Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 1, No. 1, pp. 233–241, January 2005

OPTIMAL CONTROL OF AN ECONOMIC MODEL WITH A SMALL STOCHASTIC TERM

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Abstract: Some economic models, including financial models, involve a small stochastic term. Optimal control for such models can be handled approximately, in discrete time, by considering mean and covariance. This avoids independence assumptions made in the usual Brownian motion models, and allows simple computation.

Key words: *economic model, financial model, stochastic, optimal control, discrete time*

Mathematics Subject Classification: *90C15, 93C25*

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1 Introduction

Economic models often include stochastic terms. In a model of economic growth, it may happen that the growth is largely deterministic, with a smaller, though significant, stochastic term included. The distribution of stochastic contributions may be known only up to mean and variance. When considering the optimal control of such a model, e.g. to optimize the expectation of some utility function, a detailed (and computationally heavy) stochastic optimization (e.g. of some Markovian formulation) may be less appropriate than a simpler approximate approach, where the probability distributions are described by mean and variance. Moreover, a model in discrete (rather than continuous) time does not require independence of contributions from very small disjoint time intervals.

As one application, an optimal control economic model in continuous time is modified by discretizing the time, then adding stochastic terms. In terms of the continuious-time model, the small stochastic increments are added at discrete times. The optimization can then be computed using software for continuous-time optimal control. The result shows the evolution of the expectation and standard deviation of a state variable.

Consider an economic growth model with a state function x_t $(t = 0, 1, 2, \ldots)$ describing capital (of one or several kinds) at time t, and a control function u_t $(t = 0, 1, 2, \ldots)$ describing consumption (of one or several kinds). Many such models have been proposed. Consider first a dynamic equation:

$$x_{t+1} = F(x_t - u_t,\, \epsilon_t)\ (t = 0, 1, 2, \ldots), \tag{1}$$

in which the ϵ_t terms are independently identically distributed random variables, with mean $\mathbf{E}(\epsilon_t) = 0$ and variance $\mathrm{var}(\epsilon_t) = \sigma^2$. As one example, an objective function, depending on capital:

$$J := \sum_{t=0}^{N} \rho^t U(\mathbf{E}(x_t) - \theta s_t) + \rho^N \Phi(x_N) \tag{2}$$

is to be maximized, subject to bounds on the u_t terms, namely:

$$0 < a \leq u_t \leq b \; (t = 0, 1, 2, \ldots, n). \tag{3}$$

Here ρ^t is a discount term, $U(\cdot)$ is a concave increasing utility function, $s_t := (\text{var } (x_t))^{1/2}$ is the standard deviation of x_t, θ is a constant (e.g. $\theta = 2$), and $\Phi(\cdot)$ is an endpoint term describing the more distant future. Instead of a conventional $\mathbf{E}U(x_t)$ term, the form in (2) describes a chosen lower quantile of the distribution of the state. Note that $\mathbf{E}U(x_t)$ has an implied assumption that the model will be applied a number of times, to allow averaging of the stochastic contribution. But for a "single run" model, the term in (2) may be a more appropriate description. Terms depending on consumption might also be added to the objective.

Another model is described in section 4, with objective depending on both state functions (describing money) and control functions (describing allotment of investment). Section 6 describes an inventory model, with objective depending on state function (inventory) and control functions (production and delivery), with a stochastic term in the demand.

A computation for such stochastic models often requires heavy computing, to include the details of the probability distibutions involved. However, this may be inappropriate, especially when (as usual) only limited information is available about those distributions.

This paper makes the simplifying assumption that the stochastic contributions are "small", in the sense that the deterministic contributions dominate, rather than the other way about. Consistently with this, it is also assumed that mean and variance carry sufficient information about the distributions. (Indeed, these all the data that are usually recorded and analysed.)

One possible form of the dynamic function F is:

$$F(x_t - u_t, \, \epsilon_t) = (1 + \kappa\epsilon_t) Q(x_t - u_t), \tag{4}$$

where $Q(\cdot)$ is concave increasing, and the term $\kappa\epsilon_t$ describes a stochastic variation in the growth rate of the economic system (compare Hakansson 1975).

It is remarked that many financial models are given in continuous time, with stochastic contributions based on a Wiener process (see e.g. Klebaner 1998). But this assumes the independence of stochastic contributions from disjoint time intervals, however short, an assumption unlikely to hold in the real world. The underlying stochastic process may be modified to *fractional Brownian motion* (see Cutland et al. 1995), which allows dependence between the contributions at different times. However, discrete time model can be more realistic (since the data are observed in discrete time), and may also lend itself better to numerical optimization or simulation.

2 Approximating the Stochastic Contribution

For an optimal control model, modify (1) to:

$$x_{t+1} = F(x_t, \, \epsilon_t, \, u_t) \; (t = 0, 1, 2, \ldots) \tag{5}$$

where u_t is a control variable. Denote by F_1, F_2, F_3 the partial derivatives of F with respect to its first, second and third arguments.

Assume initially that the state x_t has only a single component; a vector case is discussed later. Denote second partial derivatives similarly by F_{11}, F_{12}, etc.. Optimization will find a deterministic control variable u_t. Denote the expectation $\bar{x} = \mathbf{E}(x_t)$ and $x_t = \bar{x} + y_t$. If y_t and ϵ_t are "small", then, to a useful approximation:

$$x_{t+1} \approx F(\bar{x}_t, 0, u_t) + F_1(\bar{x}_t, 0, u_t)y_t + \frac{1}{2}F_{11}(\bar{x}_t, 0, u_t)y_t^2 + F_2(\bar{x}_t, 0, u_t)\epsilon_t. \tag{6}$$

Then:

$$\begin{aligned}\mathbf{E}(x_{t+1}) &= \mathbf{E}F(\bar{x}_t, 0, u_t) + \frac{1}{2}F_{11}(\bar{x}_t, 0, u_t)\mathbf{E}(y_t^2) \\ &= F(\bar{x}_t,\, 0,\, u_t) + \frac{1}{2}F_{11}(\bar{x}_t, 0, u_t)\mathrm{var}(x_t).\end{aligned} \tag{7}$$

Consider a random variable $w = az + bz^2 - a\bar{z} - b\bar{z}^2$, where a and b are constants, $z = \bar{z} + y$ where $\bar{z} = \mathbf{E}(z)$, and $v := \mathrm{var}(z)$. Then:

$$w = (a + 2b\bar{z})y + y^2\,; \;\; \mathbf{E}(w) = 0 + v. \tag{8}$$

Hence:

$$\begin{aligned}\mathrm{var}(w) =& \mathbf{E}((w - \;\mathbf{E}(w))^2) \\ =& \mathbf{E}[(a + 2b\bar{z})^2y^2 + b^2y^4 + b^2v^2 + 2b(a + 2b\bar{z})y^3 - 2b^2vy^2 + 2b(a + 2b\bar{z})yv] \\ =& (a + 2b\bar{z})^2v + b^2(\mathrm{var}(z^2) - 4\bar{z}\mathbf{E}(y^4) - 4\bar{z}^2v + 2v^2) + b^2v^2 \\ &+ 2b(a + 2b\bar{z})\mathbf{E}(y^3) - 2b^2v^2 \\ =& (a^2 + 4ab\bar{z})v + b^2\mathrm{var}(w^2) + b^2v^2 + 2ab\mathbf{E}\;(y^3).\end{aligned} \tag{9}$$

If z is scaled so that b is "small", then (5) and (8) with $z = x_t$ give:

$$\mathrm{var}(x_{t+1}) \approx F_1(\bar{x}_t,\, 0,\, u_t)^2(1 + 2\Gamma\bar{x}_t)\;\mathrm{var}(x_t) + F_2(\bar{x}_t,\, 0,\, u_t)^2\sigma^2, \tag{10}$$

where

$$\Gamma = (F_{11}(\bar{x}_t, 0, u_t)\bar{x}_t)/F_1(\bar{x}_t,\, 0,\, u_t).$$

Thus (7) and (10) give dynamic equations for $\mathbf{E}(x_t)$ and $\mathrm{var}(x_t)$.

3 The Vector Case

Suppose now that x_t is a column vector with p components. Then w is replaced by a (column) vector variable $\mathbf{w} = Az + z^TB^{\cdot}z$, where the superscript dot indicates a set of matrices B^j, one for eacn component of $\mathbf{w}$. Denote by $V = \mathrm{cov}(y)$ the covariance matrix of z. Let $z = \bar{z} + y$, where $\bar{z} = \mathbf{E}(z)$. Using $\mathbf{E}(y^TBy) = \bar{y}\bar{y}^T + C$, where $C = \mathbf{Tr}(B^TV)$,

$$\mathbf{w} - \mathbf{E}\mathbf{w} = Qy + y^TB^{\cdot}y - C, \tag{11}$$

$$Q = A + 2\bar{z}^{\#}B^{\cdot}\;;\; (\bar{z}^{\#}B^{\cdot})_{ij} := \sum_k \bar{z}_kB^i_{kj}. \tag{12}$$

If the elements of B are "small", then the leading terms in $\text{cov}(\mathbf{w}) = \mathbf{E}(\mathbf{w}\mathbf{w}^T)$ are those from $\mathbf{E}((Qy)(Qy)^T)$. Thus:

$$\begin{aligned}\text{cov}(\mathbf{w}) &\approx \mathbf{E}\{AyyA^T + Ayy^T(\bar{z}^{\#}B^{\cdot}) + (\bar{z}^{\#}B^{\cdot})yy^TA\} \\ &= AVA^T + AV(\bar{z}^{\#}B^{\cdot}) + (\bar{z}\#B^{\cdot})^TVA. \end{aligned} \tag{13}$$

Hence, for this vector case, the dynamic equation (10) is replaced by:

$$\text{cov}(x_{t+1}) \approx F_1\text{cov}(x_t)F_1^T + \frac{1}{2}F_1\text{cov}(x_t)(\bar{x}^{\#}B^T) + \frac{1}{2}(\bar{x}^{\#}B^{\cdot}\ \text{cov}(x_t)F_1 + F_2\text{cov}(\epsilon_t)), \tag{14}$$

where $B^{\cdot} = \frac{1}{2}F_{11}$, and the arguments of F_1, F_2 and F_{11} are $(\bar{x}_t, 0, u_t)$.

4 The Davis and Elzinga Financial Model

Davis and Elzinga (1975) proposed a model for investment in a utility company, in which the share price $P(t)$ and equity $E(t)$ per share at time t are described by differential equations:

$$\dot{P}(t) = c((1-u_1(t))rE(t) - \rho P(t)), \tag{15}$$

$$\dot{E}(t) = rE(t)(u_1(t) + u_2(t)[1 - E(t)/(1-\delta)P(t)]). \tag{16}$$

An objective function:

$$J := \int_0^T e^{-\rho t}[1-u_1(t)]E(t)dt + e^{-\rho T}P(T) \tag{17}$$

is maximized, subject to constraints $u_1(t) \geq 0,\ u_2(t) \geq 0,\ u_1(t) + u_2(t) \leq b$ on the control functions $u_1(t)$ = proportion of earnings retained and $u_2(t)$ = stock financing rate.

Denote $U(t) := P(t)/E(t)$. Then, from (15) and (16), $U(t)$ satisfies:

$$\dot{U}(t) = \alpha(t) - \beta(t)U(t) \tag{18}$$

where

$$\alpha(t) = cr(1-u_1(t)) + ru_2(t)/(1-\delta), \tag{19}$$

$$\beta(t) = c\rho + r(u_1(t) + u_2(t)). \tag{20}$$

Assume a planning period of $[0,T]$, and divide it into n equal subintervals. Denote $x_j^{Um} := U(jT/n)$ and $x_j^{Uv} := \text{var}(U(jT/n))$. For the subdivision interval $h = T/n$ sufficiently small, the differential equations (18) may be approximated by a difference equation:

$$\Delta x_j^{Um} \equiv x_{j+1}^{Um} - x_j^{Um} = h[\alpha_j - \beta_j x_j^{Um}], \tag{21}$$

$$\alpha_j = \alpha(jT/n), \quad \beta_j = \beta(jT/n). \tag{22}$$

If a stochastic term $\epsilon(t)E(t)$ is added to the right side of (15), this adds $\epsilon(t)$ to the right side of (18), hence adds ϵ_t to the right of (21), with $\epsilon_t = \epsilon(jT/n)$. Assume now that the ϵ_t are i.i.d. with $\mathbf{E}(\epsilon_t) = 0$, $\text{var}(\epsilon_t) = \sigma^2T/n$. (Thus variance σ^2 corresponds to unit time.) Then:

$$\Delta x_j^{Uv} \equiv x_{j+1}^{Uv} - x_j^{Uv} \approx -x_j^{Uv} + (1-h\beta_j)^2 x_j^{Um} + \sigma^2 \tag{23}$$

from (10), since here the F_{11} term is zero, and $F_2 = 0$. Here, both mean and variance of U are described by linear difference equations; and, since (18) is linear, the difference equation for the variance is exact.

Consider now the two differential equations (15) and (16). Denote the discretized versions of $P(t)$ and $E(t)$ by x_j^{Pm} and x_j^{Em}, and the corresponding covariance matrix for (P, E) by x_j^{PEv}. Let $\zeta = 1/(1-\delta)$. Then, since (15) is linear in $P(t)$ and $E(t)$,

$$x_{j+1}^{Pm} = x_j^{Pm} + h[cr(1 - u_1(jT/n))x_j^{Em} - c\rho x_j^{Pm}]. \tag{24}$$

Let $C_{11} = \mathrm{var}(E)$, and $C_{12} = \mathrm{cov}(P, E)$. Expanding up to quadratic terms,

$$\mathbf{E}(E^2/P) \approx 2\mathbf{E}(E)/\mathbf{E}(P) - 2C_{12}/(\mathbf{E}(P))^2 + 2\mathbf{E}(E)C_{11}/(\mathbf{E}(P))^3, \tag{25}$$

with E and P at time $t = jT/n$. Then, from (16),

$$x_{j+1}^{Em} \approx x_j^{EM} + r(u_1(jT/n) + u_2(jT/n))x_j^{Em} + r\zeta\mathbf{E}(E(jT/n)^2/P(jT/n)). \tag{26}$$

Hence, in (14), with $\bar{x}_j = (\bar{x}_j^{Pm}, \bar{x}_j^{Em}),\ \ x_j = (x_j^{Pm}, x_j^{Em})$:

$$F_1 = \begin{pmatrix} 1 - hc\rho & hcr(1-u_1) \\ hr\zeta u_2 E^2 P^{-2} & 1 - hr(u_1 + u_2) - 2hr\zeta u_2 EP^{-1} \end{pmatrix} \tag{27}$$

$$\begin{aligned} (\bar{x}^{\#} B^{\cdot}) &= \frac{1}{2} P \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2} E \begin{pmatrix} 2hr\zeta u_2 E^{2p-3} & -2hr\zeta u EP^{-2} \\ 2hr\zeta u_2 EP^{-2} & -2hr\zeta u_2 P^{-1} \end{pmatrix} \\ &= \begin{pmatrix} -hr\zeta u_2 U^{-3} & -hr\zeta u_2 U^{-2} \\ -hr\zeta u_2 U^{-2} & hr\zeta u_2 U^{-1} \end{pmatrix} \end{aligned} \tag{28}$$

Here $\zeta = 1/(1-\delta)$, and the arguments are $((x_j^{Pm}, x_j^{Em}), 0, (u_{1,j}, u_{2,j}))$. Then, from (14),

$$x_{j+1}^{PEv} \approx F_1 x_j^{PEv} F_1^T + \mathbf{S} F_1 x_j^{PEv}((\bar{x}^{\#} B^{\cdot}))^T + \sigma^2(T/n)F_2 I, \tag{29}$$

where $\mathbf{S}$ applied to a matrix denotes its symmetric part, namely the average of the matrix and its transpose, and I is a unit matrix, so $\mathrm{cov}(\epsilon_t) = \sigma^2(T/n)I$ for i.i.d. ϵ_t.

The objective function to be maximized is the expectation of (17):

$$J_n := \sum_{j=0}^{n} e^{-\rho' j}(1 - u_1(jT/n)x_j^{Em} + e^{-\rho T} x_T^{Pm}. \tag{30}$$

5 A Computational Example

For $U(t) = P(t)/E(t)$ in the Davis-Elzinga model, denote $z(t) := \mathbf{E}(U(t))$ and $v(t) := \mathrm{var}(U(t))$. The difference equations (21) and (23) can be approximated by differential equations, to allow a continuous-time optimal control program to be used. A change of time scale from $[0,T]$ to $[0,1]$ is made for the optimal control package (SCOM, see Craven and

Islam 2000); this multiplies the right hand sides by T. This gives the differential equations:

$$\begin{aligned}\dot{z}(t) &= T[\alpha(t) - \beta(t)z(t)] \\ &= T[cr(1-u_1(t)) + r\zeta u_2(t)] - T[c\rho + r(u_1(t)+u_2(t))]z(t); \qquad (31)\end{aligned}$$

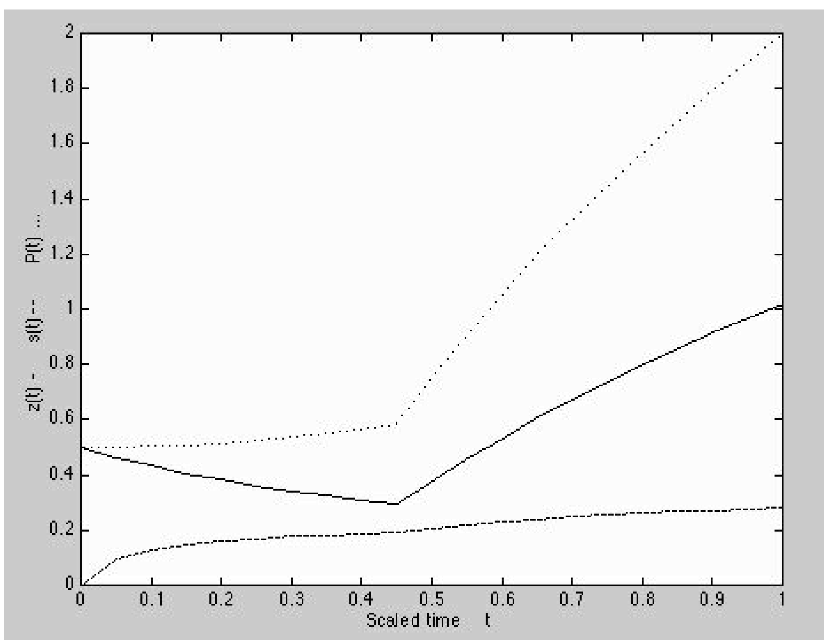
$$\begin{aligned}\dot{v}(t) &= Th^{-1}[-1 + (1-\beta(t))^2 v(t) + h^{-1}T(\sigma^2 T/n) \\ &= Th^{-1}[-2h\beta(t)(1 - \frac{1}{2}h\beta(t))]v(t) + T(T\sigma^2). \qquad (32)\end{aligned}$$

Consider the parameters $c = 1, T = 10, r = 0.2, \rho = 0.1, \delta = 0.1, b = 0.75$, and $n = 20$ subdivisions. As an approximation, the expectation of (17) is maximized, subject to the differential equations (15) and (16) (thus neglecting the stochastic term in this part), and also (31) and (32) are solved (thus calculating here the mean and variance of $P(t)/E(t)$.) The results are given in the following graphs, plotting $z(t)$ (solid line), $s(t) = v(t)^{1/2}$ (dashed line), and $P(t)$ (dotted line) against scaled time t. The optimum controls $(u_1(t), u_2(t))$ jump from (0.75, 0.00) to (0.00, 0.00) at $t = 0.45$, corresponding to a change in slope of $P(t)$ and $z(t)$. The standard deviation increases steadily with t, since the model adds a further increment at each time interval.

In Craven and Islam (2002), page 114, an economic growth model is analysed, with growth described by the Kendrick-Taylor equation (Kendrick and Taylor 1971), of the form:

$$\dot{x}(t) = ae^{\alpha t}x(t)^{\beta} - u(t) - \rho x(t) \qquad (33)$$

with a small stochastic term added to the right side. The differential equation is converted to a difference equation by discretizing the time t; the stochastic terms at discrete times t are assumed independent, with given variance; and difference equations are obtained for expectation and variance of the state $x(t)$. The optimal control results are qualitatively similar to those of the present paper.



6 A Stochastic Model for Inventory and Production

For $t = 0, 1, 2, \ldots$, denote vectors I_t = inventory at start of period t, m_t = amount manufactured during period t, y_t = amount delivered during period t, $d_t + \epsilon_t$ = demand for period t, where d_t is deterministic and ϵ_t is stochastic with $\mathbf{E}(\epsilon_t) = 0$ and $\mathrm{var}(\epsilon_t) = \sigma^2$. Let $u_t := y_t - d_t - \epsilon_t$. Consider the model:

$$\text{Max } \mathbf{E}\sum_{t=1}^{T}[\alpha_t^T y_t - \kappa_t m_t - \beta_t[I_t]_+ - \gamma_t[-I_t]_+ - \delta_t[z_t]^2] \tag{34}$$

subject to:

$$I_o = a, \quad I_{t+1} = [I_t + m_t - y_t], \quad Am_t \leq b_t, \quad m_t \geq 0 \ (t = 0, 1, \ldots, T-1). \tag{35}$$

Here α_t describes unit profits, and $\kappa_t, \beta_t, \gamma_t$ unit costs; backordering is allowed, with different unit costs for positive and negative inventory; $[v]_+ = v$ when $v \geq 0$, $= 0$ when $v < 0$. A penalty cost, with coefficient δ_t is attached to unmet demand. The deterministic part of demand is assumed to be the major part, typically involving seasonal effects, where inventory may have to built up in previous time periods to meet a high demand at certain times. Denote $u_t := y_t - d_t - \epsilon_t$, and rewrite the model as:

$$\text{Max } \mathbf{E}\sum_{t=1}^{T}[\alpha_t^T(u_t - d_t) - \kappa_t m_t - \beta_t[I_t]_+ - \gamma_t[-I_t]_+ - \delta_t[z_t]^2] \tag{36}$$

subject to:

$$I_o = a, \quad I_{t+1} = [I_t + m_t - u_t - d_t - e_t], \tag{37}$$

$$Am_t \leq b_t, \quad m_t \geq 0 \ (t = 0, 1, \ldots, T-1). \tag{38}$$

This is an optimal control model with state variables I_t and control variables m_t and u_t, and a stochastic term in the dynamic equation. If I_t, m_t and y_t have each only a single component, then the expectation $\bar{I}_t := \mathbf{E}(I_t)$ and variance $\mathrm{var}(I_t)$ may be described by (7) and (10). Here, the dynamic equation is linear, and m_t and u_t are control variables, so:

$$\bar{I}_{t+1} = [\bar{I}_t + m_t - y_t]; \ \mathrm{var}(I_{t+1}) = \mathrm{var}(I_t) + \sigma^2. \tag{39}$$

If the inventory equation is modified to:

$$I_o = a, \quad I_{t+1} = \varphi(I_t + m_t - u_t - d_t - e_t \ ), \tag{40}$$

where φ is an increasing concave function, describing reduced value for a large level of inventory. From (7), (9) and (10):

$$\bar{I}_{t+1} = [\bar{I}_t + m_t - y_t] + \beta_t \ \mathrm{var}(I_t); \tag{41}$$

$$\mathrm{var}(I_t) \approx \gamma_t \ \mathrm{var}(I_t) + \rho_t\sigma^2; \tag{42}$$

where $\beta_t = \varphi''(\cdot), \gamma_t = \varphi'(\cdot)^2 + 4\varphi'(\cdot)\varphi''(\cdot), \rho_t = \beta_t$, at argument $\bar{I}_t + m_t - y_t$.

7 Analysis of the Variance Approximation

Assume now a dynamic equation $x_{t+1} = F(x_t, e_t)$, where e_t is stochastic, and a control function has been chosen (implicit in F). Define $\hat{x}_t$ by $\hat{x}_0 = x_0, \hat{x}_{t+1} = F(\hat{x}_t, 0)$; note that $\hat{x}_t \neq \bar{x}_t := \mathbf{E}(x_t)$; but $\mathbf{E}(y_t)$, where $y_t := x_t - \hat{x}_t$, may be small. Let $\varphi_t(s) := \mathbf{E}(\exp(isy_t))$. Then $\varphi_{t+1}(s) = \mathbf{E}(\exp(isG(y_t)))$, for suitable G derived from F. Under some regularity assumptions,

$$\varphi_{t+1}(s) = \int K(s,w)\varphi_t(w)dw \quad \text{and} \quad \varphi_t(s) = \sum_{n=0}^{\infty} c_{tj}s^j, \tag{43}$$

for some kernel $K(s,w)$. Let

$$M_{mn} := \int (\partial/\partial s)^m K(s,w)dw/m!. \tag{44}$$

Then

$$c_{t+1,m} = \sum_n M_{mn}c_{tn}, \tag{45}$$

or in matrix terms, $\mathbf{c}_{t+1} = M\mathbf{c}_t$.

The approximation used previously to get a difference equation for variance relates to truncating the matrix M to five rows and columns. With $\hat{x}_t$ now subtracted, the omitted elements are small, when F is approximated by a quadratic.

Acknowledgement

A referee is thanked for several corrections.

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Manuscript received April 24, 2004
revised June 24, 2004
accepted for publication June 24, 2004

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