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THE VALUATION AND OPTIMAL STRATEGIES OF CALLABLE CONVERTIBLE BONDS

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*This paper is dedicated to Professor H. Konno on his 65th birthday with affection and respect.**

Abstract: This paper presents a simple pricing model for valuing callable convertible bonds under the setting of a coupled stopping game between the issuer (firm) and the investor (holder). We use the model to provide the valuation formula for such bonds and to explore some analytical properties of optimal conversion and call strategies as well as several numerical results by using the finite element method. Furthermore, optimal critical prices for call and conversion are examined numerically.

Key words: *asset pricing, callable discount, convertible bonds, optimal stopping time, optimal boundaries*

Mathematics Subject Classification: *91B28*

1 Introduction

Consider a zero-coupon callable convertible bond (abbreviated by CB) which permits an investor to exchange the bond for a certain number of the underlying stock at any time of his choosing and gives the issuer the right to purchase back the bond at any time for a specific amount. Since the convertible bond with such a call feature should clearly be evaluated less than the convertible bond without that feature, it is an attractive and complex security.

Brennan and Schwartz[1] provides the groundwork for the pricing model of convertible bond together with Ingersoll[4]. On the other hand, Kifer[7] paves a way of the pricing model of American options with the call feature. McConell and Schwartz[11] provides and analyzes a typical example, LYON, of the callable American option. Seko and Sawaki[12] considers the valuation of the callable American put and call options following the Kifer's approach. Kunita and Seko[8] considers a game call along the line analyzed by [12]. Longstaff and Schwartz[10] presents a new computational approach based upon the least-squares simulation for callable option. Kariya and Tsuda[6] proposes a statistical model for valuing the CB. Yagi and Sawaki[13] studies a simple binomial model for valuing the convertible bond but analyzes neither the optimal boundaries nor the decomposition of the CB price. The inspiration behind our work was the work of [1], [7] and [12]. However, these ideas cannot be transfered

*We have known Professor H. Konno for many years and have always known he is a fine man, thoughtful organizer and great teacher. This special issue is a well-deserved recognition of his excellent work, not only research achievement in OR/MS but also his leadership in the OR society in Japan.

directly but must be revised to the callable CB because the call feature does not rise the penalty lost in CBs. So far, there are not so many analytical papers which analyze the value and optimal boundaries of callable CBs although there are many empirical studies on CBs, [2] and [3].

In this paper we present a pricing model for callable CBs where the underlying stock pays dividends. In section 2 we formulate the valuation of the callable CB under the setting of a coupled stopping game based upon the Kifer's game option model[7]. In section 3 we discuss optimal call and convertible policies for the issuer and the investor, respectively. It will be shown that the CB with the call feature is worth less than the bond without and that the conversion feature increases the value of the CB to the investor. In section 4 we show that the value of the callable CB can be decomposed into four terms of the bond price, the value of the European call option with the face value as the exercise price, the early conversion premium and the callable discounted value. Section 5 provides some numerical results to recognize some analytical properties of the value of the callable and non-callable CBs by using the finite element approach. Finally, in section 6 we summarize the paper with some concluding remarks.

2 The Pricing Model for Callable Convertible Bonds

A riskless asset price B_t with the interest rate r is given by

$$dB_t = rB_t dt, \quad B_0 > 0, \ r > 0. \tag{1}$$

Let Z_t be a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ and μ, κ and $\delta(< r)$ the expected rate of return, the volatility and the constant rate of dividend payments for the firm value V_t, respectively. The process W_t defined by

$$W_t = \exp\left\{-\frac{1}{2}\left(\frac{\mu - r + \delta}{\kappa}\right)^2 t - \frac{\mu - r + \delta}{\kappa} Z_t\right\} \tag{2}$$

is a martingale [9]. Define the risk neutral probability measure $\tilde{P}$ which is equivalent to P as

$$\tilde{P}(A) = E[1_A W_T], \quad A \in \mathcal{F}_T. \tag{3}$$

Then, the stochastic differential equation of the firm value V_t can be written as the well known geometric Brownian motion

$$dV_t = (r - \delta)V_t dt + \kappa V_t d\tilde{Z}_t \tag{4}$$

where $\tilde{Z}_t$ is the standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \tilde{P})$.

Consider a firm financing stocks and convertible bonds in the capital market. The value of the firm at time t is given by

$$V_t \equiv CB(t, V_t) + mS_t^b \tag{5}$$

where $CB(t, V_t)$ is the total value of the CB with the face value F and the maturity T, and S_t^b the stock value with the number of shares m before conversion. Let S_t^a be the stock value after conversion and n the number of stocks converted from the CB. Then, equation (5) can be rewritten as

$$V_t = (n + m)S_t^a \tag{6}$$

where V_T is not affected by the conversion. Letting $z = n/(n+m)$ as the dilution ratio, the conversion value at time t $CV(t, V_t)$ is given by

$$CV(t, V_t) = nS_t^a = zV_t. \tag{7}$$

Denote $CP(t) = F$ the call price when the firm calls the CB at time t. Then the firm pays the investor $\max(zV_t, F)$.

Let σ be a call time (a stopping time) by the issuer and τ a conversion time (a stopping time) by the investor. Denote $\mathcal{T}_{t,T}$ the set of stopping times with respect to the filtration $\{\mathcal{F}_t; 0 \leq t \leq T\}$. The payoff function at the stopping time $\sigma \wedge \tau < T$ is given by

$$R(\sigma, \tau) = \max(zV_\sigma, F)1_{\{\sigma<\tau\}} + zV_\tau 1_{\{\tau\leq\sigma\}} \tag{8}$$

where $1_{\{\sigma<\tau\}} = 1$ if the issuer stops first and $1_{\{\tau\leq\sigma\}} = 1$ if the investor stops first, and the issuer pledges to pay the investor at any stopping time. At the time of $\sigma = \tau$ the investor's right dominates the issuer's one. At the maturity T the payoff function is the smaller of the value of the firm or the maximum between the conversion value and the face value of the CB, that is,

$$R(T, T) = \min(V_T, \max(zV_T, F)). \tag{9}$$

After a minor revision, we can use the following theorem given by Kifer[7] which paved a way of the pricing model for a game option. We use the model to value a callable CB as a special case.

Theorem 1. *Define*

$$J_t^v(\sigma, \tau) = \tilde{E}\left[e^{-r(\sigma\wedge\tau-t)}R(\sigma, \tau)\middle|\, V_t = v\right]. \tag{10}$$

Suppose that for all $v > 0$

$$\tilde{E}\left[\sup_{0\leq\tau\leq T} e^{-r\tau}zV_\tau \;\middle|\; V_0 = v\right] < \infty.$$

There is a unique no-arbitrage price process $CB(t, v)$ *of the game option in the sense of Black-Scholes model such that* $CB(t, v)$ *can be represented by the right continuous process*

$$\begin{aligned} CB(t, v) &= \inf_{\sigma\in\mathcal{T}_{t,T}} \sup_{\tau\in\mathcal{T}_{t,T}} J_t^v(\sigma, \tau) \\ &= \sup_{\tau\in\mathcal{T}_{t,T}} \inf_{\sigma\in\mathcal{T}_{t,T}} J_t^v(\sigma, \tau). \end{aligned} \tag{11}$$

Moreover, the optimal stopping times $\hat{\sigma}_t$ *and* $\hat{\tau}_t$ *for the issuer and the investor, respectively, are given by*

$$\begin{aligned} \hat{\sigma}_t &= \inf\{\sigma \in [t, T) \mid CB(\sigma, V_\sigma) = \max(zV_\sigma, F)\} \wedge T, \\ \hat{\tau}_t &= \inf\{\tau \in [t, T) \mid CB(\tau, V_\tau) = zV_\tau\} \wedge T. \end{aligned} \tag{12}$$

Furthermore, there exists a self-financing portfolio π^* *such that* $(\hat{\sigma}, \pi^*)$ *is a hedge against the callable contingent claim with initial capital* $J_0^v(\hat{\sigma}, \tau) = CB(0, v)$ *for all* τ *and with* $\tilde{P}$*-probability such portfolio* π^* *is unique.*

Proof should be referred to Kifer[7] after putting $Y_t = zV_t$ and $X_t = \max(zV_t, F)$ as the payoff function of the investor and issuer, respectively.

3 Optimal Call and Conversion Policies and Their Optimal Boundaries to Value the Callable CB

To value the callable CB we must identify the call and conversion policies for the firm and the investor to follow. In according to the spirit of Brennan and Schwartz[1] we assume that it is optimal for the investor to maximize the value of the CB and for the firm to minimize the value of CB at each stopping time.

Optimal conversion policy The investor will never convert the CB when its value is greater than the conversion value given by (7) because the investor seeks to maximize the value of the CB.

Optimal call policy The firm will never call the CB when its value is less than the greater of the call price or its conversion value because the firm wishes to minimize the value of the CB.

It follows from these arguments that the value of the callable CB must satisfy the inequalities

$$zV_t \leq CB(t, V_t) \leq \max(zV_t, F) \tag{13}$$

which provides the lower and upper bounds of the CB value and allows the firm to declare the bankruptcy in the maximum of zV_t and F. If $V_t \geq F/z$ for some t, we have from equation(13)

$$CB(t, V_t) = zV_t. \tag{14}$$

Define the stopping region $\mathcal{S}^f$ for the firm, $\mathcal{S}^i$ for the investor and the continuation region $\mathcal{C}$ for the both of them as follows;

$$\begin{aligned} \mathcal{S}^f &= \{(t, v) \mid CB(t, v) = \max(zv, F)\} \\ \mathcal{S}^i &= \{(t, v) \mid CB(t, v) = zv\} \\ \mathcal{C} &= \{(t, v) \mid zv < CB(t, v) < \max(zv, F)\}. \end{aligned}$$

On the other hand, defining the value of non-callable CB by $\overline{CB}(t, v)$, the stopping region $\overline{\mathcal{S}}^i$ and the continuation region $\overline{\mathcal{C}}$ for the investor is

$$\begin{aligned} \overline{\mathcal{S}}^i &= \{(t, v) \mid \overline{CB}(t, v) = zv\} \\ \overline{\mathcal{C}} &= \{(t, v) \mid \overline{CB}(t, v) > zv\} \end{aligned}$$

because the non-callable CB allows only the investor to exercise the right for the conversion. For each t define

$$\mathcal{S}_t^f = \{v| \ (t, v) \in \mathcal{S}^f\}, \quad \mathcal{S}_t^i = \{v| \ (t, v) \in \mathcal{S}^i\}, \quad \mathcal{C}_t = \{v| \ (t, v) \in \mathcal{C}\}$$

and similarly

$$\overline{\mathcal{S}}_t^i = \{v| \ (t, v) \in \overline{\mathcal{S}}^i\}, \quad \overline{\mathcal{C}}_t = \{v| \ (t, v) \in \overline{\mathcal{C}}\}.$$

Proposition 2. *The value of callable CB is less than or equal to the value of non-callable, that is,*

$$CB(t, v) \leq \overline{CB}(t, v) \quad \textit{for all } t \textit{ and } v \tag{15}$$

and moreover,

$$(\mathcal{S}_t^f \cup \mathcal{S}_t^i) \supseteq \overline{\mathcal{S}}_t^i \quad \textit{for all } t. \tag{16}$$

The proof immediately follows from Theorem 1 because only the investor seeks to maximize the value of CB in equation (12). Equation (16) follows from equation (15).

The optimal call boundary for the firm can be defined as the graph of $v_t^f \equiv \inf\{v|\ v \in \mathcal{S}_t^f\}$. Similarly, the optimal conversion boundary for the investor is the graph of $v_t^i \equiv \inf\{v|\ v \in \mathcal{S}_t^i\}$, $t \in [0, T]$.

Set $v_t^* = \min(v_t^f, v_t^i)$ and similarly $\overline{v}_t^i$ the optimal conversion boundary for the non-callable CB.

Theorem 3. *The following relationships hold between the callable and non-callable CB,*

(i) $\mathcal{S}_t^f = [v_t^f, \infty)$, $\mathcal{S}_t^i = [v_t^i, \infty)$, $\mathcal{C}_t = [0, v_t^*)$

(ii) $\overline{\mathcal{S}}_t^i = [\overline{v}_t^i, \infty)$, $\overline{\mathcal{C}}_t = [0, \overline{v}_t^i)$

(iii) $v_t^* \leq \overline{v}_t^i$, $v_t^f = F/z$ *and* $v_t^* \leq F/z$.

Proof. (i) and (ii) follow from the definitions of $\mathcal{S}_t^f$, $\mathcal{S}_t^i$ and $\mathcal{C}_t$, $\overline{\mathcal{S}}_t^i$ and $\overline{\mathcal{C}}_t$. Inequality $v_t^* \leq \overline{v}_t^i$ in property (iii) can easily be proved from Proposition 2. To show equality $v_t^f = F/z$ proof follows a contradiction. Assume that there exists an optimal call boundary v_t in the region satisfy $v < F/z$, and define $\dot{v} = F/z$. Then, from the definition of $\mathcal{S}^f$ we have $CB(t, v_t^{'}) = F$ for some $v_t^{'} < \dot{v}$. On the other hand, if the optimal call boundary $\dot{v}$ is followed, $CB(t, v_t^{'}) < F$ for $v_t^{'} < \dot{v}$ which contradicts the fact that the firm wishes to minimize the value of CB. This implies that an optimal call boundary must satisfy $v \geq F/z$. From equation (14) we obtain $CB(t, v) = zv$ which can be rewritten as $v_t^f = F/z$. Property $v_t^* \leq F/z$ immediately follows from the definition of v_t^* and equality $v_t^f = F/z$. □

Note that the optimal call and conversion policies depend upon the conversion ratio, the value of the CB and the time remained before the maturity.

4 A Decomposition of the Value of the Convertible Bond

Ingersoll[4] has shown that it is optimal for the investor not to convert the non-callable CB before the maturity when the stock pays no dividend, and hence the value of the non-callable CB can be presented in term of the sum of the bond price and the price of the European call option when the CB pays no coupon. This means that when the stock pays a constant dividend, the value of the non-callable CB can be rewritten as the sum of the bond price and the price of American call option.

Theorem 4. *Let $B(t, v)$ be the bond price when the value of firm is v at time t and $\overline{C}$ the price of the American call option. Then, the value of the non-callable CB can be decomposed as follows;*

$$\begin{aligned} \overline{CB}(t, v) &= B(t, v) + \overline{C}(t, v; B(t, v)) \\ &= B(t, v) + C_e(t, v; F) + \overline{p}(t, v) \end{aligned} \tag{17}$$

where $C_e(t, v; F)$ is the European call option when the exercise price is the face value with the underwriting asset value zv and $\overline{p}(t, v)$ is the premium of the early conversion given by

$$\overline{p}(t, v) = \tilde{E}\left[\int_t^T e^{-r(s-t)} \delta z V_s 1_{\{\overline{v}_s^i \leq V_s\}} ds \Big| V_t = v\right]. \tag{18}$$

Proof. It is well known from Ingersoll[4] that we have

$$\overline{CB}(t,v) = B(t,v) + \overline{C}(t,v).$$

$\overline{C}(t,v)$ is once continuously differentiable in t and twice continuously differentiable in v because it is the price of American Call option. Putting $f(t,v) = e^{-rt}\overline{C}(t,v)$ and applying the Ito's lemma we obtain

$$f(T,V_T) = f(t,v) + \int_t^T \frac{\partial f}{\partial v} dV_s + \int_t^T \left(\frac{1}{2}\kappa^2 V_s^2 \frac{\partial^2 f}{\partial v^2} + \frac{\partial f}{\partial s} \right) ds. \tag{19}$$

Therefore,

$$\begin{aligned} & e^{-rT}\overline{C}(T,V_T) \\ &= e^{-rt}\overline{C}(t,v) + \int_t^T e^{-rs}\frac{\partial \overline{C}}{\partial v} dV_s + \int_t^T e^{-rs}\left(\frac{1}{2}\kappa^2 V_s^2 \frac{\partial^2 \overline{C}}{\partial v^2} - r\overline{C} + \frac{\partial \overline{C}}{\partial s} \right) ds \\ &= e^{-rt}\overline{C}(t,v) + \int_t^T e^{-rs}\kappa V_s \frac{\partial \overline{C}}{\partial v} d\tilde{Z}_s \\ &+ \int_t^T e^{-rs}\left(\frac{1}{2}\kappa^2 V_s^2 \frac{\partial^2 \overline{C}}{\partial v^2} + (r-\delta)V_s\frac{\partial \overline{C}}{\partial v} - r\overline{C} + \frac{\partial \overline{C}}{\partial s} \right) ds. \end{aligned} \tag{20}$$

Note that

$$\begin{aligned} & \frac{1}{2}\kappa^2 v^2 \frac{\partial^2 \overline{C}}{\partial v^2} + (r-\delta)v\frac{\partial \overline{C}}{\partial v} - r\overline{C} + \frac{\partial \overline{C}}{\partial t} \\ &= \begin{cases} 0, & (t,v) \in \overline{\mathcal{C}} \\ -\delta z v - \left(\frac{1}{2}\kappa^2 v^2 \frac{\partial^2 B}{\partial v^2} + (r-\delta)v\frac{\partial B}{\partial v} - rB + \frac{\partial B}{\partial t} \right), & (t,v) \in \overline{\mathcal{S}}^i \end{cases} \end{aligned}$$

and

$$\frac{1}{2}\kappa^2 v^2 \frac{\partial^2 B}{\partial v^2} + (r-\delta)v\frac{\partial B}{\partial v} - rB + \frac{\partial B}{\partial t} = 0.$$

Equation (20) can be rearranged as

$$e^{-rT}\overline{C}(T,V_T) = e^{-rt}\overline{C}(t,v) + \int_t^T e^{-rs}\kappa V_s \frac{\partial \overline{C}}{\partial v} d\tilde{Z}_s - \int_t^T e^{-rs}\delta z V_s 1_{\{\overline{v}_s^i \le V_s\}} ds.$$

After multiplying by e^{rt} and taking the expectation conditioning on $V_t = v$, we have

$$\tilde{E}[e^{-r(T-t)}\overline{C}(T,V_T) \mid V_t = v] = \overline{C}(t,v) - \tilde{E}\left[\int_t^T e^{-r(s-t)}\delta z V_s 1_{\{\overline{v}_s^i \le V_s\}} ds \;\middle|\; V_t = v \right].$$

□

Next, we present the decomposition of the value of the callable CB. Let $C(t,v)$ be the value of conversion option which is the difference between the value of the callable CB and the bond price, that is,

$$C(t,v) = CB(t,v) - B(t,v). \tag{21}$$

Lemma 5. *The value of conversion option $C(t,v)$ satisfies*

$$\frac{\partial C}{\partial v}(t, v_t^{*-}) \leq \frac{\partial C}{\partial v}(t, v_t^{*+}) \tag{22}$$

at $v = v_t^*$.

Proof. It is obvious that $\frac{\partial CB}{\partial v}(t, v_t^{*+}) = z$ at the boundary in the stopping region $\mathcal{S}^f \cup \mathcal{S}^i$. If $\frac{\partial CB}{\partial v}(t, v_t^{*-}) > z$ in the continuation region, the value of CB could not satisfy equation (7). That is, we must have $\frac{\partial CB}{\partial v}(t, v_t^{*-}) \leq z$. Since $B(t,\cdot)$ is differentiable, $\frac{\partial B}{\partial v}(t, v_t^{*-}) = \frac{\partial B}{\partial v}(t, v_t^{*+})$ holds. Hence, we obtain

$$\begin{aligned}
\frac{\partial C}{\partial v}(t, v_t^{*-}) &= \frac{\partial CB}{\partial v}(t, v_t^{*-}) - \frac{\partial B}{\partial v}(t, v_t^{*-}) \\
&\leq z - \frac{\partial B}{\partial v}(t, v_t^{*-}) \\
&= \frac{\partial CB}{\partial v}(t, v_t^{*+}) - \frac{\partial B}{\partial v}(t, v_t^{*+}) \\
&= \frac{\partial C}{\partial v}(t, v_t^{*+}).
\end{aligned}$$

□

Proposition 6. *The value of the conversion option $C(t,v)$ is convex and increasing in v for each t.*

Proof. For $v \leq F/z$ the conversion option can be considered as an American knock out call option where the exercise price is $B(t,v)$ and the investor may receive the payoff $(F - B(t, F/z))$ whenever the firm value V_t hits the value F/z with the conversion value zv. Then, we know from Kunita and Seko [8] that the value of the conversion option is increasing and convex in v. For $v > F/z$ we have from $\frac{\partial^2 B}{\partial v^2} \leq 0$ that

$$\frac{\partial^2 C}{\partial v^2} = -\frac{\partial^2 B}{\partial v^2} \geq 0$$

which implies together with Lemma 5 that Proposition 6 holds. □

Theorem 7. *The value of the callable CB can be represented by the following decomposition;*

$$\begin{aligned}
CB(t,v) &= B(t,v) + C(t,v) \\
&= B(t,v) + C_e(t, zv; F) + p(t,v) - d(t,v)
\end{aligned} \tag{23}$$

where

$$p(t,v) = \tilde{E}\left[\int_t^T e^{-r(s-t)} \delta z V_s 1_{\{v_s^* \leq V_s\}} ds \middle| V_t = v\right] \tag{24}$$

and

$$d(t,v) = \tilde{E}\left[\int_t^T e^{-r(s-t)} \left(\frac{\partial C}{\partial v}(s, v_s^{*+}) - \frac{\partial C}{\partial v}(s, v_s^{*-})\right) dL_s^v(v_s^*) \middle| V_t = v\right] \tag{25}$$

where we can call $p(t,v)$ the early conversion premium and $d(t,v)$ the callable discounted value, respectively, and $L_t^v(v_t^*)$ the local time of V_t at the level v_t^* in the time interval $[0,t]$.

Proof. Put $g(t,v)=e^{-rt}C(t,v)$ and applying the generalized Ito's lemma for convex functions [5]. Then we obtain

$$\begin{aligned} g(T,V_T) &= g(t,v)+\int_t^T \frac{\partial g}{\partial v}dV_s+\int_t^T\left(\frac{1}{2}\kappa^2V_s^2\frac{\partial^2 g}{\partial v^2}+\frac{\partial g}{\partial s}\right)ds \\ &\quad +\int_t^T\left(\frac{\partial g}{\partial v}(s,v_s^{*+})-\frac{\partial g}{\partial v}(s,v_s^{*-})\right)dL_s^v(v_s^*). \end{aligned} \tag{26}$$

Note from the free boundary conditions that

$$\frac{1}{2}\kappa^2v^2\frac{\partial^2 C}{\partial v^2}+(r-\delta)v\frac{\partial C}{\partial v}-rC+\frac{\partial C}{\partial t}=\begin{cases} 0, & (t,v)\in\mathcal{C} \\ -\delta zv, & (t,v)\in\mathcal{S}^i\cup\mathcal{S}^f. \end{cases} \tag{27}$$

Therefore, we have

$$\begin{aligned} e^{-rT}C(T,V_T) &= e^{-rt}C(t,v)+\int_t^T e^{-rs}\kappa V_s\frac{\partial C}{\partial v}d\tilde{Z}_s-\int_t^T e^{-rs}\delta zV_s 1_{\{v_s^*\le V_s\}}ds \\ &\quad +\int_t^T e^{-rs}\left(\frac{\partial C}{\partial v}(s,v_s^{*+})-\frac{\partial C}{\partial v}(s,v_s^{*-})\right)dL_s^v(v_s^*). \end{aligned}$$

Multiplying by e^{rt} and taking the expectation conditioning on $V_t=v$, we obtain

$$\begin{aligned} \tilde{E}[e^{-r(T-t)}C(T,V_T)\mid V_t=v] &= C(t,v)-\tilde{E}\left[\int_t^T e^{-r(s-t)}\delta zV_s 1_{\{v_s^*\le V_s\}}ds\;\middle|\; V_t=v\right] \\ &\quad +\tilde{E}\left[\int_t^T e^{-r(s-t)}\left(\frac{\partial C}{\partial v}(s,v_s^{*+})-\frac{\partial C}{\partial v}(s,v_s^{*-})\right)dL_s^v(v_s^*)\right] \end{aligned} \tag{28}$$

which can be rewritten as in equation (23). □

Corollary 8. *The early conversion premium for the callable CB is more than or equal to the one for the non-callable CB, that is,* $p(t,v)\ge\overline{p}(t,v)$ *for all* t, v *and hence* $d(t,v)\ge 0$ *for all* t, v.

Proof. From definition of v_t^* and Theorem 3 (iii) we have $v_t^*\le\overline{v}_t^i$ for all t. Then, from equation (18) and (24) we obtain $p(t,v)\ge\overline{p}(t,v)$ for all t, v because the conditional expectations should be taken over the time interval $[t,T]$. □

5 Numerical Examples

In this section we present the numerical valuation of the callable and non-callable CBs, and the optimal call and conversion boundaries through a simple finite element method. Table 1 shows the data we use to evaluate the values of the CB.

Figure 1 demonstrates the relationship among callable CB, non-callable CB and plain bond as the functions of the initial firm value v to confirm inequalities (13) and (15). It also illustrates how much cheaper the callable CB is than non-callable one and the plain bond, respectively. For the data given in Table 1 the optimal boundary is given by $v_0^*=v_0^f=F/z$, which can be recognized numerically. Figure 2 shows the callable discount value which is the distance between the callable and non-callable CB's. It is also important to observe the

Table 1: Data

Face value (Call price) F	100
Maturity T	3
Conversion ratio z	0.4
Interest rate r	0.05
Dividend rate δ	0.02
Volatility κ	0.3

value of the conversion option C possessing the properties that C is increasing and convex in v but not smooth at the value of $v_0^* = v_0^f$. Figure 3 shows the behavior of the optimal conversion and call boundaries for the non-callable and the callable CBs, respectively, for the different levels of t. It can be seen from Figure 3 that optimal boundary v_t^* is less than or equal to F/z for all t, which is given by equation (16).

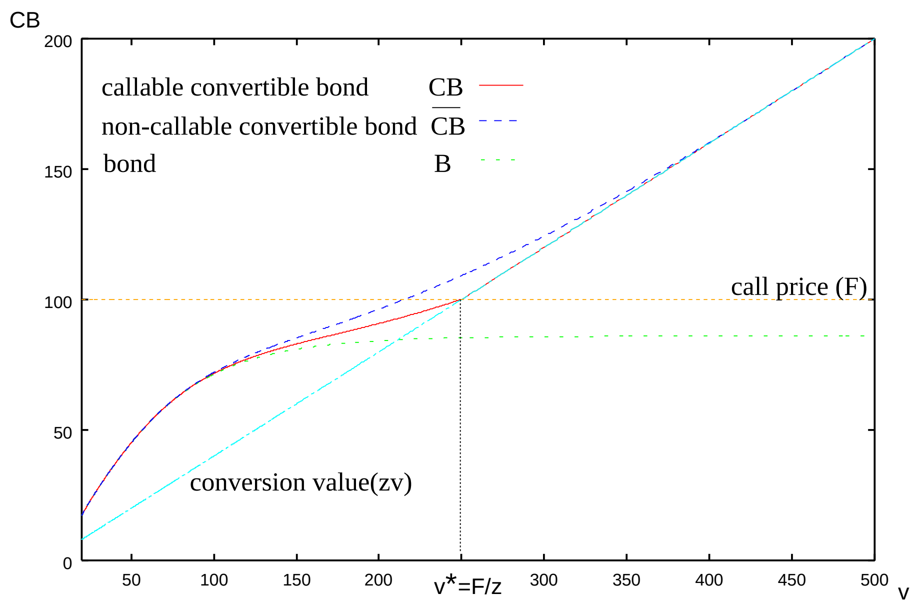

Figure 1: The value of CB

6 Summary and Concluding Remarks

In this paper we have studied the pricing of the callable convertible bonds and the optimal call and conversion policies of the callable CB by means of a coupled stopping game between the issuer and the investor. We have shown that the value of the callable CB can be decomposed into the sum of the bond price, the value of the European call and the difference between the early conversion premium and the callable discounted value. Furthermore, we have embodied the optimal conversion and call boundaries numerically computed by the finite differential method.

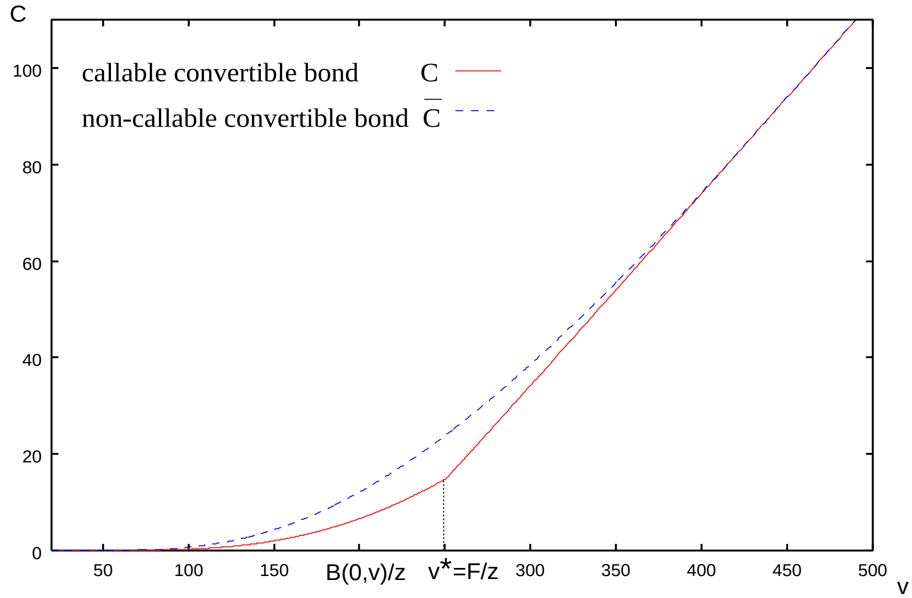

Figure 2: The value of the conversion option

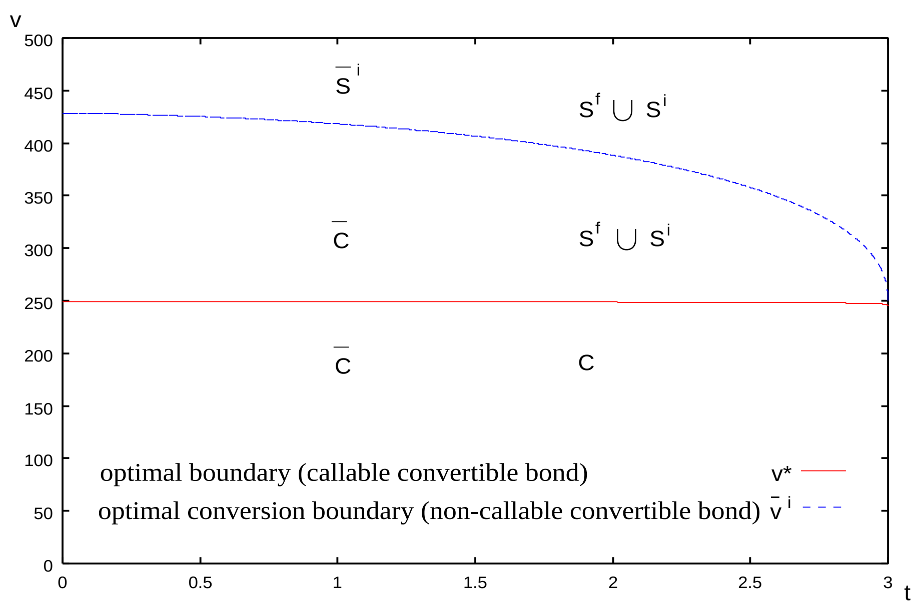

Figure 3: Optimal boundaries of the CB

We assume that the bond has no coupon payment beside the stock pays a constant dividend. Further investigation is left for future research when the bond pays coupons to the bondholder because the bond usually pays coupons. It is interesting to investigate how the coupon payments affect the value and optimal boundaries of the callable CB. What happens to the value of the CB if the call time is restricted ? For example, the firm is allowed to call only in some certain time interval. Considering these questions leads more practical evaluation of existing financial commodities traded in the actual market.

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