to be an *efficient point for* (MC) if there is no point $\boldsymbol{y} \in D$ such that $C\boldsymbol{y} \geqq C\boldsymbol{x}$ and $C\boldsymbol{y} \neq C\boldsymbol{x}$. The *efficient set for* (MC), denoted by D_E, is the set of all efficient points for (MC), i.e.,

$$D_E = \{\, \boldsymbol{x} \in D \mid \nexists \boldsymbol{y} \in D : C\boldsymbol{y} \geqq C\boldsymbol{x},\ C\boldsymbol{y} \neq C\boldsymbol{x} \,\}. \tag{2.6}$$

We assume that D is bounded for simplicity. The *linear optimization problem over the efficient set* is

$$(P_E) \qquad \left|\ \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{p}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in D_E, \end{array}\right.$$

where $\boldsymbol{p} \in \mathrm{R}_n$. Note that (mmF) is (P_E) with $D = X$ and $C = I$, hence $D_E = X_M$, and $\boldsymbol{p} = \boldsymbol{d}$. Figure 1 shows a two-dimensional example of the problem (P_E), where $\boldsymbol{c}^i$ is the ith row of C for $i = 1, 2$. The efficient set D_E is depicted by bold lines.

Since Philip first considered (P_E) and proposed an algorithm based on local search and cutting plane technique in [25], a number of papers followed his work. The overview about the efficient set and several algorithms for (P_E) can be found in Yamamoto [37]. For the details about (P_E), the reader should refer to White [36], Sawaragi-Nakayama-Tanino [27] and Steuer [31]. The mathematical structure of the efficient set is studied in Naccache [24], Benson [7] and Hu-Sun [17]. The method enumerating the efficient vertices for (MC) can be found in Ecker-Kouada [10, 11]. For solution methods for (P_E), see Benson [4–6], Bolintineanu [8], Ecker-Song [12], Fülöp [20], Dauer-Fosnaugh [9], Thach-Konno-Yokota [32], Sayin [28], Phong-Tuyen [26], Thoai [33], Muu-Luc [22] and An-Tao-Thoai [3]. D.C. optimization approaches for (P_E) can be seen in An-Tao-Muu [1, 2].

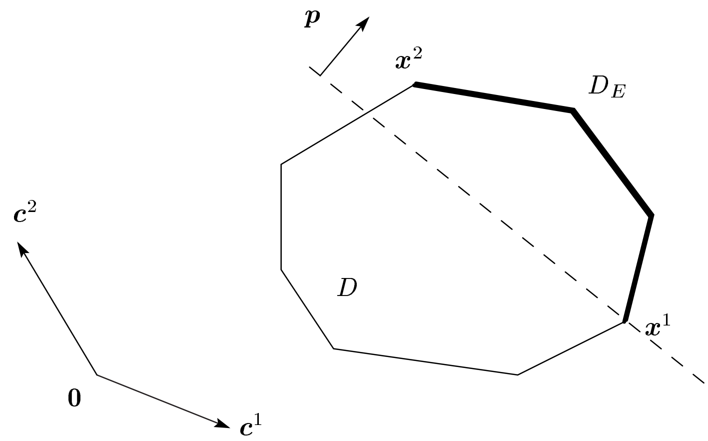


Figure 1: A two-dimensional example of (P_E)

$\boxed{2.1}$ Gap Function and Some Fundamental Theorems

We define the *gap function* $g : \mathrm{R}^n \to \mathrm{R} \cup \{-\infty\}$ by

$$g(\boldsymbol{x}) = \max\{\, \boldsymbol{e}C\boldsymbol{y} \mid \boldsymbol{y} \in D,\ C\boldsymbol{y} \geqq C\boldsymbol{x}\,\} - \boldsymbol{e}C\boldsymbol{x}. \tag{2.7}$$

If there is no point $\boldsymbol{y} \in D$ such that $C\boldsymbol{y} \geqq C\boldsymbol{x}$ then $g(\boldsymbol{x}) = -\infty$. Note that $g(\boldsymbol{x}) \geqq 0$ for all $\boldsymbol{x} \in D$ and g is a concave and piece-wise linear function (see Figure 2). It is easily seen that

$$D_E = \{\, \boldsymbol{x} \in D \mid g(\boldsymbol{x}) \leqq 0\,\}. \tag{2.8}$$

Indeed, a point $\boldsymbol{x} \in D$ is in D_E if and only if $C\boldsymbol{y} = C\boldsymbol{x}$ for all $\boldsymbol{y} \in D$ such that $C\boldsymbol{y} \geqq C\boldsymbol{x}$, that is equivalent to $g(\boldsymbol{x}) \leqq 0$. Then the alternative form of (P_E) is

$$(P_E) \qquad \left|\begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{p}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in D,\ -g(\boldsymbol{x}) \geqq 0. \end{array}\right.$$

Since $-g$ is a convex function, the inequality constraint $-g(\boldsymbol{x}) \geqq 0$ is called a *reverse convex constraint*, and hence (P_E) is a *linear reverse convex problem*, which is one of D.C. optimization problems (See Tuy [35] and Horst-Tuy [16]). Moreover, it is known that the above alternative form of (P_E) can be cast into the problem

$$(P_E(t)) \qquad \left|\begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{p}\boldsymbol{x} + tg(\boldsymbol{x}) \\ \text{s.t.} & \boldsymbol{x} \in D, \end{array}\right.$$

where $t > 0$ is an exact penalty parameter. Note that a tight exact penalty parameter is given in e.g. An-Tao-Muu [1] and Dauer-Fosnaugh [9]. When we consider (mmF), the problem

$$(mmF(t)) \qquad \left|\begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{d}\boldsymbol{x} + tg(\boldsymbol{x}) \\ \text{s.t.} & \boldsymbol{x} \in X, \end{array}\right.$$

is equivalent to (mmF) for any $t > t^* = \max\{\, \boldsymbol{dx} \mid \boldsymbol{x} \in X \,\} - \min\{\, \boldsymbol{dx} \mid \boldsymbol{x} \in X \,\}$ (See Muu-Shi [23]).

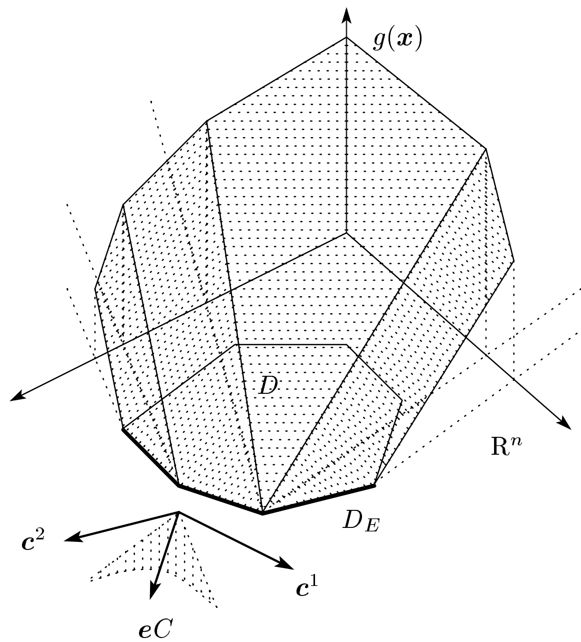


Figure 2: An example of the gap function g

We introduce some important theorems about (P_E), whose proofs can be found in e.g. Steuer [31] and White [36]. We will outline some of the proofs to make this paper self-contained.

Theorem 2.1. *The point $\bar{\boldsymbol{x}} \in D$ is an efficient point for (MC) if and only if there exists $\boldsymbol{\lambda} \in \mathrm{R}_{p++}$ such that $\bar{\boldsymbol{x}}$ is an optimal solution of the single criterion problem*

$$(SC(\boldsymbol{\lambda})) \qquad \left|\begin{array}{ll} \max\limits_{\boldsymbol{x}} & \boldsymbol{\lambda} C\boldsymbol{x} \\ s.t. & \boldsymbol{x} \in D. \end{array}\right.$$

Furthermore, there exists $M > 0$ such that we can replace the above R_{p++} with

$$\Lambda = \{\, \boldsymbol{\lambda} \in \mathrm{R}_{p++} \mid \boldsymbol{\lambda} \geqq \boldsymbol{e},\ \boldsymbol{\lambda}\boldsymbol{1} = M \,\}. \tag{2.9}$$

Proof. See Appendix A. □

By Theorem 2.1, we have only to choose $\boldsymbol{\lambda} \in \Lambda$ and solve $(SC(\boldsymbol{\lambda}))$ to obtain an initial point in $D_E \cap D_V$. It was shown in Shigeno-Takahashi-Yamamoto [30] that n^2 suffices for M of (2.9), when we consider (mmF). The following theorem is also well known.

Theorem 2.2. *The set D_E is a connected union of the several faces of D.*

See Steuer [31] and Naccache [24] for the detail.

$\boxed{3}$ D.C. Optimization

A set S is said to be a *D.C. set* (difference of two convex sets) if $S = Q \backslash R$ for two convex sets Q and R. Similarly, a function f is said to be a *D.C. function* if $f = q - r$ for two convex functions q and r. The optimization problem described in terms of D.C. sets and/or D.C. functions is called a *D.C. optimization problem*. D.C. optimization problem covers many of nonlinear programming problems such as location planning problem, engineering design problem, multilevel programming problem, and optimization problem over the efficient set.

Let $D = \{ \boldsymbol{x} \in \mathrm{R}^n \mid f(\boldsymbol{x}) \leqq 0 \}$, where $f : \mathrm{R}^n \to \mathrm{R} \cup \{+\infty\}$ is a convex function. We assume that D is bounded for simplicity. Let $h : \mathrm{R}^n \to \mathrm{R} \cup \{+\infty\}$ be a convex function and assume

$$\operatorname{int} \{ \boldsymbol{x} \in \mathrm{R}^n \mid h(\boldsymbol{x}) \leqq 0 \} = \{ \boldsymbol{x} \in \mathrm{R}^n \mid h(\boldsymbol{x}) < 0 \}. \tag{3.10}$$

In this section, we focus on a *canonical form D.C. problem*:

$$(CDC) \qquad \left| \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{p}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in D, \ h(\boldsymbol{x}) \geqq 0, \end{array} \right.$$

where $\boldsymbol{p} \in \mathrm{R}_n$. Letting $H = \{ \boldsymbol{x} \in \mathrm{R}^n \mid h(\boldsymbol{x}) \leqq 0 \}$, (CDC) can be written as

$$(CDC) \qquad \left| \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{p}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in D \backslash \operatorname{int} H. \end{array} \right.$$

In the case where D is a polyhedron given by $D = \{ \boldsymbol{x} \in \mathrm{R}^n \mid B\boldsymbol{x} \leqq \boldsymbol{z}, \ \boldsymbol{x} \geqq \boldsymbol{0} \}$ with $B \in \mathrm{R}^{m \times n}$ and $\boldsymbol{z} \in \mathrm{R}^m$, the above problem is called a *linear reverse convex problem* $(LRCP)$. Figure 3 shows two-dimensional examples of the problems (CDC) and $(LRCP)$, respectively. Below we explain the cut and split method for $(LRCP)$. For other algorithms on D.C. optimization, the reader should refer to Tuy [34,35] and Horst-Tuy [16].

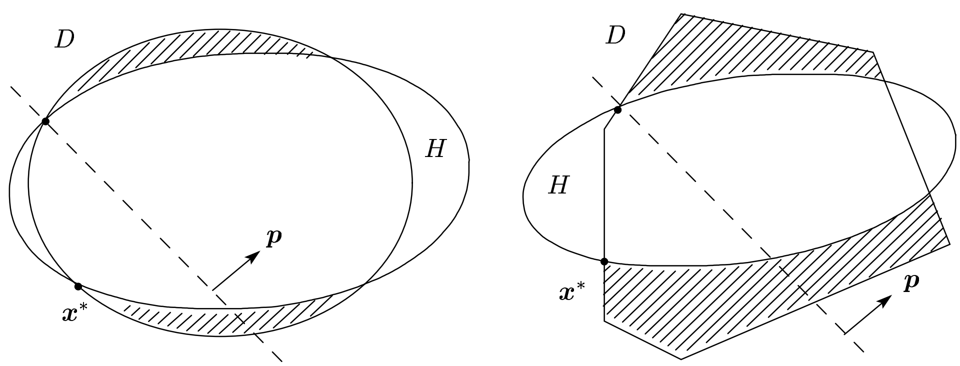


Figure 3: Two-dimensional examples of (CDC) and $(LRCP)$

$\boxed{3.1}$ Cut and Split Method

To apply the cut and split (CS for short) method for $(LRCP)$ we assume that

$$D \subseteqq \mathrm{R}^n_+ \text{ and } \mathbf{0} \in D_V \cap \operatorname{int} H. \tag{3.11}$$

In the first step of the CS method for $(LRCP)$, we find an initial feasible solution $\bar{\boldsymbol{x}}$, which serves as an initial incumbent, and set up the family $\mathcal{S}$ of polyhedral cones, initially $\mathcal{S} := \{\mathrm{R}^n_+\}$. In each iteration, we calculate a lower bound with respect to K for each cone $K \in \mathcal{S}$, then we split the cone whose lower bound is minimum until the optimality condition is met. To calculate a lower bound we define a *concavity cut for* $K \backslash H$ as follows. Given a polyhedral cone K with a vertex at $\mathbf{0}$ and exactly n extreme rays, let $\boldsymbol{u}^i$ denote the intersection point of ∂H and the ith extreme ray of K for $i = 1, \dots, n$. The *concavity cut* $l_K(\boldsymbol{x}) \geqq 0$ *for* $K \backslash H$ is given by the linear function $l_K : \mathrm{R}^n \to \mathrm{R}$ such that

$$l_K(\boldsymbol{x}) = \boldsymbol{e}U^{-1}\boldsymbol{x} - 1, \tag{3.12}$$

where $U = [\boldsymbol{u}^1, \cdots, \boldsymbol{u}^n]$. Note that $l_K(\mathbf{0}) = -1$ and $l_K(\boldsymbol{u}^i) = 0$ for $i = 1, \dots, n$. Figure 4 shows an example of the concavity cut. See Horst-Tuy [16] for the detail.

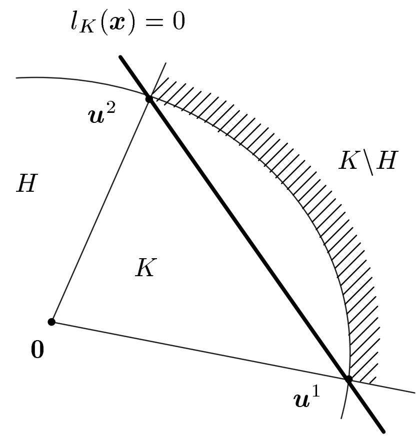


Figure 4: The concavity cut $l_K(\boldsymbol{x}) \geqq 0$ for $K \backslash H$

For each $K \in \mathcal{S}$ we solve the linear programming problem

$$(LP(K)) \quad \left| \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{p}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in D \cap K, \ l_K(\boldsymbol{x}) \geqq 0, \end{array} \right.$$

to obtain a solution $\boldsymbol{\omega}^K$ with the optimal value $\boldsymbol{p}\boldsymbol{\omega}^K$, which is a lower bound with respect to K. If $h(\boldsymbol{\omega}^K) \geqq 0$, i.e., $\boldsymbol{\omega}^K \in D_E$, and $\boldsymbol{p}\boldsymbol{\omega}^K < \boldsymbol{p}\bar{\boldsymbol{x}}$ for some K then we update the incumbent $\bar{\boldsymbol{x}}$ to $\boldsymbol{\omega}^K$. When there remain cones after discarding the cones K with $\boldsymbol{p}\boldsymbol{\omega}^K \geqq \boldsymbol{p}\bar{\boldsymbol{x}}$ if any, then we choose one of them and perform either the $\boldsymbol{\omega}$-subdivision or the bisection, which are defined below, and go to the next iteration.

For a cone K generated by n extreme rays with directions $\boldsymbol{r}^1, \dots, \boldsymbol{r}^n$, let $\boldsymbol{\omega} \in K$ be a point such that $\boldsymbol{\omega} = \sum_{j \in J} \theta_j \boldsymbol{r}^j$ for some $\theta_j > 0$ for $j \in J$, where $J \subseteqq \{1, \dots, n\}$ is the index set of at least two elements, i.e., $|J| \geqq 2$. Let K_j denote the cone generated by $\{\boldsymbol{r}^1, \dots, \boldsymbol{r}^n\} \backslash \{\boldsymbol{r}^j\} \cup \{\boldsymbol{\omega}\}$ for each $j \in J$. The cone K is then split into $|J|$ cones K_j.

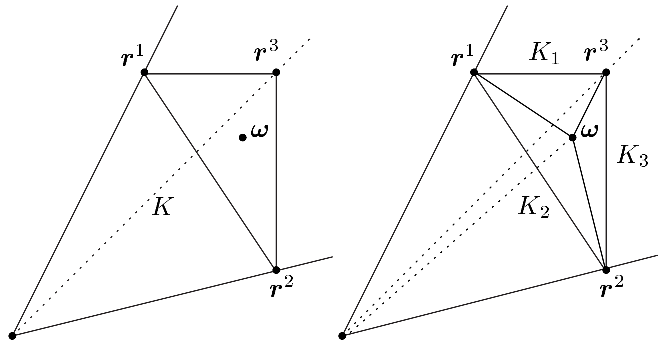


Figure 5: The $\boldsymbol{\omega}$-subdivision

This splitting is called the $\boldsymbol{\omega}$*-subdivision* (See Figure 5). When $\boldsymbol{\omega} = (\boldsymbol{r} + \boldsymbol{r}')/2$ for $\boldsymbol{r}, \boldsymbol{r}' \in \{\boldsymbol{r}^1, \dots, \boldsymbol{r}^n\}$ such that $\|\boldsymbol{r} - \boldsymbol{r}'\|$ is maximum, the subdivision is called the *bisection*.

The CS method is described as follows.

/** CS method **/

$\langle 0 \rangle$ (initialization) Find an initial feasible solution $\bar{\boldsymbol{x}}$ of $(LRCP)$. Set $K_0 := \mathrm{R}^n_+$, $\mathcal{S} := \{K_0\}$, $\mathcal{R} := \mathcal{S}$, and $k := 0$.

$\langle k \rangle$ (iteration k) For each $K \in \mathcal{S}$, solve $\beta_K := \min\{\, \boldsymbol{px} \mid \boldsymbol{x} \in D \cap K,\ l_K(\boldsymbol{x}) \geqq 0 \,\}$ to obtain a solution $\boldsymbol{\omega}^K$, where $l_K(\boldsymbol{x}) \geqq 0$ is the concavity cut for $K \backslash H$.

- $\langle k1 \rangle$ (update) Solve $\beta^* := \min\{\, \beta_K \mid K \in \mathcal{S},\ h(\boldsymbol{\omega}^K) \geqq 0 \,\}$ to obtain the cone K^*. If $\beta^* < \boldsymbol{p}\bar{\boldsymbol{x}}$ then set $\bar{\boldsymbol{x}} := \boldsymbol{\omega}^{K^*}$.
- $\langle k2 \rangle$ (termination) Let $\mathcal{R}' := \{\, K \in \mathcal{R} \mid \beta_K < \boldsymbol{p}\bar{\boldsymbol{x}} \,\}$. If $\mathcal{R}' = \emptyset$ then stop, since $\bar{\boldsymbol{x}}$ solves $(LRCP)$.
- $\langle k3 \rangle$ (subdivision) Solve $\min\{\, \beta_K \mid K \in \mathcal{R}' \,\}$ to obtain the cone K^{**}. Perform either the $\boldsymbol{\omega}$-subdivision for some $\boldsymbol{\omega} \in K^{**}$ or the bisection on K^{**}. Let $\mathcal{S}^{**}$ be the partition of K^{**}. Set $\mathcal{S} := \mathcal{S}^{**}$, $\mathcal{R} := \mathcal{S}^{**} \cup (\mathcal{R}' \backslash \{K^{**}\})$, $k := k+1$ and go to $\langle k \rangle$.

The convergence of the CS method critically depends on the subdivision rule of K^{**}. Let us denote $\boldsymbol{\omega}^{K^{**}}$ for iteration k by $\boldsymbol{\omega}^k$. If the subdivision is *exhaustive*, namely any nested sequence of cones generated in the algorithm will shrink to a ray, there is at least one accumulation point $\boldsymbol{\omega}^*$ of $\{\boldsymbol{\omega}^k\}$ contained in ∂H, and hence $\boldsymbol{\omega}^*$ is feasible. Since $\boldsymbol{p}\boldsymbol{\omega}^k$ is a lower bound of $(LRCP)$ for all k, $\boldsymbol{\omega}^*$ is an optimal solution of $(LRCP)$.

4 Cut and Split Methods for (mmF)

In this section we apply the cut and split method to (mmF). The problem (mmF) is written as

$$(mmF) \qquad \left|\begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{dx} \\ \text{s.t.} & \boldsymbol{x} \in X_M, \end{array}\right.$$

where $X = \{\, \boldsymbol{x} \in \mathrm{R}^n \mid A\boldsymbol{x} = \boldsymbol{0},\ \boldsymbol{0} \leqq \boldsymbol{x} \leqq \boldsymbol{c} \,\}$ and $X_M = \{\, \boldsymbol{x} \in X \mid \nexists \boldsymbol{y} \in X\ :\ \boldsymbol{y} \geqq \boldsymbol{x},\ \boldsymbol{y} \neq \boldsymbol{x} \,\}$. Using the gap function

$$g(\boldsymbol{x}) = \max\{\, \boldsymbol{e}\boldsymbol{y} \mid \boldsymbol{y} \in X,\ \boldsymbol{y} \geqq \boldsymbol{x} \,\} - \boldsymbol{e}\boldsymbol{x}, \tag{4.13}$$

and the set $G = \{\, \boldsymbol{x} \in \mathrm{R}^n \mid -g(\boldsymbol{x}) \leqq 0 \,\}$, (mmF) is also written as

$$(mmF) \qquad \left|\ \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{d}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in X \backslash \operatorname{int} G. \end{array}\right.$$

We exclude the trivial case where $X = \{\boldsymbol{0}\}$. This implies that $\boldsymbol{0} \notin X_M$ or $-g(\boldsymbol{0}) < 0$. We also assume that there is no t-s-path, which means $\boldsymbol{d}\boldsymbol{x} \geqq 0$ for all $\boldsymbol{x} \in X$. Note that all vertices of X are integer vectors by the total unimodularity of A and the integrality of $\boldsymbol{c}$. If $\boldsymbol{x} \in X$ is an integer vector, then $g(\boldsymbol{x})$ takes an integer. As stated in Subsection 2.1, $g(\boldsymbol{x}) \geqq 0$ for all $\boldsymbol{x} \in X$. Additionally we have the following lemma.

Lemma 4.1. *$g(\boldsymbol{x}) > 0$ for all points $\boldsymbol{x}$ in the relative interior of X.*

Proof. Let $\boldsymbol{x}$ be a point in the relative interior of X, i.e., $A\boldsymbol{x} = \boldsymbol{0}$ and $\boldsymbol{0} < \boldsymbol{x} < \boldsymbol{c}$. Letting $\boldsymbol{x}' = (1+\varepsilon)\boldsymbol{x}$ for a sufficiently small $\varepsilon > 0$, we see that $A\boldsymbol{x}' = \boldsymbol{0}$ and $\boldsymbol{0} < \boldsymbol{x}' < \boldsymbol{c}$, i.e., $\boldsymbol{x}' \in X$. Therefore $g(\boldsymbol{x}) \geqq \boldsymbol{e}(\boldsymbol{x}' - \boldsymbol{x}) = \varepsilon \boldsymbol{e}\boldsymbol{x} > 0$. □

The following corollary is a direct consequence of Lemma 4.1.

Corollary 4.2. *$X_M \subseteqq \partial X$.*

4.1 Extension of Gap Function

The *domain of* g, denoted by $\operatorname{dom} g$, is given by $\operatorname{dom} g = \{\, \boldsymbol{x} \in \mathrm{R}^n \mid g(\boldsymbol{x}) > -\infty \,\}$. Since $g(\boldsymbol{v}) = -\infty$ if there is no point $\boldsymbol{y} \in X$ such that $\boldsymbol{y} \geqq \boldsymbol{v}$, no information is available about how far the point $\boldsymbol{v}$ is from the domain of g. In this subsection we extend the gap function g to R^n. The gap function $g(\boldsymbol{x})$ of (4.13) is given by the optimal value of the problem

$$(P_G(\boldsymbol{x})) \qquad \left|\ \begin{array}{ll} \max\limits_{\boldsymbol{y}} & \boldsymbol{e}\boldsymbol{y} - \boldsymbol{e}\boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{y} = \boldsymbol{0},\ \boldsymbol{0} \leqq \boldsymbol{y} \leqq \boldsymbol{c}, \\ & \boldsymbol{y} \geqq \boldsymbol{x}, \end{array}\right.$$

whose dual problem is

$$(D_G(\boldsymbol{x})) \qquad \left|\ \begin{array}{ll} \min\limits_{\boldsymbol{\pi},\boldsymbol{\alpha},\boldsymbol{\beta}} & \boldsymbol{\alpha}\boldsymbol{c} - \boldsymbol{\beta}\boldsymbol{x} - \boldsymbol{e}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\pi}A + \boldsymbol{\alpha} - \boldsymbol{\beta} \geqq \boldsymbol{e}, \\ & \boldsymbol{\alpha}, \boldsymbol{\beta} \geqq \boldsymbol{0}. \end{array}\right.$$

Note that $(D_G(\boldsymbol{x}))$ is always feasible, e.g. take $\boldsymbol{\pi} = \boldsymbol{\beta} = \boldsymbol{0}$ and $\boldsymbol{\alpha} \geqq \boldsymbol{e}$. Therefore, $(P_G(\boldsymbol{x}))$ is infeasible if and only if $(D_G(\boldsymbol{x}))$ is unbounded. Adding the upper bound constraints $\boldsymbol{\beta} \leqq \bar{\boldsymbol{\beta}}$ to $(D_G(\boldsymbol{x}))$ yields the following problem

$$(\overline{D_G(\boldsymbol{x})}) \qquad \left|\ \begin{array}{ll} \min\limits_{\boldsymbol{\pi},\boldsymbol{\alpha},\boldsymbol{\beta}} & \boldsymbol{\alpha}\boldsymbol{c} - \boldsymbol{\beta}\boldsymbol{x} - \boldsymbol{e}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\pi}A + \boldsymbol{\alpha} - \boldsymbol{\beta} \geqq \boldsymbol{e}, \\ & \boldsymbol{\alpha} \geqq \boldsymbol{0},\ \boldsymbol{0} \leqq \boldsymbol{\beta} \leqq \bar{\boldsymbol{\beta}}, \end{array}\right.$$

where $\bar{\boldsymbol{\beta}} \geqq \mathbf{0}$ will be specified in the following theorem. The dual problem of $(\overline{D_G(\boldsymbol{x})})$ is

$$(\overline{P_G(\boldsymbol{x})}) \quad \left| \begin{array}{ll} \max\limits_{\boldsymbol{y},\boldsymbol{t}} & \boldsymbol{e}\boldsymbol{y} - \boldsymbol{e}\boldsymbol{x} - \bar{\boldsymbol{\beta}}\boldsymbol{t} \\ \text{s.t.} & A\boldsymbol{y} = \mathbf{0},\ \mathbf{0} \leqq \boldsymbol{y} \leqq \boldsymbol{c}, \\ & \boldsymbol{y} + \boldsymbol{t} \geqq \boldsymbol{x},\ \boldsymbol{t} \geqq \mathbf{0}. \end{array} \right.$$

Then we define the *extended gap function* $\bar{g} : \mathrm{R}^n \to \mathrm{R}$ as

$$\bar{g}(\boldsymbol{x}) = \max\{\, \boldsymbol{e}\boldsymbol{y} - \bar{\boldsymbol{\beta}}\boldsymbol{t} \mid \boldsymbol{y} \in X,\ \boldsymbol{y} + \boldsymbol{t} \geqq \boldsymbol{x},\ \boldsymbol{t} \geqq \mathbf{0} \,\} - \boldsymbol{e}\boldsymbol{x}. \tag{4.14}$$

Theorem 4.3.

(i) *The domain of* $\bar{g}$ *is* R^n.

(ii) *If* $\bar{\boldsymbol{\beta}} \geqq n\boldsymbol{e}$ *then* $\bar{g} = g$ *on the domain of* g.

Proof. (i) For any $\boldsymbol{x} \in \mathrm{R}^n$, $(\overline{D_G(\boldsymbol{x})})$ has a feasible solution and the objective function is bounded. By the duality theorem of linear programming there is an optimal value of $(\overline{P_G(\boldsymbol{x})})$, and hence $\bar{g}(\boldsymbol{x}) > -\infty$ for any $\boldsymbol{x} \in \mathrm{R}^n$.
(ii) Let Ω and $\bar{\Omega}$ denote the feasible sets of $(D_G(\boldsymbol{x}))$ and $(\overline{D_G(\boldsymbol{x})})$, respectively, i.e.,

$$\begin{aligned} \Omega &= \{\, (\boldsymbol{\pi}, \boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathrm{R}_{m+2n} \mid \boldsymbol{\pi}A + \boldsymbol{\alpha} - \boldsymbol{\beta} \geqq \boldsymbol{e},\ \boldsymbol{\alpha}, \boldsymbol{\beta} \geqq \mathbf{0} \,\},\ \text{and} \\ \bar{\Omega} &= \{\, (\boldsymbol{\pi}, \boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathrm{R}_{m+2n} \mid \boldsymbol{\pi}A + \boldsymbol{\alpha} - \boldsymbol{\beta} \geqq \boldsymbol{e},\ \boldsymbol{\alpha} \geqq \mathbf{0},\ \mathbf{0} \leqq \boldsymbol{\beta} \leqq \bar{\boldsymbol{\beta}} \,\}. \end{aligned}$$

By the theory of linear programming, if every vertex $\boldsymbol{v}$ of Ω satisfies $\boldsymbol{v} \leqq \bar{\boldsymbol{\beta}}$ then we have $\bar{g}(\boldsymbol{x}) = g(\boldsymbol{x})$ for all $\boldsymbol{x}$ in the domain of g. Replacing $\boldsymbol{\pi}$ by $\boldsymbol{\pi}^1 - \boldsymbol{\pi}^2$ with $\boldsymbol{\pi}^1, \boldsymbol{\pi}^2 \geqq \mathbf{0}$ and introducing a slack variable vector $\boldsymbol{\gamma} \geqq \mathbf{0}$, Ω is rewritten as

$$\Omega = \left\{ \begin{pmatrix} (\boldsymbol{\pi}^1)^\top \\ (\boldsymbol{\pi}^2)^\top \\ \boldsymbol{\alpha}^\top \\ \boldsymbol{\beta}^\top \\ \boldsymbol{\gamma}^\top \end{pmatrix} \;\middle|\; \begin{pmatrix} A^\top & -A^\top & I & -I & -I \end{pmatrix} \begin{pmatrix} (\boldsymbol{\pi}^1)^\top \\ (\boldsymbol{\pi}^2)^\top \\ \boldsymbol{\alpha}^\top \\ \boldsymbol{\beta}^\top \\ \boldsymbol{\gamma}^\top \end{pmatrix} = \mathbf{1},\ \begin{pmatrix} (\boldsymbol{\pi}^1)^\top \\ (\boldsymbol{\pi}^2)^\top \\ \boldsymbol{\alpha}^\top \\ \boldsymbol{\beta}^\top \\ \boldsymbol{\gamma}^\top \end{pmatrix} \geqq \mathbf{0} \right\}.$$

Let $\boldsymbol{v}$ be a vertex of Ω. Then it is a basic solution of the system defining Ω, i.e., $\boldsymbol{v} = (\boldsymbol{w}^B, \boldsymbol{w}^N) = (B^{-1}\mathbf{1}, \mathbf{0})$ for some nonsingular $n \times n$ submatrix B of $(A^\top\ -A^\top\ I\ -I\ -I)$. Since the incidence matrix A is totally unimodular, i.e., each subdeterminant of A is -1, 0, or $+1$, so is $(A^\top\ -A^\top\ I\ -I\ -I)$. Therefore the matrix B^{-1} is composed of -1, 0 and $+1$, and hence $B^{-1}\mathbf{1} \leqq n\mathbf{1}$. This completes the proof. □

Theorem 4.3 yields an equivalent form of (mmF). Namely, fixing $\bar{\boldsymbol{\beta}} = n\boldsymbol{e}$ we can reformulate the problem (mmF) as

$$(mmF) \quad \left| \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{d}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in X \backslash \operatorname{int} \bar{G}, \end{array} \right.$$

where $\bar{G} = \{\, \boldsymbol{x} \in \mathrm{R}^n \mid -\bar{g}(\boldsymbol{x}) \leqq 0 \,\}$.

4.2 Local Search

For $\boldsymbol{v} \in X_M \cap X_V$, we define the set of all efficient vertices linked to $\boldsymbol{v}$ by

$$\begin{aligned} N_M(\boldsymbol{v}) &= \{\, \boldsymbol{v}' \in X_M \cap X_V \mid [\boldsymbol{v}, \boldsymbol{v}'] \text{ is an edge of } X \,\} \\ &= \{\, \boldsymbol{v}' \in X_V \mid [\boldsymbol{v}, \boldsymbol{v}'] \text{ is an edge of } X \text{ and } -\bar{g}(\boldsymbol{v}') \geqq 0 \,\}. \end{aligned} \tag{4.15}$$

When we find a feasible solution $\boldsymbol{w} \in X_M$, we apply the Local Search procedure starting with $\boldsymbol{w}$ (LS($\boldsymbol{w}$) for short) for further improvement.

The procedure is described as follows.

/** LS($\boldsymbol{w}$) procedure **/

$\langle 0 \rangle$ (initialization) If $\boldsymbol{w} \notin X_V$ then solve $\min\{\, \boldsymbol{dx} \mid \boldsymbol{x} \in F \,\}$, where F is the face of X containing $\boldsymbol{w}$ in its relative interior, to obtain a vertex $\boldsymbol{v}^0 \in X_M \cap X_V$, otherwise set $\boldsymbol{v}^0 := \boldsymbol{w}$. Set $k := 0$.

$\langle k \rangle$ (iteration k) Solve $\min\{\, \boldsymbol{dv} \mid \boldsymbol{v} \in N_M(\boldsymbol{v}^k) \,\}$ to obtain a solution $\boldsymbol{v}^*$. If $\boldsymbol{dv}^* \geqq \boldsymbol{dv}^k$ then stop, $\boldsymbol{v}^k$ is the local optimal vertex of (mmF). Otherwise set $\boldsymbol{v}^{k+1} := \boldsymbol{v}^*$, $k := k+1$ and go to $\langle k \rangle$.

Remark 4.4. If $\boldsymbol{w} \in X_M$, the face F of X containing $\boldsymbol{w}$ in its relative interior is contained in X_M by Theorem 2.2.

$\boxed{4.3}$ Cut and Split Method for (mmF)

We can directly apply the CS method to (mmF) since it satisfies the assumptions of (3.11), i.e.,

$$X \subseteqq \mathrm{R}^n_+ \text{ and } \mathbf{0} \in X_V \cap \operatorname{int} \bar{G}. \tag{4.16}$$

To make the algorithm more efficient we combine the LS($\boldsymbol{w}$) procedure with the CS method, namely we apply the LS($\boldsymbol{w}$) procedure to obtain a tighter upper bound every time we find a feasible solution $\boldsymbol{w} \in X_M$.

The CS method for (mmF) is described as follows.

/** CS method for (mmF) **/

$\langle 0 \rangle$ (initialization) Find an initial feasible vertex $\boldsymbol{w}^0 \in X_M \cap X_V$ of (mmF). If $N_M(\boldsymbol{w}^0) = \emptyset$ then stop. We see that $\boldsymbol{w}^0$ is the unique optimal solution of (mmF). Otherwise, apply the LS($\boldsymbol{w}^0$) procedure to obtain a local optimal vertex $\bar{\boldsymbol{x}} \in X_M \cap X_V$. Set $K_0 := \mathrm{R}^n_+$, $\mathcal{S} := \{K_0\}$, $\mathcal{R} := \mathcal{S}$, $\gamma := 0$ and $k := 0$.

$\langle k \rangle$ (iteration k) For each $K \in \mathcal{S}$, solve $\beta_K := \min\{\, \boldsymbol{dx} \mid \boldsymbol{x} \in X \cap K,\ l_K(\boldsymbol{x}) \geqq 0 \,\}$ to obtain a solution $\boldsymbol{\omega}^K$, where $l_K(\boldsymbol{x}) \geqq 0$ is the concavity cut for $K \backslash \bar{G}$.

- $\langle k1 \rangle$ (update) Set $\mathcal{L} := \{\, K \in \mathcal{S} \mid \boldsymbol{\omega}^K \in X_M \,\}$. If $\mathcal{L} \neq \emptyset$ then apply the LS($\boldsymbol{\omega}^K$) procedure to obtain a local optimal vertex $\boldsymbol{v}^K$ for each $K \in \mathcal{L}$. Solve $\min\{\, \boldsymbol{dv}^K \mid K \in \mathcal{L} \,\}$ to obtain the cone K^*. If $\boldsymbol{dv}^{K^*} < \boldsymbol{d}\bar{\boldsymbol{x}}$, set $\bar{\boldsymbol{x}} := \boldsymbol{v}^{K^*}$.
- $\langle k2 \rangle$ (termination) Set $\mathcal{R}' := \{\, K \in \mathcal{R} \mid \beta_K < \boldsymbol{d}\bar{\boldsymbol{x}} \,\}$. If $\mathcal{R}' = \emptyset$ or $\boldsymbol{d}\bar{\boldsymbol{x}} - \gamma < 1$ then stop, since $\bar{\boldsymbol{x}}$ solves (mmF).

$\langle k3 \rangle$ (subdivision) Solve $\min\{\beta_K \mid K \in \mathcal{R}'\}$ to obtain the cone K^{**}. If $\beta_{K^{**}} > \gamma$ then set $\gamma := \beta_{K^{**}}$. Let $\mathcal{S}^{**}$ be the partition of K^{**} obtained by the $\boldsymbol{\omega}$-subdivision on K^{**} for some direction $\boldsymbol{\omega} \in K^{**}$, or the bisection. Set $\mathcal{S} := \mathcal{S}^{**}$, $\mathcal{R} := \mathcal{S}^{**} \cup (\mathcal{R}' \backslash \{K^{**}\})$, $k := k+1$ and go to $\langle k \rangle$.

Remark 4.5. In Step $k3$ the direction $\boldsymbol{\omega} \in K^{**}$ is obtained as follows. Suppose that K^{**} is generated by the directions $\boldsymbol{u}^1, \ldots, \boldsymbol{u}^n$ such that $\boldsymbol{u}^i \in \partial \bar{G}$, initially $\boldsymbol{u}^i = \alpha^* \boldsymbol{e}^i$ with $\alpha^* = \max\{\alpha \mid -\bar{g}(\alpha \boldsymbol{e}^i) \leqq 0\}$ for each $i = 1, \ldots, n$. The function $l_{K^{**}} : \mathrm{R}^n \to \mathrm{R}$ defining the concavity cut $l_{K^{**}}(\boldsymbol{x}) \geqq 0$ for $K^{**} \backslash \bar{G}$ is given by $l_{K^{**}}(\boldsymbol{x}) = \boldsymbol{e} U^{-1} \boldsymbol{x} - 1$, where $U = [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^n]$. Here we solve

$$\eta = \max\{ l_{K^{**}}(\boldsymbol{y}) \mid \boldsymbol{y} \in X \cap K^{**} \}, \tag{4.17}$$

to obtain a solution $\boldsymbol{y}^*$. Since K^{**} is in $\mathcal{R}'$, i.e., $\beta_{K^{**}} < \boldsymbol{d}\bar{\boldsymbol{x}}$, we see that $\eta \geqq 0$. If $\eta > 0$ then we perform $\boldsymbol{\omega}$-subdivision on K^{**} with $\boldsymbol{\omega} = \boldsymbol{y}^*$. If $\eta = 0$ then we discard K^{**} from $\mathcal{R}'$ and go back to Step $k2$. In this case there is no point $\boldsymbol{v} \in X_V \cap K^{**}$ such that $\boldsymbol{v} \neq \boldsymbol{u}^i$ for each $i = 1, \ldots, n$. Then we can discard K^{**} from further consideration, because at least one vertex of X solves (mmF).

Every time we obtain an optimal solution $\boldsymbol{y}^*$ of (4.17) with $\eta > 0$, we can perform $\boldsymbol{y}^*$-subdivision on K^{**}. This assertion follows from the following theorem.

Theorem 4.6. *Let K^{**} be a cone generated by the directions $\boldsymbol{u}^1, \ldots, \boldsymbol{u}^n$ such that $\boldsymbol{u}^i \in \partial \bar{G}$ for each $i = 1, \ldots, n$, and $\boldsymbol{y}^*$ be an optimal solution of* (4.17) *with $\eta > 0$. Then $\boldsymbol{y}^* \neq \alpha \boldsymbol{u}^i$ for any $i = 1, \ldots, n$ and for any $\alpha > 0$.*

Proof. Assume that $\boldsymbol{y}^*$ lies on an extreme ray of K^{**}, i.e., $\boldsymbol{y}^* = \alpha \boldsymbol{u}^j$ for some $\alpha > 0$ and $\boldsymbol{u}^j$. Since $0 < \eta = l_{K^{**}}(\boldsymbol{y}^*) = \boldsymbol{e} U^{-1}(\alpha \boldsymbol{u}^j) - 1 = \alpha - 1$, we have $\alpha > 1$. By the choice of $\boldsymbol{u}^j$, we have $\bar{g}((1+\varepsilon)\boldsymbol{u}^j) < 0$ for any $\varepsilon > 0$. Therefore we have $\bar{g}(\boldsymbol{y}^*) < 0$. On the other hand, $\boldsymbol{y}^* \in X \cap K^{**} \subseteqq K$, which implies $\bar{g}(\boldsymbol{y}^*) \geqq 0$. This is a contradiction. □

Furthermore the following assertion is also available in this subdivision rule.

Theorem 4.7. *Let K be the cone generated by the directions $\boldsymbol{u}^1, \ldots, \boldsymbol{u}^n$ and $U = [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^n]$. If $\boldsymbol{u}^i \in X_M$ for each $i = 1, \ldots, n$ then some $\boldsymbol{u}^j$ solves* $\min\{\boldsymbol{d}\boldsymbol{x} \mid \boldsymbol{x} \in X \cap K,\ l_K(\boldsymbol{x}) \geqq 0\}$.

Proof. Let $\boldsymbol{\omega}$ be an optimal solution of $\min\{\boldsymbol{d}\boldsymbol{x} \mid \boldsymbol{x} \in X \cap K,\ l_K(\boldsymbol{x}) \geqq 0\}$. Since $\boldsymbol{\omega} \in K$, there are nonnegative numbers $\mu_1, \ldots, \mu_n$ such that $\boldsymbol{\omega} = \sum_{i=1}^n \mu_i \boldsymbol{u}^i$. Also we see that $0 \leqq l_K(\boldsymbol{\omega}) = \boldsymbol{e} U^{-1}(\sum_{i=1}^n \mu_i \boldsymbol{u}^i) - 1 = \sum_{i=1}^n \mu_i - 1$, and hence $\sum_{i=1}^n \mu_i \geqq 1$. Note that $\boldsymbol{d}\boldsymbol{x} \geqq 0$ for all $\boldsymbol{x} \in X$ by the assumption that a given network has no t-s-path. Let $\boldsymbol{u}^j$ attain $\min\{\boldsymbol{d}\boldsymbol{u}^i \mid i = 1, \ldots, n\}$. Then $\boldsymbol{d}\boldsymbol{u}^j \leqq \sum_{i=1}^n \mu_i \boldsymbol{d}\boldsymbol{u}^i = \boldsymbol{d}\boldsymbol{\omega}$, in other words, $\boldsymbol{u}^j \in X_M$ solves $\min\{\boldsymbol{d}\boldsymbol{x} \mid \boldsymbol{x} \in X \cap K,\ l_K(\boldsymbol{x}) \geqq 0\}$. □

We see that the set $\{\boldsymbol{x} \in K^{**} \mid l_{K^{**}}(\boldsymbol{x}) > 0\}$ does not contain a vertex of X when η of (4.17) is zero. Suppose that an oracle is available that provides a vertex of X in $\{\boldsymbol{x} \in K^{**} \mid l_{K^{**}}(\boldsymbol{x}) > 0\}$ whenever there are some, and take the vertex as the direction $\boldsymbol{\omega}$ in Step $k3$. Then, owing to the finiteness of X_V, the $\boldsymbol{\omega}$-subdivision is repeated at most $|X_V|$ times, and hence the CS method terminates after finitely many iterations. However, the oracle is costly and the authors estimate it NP-complete to check whether $X_V \cap \{\boldsymbol{x} \in K^{**} \mid l_{K^{**}}(\boldsymbol{x}) > 0\}$ is not empty. See Freund-Orlin [13].

We illustrate the CS method for (mmF) in Figure 6, in which we use a two-dimensional general polyhedron $X = \{\, \boldsymbol{x} \in \mathrm{R}^2 \mid B\boldsymbol{x} \leqq \boldsymbol{z},\ \boldsymbol{x} \geqq \boldsymbol{0} \,\}$ instead of $X = \{\, \boldsymbol{x} \in \mathrm{R}^n \mid A\boldsymbol{x} = \boldsymbol{0},\ \boldsymbol{0} \leqq \boldsymbol{x} \leqq \boldsymbol{c} \,\}$ since the latter is unsuitable for illustration. We first obtain a local optimal vertex $\bar{\boldsymbol{x}} \in X_M \cap X_V$ and set $K_0 := \mathrm{R}^n_+$ (See (a)). We determine the concavity cut $l_{K_0}(\boldsymbol{x}) \geqq 0$ for $K_0 \backslash \bar{G}$ and obtain a point $\boldsymbol{\omega}^{K_0}$ (See (b)). The value $\boldsymbol{d}\boldsymbol{\omega}^{K_0}$ is a lower bound with respect to K_0. Since $\boldsymbol{\omega}^{K_0} \notin X_M$ and $\boldsymbol{d}\boldsymbol{\omega}^{K_0} < \boldsymbol{d}\bar{\boldsymbol{x}}$, we split the cone K_0 into K_1 and K_2 (See (c)). In the next iteration, points $\boldsymbol{\omega}^{K_1}$ and $\boldsymbol{\omega}^{K_2}$ are obtained (See (d)). We see that $\boldsymbol{\omega}^{K_2} \in X_M$, and hence apply the LS($\boldsymbol{\omega}^{K_2}$) procedure to obtain a better point $\boldsymbol{v}^{K_2}$ and update the incumbent $\bar{\boldsymbol{x}}$ to $\boldsymbol{v}^{K_2}$. The cone K_2 is discarded, since $\boldsymbol{d}\boldsymbol{\omega}^{K_2} \geqq \boldsymbol{d}\bar{\boldsymbol{x}}$. Meanwhile K_1 is split into K_3 and K_4 since $\boldsymbol{\omega}^{K_1} \notin X_M$ and $\boldsymbol{d}\boldsymbol{\omega}^{K_1} < \boldsymbol{d}\bar{\boldsymbol{x}}$ (See (e)). We obtain points $\boldsymbol{\omega}^{K_3}$ and $\boldsymbol{\omega}^{K_4}$ in the next iteration (See (f)) and continue the algorithm.

$\boxed{5}$ Conclusion and Further Works

Computational experiment should be carried out to verify the efficiency of the algorithm we proposed in this paper. In many problems formulated as (P_E) the criterion matrix C has quite a small number p of rows. Some sophisticated algorithms for (P_E) take advantage of this property. However, in (mmF) the number p is equal to the number of arcs, i.e., $p = n$. Therefore it is likely that a primitive method surpasses some sophisticated algorithms. The comparative studies of several algorithms for (mmF) are significant and required for further works.

If we can reduce the number of variables in (mmF), we can make the CS method more efficiently. The dimension of $X = \{\, \boldsymbol{x} \in \mathrm{R}^n \mid A\boldsymbol{x} = \boldsymbol{0},\ \boldsymbol{0} \leqq \boldsymbol{x} \leqq \boldsymbol{c} \,\}$ is much smaller than the number of variables in many cases. This property seems to be useful, however, there is still some difficulties to overcome (See Appendix B).

Appendix A

The Proof of Theorem 2.1

Proof. ($\Leftarrow$) Assume that $\bar{\boldsymbol{x}} \in D$ is not an efficient point for (MC). There exists $\boldsymbol{y} \in D$ such that $C\boldsymbol{y} \geqq C\boldsymbol{x}$ and $C\boldsymbol{y} \neq C\boldsymbol{x}$. Then, $\bar{\boldsymbol{x}}$ is not an optimal solution of $(SC(\boldsymbol{\lambda}))$ for any $\boldsymbol{\lambda} \in \mathrm{R}_{p++}$. ($\Rightarrow$) Suppose $\bar{\boldsymbol{x}} \in D$ is an efficient point for (MC). Let $L_{\bar{\boldsymbol{x}}} = \mathrm{diag}\{l_1, \ldots, l_n\}$, where

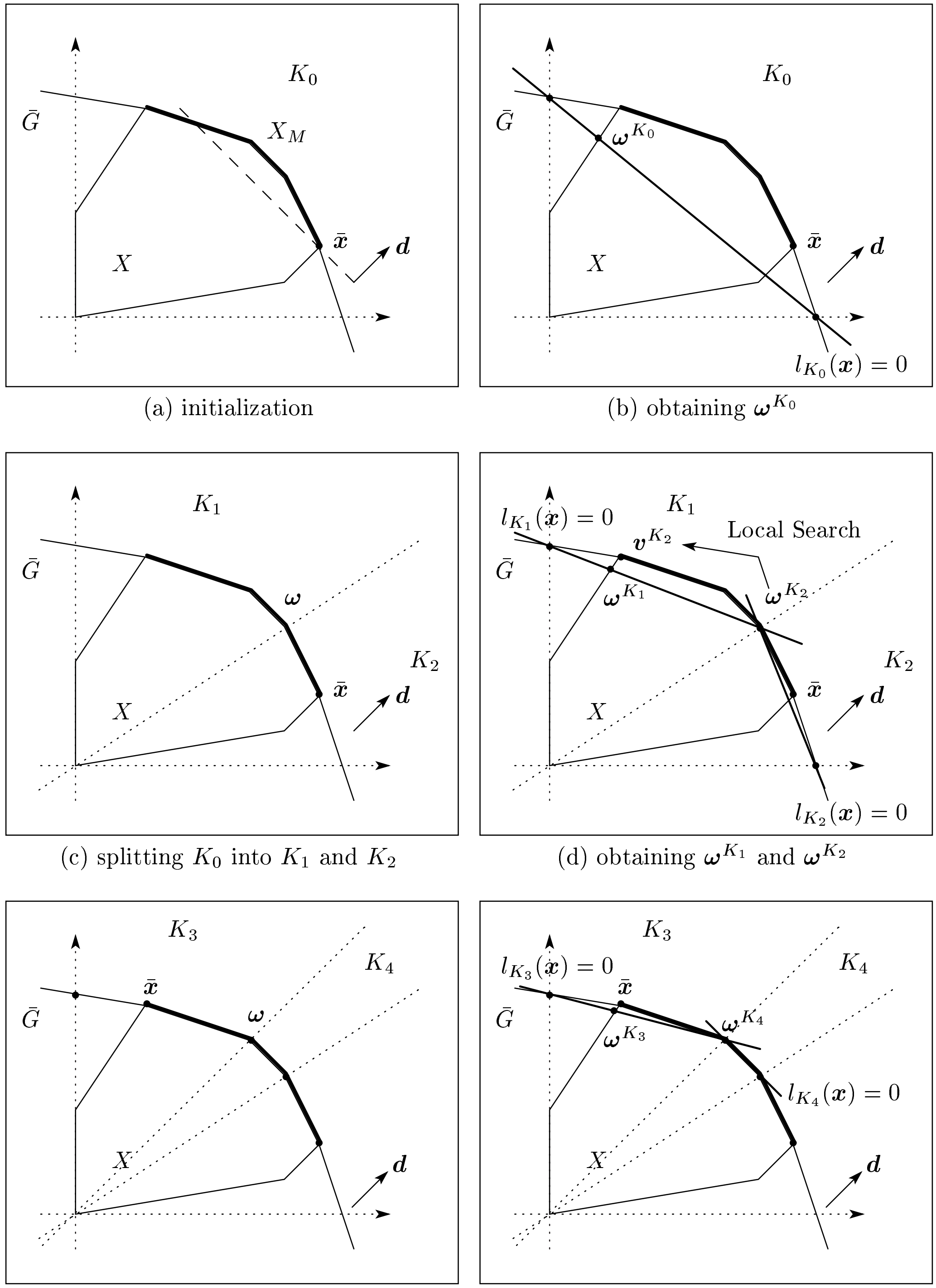
$$l_i = \begin{cases} 1 & \text{if } \bar{x}_i = 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } i = 1, \ldots n. \tag{5.18}$$

If there exists a vector $\boldsymbol{u} \in \mathrm{R}^n$ satisfies the system

$$C\boldsymbol{u} \geqq \boldsymbol{0},\ C\boldsymbol{u} \neq \boldsymbol{0},\ L_{\bar{\boldsymbol{x}}}\boldsymbol{u} \geqq \boldsymbol{0},\ B\boldsymbol{u} = \boldsymbol{0}, \tag{5.19}$$

setting $\boldsymbol{x} = \bar{\boldsymbol{x}} + \theta\boldsymbol{u}$ for a sufficiently small $\theta > 0$, we see $\boldsymbol{x} \in D$ satisfies $C\boldsymbol{x} \geqq C\bar{\boldsymbol{x}}$ and $C\boldsymbol{x} \neq C\bar{\boldsymbol{x}}$. This contradicts that $\bar{\boldsymbol{x}}$ is an efficient point for (MC). Then, there is no vector $\boldsymbol{u} \in \mathrm{R}^n$ satisfies the system (5.19). Applying the Tucker's alternative theorem (See Mangasarian [21]), there are vectors $\boldsymbol{\lambda} \in \mathrm{R}_p$, $\boldsymbol{\mu} \in \mathrm{R}_n$ and $\boldsymbol{\nu} \in \mathrm{R}_m$ such that

$$\boldsymbol{\lambda}C + \boldsymbol{\mu}L_{\bar{\boldsymbol{x}}} + \boldsymbol{\nu}B = \boldsymbol{0},\ \boldsymbol{\lambda} > \boldsymbol{0},\ \boldsymbol{\mu} \geqq \boldsymbol{0}. \tag{5.20}$$



(a) initialization

(b) obtaining $\boldsymbol{\omega}^{K_0}$

(c) splitting K_0 into K_1 and K_2

(d) obtaining $\boldsymbol{\omega}^{K_1}$ and $\boldsymbol{\omega}^{K_2}$

(e) splitting K_1 into K_3 and K_4

(f) obtaining $\boldsymbol{\omega}^{K_3}$ and $\boldsymbol{\omega}^{K_4}$

Figure 6: An example of the CS method for (mmF)

For any $\boldsymbol{x} \in \mathrm{R}^n$, we see that

$$\boldsymbol{\lambda} C(\bar{\boldsymbol{x}} - \boldsymbol{x}) + \boldsymbol{\mu} L_{\bar{\boldsymbol{x}}}(\bar{\boldsymbol{x}} - \boldsymbol{x}) + \boldsymbol{\nu} B(\bar{\boldsymbol{x}} - \boldsymbol{x}) = 0. \tag{5.21}$$

Therefore for any $\boldsymbol{x} \in D$, we have $\boldsymbol{\mu} L_{\bar{\boldsymbol{x}}}(\bar{\boldsymbol{x}} - \boldsymbol{x}) \leqq 0$ and $\boldsymbol{\nu} B(\bar{\boldsymbol{x}} - \boldsymbol{x}) = 0$, and hence $\boldsymbol{\lambda} C\bar{\boldsymbol{x}} \geqq \boldsymbol{\lambda} C\boldsymbol{x}$. It remains to proof the assertion that there exists $M > 0$ such that we can replace R_{p++} with Λ of (2.9). By the theory of the parametric linear program, D_E is the union of finitely many faces $F_1, \cdots, F_K$ of D such that F_k is the optimal face of $(SC(\boldsymbol{\lambda}^k))$ for some $\boldsymbol{\lambda}^k \in \mathrm{R}_{p++}$. Let $\alpha_k = 1/\min\{\, \lambda_i^k \mid i = 1, \dots, p \,\}$ for $k = 1, \dots, K$ and $M = \max\{\, \alpha_k(\boldsymbol{\lambda}^k \mathbf{1}) \mid k = 1, \dots, K \,\}$. Then $\bar{\boldsymbol{\lambda}}^k = (M/\boldsymbol{\lambda}^k\mathbf{1})\boldsymbol{\lambda}^k \in \Lambda$ and F^k remains the optimal face of $(SC(\bar{\boldsymbol{\lambda}}^k))$ for each $k = 1, \dots, K$. □

Appendix B

An idea to reduce the number of variables

After explaining a possible way to reduce the number of variables of (mmF) we demonstrate that it does not work well due to the high degeneracy of the problem.

An idea to reduce the variables is a combination of primal and dual representations of the feasible region $X = \{\, \boldsymbol{x} \in \mathrm{R}^n \mid A\boldsymbol{x} = \mathbf{0},\ \mathbf{0} \leqq \boldsymbol{x} \leqq \boldsymbol{c} \,\}$. First choose a vertex $\boldsymbol{v}^0 \in X_V \cap \operatorname{int} \bar{G}$ and enumerate all, say q adjacent vertices $\boldsymbol{v}^1, \dots, \boldsymbol{v}^q \in X_V$ linked to $\boldsymbol{v}^0$. Then X can be rewritten as

$$X = \left\{ \boldsymbol{x} \in \mathrm{R}^n \;\middle|\; \boldsymbol{x} = \boldsymbol{v}^0 + \sum_{i=1}^{q} \mu_i(\boldsymbol{v}^i - \boldsymbol{v}^0), \mathbf{0} \leqq \boldsymbol{x} \leqq \boldsymbol{c}, \mu_i \geqq 0 \text{ for all } i = 1, \dots, q \right\}.$$

We define $X^R = \{\, \boldsymbol{\mu} \in \mathrm{R}^q \mid \mathbf{0} \leqq \boldsymbol{v}^0 + V\boldsymbol{\mu} \leqq \boldsymbol{c},\ \boldsymbol{\mu} \geqq \mathbf{0} \,\}$, where $V = [\boldsymbol{v}^1 - \boldsymbol{v}^0, \dots, \boldsymbol{v}^q - \boldsymbol{v}^0] \in \mathrm{R}^{n \times q}$, and $X_M^R = \{\, \boldsymbol{\mu} \in X^R \mid -\bar{g}^R(\boldsymbol{\mu}) \geqq 0 \,\}$, where

$$\bar{g}^R(\boldsymbol{\mu}) = \max\{\, \boldsymbol{e}V\boldsymbol{\nu} - n\boldsymbol{et} \mid \boldsymbol{\nu} \in X^R,\ V\boldsymbol{\nu} + \boldsymbol{t} \geqq V\boldsymbol{\mu},\ \boldsymbol{t} \geqq \mathbf{0} \,\} - \boldsymbol{e}V\boldsymbol{\mu}.$$

Then the problem (mmF) is equivalent to

$$(mmF_R) \qquad \left|\ \begin{array}{ll} \min\limits_{\boldsymbol{\mu}} & \boldsymbol{d}(\boldsymbol{v}^0 + V\boldsymbol{\mu}) \\ \text{s.t.} & \boldsymbol{\mu} \in X_M^R. \end{array}\right.$$

If $q < n$, (mmF_R) is worth considering, however, q can be much larger than n in spite of the low dimensionality of X. Take the network in Figure 7 with unit capacity for all arcs, i.e., $\boldsymbol{c} = \boldsymbol{e}$, and take the origin $\mathbf{0}$ as $\boldsymbol{v}^0 \in X_V \cap \operatorname{int} \bar{G}$. Then we see that the unit flow conveyed along a simple path from source to sink is a vertex of X linked to the origin. This means that q is as many as the simple paths from source to sink, which amounts to $3^3 = 27$ in this example while $n = 15$ and $\dim X = 13$.

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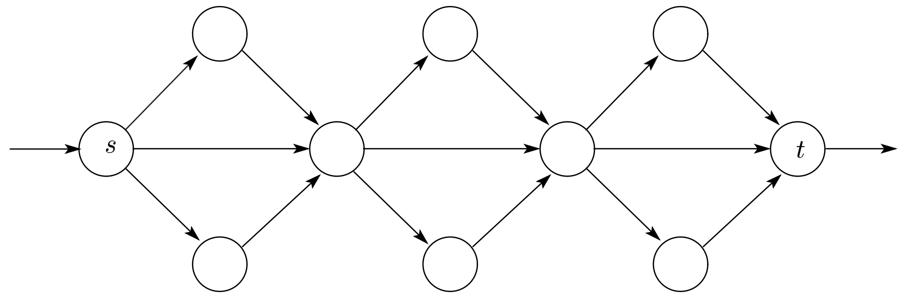

Figure 7: The case where q is larger than n

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