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# SOCIAL WELFARE FUNCTION FOR RESTRICTED INDIVIDUAL PREFERENCE *

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*Dedicated to Professor Hiroshi Konno on his 65th birthday.*

**Abstract:** We consider the social preference ordering in a society where each individual's preference domain is restricted to a subset of the whole set of alternatives. We show that the social welfare function satisfying unrestricted domain property, independence of irrelevant alternatives and weak Pareto principle is always dictatorial when at least one individual is entitled to express his/her preference on the whole set of alternatives.

**Key words:** *social welfare function, voting, impossibility theorem, mutual evaluation*

**Mathematics Subject Classification:** *91B14, 91B08, 91B12*

## 1 Introduction

We often encounter the problem of aggregating opinions of individuals in a society. Arrow [2] introduced the social choice theory for this problem, and gave the monumental impossibility theorem: every social welfare function that satisfies unrestricted domain property, independence of irrelevant alternatives and weak Pareto principle is dictatorial. From then onward, the difficulty of the problem has been well recognized, and a variety of impossibility theorems in Arrow's framework have been developed. The reader is recommended referring to Sen [15].

This paper studies the existence and properties of a social welfare function when individual preference domain is restricted: one expresses one's preference on one's alternative set that is a subset of the whole set of alternatives. This modification can be viewed as a relaxation of the unrestricted domain property in Arrow's framework. For relaxation of the unrestricted domain property, there are many researches such as Blair and Muller [3], Bordes and Le Breton [4], Fishburn and Kelly [5], Kalai, Muller and Satterthwaite [9] and Redekop [11]. In particular, Kalai and Muller [8], Ritz [12, 13] and Gaertner [6] deal with a restriction on permissible preferences for individuals instead of profile restriction.

Ando, Ohara, and Yamamoto [1] consider a society where individuals evaluate mutually. The set of alternatives coincides with the set of all individuals, and each individual expresses his/her preference ordering on the whole set of individuals except him/herself. They studied properties of the social welfare function, and proved that an outcome of social welfare

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function satisfying unrestricted domain property and weak Pareto principle can be cyclic, hence cannot be a preference order, meaning the nonexistence of social welfare function.

In order to avoid the paradoxical outcome of Ando, Ohara, and Yamamoto [1], we add several individuals who express their preferences on the whole set of alternatives. Unlike their model, the set of alternatives can be any finite set instead of the set of individuals. Accordingly, unrestricted domain property, independence of irrelevant alternatives and weak Pareto principle are defined. We will show in Theorem 3.1 that the social welfare function satisfying these three axioms is dictatorial. More precisely, someone who expresses his/her preference on the whole set of alternatives is a dictator. This model applies, for example, to a vote in a class. Take a vote for a varsity team of the class. Each pupil submits his/her preference on all of his/her classmates except him/herself. The teacher is entitled to express his/her preference on all pupils in the class. The main theorem implies that the teacher is a dictator under the three axioms.

In Section 2, the framework of the model and notations are introduced. In Section 3, we give the main theorem. In Section 4, a special case is discussed. In Section 5, we show another proof of the main theorem by utilizing Arrow's impossibility theorem. Section 5 summarizes the results.

## 2 Notations and Framework

Let us denote the finite set of alternatives by $X$ and assume that there are at least three alternatives, i.e., $|X| \geq 3$. A binary relation $\succsim$ on $X$ is called a *preference ordering* or simply *preference* if it satisfies the following conditions:

(i) reflexivity : $x \succsim x$ holds for any alternative $x \in X$,

(ii) completeness : $x \succsim y$, $y \succsim x$ or both hold for any pair of alternatives $x, y \in X$,

(iii) transitivity : if $x \succsim y$ and $y \succsim z$, then $x \succsim z$ holds for any alternatives $x, y, z \in X$.

We write $x \sim y$ when both $x \succsim y$ and $y \succsim x$ hold while we write $x \succ y$ when $x \succsim y$ and $y \not\succsim x$. For a set $Y \subseteq X$ of alternatives, we denote by $\succsim|Y$ the restriction of binary relation $\succsim$ on $X$ to $Y$, i.e., $\succsim|Y$ is defined on $Y \times Y$ and $x \succsim|Y y$ if and only if $x \succsim y$ and $x, y \in Y$.

Let $N = \{1, ..., n\}$ be the finite set of all individuals and assume that $n \geq 2$. In Arrow's framework, each individual is interested in all of the alternatives, and hence his/her preference is defined as a preference ordering on the whole set $X$. However, there might be some individuals who are not interested in all the alternatives. To express such a situation we consider the set of alternatives that are of interest to individual $i$, and denote it by $X_i$. Then individual $i$ has his/her preference on $X_i$. We assume that $|X_i| \geq 2$ for all $i \in N$. We denote by $\mathcal{W}_i$ the set of all preference orderings defined on $X_i$. Let $\mathcal{P}$ be a subset of $\mathcal{W}_1 \times \cdots \times \mathcal{W}_n$. We call an element $p \in \mathcal{P}$ a *profile*, and denote by $\succsim_i^p$ the *preference of individual $i$ at profile $p$*. A *social welfare function*, which will be denoted by $f$ hereafter, is a mapping that assigns a preference ordering on $X$ to a profile $p \in \mathcal{P}$, i.e., $f : \mathcal{P} \to \mathcal{W}$, where $\mathcal{W}$ is the set of all preference orderings on $X$. We denote by $\succsim^{f(p)}$ the preference ordering determined by $f$ at profile $p \in \mathcal{P}$.

Some axioms on the social welfare function are introduced. The first axiom means that each individual is allowed to have any preference he/she wishes.

**Axiom 2.1 (Unrestricted Domain Property (UDP)).** The domain $\mathcal{P}$ of the social welfare function $f$ is $\mathcal{P} = \mathcal{W}_1 \times \cdots \times \mathcal{W}_n$.

Given a set $A \subseteq X$ of alternatives, let $N(A)$ be the set of all individuals whose preference domain contains $A$, i.e.,

$$N(A) = \{\, i \in N \mid A \subseteq X_i \,\}.$$

**Axiom 2.2 (Independence of Irrelevant Alternatives (IIA)).** If the property that

$$\succsim_i^p|\{x,y\} = \succsim_i^q|\{x,y\} \text{ for all } i \in N(\{x,y\}) \text{ implies } \succsim^{f(p)}|\{x,y\} = \succsim^{f(q)}|\{x,y\}$$

holds for any pair of distinct alternatives $x, y \in X$ and for any pair of distinct profiles $p, q \in \mathcal{P}$, then the social welfare function $f$ is said to satisfy *independence of irrelevant alternatives*.

**Axiom 2.3 (Weak Pareto Principle (WPP)).** If the property that

$$x \succ_i^p y \text{ for all } i \in N(\{x,y\}) \text{ implies } x \succ^{f(p)} y$$

holds for any pair of distinct alternatives $x, y \in X$ and for any profile $p \in \mathcal{P}$, then the social welfare function $f$ is said to have *weak Pareto principle*.

**Definition 2.4 (Dictator).** An individual $i \in N$ is called a *dictator* if $x \succ_i^p y$ implies $x \succ^{f(p)} y$ for any pair of distinct alternatives $x, y \in X_i$ and for any profile $p \in \mathcal{P}$. If there exists a dictator, then the social welfare function $f$ is said to be *dictatorial*.

## 3 Impossibility Theorem

We will prove in this section Theorem 3.1 that the social welfare function satisfying Axioms (UDP), (IIA) and (WPP) is dictatorial in the presence of an individual in $N(X)$ by showing that someone in $N(X)$ is a dictator.

**Theorem 3.1.** *Suppose that $N(X) \neq \emptyset$. If the social welfare function $f$ satisfies Axioms (UDP), (IIA) and (WPP), then there exists a dictator in $N(X)$.*

We first define $(x \succ y)$-*decisive coalition* and *decisive coalition* for the proof of Theorem 3.1.

**Definition 3.2 ($(x \succ y)$-decisive coalition).** Let $x, y \in X$ be a pair of distinct alternatives. A nonempty subset of individuals $M \subseteq N(\{x,y\})$ is said to be an $(x \succ y)$-*decisive coalition* if for any profile $p \in \mathcal{P}$

$$x \succ_i^p y \text{ for all } i \in M \text{ and } y \succ_j^p x \text{ for all } j \in N(\{x,y\}) \setminus M \text{ imply } x \succ^{f(p)} y.$$

**Definition 3.3 (decisive coalition).** A nonempty subset of individuals $M \subseteq N$ is said to be a *decisive coalition* if $M$ is an $(x \succ y)$-decisive coalition for some pair of distinct alternatives $x, y \in X$.

**Lemma 3.4.** *Assume Axiom (IIA) and let $M \subseteq N$ be a nonempty subset of $N(\{x,y\})$ for some pair of distinct alternatives $x, y \in X$. If there is a profile $p \in \mathcal{P}$ such that*

$$x \succ_i^p y \text{ for all } i \in M,\ y \succ_j^p x \text{ for all } j \in N(\{x,y\}) \setminus M \text{ and } x \succ^{f(p)} y,$$

*then $M$ is an $(x \succ y)$-decisive coalition.*

*Proof.* Let $q$ be an arbitrary profile such that $x \succ_i^q y$ for $i \in M$ and $y \succ_j^q x$ for $j \in N(\{x,y\}) \setminus M$. Then $\succsim_i^p|\{x,y\} = \succsim_i^q|\{x,y\}$ for all $i \in N(\{x,y\})$. Applying Axiom (IIA), we have $x \succ^{f(q)} y$, meaning that $M$ is an $(x \succ y)$-decisive coalition. □

**Lemma 3.5.** *Suppose that $N(X) \neq \emptyset$ and Axioms (UDP), (IIA) and (WPP). Then any $(x \succ y)$-decisive coalition contains an individual $i$ with $X_i \setminus \{x,y\} \neq \emptyset$.*

*Proof.* Let $M$ be an $(x \succ y)$-decisive coalition and assume that $X_i = \{x,y\}$ for all $i \in M$. Choose an arbitrary alternative, say $z$, in $X \setminus \{x,y\}$. Then $M$ intersects none of $N(\{x,y,z\})$, $N(\{y,z\})$ and $N(\{x,z\})$, each of which contains $N(X)$ and hence is nonempty. Let $p \in \mathcal{P}$ be a profile such that

$$\begin{array}{ll} x \succ_i^p y & \text{for } i \in M, \\ y \succ_i^p z \succ_i^p x & \text{for } i \in N(\{x,y,z\}), \\ y \succ_i^p z & \text{for } i \in N(\{y,z\}) \setminus N(\{x,y,z\}), \\ z \succ_i^p x & \text{for } i \in N(\{x,z\}) \setminus N(\{x,y,z\}) \text{ and} \\ y \succ_i^p x & \text{for } i \in N(\{x,y\}) \setminus (M \cup N(\{x,y,z\})). \end{array}$$

Since $M$ is an $(x \succ y)$-decisive coalition, we have

$$x \succ^{f(p)} y. \tag{3.1}$$

Concerning the pair of $y$ and $z$, $y \succ_i^p z$ holds for all $i \in N(\{y,z\})$, implying

$$y \succ^{f(p)} z \tag{3.2}$$

by Axiom (WPP). In the same way we see

$$z \succ^{f(p)} x. \tag{3.3}$$

Clearly (3.2) and (3.3) together contradict (3.1). □

**Lemma 3.6.** *Suppose that $N(X) \neq \emptyset$ and Axioms (UDP), (IIA) and (WPP). Then there is a decisive coalition consisting of a single individual.*

*Proof.* For a pair of distinct alternatives $x, y \in X$, $N(\{x,y\})$ is clearly an $(x \succ y)$-decisive coalition from Axiom (WPP). Therefore there is at least one decisive coalition.

Let $M$ be a decisive coalition that is minimal with respect to set inclusion partial order, and suppose that it is an $(x \succ y)$-decisive coalition. We will show that the assumption $|M| \geq 2$ leads to a contradiction. We have seen in Lemma 3.5 that $X_i \setminus \{x,y\} \neq \emptyset$ for some individual $i \in M$. Let $z$ be an arbitrary alternative in $X_i \setminus \{x,y\}$. For $i \in M$, $z \in X_i \setminus \{x,y\}$ and $M \setminus \{i\} \neq \emptyset$ thus constructed, we consider a profile $p \in \mathcal{P}$ such that

$$\begin{array}{ll} z \succ_i^p x \succ_i^p y, & \\ x \succ_j^p y \succ_j^p z & \text{for } j \in (M \setminus \{i\}) \cap N(\{x,y,z\}), \\ x \succ_j^p y & \text{for } j \in (M \setminus \{i\}) \setminus N(\{x,y,z\}), \\ y \succ_j^p z \succ_j^p x & \text{for } j \in (N \setminus M) \cap N(\{x,y,z\}), \\ y \succ_j^p x & \text{for } j \in (N \setminus M) \cap (N(\{x,y\}) \setminus N(\{x,y,z\})), \\ y \succ_j^p z & \text{for } j \in (N \setminus M) \cap (N(\{y,z\}) \setminus N(\{x,y,z\})) \text{ and} \\ z \succ_j^p x & \text{for } j \in (N \setminus M) \cap (N(\{x,z\}) \setminus N(\{x,y,z\})). \end{array}$$

Since $M$ is an $(x \succ y)$-decisive coalition, we have

$$x \succ^{f(p)} y. \tag{3.4}$$

The following two cases are possible.
Case A: $z \succ^{f(p)} y$.

Since $z \succ_i^p y$ and $y \succ_j^p z$ for all $j \in N(\{y,z\}) \setminus \{i\}$, we conclude that $\{i\}$ alone is a $(z \succ y)$-decisive coalition from Lemma 3.4. This contradicts the minimality assumption of $M$.
Case B: $y \succsim^{f(p)} z$.

First note that

$$x \succ^{f(p)} z \tag{3.5}$$

by (3.4) and the transitivity. We will show that $(M \setminus \{i\}) \cap N(\{x,y,z\})$ is an $(x \succ z)$-decisive coalition. Suppose that $(M \setminus \{i\}) \cap N(\{x,y,z\}) = \emptyset$. Then $z \succ_i^p x$ for all $i \in N(\{x,z\})$. This implies $z \succ^{f(p)} x$ by Axiom (WPP), which contradicts (3.5). Therefore $(M \setminus \{i\}) \cap N(\{x,y,z\}) \neq \emptyset$. By the construction of profile $p$ and (3.5) we see that $(M \setminus \{i\}) \cap N(\{x,y,z\})$ is an $(x \succ z)$-decisive coalition and this fact again contradicts the minimality of $M$. □

**Lemma 3.7.** *Suppose that $N(X) \neq \emptyset$ and Axioms (UDP), (IIA) and (WPP). Then there is an individual in $N(X)$ who alone forms a decisive coalition.*

*Proof.* Let $\{i\}$ be a decisive coalition demonstrated in Lemma 3.6, and assume that it is an $(x \succ y)$-decisive coalition. Note that $x, y \in X_i$. Suppose that $i \notin N(X)$, i.e., $X \setminus X_i \neq \emptyset$, and let $z$ be an arbitrary alternative in $X \setminus X_i$. Note also that $z$ is distinct from $x$ or $y$. Now consider a profile $p \in \mathcal{P}$ such that

$$\begin{array}{ll} x \succ_i^p y, & \\ y \succ_j^p z \succ_j^p x & \text{for } j \in N(\{x,y,z\}), \\ y \succ_j^p x & \text{for } j \in N(\{x,y\}) \setminus (\{i\} \cup N(\{x,y,z\})), \\ y \succ_j^p z & \text{for } j \in N(\{y,z\}) \setminus N(\{x,y,z\}) \text{ and} \\ z \succ_j^p x & \text{for } j \in N(\{x,z\}) \setminus N(\{x,y,z\}). \end{array}$$

Since $\{i\}$ is an $(x \succ y)$-decisive coalition, we have

$$x \succ^{f(p)} y. \tag{3.6}$$

We also have $y \succ^{f(p)} z$ and $z \succ^{f(p)} x$ by Axiom (WPP). This is contrary to (3.6) by the transitivity. Therefore we conclude that $i \in N(X)$. □

**Lemma 3.8.** *Suppose that $N(X) \neq \emptyset$ and Axioms (UDP), (IIA) and (WPP). If an individual in $N(X)$ forms a decisive coalition, it is an $(x \succ y)$-decisive coalition for any pair of distinct alternatives $x, y \in X$.*

*Proof.* Suppose that $i \in N(X)$ forms a $(u \succ v)$-decisive coalition, and let $w$ be an arbitrary alternative distinct from $u$ or $v$. We first show that $\{i\}$ is a $(u \succ w)$-decisive coalition and then show that it is a $(w \succ v)$-decisive coalition.
(I) $\{i\}$ is a $(u \succ w)$-decisive coalition:

First consider a profile $p \in \mathcal{P}$ such that

$$\begin{array}{ll} u \succ_i^p v \succ_i^p w, & \\ v \succ_j^p w \succ_j^p u & \text{for } j \in N(\{u,v,w\}) \setminus \{i\}, \\ v \succ_j^p u & \text{for } j \in N(\{u,v\}) \setminus N(\{u,v,w\}), \\ v \succ_j^p w & \text{for } j \in N(\{v,w\}) \setminus N(\{u,v,w\}) \text{ and} \\ w \succ_j^p u & \text{for } j \in N(\{u,w\}) \setminus N(\{u,v,w\}). \end{array}$$

Since $\{i\}$ is a $(u \succ v)$-decisive coalition, we see

$$u \succ^{f(p)} v. \tag{3.7}$$

Note that $v \succ^{f(p)} w$ by Axiom (WPP) since $v \succ_j^p w$ for all $j \in N(\{v,w\})$. This together with (3.7) implies $u \succ^{f(p)} w$ by the transitivity. Since $u \succ_i^p w$ and $w \succ_j^p u$ for all $j \in N(\{u,w\}) \setminus \{i\}$, we conclude that $\{i\}$ is a $(u \succ w)$-decisive coalition from Lemma 3.4.

(II) $\{i\}$ is a $(w \succ v)$-decisive coalition:

Next, consider a profile $q \in \mathcal{P}$ such that

$$\begin{array}{ll} w \succ_i^q u \succ_i^q v, & \\ v \succ_j^q w \succ_j^q u & \text{for } j \in N(\{u,v,w\}) \setminus \{i\}, \\ v \succ_j^q u & \text{for } j \in N(\{u,v\}) \setminus N(\{u,v,w\}), \\ v \succ_j^q w & \text{for } j \in N(\{v,w\}) \setminus N(\{u,v,w\}) \text{ and} \\ w \succ_j^q u & \text{for } j \in N(\{u,w\}) \setminus N(\{u,v,w\}). \end{array}$$

Since $\{i\}$ is a $(u \succ v)$-decisive coalition, we have $u \succ^{f(q)} v$. Furthermore, from Axiom (WPP), we also have $w \succ^{f(q)} u$, and hence by the transitivity, we have $w \succ^{f(q)} v$. Observe that $w \succ_i^q v$ and $v \succ_j^q w$ for all $j \in N(\{v,w\}) \setminus \{i\}$. This means that $\{i\}$ is a $(w \succ v)$-decisive coalition from Lemma 3.4.

Now let $x$ and $y$ be two distinct alternatives. When $y \neq u$, $\{i\}$ is also a $(u \succ y)$-decisive coalition by the argument (I). Applying the argument (II) we see that $\{i\}$ is an $(x \succ y)$-decisive coalition. When $y = u$, choose an arbitrary $w \in X \setminus \{u,v\}$. Then by the argument (II), $\{i\}$ is a $(w \succ v)$-decisive coalition. Applying the arguments (II) and (I) repeatedly, we see that $\{i\}$ is a $(w \succ y)$-decisive coalition and then it is an $(x \succ y)$-decisive coalition. □

*Proof of Theorem 3.1.*

We have shown that there is an individual, say $i$, in $N(X)$ who alone forms a decisive coalition in Lemma 3.7. Take an arbitrary pair of distinct alternatives $x, y \in X$, and a profile $p \in \mathcal{P}$ such that $x \succ_i^p y$. We will show that $x \succ^{f(p)} y$. Let

$$\begin{array}{l} N_1 = \{\, j \in N(\{x,y\}) \setminus \{i\} \mid x \succ_j^p y \,\}, \\ N_2 = \{\, j \in N(\{x,y\}) \setminus \{i\} \mid y \succ_j^p x \,\}, \\ N_3 = \{\, j \in N(\{x,y\}) \setminus \{i\} \mid x \sim_j^p y \,\}, \text{ and} \\ N_4 = N \setminus N(\{x,y\}). \end{array}$$

Choose an alternative $z \in X \setminus \{x,y\}$ arbitrarily, and consider the following profile $q \in \mathcal{P}$ such that

$$\begin{array}{ll} x \succ_i^q z \succ_i^q y, & \\ z \succ_j^q x \succ_j^q y & \text{for } j \in N_1 \cap N(\{x,y,z\}), \\ x \succ_j^q y & \text{for } j \in N_1 \setminus N(\{x,y,z\}), \\ z \succ_j^q y \succ_j^q x & \text{for } j \in N_2 \cap N(\{x,y,z\}), \\ y \succ_j^q x & \text{for } j \in N_2 \setminus N(\{x,y,z\}), \\ z \succ_j^q x \sim_j^q y & \text{for } j \in N_3 \cap N(\{x,y,z\}), \\ x \sim_j^q y & \text{for } j \in N_3 \setminus N(\{x,y,z\}), \\ z \succ_j^q y & \text{for } j \in N_4 \cap N(\{y,z\}) \text{ and} \\ z \succ_j^q x & \text{for } j \in N_4 \cap N(\{x,z\}). \end{array}$$

Note that $\succsim_j^p|\{x,y\} = \succsim_j^q|\{x,y\}$ for all $j \in N(\{x,y\})$. Since $\{i\}$ is an $(x \succ z)$-decisive coalition from Lemma 3.8, we see $x \succ^{f(q)} z$. By Axiom (WPP), we also have $z \succ^{f(q)} y$. Then $x \succ^{f(q)} y$ by the transitivity. Applying Axiom (IIA) we conclude that $x \succ^{f(p)} y$, meaning that the individual $i$ is a dictator. □

## 4 Special Case

Ando, Ohara and Yamamoto [1] consider a social preference ordering in a situation of mutual evaluation. Each individual evaluates all individuals in the society but him/herself. Namely, the set of alternatives coincides with the set of all individuals in the society, $X = N$, and individual $i$'s preference domain $X_i$ is given by $X_i = N \setminus \{i\}$. They show an impossibility theorem in this situation. One of the crucial roles in their argument is played by the "cyclic profile" $c$ which is the profile defined by

$$\begin{aligned}
&2 \succ_1^c 3 \succ_1^c \cdots \succ_1^c n \\
&i+1 \succ_i^c i+2 \succ_i^c \cdots \succ_i^c n-1 \succ_i^c n \succ_i^c 1 \succ_i^c \cdots \succ_i^c i-1 \quad \text{for } i = 2, \cdots, n-1 \\
&1 \succ_n^c 2 \succ_n^c \cdots \succ_n^c n-1.
\end{aligned}$$

It is readily seen that assuming Axiom (WPP) would lead to a social preference $\succsim^{f(c)}$ such that

$$1 \succ^{f(c)} 2 \succ^{f(c)} \cdots \succ^{f(c)} n-1 \succ^{f(c)} n \succ^{f(c)} 1$$

which is not a preference ordering. Hence the social welfare function is impossible. They show that relaxing Axiom (WPP) in several ways would not lead to a positive result under Axioms (UDP) and (IIA). To exclude the controversial cyclic profile we add an individual who is entitled to evaluate all the individuals in the society. Then from Theorem 3.1 we see that the social welfare function is dictatorial and the added individual is a dictator.

## 5 Alternative Proof of Theorem 3.1

The argument in the previous section is based on the decisive coalition and hence it does not help clarify the relationship between Arrow's and our own impossibility theorems. In this last section we give an alternative proof of Theorem 3.1, which will shed a light on the relationship.

Let $\mathcal{Q} = \mathcal{W}^n$. For each profile $q \in \mathcal{Q}$, let

$$r(q) = (\succsim_1^q|X_1, \succsim_2^q|X_2, \cdots, \succsim_n^q|X_n).$$

That is to say, $r(q)$ is the restriction of profile $q \in \mathcal{Q}$ to $\mathcal{W}_1 \times \cdots \times \mathcal{W}_n$. We define $g : \mathcal{Q} \to \mathcal{W}$ by the social welfare function $f$ satisfying Axioms (UDP), (IIA) and (WPP) in the previous sections as

$$g(q) = f(r(q)) \text{ for each } q \in \mathcal{Q}. \tag{5.1}$$

**Lemma 5.1.** *The function $g$ satisfies the Independence of Irrelevant Alternatives in Arrow's sense, i.e.,*

$$\succsim_i^p|\{x,y\} = \succsim_i^q|\{x,y\} \text{ for all } i \in N \text{ implies } \succsim^{g(p)}|\{x,y\} = \succsim^{g(q)}|\{x,y\}$$

*holds for any pair of distinct alternatives $x, y \in X$ and for any pair of distinct profiles $p, q \in \mathcal{Q}$.*

*Proof.* Take a pair of distinct alternatives $x, y \in X$, and a pair of distinct profiles $p, q \in \mathcal{Q}$ such that

$$\succsim_i^p |\{x,y\} = \succsim_i^q |\{x,y\} \text{ for all } i \in N.$$

Then clearly

$$\succsim_i^{r(p)} |\{x,y\} = \succsim_i^{r(q)} |\{x,y\} \text{ for all } i \in N(\{x,y\}).$$

By Axiom (IIA) concerning $f$, we obtain

$$\succsim^{f(r(p))} |\{x,y\} = \succsim^{f(r(q))} |\{x,y\},$$

which implies by (5.1) that

$$\succsim^{g(p)} |\{x,y\} = \succsim^{g(q)} |\{x,y\}.$$

□

**Lemma 5.2.** *The function $g$ satisfies the weak Pareto principle in Arrow's sense, i.e.,*

$$x \succ_i^q y \text{ for all } i \in N \text{ implies } x \succ^{g(q)} y$$

*holds for any pair of distinct alternatives $x, y \in X$ and for any profile $q \in \mathcal{Q}$.*

*Proof.* Take a pair of distinct alternatives $x, y \in X$, and a profile $q \in \mathcal{Q}$ such that

$$x \succ_i^q y \text{ for all } i \in N.$$

Then clearly

$$x \succ_i^{r(q)} y \text{ for all } i \in N(\{x,y\}).$$

By Axiom (WPP) concerning $f$, we obtain

$$x \succ^{f(r(q))} y,$$

which implies by (5.1) that

$$x \succ^{g(q)} y.$$

□

*Alternative proof of Theorem 3.1.*

Since $g$ defined above satisfies unrestricted domain property, independence of irrelevant alternatives and weak Pareto principle in Arrow's sense, we see that $g$ is dictatorial by Arrow's impossibility theorem. Namely, there is an individual $i \in N$ such that

$$x \succ_i^q y \text{ implies } x \succ^{g(q)} y$$

for any pair of distinct alternatives $x, y \in X$ and for any profile $q \in \mathcal{Q}$.

For a profile $p \in \mathcal{P}$ suppose that $x \succ_i^p y$ holds for $x, y \in X_i$. By the definition of $\mathcal{Q}$ there is a profile $q \in \mathcal{Q}$ such that $r(q) = p$. Note that $x \succ_i^q y$. Since individual $i$ is a dictator of $g$, we see $x \succ^{g(q)} y$. By the definition (5.1) of $g$, this implies $x \succ^{f(r(q))} y$, or equivalently $x \succ^{f(p)} y$. Therefore we conclude that $f$ is dictatorial.

Finally, we will show that there exists a dictator of $f$ in $N(X)$. Assume that individual $i \in N \setminus N(X)$ is a dictator of $f$. Choose an arbitrary pair of distinct alternatives, say $x$ and $y$, in $X_i$ and an arbitrary alternative, say $z$, in $X \setminus X_i$. We consider a profile $p \in \mathcal{P}$ such that

$$\begin{array}{ll} x \succ_i^p y, & \\ y \succ_j^p z \succ_j^p x & \text{for } j \in N(\{x,y,z\}), \\ y \succ_j^p z & \text{for } j \in N(\{y,z\}) \setminus N(\{x,y,z\}) \text{ and} \\ z \succ_j^p x & \text{for } j \in N(\{x,z\}) \setminus N(\{x,y,z\}). \end{array}$$

Since individual $i$ is dictator, we have

$$x \succ^{f(p)} y. \tag{5.2}$$

Concerning the pair of $y$ and $z$, $y \succ_j^p z$ holds for all $j \in N(\{y,z\})$, implying

$$y \succ^{f(p)} z \tag{5.3}$$

by Axiom (WPP). In the same way we see

$$z \succ^{f(p)} x. \tag{5.4}$$

Clearly (5.3) and (5.4) together contradict (5.2) by the transitivity. Therefore we conclude that the dictator of $f$ is in $N(X)$. □

## 6 Concluding Remarks

We consider a society where each individual's preference domain is restricted to a subset of the whole set of alternatives. We have shown the impossibility theorem that every social welfare function satisfying unrestricted domain property, independence of irrelevant alternatives and weak Pareto principle is dictatorial whenever at least one individual has an unrestricted preference domain.

One of possible future research themes would be strategy-proofness (see [7, 14]). A natural question to answer would be "is a nonmanipulable voting always dictatorial when individuals have restricted preference domain?" such as mutual evaluation situation.

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