Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 1, No. 2, pp. 347−355, May 2005

NON-DISCRIMINATING PRICES IN LINEAR PROGRAMS WITH A REVERSE CONVEX CONSTRAINT

P.T. Thach

This article is dedicated to Professor Hiroshi Konno on the occasion of his 65th anniversary.

Abstract: In this paper we extend the concept of non-discrimi- nating prices to a problem with a hierarchical structure in which the sublevel holds linear constraints and the central level holds a reverse convex constraint. The objective is a linear function to be minimized. In the study of the relationship between prices and characteristics of optimal solutions we prove that the existence of an optimal non-discriminating price is equivalent to the convexity of the set of optimal solutions. On the basis of this optimal price one can linearize the problem, provided that an optimal solution to the dual is known.

Key words: *price, linear program, reverse convex constraint, linearization*

Mathematics Subject Classification: *90C30*

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1 Introduction

It is well known that the Simplex Method using multipliers for linear programs can be interpreted as a pricing mechanism, and this pricing mechanism is especially efficient for practical problems in the presence of certain hierarchical structures (Refs. [2, 3]). The simplex multipliers obtained by pricing out the constraints at the central level can be used to define a price function which coordinates the activities at the sublevel. So far there have been great research attempts to extend this result to more general applications (e.g., Refs. [1, 4, 5, 6, 12, 13, 15]). Indeed, the Lagrange multiplier methods have been deeply developed for nonlinear programs. For quite general problems the class of nondecreasing functions is involved to define price functions (cf. [6, 13]), however there is a difficulty about the trade-off between the simplicity of price functions and the existence of the optimal one. Behind this kind of difficulty is the duality gap between the optimal values in a primal-dual pair for nonconvex problems if we stick to convex duality. In the streamline of these researches this paper presents an alternative price concept to a special problem in which the central level holds a constraint given by the complement of a convex set. More concretely, the constraint at the central level is of the form

$$g(x) \ \geq \ 0,$$

where $x = (x_1, x_2, \dots, x_n)$ is a vector of activities in R^n and g is a convex function defined on R^n. Let x' and x'' be vectors of activities. If we call a compromise between x' and x'' a vector x such that $x = \mu' x' + \mu'' x''$, $\mu' \geq 0$, $\mu'' \geq 0$, and $\mu' + \mu'' = 1$, then the constraint given above assures that any compromise between infeasible vectors x' and x'' is also infeasible. Applications to this type of constraint can be found in the literature (cf. Refs. [2], [7] - [12], [14]). The problem under our consideration is as follows

$$\begin{array}{ll} \text{Minimize} & cx, \\ \text{subject to} & g(x) \geq 0, \\ & Ax \geq a, \end{array} \tag{1}$$

where A is a $m \times n$-matrix, $a \in R^m$, $c \in R^n$, $c \neq 0$, $x \in R^n$, g is a convex function defined on R^n such that $g(0) < 0$ and cx denotes the inner product of vectors c and x. Throughout this paper we assume that this problem is solvable, i.e., there is a feasible solution at which the objective function achieves its minimum value.

In order to extend the price concept to the nonlinear constraint in the above problem let us define

$$f(x) \;=\; \begin{cases} cx & \text{if } Ax \geq a, \\ +\infty & \text{otherwise.} \end{cases}$$

The problem is then considered as minimizing $f(x)$ on the domain $\{x \in R^n |\; g(x) \geq 0\}$:

$$\begin{array}{ll} \text{Minimize} & f(x), \\ \text{subject to} & g(x) \geq 0. \end{array} \tag{2}$$

Using a vector $y = (y_1, y_2, \ldots, y_n)$ in the dual space of the activity space as a vector of prices we can represent the constraint in (2) as follows

$$yx \geq 1, \qquad y \in \Omega,$$

where

$$\Omega \;=\; \{y \in R^n |\; yx \leq 1 \quad \forall x:\; g(x) \leq 0\}.$$

Thus, a vector $x = (x_1, x_2, \ldots, x_n)$ is feasible to the constraint $g(x) \geq 0$ if and only if there is $y \in \Omega$ such that $yx \geq 1$. In particular, if $y \in \Omega$ then every vector x such that $yx \geq 1$ is feasible to the constraint at the central level. A vector y of prices is called feasible if $y \in \Omega$.

For a given feasible vector $\overline{y}$ of prices let us consider the following criterion on the vector $\overline{x}$ of activities and a multiplier δ:

(D) $f(\overline{x}) - \delta(\overline{y}\,\overline{x} - 1) \;=\; \min_x \{f(x) - \delta(\overline{y}x - 1)\}$;

(C) $\overline{y}\,\overline{x} - 1 \geq 0$, $\delta \geq 0$, $\delta(\overline{y}\,\overline{x} - 1) = 0$.

The first criterion is a decomposition principle which tells us that $\overline{x}$ is an optimal solution at the sublevel according to the vector $\overline{y}$ of prices. The second is the complementarity for the constraint at the central level using vector $\overline{y}$ of prices. In [12] a concept of optimal price vector was used for the case in which (D) and (C) give a sufficient criterion for the optimality of activity vector $\overline{x}$. Namely a feasible vector $\overline{y}$ of prices is called *optimal* if any feasible vector $\overline{x}$ of activities satisfying (D) and (C) is optimal to (2). Since by dualization an optimal vector of prices is an optimal solution to a dual program which is a quasiconcave minimization over a convex set, cutting-plane methods can be used to find optimal vectors of prices (cf. [12]). In this paper we are interested in the question if there exists an optimal vector $\overline{y}$ of prices such that the criterion of (D) and (C) is both sufficient and necessary for the optimality. In section 2 we provide a condition for the existence of such a vector $\overline{y}$ and its dual interpretation. Concluding remarks are given in section 3.

2 Optimal Non-Discriminating Prices

Before stating a definition of optimal non-discriminating prices let us consider the following example.

Example 1. In a working plan of two factories, x_i $(i = 1, 2)$ denotes the activity of factory i, and c_i $(i = 1, 2)$ denotes the operating cost for each one unit of activity i. Both c_1 and c_2 have a positive value. We assume that the two factories share 10 units of common resource and each one activity unit of either factory 1 or factory 2 consumes one unit of resource. Besides these data, the power of the working plan is measured by the square norm of activities

$$p(x_1, x_2) \;=\; x_1^2 + x_2^2,$$

and the constraint for the working plan power is

$$p(x_1, x_2) \;\geq\; 100.$$

The problem is to minimize the total operating cost under the constraint for the working plan power, the constraint of common resource and the constraint of nonnegative activities:

$$\begin{array}{ll} \text{Minimize} & c_1x_1 + c_2x_2, \\ \text{subject to} & x_1^2 + x_2^2 \geq 100, \\ & x_1 + x_2 \leq 10,\ x_1 \geq 0,\ x_2 \geq 0. \end{array}$$

There are only two feasible solution to this program:

$$x^1 \;=\; (10, 0), \qquad x^2 \;=\; (0, 10).$$

The solution x^1 is favoured by factory 1, while x^2 is favoured by factory 2. Assume that $c_1 = c_2$. Then, both x^1 and x^2 are optimal, and it can be checked that there are only two optimal vectors of prices:

$$y^1 \;=\; (0.1, 0), \qquad y^2 \;=\; (0, 0.1).$$

With the price y^1 the criterion of (D) and (C) detects the optimal solution x^1 but fails to detect the optimal solution x^2. Similarly, the price y^2 detects x^2 but fails to detect x^1. Since $c_1 = c_2$, the two factories share the same production technology. Therefore, the price y^1 discriminates x^2 (which is favourable to factory 2), while y^2 discriminates x^1 (which is favourable to factory 1). Behind this discriminating phenomenon is the fact that the criterion of (D) and (C) is only sufficient for optimality but not necessary.

Now we give a definition of non-discriminating prices. An optimal vector $\overline{y} \in \Omega$ of prices is called *non-discriminating* if the criterion of (D) and (C) is necessary and sufficient for the optimality of $\overline{x}$, or in the other words, $\overline{x}$ solves (2) if and only if there is δ such that $\overline{x}$ and δ satisfy the criterion of (D) and (C).

Since (D) and (C) together are a necessary and sufficient condition for optimality of $\min\{cx | Ax \leq a, \overline{y}x \geq 1\}$, the set of vectors $\overline{x}$ satisfying (D) and (C) is convex. Therefore, it follows from the definition that if there is an optimal non-discriminating vector of prices then the set of optimal solutions to (2) is convex. The inverse direction is also true and will be proved in the subsequent theorem.

In the space of price vectors let us consider a dual problem of (2):

$$\begin{array}{ll} \text{Minimize} & f^c(y), \\ \text{subject to} & y \in \Omega, \end{array} \tag{3}$$

where f^c is a quasi-conjugation of f which is defined at any $y \neq 0$ as follows

$$f^c(y) \;=\; \inf\{f(x)|\; yx \geq 1\},$$

(cf. Ref. [11]). The quasi-conjugation f^c is quasiconcave on R^n. By the duality relationship (Ref. [11]), the dual problem (3) is solvable and

$$\min(2) \;=\; \min(3).$$

Denote by α the above finite optimal value. If y solves the dual (3), then the following set

$$S(y) \;=\; \{x|Ax \geq a, cx = \alpha \text{ and } yx \geq 1\}$$

is nonempty and any $x \in S(y)$ is optimal to the primal (2) (Ref. [11]).

An optimal solution $\overline{y}$ of (3) is called *comprehensive* if the set $S(\overline{y})$ includes the set $S(y)$ for any optimal solution y of (3), or in the other words one has the following deduction for any optimal solution y of (3)

$$\left.\begin{array}{l} Ax \geq a \\ yx \geq 1 \\ cx = \alpha \end{array}\right\} \Longrightarrow \overline{y}x \geq 1. \tag{4}$$

It is obvious that if problem (3) has a unique optimal solution $\overline{y}$ then $\overline{y}$ is a comprehensive optimal solution.

Theorem 1. *The following three conditions are equivalent.*

(i) *The set of optimal solutions to the primal problem (2) is convex;*

(ii) *There is a comprehensive optimal solution to the dual problem (3);*

(iii) *There is an optimal non-discriminating vector of prices.*

Furthermore, y is a comprehensive optimal solution to (3) if and only if it is an optimal non-discriminating vector of prices.

Proof. (i) $\Rightarrow$ (ii): Suppose that the set $\overline{X}$ of optimal solutions to (2) is convex. Since $\overline{X}$ has no common solution with the open convex set $\{x \in R^n|\; g(x) < 0\}$, by separation theorem there is $\overline{y} \in \Omega$ such that

$$\overline{y}x \;\geq\; 1 \qquad \forall x \in \overline{X}. \tag{5}$$

Let $\overline{x} \in \overline{X}$, then

$$\begin{aligned} f^c(\overline{y}) &= \min\{cx \mid\; Ax \geq a,\; \overline{y}x \geq 1\} \\ &\leq c\overline{x} \quad \text{(by (5))} \\ &= \alpha, \end{aligned}$$

which implies that $\overline{y}$ is an optimal solution to (3). Let y be any optimal solution of (3). Then,

$$\begin{aligned} \alpha &= f^c(y) \\ &= \min\{cx|\; Ax \geq a,\; yx \geq 1\}. \end{aligned} \tag{6}$$

Let $\overline{x}$ be a solution such that $A\overline{x} \geq a$, $y\overline{x} \geq 1$, and $c\overline{x} = \alpha$. Then, $\overline{x}$ solves the linear program (6), and therefore $\overline{x}$ solves (2), i.e., $\overline{x} \in \overline{X}$. It follows from (5) that $\overline{y}\,\overline{x} \geq 1$. So, $\overline{y}$ satisfies condition (4) of comprehensiveness.

(ii) $\Leftrightarrow$ (iii): Let $\overline{y}$ be a comprehensive optimal solution of (3). It has been known that for any optimal solution x' to (2) there is an optimal solution y' to (3) such that $y'x' \geq 1$ (Ref. [11]). Since $\overline{y}$ is a comprehensive optimal solution, this implies $\overline{y}x' \geq 1$. Thus, $\overline{y}$ is a comprehensive optimal solution of (3) if and only if $\overline{y}x' \geq 1$ for any optimal solution x' to (2). So, the set $\overline{X}$ of optimal solutions to (2) coincides with the set of optimal solutions to the following linear program

$$\min\{cx|\ Ax \geq a,\ \overline{y}x \geq 1\},$$

for which conditions (D) and (C) give an optimal criterion. Therefore $\overline{y}$ is an optimal non-discriminating vector of prices if and only if $\overline{y}$ is a comprehensive optimal solution to (3).

The proof of "(iii) $\Rightarrow$ (i)" is straightforward. □

In several applications problem 1 has a discrete structure, i.e., it has a finite number of feasible solutions (e.g., example 1). Then the number of its optimal solutions is finite. Therefore, it is straightforward from Theorem 1 that there is an optimal non-discriminating vector of prices if and only if the problem with a discrete structure yields a unique optimal solution.

Example 2. We reconsider the problem in example 1 where the operating cost c_1 is less than the operating cost c_2. In this case, $x^1 = (10, 0)$ is the unique optimal solution. Therefore the optimal price $y^1 = (0.1, 0)$ is non-discriminating because with y^1 the criterion of (D) and (C) is necessay and sufficient for the optimality. In an economic interpretation the price y^1 is favourable to factory 1, but it is non-discriminating because the production technology of factory 1 is more efficient than that of factory 2 ($c_1 < c_2$).

Before closing this section we discuss the comprehensiveness under the restricted assumption that $a = (a_1, a_2, \ldots, a_m) > 0$. This assumption could be satisfied in production problems with positive lower-bound constraints of activities and constraints of positive demands.

As before we call y a compromise between vectors $y^1, y^2, \ldots, y^s$ ($y^i \in R^n$) if there is a representation

$$y = \sum_{i=1}^{s} \mu_i y^i,$$

where $\mu_i \geq 0$, $i = 1, 2, \ldots, s$ and $\sum_{i=1}^{s} \mu_i = 1$. In the above representation if $\mu_i > 0$ then the compromise y is said proper to y^i.

Theorem 2. *Assume that $a_i > 0$, $i = 1, 2, \ldots, m$. Set $c' = \frac{1}{\alpha}c$, and $A'_i = \frac{1}{a_i}A_i$, $i = 1, 2, \ldots, m$ where A_i is the i-th row of the matrix A. Then*

(i) *If $c' \in \Omega$ then c' is a comprehensive optimal solution to (3);*

(ii) *In general, $\overline{y} \in \Omega$ is a comprehensive optimal solution to (3) if and only if for any optimal solution y to (3) there is a compromise between c' and $\overline{y}$, proper to $\overline{y}$, which is also a compromise between A'_i, $i = 1, 2, \ldots, m$, and y.*

Proof. Let y be an optimal solution to (3). Then

$$\begin{aligned}\alpha &= f^c(y)\\ &= \min\{cx|\ Ax \geq a,\ yx \geq 1\}.\end{aligned}$$

Suppose that $\overline{x}$ solves the above linear program:

$$A\overline{x} \geq a, \quad y\overline{x} \geq 1, \quad c\overline{x} = \alpha. \tag{7}$$

Since $a_i > 0$, $i = 1, 2, \ldots, m$, the open half line $\{\theta\overline{x}|\ \theta > 1\}$ is entirely contained in the interior of the domain $\{x \in R^n|\ Ax \geq a,\ yx \geq 1\}$:

$$\begin{aligned}A\theta\overline{x} &= \theta A\overline{x} > a\\ y\theta\overline{x} &= \theta y\overline{x} > 1 \qquad \forall\theta > 1.\end{aligned}$$

If $\alpha < 0$, we see $c(\theta\overline{x}) = \theta\alpha < \alpha$ for $\theta > 1$, which is a contradiction. Suppose that $\alpha = 0$. Take some $\theta > 1$ and a small neighborhood of $\theta\overline{x}$ that is contained in the interior of the domain. By the linearity of the objective function, we could find a point with a negative objective function value in the neighborhood. This is again a contradiction. Therefore,

$$\alpha = c\overline{x} > 0.$$

We assume now that c' $(= \frac{1}{\alpha}c)$ belongs to Ω. Then,

$$\begin{aligned}f^c(c') &= \min\{cx|\ Ax \geq a,\ c'x \geq 1\}\\ &= \min\{cx|\ Ax \geq a,\ cx \geq \alpha\}\\ &\leq c\overline{x} \qquad \text{(from (7))}\\ &= \alpha.\end{aligned}$$

Thus, c' is optimal to (3). To see the comprehensiveness of c' we take any optimal solution y of (3) and confirm that

$$\begin{aligned}f^c(y) &= \min\{cx|\ Ax \geq a,\ yx \geq 1\}\\ &= \alpha.\end{aligned}$$

This implies

$$\left.\begin{aligned}Ax \geq a\\ yx \geq 1\end{aligned}\right\} \Longrightarrow cx \geq \alpha.$$

Since $c' = \frac{1}{\alpha}c$, one then obtains

$$\left.\begin{aligned}Ax \geq a\\ yx \geq 1\end{aligned}\right\} \Longrightarrow c'x \geq 1.$$

In particular, c' satisfies condition (4) of comprehensiveness. We thus completed the proof for the first part of the theorem.

Let $\overline{y} \in \Omega$ be a solution such that for any optimal solution y to (3) there is a compromise between c' and $\overline{y}$, proper to $\overline{y}$, which is also a compromise between A_i', $i = 1, 2, \ldots, m$, and y:

$$\begin{aligned}\mu^{\overline{y}}\overline{y} + \mu^c c' &= \sum_{i=1}^m \mu_i^A A_i' + \mu^y y,\\ \mu^{\overline{y}} + \mu^c &= 1,\ \mu^{\overline{y}} > 0,\ \mu^c \geq 0,\\ \sum_{i=1}^m \mu_i^A + \mu^y &= 1,\ \mu_i^A \geq 0,\ i = 1, 2, \ldots, m,\ \mu^y \geq 0.\end{aligned}$$

The first equation in the above display is equivalent to

$$\mu^{\overline{y}}\overline{y} \; = \; \sum_{i=1}^{m} \mu_i^A A_i' + \mu^y y - \mu^c c',$$

or to the following representation of $\overline{y}$:

$$\overline{y} \; = \; \sum_{i=1}^{m} \frac{\mu_i^A}{\mu^{\overline{y}}} A_i' + \frac{\mu^y}{\mu^{\overline{y}}} y - \frac{\mu^c}{\mu^{\overline{y}}} c'.$$

Since

$$\begin{array}{rcl} \frac{\mu_i^A}{\mu^{\overline{y}}} & \geq & 0, \; i = 1, 2, \ldots, m, \\ \frac{\mu^y}{\mu^{\overline{y}}} & \geq & 0, \; \frac{\mu^c}{\mu^{\overline{y}}} \; \geq \; 0, \\ 1 & = & \sum_{i=1}^{m} \frac{\mu_i^A}{\mu^{\overline{y}}} + \frac{\mu^y}{\mu^{\overline{y}}} - \frac{\mu^c}{\mu^{\overline{y}}}, \end{array}$$

the above representation of $\overline{y}$ is equivalent to the following deduction

$$\left. \begin{array}{l} A_i' x \geq 1, \; i = 1, 2, \ldots, m \\ yx \geq 1 \\ c'x \leq 1 \end{array} \right\} \Longrightarrow \; \overline{y}x \geq 1.$$

Since $A_i' = \frac{1}{a_i} A_i$, $c' = \frac{1}{\alpha} c$ and y solves (3), this deduction reduces to the definition (4) of the comprehensiveness for $\overline{y}$. □

3 Concluding Remarks

In the previous section we studied conditions for an optimal vector of prices which are non-discriminating in the sense that they set up an equal pricing value over the set $\overline{X}$ of optimal solutions to (2). As we have shown, these conditions are equivalent to the convexity of $\overline{X}$. Under this convexity property we can linearlize the problem, provided that an optimal vector of non-discriminating prices is available, or equivalently, an comprehensive optimal solution to the dual (3) is known. Indeed, for a given comprehensive optimal solution $\overline{y}$ to the dual (3) we can solve the problem (2) by solving the criterion of (D) and (C) that is just a linear program. If $\overline{X}$ is nonconvex, we have also shown that there does not exist an optimal non-discriminating vector of prices. The concept of prices developed in this case leads to optimal prices which set up unequal pricing values on $\overline{X}$. For computational issues we can, in principle, extend the well known pricing mechanism in linear programs to obtain a decomposition algorithm for problem (2), in which the sublevel solves linear programs and the central level solves relaxed dual problems of (3) (Ref. [12]). The interaction between the central level and the sublevel is carried out by the set of prices and the set of activities corresponding to these given prices. If the working dimension of (3) is limited, then this interaction can be practical from computational point of view (Ref. [8]).

Acknowledgements

The author would like to thank Prof. Takahito Kuno and the referees for helpful comments and suggestions in the revision of the paper.

References

[1] R.E. Burkard, H.W. Hamacher and J. Tind, On general decomposition schemes in mathematical programming, *Math. Prog. Study* 24 (1985) 238–252.

[2] G.B. Dantzig, *Linear Programming and Extensions*, Eleventh printing, Princeton University Press, Princeton, N.J., 1998.

[3] G.B. Dantzig, and P. Wolfe, Decomposition principle for linear programs, *Oper. Res.* 8 (1960) 101–111.

[4] H. Everett, Generalized Lagrange multiplier method for solving problems of optimum allocation of resources, *Oper. Res.* 11 (1963) 399–417.

[5] A.M. Geoffrion, *Primal Resource-Directive Approaches for Optimizing Nonlinear Decomposition Systems*, working paper 141, the RANK Corporation (Santa Monica, CA, 1968).

[6] F.J. Gould, Extensions of Lagrange multipliers in nonlinear programming, *SIAM J. Appl. Math.* 17 (1969) 1280–1297.

[7] R.J. Hillestad and S.E. Jacobsen, Linear programs with an additional reverse convex constraint, *Appl. Math. Optim.* 6 (1980) 257–269.

[8] H. Konno, P.T. Thach and H. Tuy, *Optimization on Low Rank Nonconvex Structures*, Kluwer Academic Publishers, Dordrecht, 1997.

[9] T. Kuno, H. Konno and Y. Yamamoto, A parametric successive underestimation method for convex programming problems with an additional convex multiplicative constraint, *J. Oper. Res. Soc. Japan* 35 (1992) 290–299.

[10] J.B. Rosen, Iterative solution of nonlinear optimal control problems, *SIAM J. Control* 4 (1966) 223–244.

[11] P.T. Thach, Quasiconjugate of functions, duality relationship between quasiconvex minimization under a reverse convex constraint and quasiconvex maximization under a convex constraint and its applications, *J. Math. Anal. Appl.* 159 (1991) 299–322.

[12] P.T. Thach and H. Konno, A generalized Dantzig-Wolfe decomposition principle for a class of nonconvex programming problems, *Math. Program.* Series B, 62 (1993) 239–260.

[13] J. Tind and L.A. Wolsey, An elementary survey of general duality theory in mathematical programming, *Math. Program.* 21 (1981) 241–261.

[14] H. Tuy, Convex programs with an additional reverse convex constraint, *J. Optim. Theory Appl.* 52 (1987) 463–486.

[15] L.A. Wolsey, Integer programming duality: price functions and sensivity analysis, *Math. Program.* 20 (1981) 173–195.

Manuscript received 31 May 2004
revised 9 November 2004
accepted for publication 22 December 2004

P.T. Thach
Institute of Mathematics, 18 Hoang Quoc Viet, 10307 Hanoi, Vietnam
E-mail address: ptthach@thevinh.ncst.ac.vn