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NONLINEAR FUNCTIONAL ERROR BOUNDS

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This work is dedicated to Professor R. Tyrrell Rockafellar on the occasion of his 70th birthday.

Abstract: A functional error bound of a set C is a bound on the distance from a point to the set involving the function that defines the set C with certain properties. Examples of such sets include the solution set of a functional inequality, the set of all minimizers or the set of stationary points. In this paper, we derive sufficient conditions for the existence of nonlinear functional error bounds for the solution set of an functional inequality and the set of subdifferential stationary points involving a lower semicontinuous function defined on a Banach space. It is also shown that error bound conditions for a functional inequality become necessary if the function is convex. Applying the error bound conditions to the set of all minimizers of the function, we obtain the conditions for the existence of the ψ-weak sharp minima.

Key words: *error bounds, nonlinear error bounds, calmness of set-valued maps, nonlinear weak sharp minima, upper Lipschitz set-valued maps, subdifferential stationary sets*

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1 Introduction

Let X be a Banach space, $D \subset X$ closed and $f : X \to \overline{\mathbb{R}}$ a proper lower semicontinuous function. Consider the problem of finding x such that

$$f(x) \le 0, \quad x \in D \subset X. \tag{1}$$

Let the solution set of (1) to be $S := \{x \in X : f(x) \le 0,\, x \in D\}$.

Definition 1.1. *The solution set S is said to have a linear error bound if there exists $\tau > 0$ such that*

$$\mathrm{d}(x, S) \le \tau[f(x)]_+ \quad \forall\, x \in D$$

where $[f(x)]_+ := \max\{f(x), 0\}$

Hoffman and many other authors have shown that a linear error bound holds for a system of linear inequalities and equalities. Putting into the frame work of the problem (1), the solution set S always has a linear error bound if g is defined as the maximum of a finite set of affine functions, for instance, $f(x) := \max\{Ax - b, Bx - d, -Bx + d\}$ with $A : X \to \mathbb{R}^m$ and $B : X \to \mathbb{R}^l$ continuous linear operators. Deng [9] showed that for a convex function g

that satisfies the Slater's condition, namely there exists $x \in D$ such that $f(x) < 0$, a linear error bound exists. For a lower semicontinuous function, an important sufficient condition for S to have a linear error bound is that the subdifferential of f at every point outside of S is bounded away from 0 by a constant (See [23], [24] and [2]). K.F. Ng and X.Y. Zheng [15] studied the conditions for existence of linear error bounds using Dini derivatives. The Dini derivatives are also used in the conditions in [24]. In fact if we define $g(x) := \max\{f(x), 0\}$ the existence of a linear error bound for (1) is equivalent to all minimizers of the function g forming a set of weak sharp minima [6]. Intuitively, a linear error bound exists if the function f ascends at a linear rate when x moves away from S.

Of course a linear error bound will not hold for many functions such as $f(x) := \|x\|^2$. Z.Q. Luo and J.S. Pang [14] have studied nonlinear error bounds for analytic systems and quadratic systems. In this paper, we study the sufficient conditions for the existence of general nonlinear error bounds using a general notion of subdifferential that satisfies a set of most basic properties, these conditions are met by many commonly subdifferentials. This practice has been quite common in literatures. In our studies, we also make a use of an abstract directional derivative. It is shown that when the function g is convex, our sufficient conditions become necessary as well. The discussions on the nonlinear weak sharp minima are presented in the later part of the paper. As in many existing literatures on error bounds, the crucial instrument for establishing the sufficient conditions for error bounds is the Ekeland's Variational Principle.

We introduce the following definition.

Definition 1.2. *For a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\psi(0) = 0$, the problem (1) is said to have a ψ-error bound if*

$$\psi(\,\mathrm{d}(x, S)) \leq [f(x)]_+ \quad \forall\, x \in D. \tag{2}$$

When (2) holds for $\psi(t) = t^\kappa$, we say the system (1) has an error bound of order κ.

In the sequel, X is a Banach space with its norm $\|\cdot\|$ unless specified otherwise. The topological dual space of X is denoted by X^* with its norm $\|\cdot\|_*$. For a point $\bar{x} \in X$ we use $\mathbb{B}(\bar{x}, \varepsilon)$ to denote the closed ε-ball centered at $\bar{x}$. If f is a function defined on X, $\mathrm{dom} f := \{x \in X : f(x) < \infty\}$ is the domain of f. $\mathbb{R}$ is the set of all real numbers and $\mathbb{R}_+ := \{\mu \in \mathbb{R} : \mu \geq 0\}$. For a point $x \in X$ and a subset $D \subset X$, the distance from x to D is $\mathrm{d}(x, D) := \inf\{\|x' - x\| : x' \in D\}$, we adopt the convention that $\mathrm{d}(x, \emptyset) = \infty$ and $D + \emptyset = \emptyset$. Given two subsets C and D of X, $C \backslash D := \{x \in C : x \notin D\}$.

The paper is organized as follows. In Section 2, we derive our main theorem and obtain sufficient conditions for nonlinear error bounds of lower semicontinuous functional inequality systems using abstract subdifferentials and directional derivatives. In Section 3 we establish necessary conditions for convex functional inequality systems. In Section 4 we study the relationship of the calmness properties of the inverse subdifferential mappings and the error bounds on the set of subdifferential stationary set $U = \{x \in X : 0 \in \partial f(x)\}$. We will present the definitions for a general notion of nonlinear weak sharp minima of a lower semicontinuous function and discuss the sufficient conditions in the Section 5. In the last section we provide some concluding remarks.

2 Nonlinear Error Bounds

Consider a proper lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ that is bounded from below.

We use $\partial f : X \rightrightarrows X^*$ to denote any subdifferential of f defined on a Banach space that satisfies the following set of assumptions:

(a) If x is a local minimizer of f, then $0 \in \partial f(x)$;
(b) For any $x \in (\mathrm{dom} f \cap \mathrm{dom} g)$ and a convex function $g : X \to \overline{\mathbb{R}}$, one has

$$\partial(f+g)(x) \subset \partial f(x) + \partial g(x);$$

(c) If f is convex and finite near a point $x \in \mathrm{dom} f$, then ∂f coincides with the subdifferential of f in the sense of convex analysis.

Note that the Clarke subdifferential [7], Michel-Penot subdifferential and limiting subdifferential [4], Mordukhovich subdifferential [13] and of course the subdifferential in convex analysis [21] all satisfy these conditions in the appropriate context.

The following theorem was inspired by the results of O. Cornejo, A. Jourani and C. Zălinescu [8] and Penot [18].

Theorem 2.1. *Let C be a closed nonempty subset of X and $\bar{x} \in C$. If there exists $\varepsilon > 0$ and a function $\phi : \mathbb{R} \to \mathbb{R}$ nondecreasing on $[0, \varepsilon]$ such that*

$$\phi(\mathrm{d}(x, C)) \le \|v\|_* \quad \forall v \in \partial f(x) \quad \forall x \in \mathbb{B}(\bar{x}, 2\varepsilon), \tag{3}$$

then for any function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\psi(0) = 0$ and for each $t \in (0, \varepsilon]$ there exists $0 < \lambda \le t$ such that

$$\psi(t) \le \lambda \phi(t - \lambda), \tag{4}$$

the following inequality holds

$$\psi(\mathrm{d}(x, C)) \le f(x) - \inf f, \quad \forall x \in \mathbb{B}(\bar{x}, \varepsilon). \tag{5}$$

Proof. If (5) were not true, then there exist $x_0 \in \mathbb{B}(\bar{x}, \varepsilon)$ with $x_0 \notin C$ and $c_0 < 1$ such that

$$f(x_0) < \inf f + c_0 \psi(\mathrm{d}(x_0, C)).$$

Choose λ such that $0 < \lambda \le \mathrm{d}(x_0, C)$ and (4) holds for $t := \mathrm{d}(x_0, C)$. Observe that $\mathrm{d}(x_0, C) \le \|x_0 - \bar{x}\| \le \varepsilon$. By Ekeland's Variational Principle [10], there exists $x' \in X$ such that

$$\|x' - x_0\| \le \lambda; \tag{6}$$

$$f(x') \le f(y) + (c_0/\lambda)\psi(\mathrm{d}(x_0, C))\|x' - y\|, \forall y \in X. \tag{7}$$

It follows that there exists $v^* \in \partial f(x')$ such that $\|v^*\|_* \le (c_0/\lambda)\psi(\mathrm{d}(x_0, C))$.

On the other hand, we have

$$\mathrm{d}(x', C) \ge \mathrm{d}(x_0, C) - \|x' - x_0\| \ge \mathrm{d}(x_0, C) - \lambda \ge 0,$$

and

$$\|x' - \bar{x}\| \le \|x' - x_0\| + \|x_0 - \bar{x}\| \le \lambda + \varepsilon \le 2\varepsilon$$

so $x' \in \mathbb{B}(\bar{x}, 2\varepsilon)$. Consequently from (3), (4) and the nondecreasing property of ϕ, we have

$$\begin{aligned} \|v^*\|_* &\ge \phi(\mathrm{d}(x', C)) \ge \phi(\mathrm{d}(x_0, C) - \lambda) \ge (1/\lambda)\psi(\mathrm{d}(x_0, C)) \\ &> (c_0/\lambda)\psi(\mathrm{d}(x_0, C)) \ge \|v^*\|_*. \end{aligned} \tag{8}$$

The contradiction indicates that (5) must be true. □

Remarks.

(a) The set C can be an arbitrary set. It does not have to be the set of all minimizers S of f. For instance, we can let C be the set of subdifferential stationary points of f or a sublevel set of f.

(b) The spirit of the above theorem is that a growth function ϕ on the map $x \mapsto \mathrm{d}(0, \partial f(x))$, which maybe easier to obtain, can be used to define a growth function on the function f itself.

(c) To achieve stronger result in the inequality (5), one would choose ψ as large as possible. Here are some examples for the ϕ-ψ function pair that satisfy the conditions in Theorem 2.1: $\phi(t) = at^{\kappa}$, $\psi(t) = \sigma t^{\kappa+1}$ where $\sigma = a\kappa^{\kappa}/(1+\kappa)^{1+\kappa}$, $a > 0$ and $\kappa > 0$; $\phi(t) \equiv a$, $\psi(t) = at$ or in general one may choose

$$\psi(t) := \sup_{0 \le \gamma \le 1} (1-\gamma)t\phi(\gamma t).$$

(d) If in addition to the assumptions in Theorem 2.1 2.1, ϕ is continuous near 0, then the function ψ can be chosen to be

$$\psi(t) := \max_{0 \le \lambda \le t} \lambda\phi(t-\lambda)$$

(e) The results in the above theorem are presented as a local property around the point $\bar{x}$. Similar results can be obtained if $\mathbb{B}(\bar{x}, \varepsilon)$ is replaced by $C + \varepsilon\mathbb{B}$ (See [8, 18]).

(e) The function satisfying (5) is said to be (locally) ψ-conditioned around $\bar{x}$. See [8] and the references therein.

(f) The connections of this theorem to the conditioning of functions and the asymptotical behaviors of f can be found in [18].

One advantage of the above Theorem is that C can be chosen to represent various subsets defined by different problems. We now apply different choices of the function ψ and the subset C to derive various error bound results. First we consider a proper lower semicontinuous function $g : X \to \overline{\mathbb{R}}$ and let S be the solution set of the inequality $g(x) \le 0$, namely

$$S := \{x \in X : g(x) \le 0\}. \tag{9}$$

The following theorem gives a sufficient condition on the ψ-error bound of the system (9).

Theorem 2.2. *If there exists $\varepsilon > 0$ and a function $\phi : \mathbb{R} \to \mathbb{R}$ nondecreasing on $[0, \varepsilon]$ such that for each $x \in \mathbb{B}(\bar{x}, 2\varepsilon)$ with $x \notin S$,*

$$\phi(\mathrm{d}(x, S)) \le \|v\|_* \quad \forall\, v \in \partial g(x), \tag{10}$$

then for each $x \in \mathbb{B}(\bar{x}, \varepsilon)$,

$$\psi(\mathrm{d}(x, S)) \le [g(x)]_+ \tag{11}$$

where $[g(x)]_+ := \max\{g(x), 0\}$ and

$$\psi(t) := \sup_{0<\gamma<1} (1-\gamma)t\phi(\gamma t).$$

Proof. Choose the subset $C := S$ and $f(x) := [g(x)]_+$ in the Theorem 2.1. Set $\lambda := (1-\gamma)t$ and $\psi_\gamma(t) := (1-\gamma)t\phi(\gamma t)$ in (4), then by Theorem 2.1 we have for each $x \in \mathbb{B}(\bar{x}, \varepsilon)$,

$$\psi_\gamma(\mathrm{d}(x,S)) \le [g(x)]_+ \quad \forall\, \gamma \in (0,1). \tag{12}$$

Take a supremum on (12) over $\gamma \in (0,1)$ we obtain (11). □

Now if, in Theorem 2.1, take $C := S$, $f(x) := [g(x)]_+$ and $\phi(t) \equiv \mu$ for a constant μ, we obtain a result similar to a sufficient condition by Z. Wu and J.J. Ye [23] (See also [2]): if the subdifferential of g near $\bar{x}$ is bounded away from 0 by a constant, then a linear error bound exists for the system (9). Specifically, we have

Corollary 2.3. *Assume $\bar{x} \in S$. If there exist $\varepsilon > 0$ and a constant $\mu > 0$ such that*

$$\|v\|_* \ge \mu \quad \forall\, v \in \partial g(x) \quad \forall\, x \in \mathbb{B}(\bar{x}, 2\varepsilon)\backslash S,$$

then the system (9) has a linear error bound:

$$\mathrm{d}(x,S) \le \mu^{-1}[g(x)]_+ \quad \forall\, x \in \mathbb{B}(\bar{x}, \varepsilon). \tag{13}$$

Proof. In Theorem 2.2, let $\phi(t) \equiv \mu$ for $t > 0$ and $\phi(0) = 0$. It is clear that $\psi(t) := t\phi(0) = \mu t$ and (13) follows immediately. □

The following corollary gives a result for exponential error bound for the system (9). Such error bounds have also been studied by Z. Wu and J. J. Ye [24]. Our conditions here are different from theirs.

Corollary 2.4. *Assume $\bar{x} \in S$. If there exist $\varepsilon > 0$, $\mu > 0$ and $\kappa > 0$ such that*

$$\mu[\mathrm{d}(x,S)]^\kappa \le \|v\|_* \quad \forall\, v \in \partial g(x), \forall\, x \in \mathbb{B}(\bar{x}, 2\varepsilon), \tag{14}$$

then inequality system (9) has an error bound of order $\kappa + 1$:

$$\sigma[\mathrm{d}(x,S)]^{\kappa+1} \le [g(x)]_+ \quad \forall\, x \in \mathbb{B}(\bar{x}, \varepsilon) \tag{15}$$

where $\sigma = \mu\kappa^\kappa/(1+\kappa)^{\kappa+1}$.

Proof. In Theorem 2.2, let $\phi(t) := \mu t^\kappa$. Then

$$\psi(t) = \sup_{0<\gamma<1} \mu(1-\gamma)\gamma^\kappa t^{\kappa+1} = \sigma t^{\kappa+1}.$$

The rest follows from Theorem 2.2. □

Analogous to our approach with the subdifferentials, we consider all directional derivatives with certain set of properties. A directional derivative of f at $x \in X$ in the direction of $w \in X$, denoted by $\mathrm{d}f(x; w)$ is a function on $X \times X$ satisfying the following set of conditions:

(a) If x is a local minimizer of f, then $\mathrm{d}f(x; w) \ge 0$ for all $w \in X$;
(b) For a convex function g, one has $\mathrm{d}(f+g)(x; w) \le \mathrm{d}f(x; w) + \mathrm{d}g(x; w)$;
(c) If $h(x) := \|x - x_0\|$ for some $x_0 \in X$, then $\mathrm{d}h(x_0; w) \le \|w\|$;
(d) For a scalar $\lambda \ge 0$, $\mathrm{d}(\lambda f) = \lambda\, \mathrm{d}f$.
(e) $\mathrm{d}f(x; 0) = 0$.

Many well known directional derivatives do possess these basic assumptions.

The following theorem provides a set of sufficient conditions for error bounds using the directional derivatives, it is basically a variant of Theorem 2.1.

Theorem 2.5. *Let C be a nonempty subset of X and $\bar{x} \in C$. If for a given $\varepsilon > 0$, the function $\phi : \mathbb{R} \to \mathbb{R}$ is nondecreasing on $[0, 2\varepsilon]$ and for each $x \in \mathbb{B}(\bar{x}, 2\varepsilon)$ there exists at least one direction $w_0 \in X$ such that $w_0 \neq 0$ and*

$$\mathrm{d}f(x; w_0) \leq -\|w_0\|\phi(\mathrm{d}(x, C)), \tag{16}$$

then for any function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\psi(0) = 0$ and for each $t \in (0, \varepsilon]$ there exists $\lambda \leq t$ such that

$$\psi(t) \leq \lambda\phi(t - \lambda), \tag{17}$$

one has

$$\psi(\mathrm{d}(x, C)) \leq f(x) - \inf f \quad \forall\, x \in \mathbb{B}(\bar{x}, \varepsilon). \tag{18}$$

Proof. The proof is analogous to that of Theorem 2.1. Suppose (18) were not true. From the proof of Theorem 2.1, we have (7). It follows that for all $w \in X$,

$$0 \leq \mathrm{d}f(x'; w) + (c_0/\lambda)\psi(\mathrm{d}(x_0, C))\|w\|.$$

Combining this with (8), we have

$$0 \leq \mathrm{d}f(x'; w) + c_0\phi(\mathrm{d}(x_0, C))\|w\| < \mathrm{d}f(x'; w) + \phi(\mathrm{d}(x_0, C))\|w\|, \tag{19}$$

contradicting with (16). Therefore (16) must be true. □

Letting $C := S$, $\phi(x) \equiv \mu$, a constant function, and $f(x) := \max\{g(x), 0\}$ we obtain a result similar to that of Theorem 4 of [24].

Corollary 2.6. *Assume $\bar{x} \in S$. Suppose $\varepsilon > 0$ and $\mu > 0$ are constants such that there exist at least one direction $w_0 \in X$ for each $x \in \mathbb{B}(\bar{x}, 2\varepsilon)$ satisfying*

$$\mathrm{d}f(x; w_0) \leq -\mu\|w_0\|,$$

then the system (9) has a linear error bound:

$$\mathrm{d}(x, S) \leq \mu^{-1}[g(x)]_+ \quad \forall\, x \in \mathbb{B}(\bar{x}, \varepsilon). \tag{20}$$

Here we also state a corollary dealing with the exponential error bounds analogous to Corollary 2.4 using the directional derivatives.

Corollary 2.7. *Assume $\bar{x} \in S$. Suppose $\varepsilon > 0$, $\mu > 0$ and $\kappa > 0$ are constants such that there exist at least one direction $w_0 \in X$ for each $x \in \mathbb{B}(\bar{x}, 2\varepsilon)$ satisfying*

$$\mathrm{d}f(x; w_0) \leq -\mu\|w_0\|[\mathrm{d}(x, S)]^\kappa. \tag{21}$$

Then the system (9) has an error bound of order $\kappa + 1$:

$$\sigma[\mathrm{d}(x, S)]^{\kappa+1} \leq [g(x)]_+ \quad \forall\, x \in \mathbb{B}(\bar{x}, \varepsilon) \tag{22}$$

where $\sigma = \mu\kappa^\kappa/(\kappa + 1)^{\kappa+1}$.

Proof. In Theorem 2.5, let $C := S$, $\phi(x) = \mu t^\kappa$ and $f(x) := \max\{g(x), 0\}$. □

3 Error Bounds for Convex Inequalities

In the previous sections, we have established a series of sufficient conditions for nonlinear error bounds. We now consider a convex function $f : X \to \overline{\mathbb{R}}$ and discuss the necessary conditions for nonlinear error bounds of inequalities involving convex functions.

In the following theorem, $f : X \to \overline{\mathbb{R}}$ is a proper lower semicontinuous convex function that is bounded from below. For convex functions, our abstract subdifferential ∂f reverts to the usual partial differential in the convex analysis.

Let E be the set of all minimizers of f over X. To avoid triviality, we assume $E \neq \emptyset$.

Theorem 3.1. *Assume $\bar{x} \in E$. If there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\psi(0) = 0$ such that*

$$\psi(\,\mathrm{d}(x, E)) \leq f(x) - \inf f \quad \forall\, x \in \mathbb{B}(\bar{x}, \varepsilon), \tag{23}$$

then

$$\phi(\,\mathrm{d}(x, E)) \leq \|v^*\|_* \quad \forall\, v^* \in \partial f(x) \quad \forall\, x \in \mathbb{B}(\bar{x}, \varepsilon) \tag{24}$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function satisfying $\phi(0) = 0$ and $\phi(t) \leq \psi(t)/t$ for all $t \in (0, \varepsilon)$.

Proof. For any $x \in \mathbb{B}(\bar{x}, \varepsilon)$ with $x \notin E$ and $v^* \in \partial f(x)$, one has by the convex analysis that

$$\langle x' - x, v^* \rangle \leq f(x') - f(x) \quad \forall\, x' \in X.$$

Choose $x' \in E$, we have

$$\psi(\,\mathrm{d}(x, E)) \leq f(x) - f(x') \leq \langle x - x', v^* \rangle \leq \|x - x'\| \|v^*\|_*.$$

Take an infimum of $x' \in E$, we get

$$\psi(\,\mathrm{d}(x, E)) \leq \|v^*\|_*\, \mathrm{d}(x, E)$$

with $\mathrm{d}(x, E) \leq \|x - \bar{x}\| \leq \varepsilon$. It follows that

$$\phi(\,\mathrm{d}(x, E)) \leq \psi(\,\mathrm{d}(x, E))/\,\mathrm{d}(x, E) \leq \|v^*\|_*$$

We conclude the proof. □

In the rest of this section, $g : X \to \overline{\mathbb{R}}$ is a proper lower semicontinuous and convex function bounded from below. As before, let $S := \{x \in X : g(x) \leq 0\}$ and assume $S \neq \emptyset$.

Corollary 3.2. *Let $\bar{x} \in S$. If there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\psi(0) = 0$ such that*

$$\psi(\,\mathrm{d}(x, S)) \leq [g(x)]_+ \quad \forall\, x \in \mathbb{B}(\bar{x}, \varepsilon), \tag{25}$$

then

$$\phi(\,\mathrm{d}(x, S)) \leq \|v^*\|_* \quad \forall\, v^* \in \partial g(x), \forall\, x \in \mathbb{B}(\bar{x}, \varepsilon) \backslash S \tag{26}$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function satisfying $\phi(0) = 0$ and $\phi(t) \leq \psi(t)/t$ for all $t \in (0, \varepsilon)$.

Proof. In Theorem 3.1, define $f(x) := \max\{g(x), 0\}$. It suffices to notice that for $x \in S$, (26) is trivial and for $x \notin S$, $\partial f(x) = \partial g(x)$. □

Corollary 3.3. *Let $\bar{x} \in S$. If there exist $\kappa > 0$ and $\mu > 0$ such that*

$$\mu[\mathrm{d}(x,S)]^{\kappa} \leq [g(x)]_{+} \quad \forall\, x \in \mathbb{B}(\bar{x},\varepsilon), \tag{27}$$

then

$$\mu[\mathrm{d}(x,S)]^{\kappa-1} \leq \|v^*\|_* \quad \forall\, v^* \in \partial g(x),\, \forall\, x \in \mathbb{B}(\bar{x},\varepsilon)\backslash S. \tag{28}$$

Proof. Let $\psi(t) := \mu t^{\kappa}$. □

In the following corollary, we establish the necessary and sufficient condition for linear error bounds of the functional system (9): The solution set S of the system $g(x) \leq 0$ has a linear error bound if and only if the subdifferential of g is bounded away from 0 by a constant near S. The corollary deals with the local error bounds, but it becomes a global one when $\varepsilon = \infty$.

Corollary 3.4. *Let $\bar{x} \in S$. There exist $\kappa > 0$ and $\mu > 0$ such that*

$$\mathrm{d}(x,S) \leq \mu^{-1}[g(x)]_{+} \quad \forall\, x \in \mathbb{B}(\bar{x},\varepsilon), \tag{29}$$

if and only if

$$\mu \leq \|v^*\|_* \quad \forall\, v^* \in \partial g(x),\, \forall\, x \in \mathbb{B}(\bar{x},\varepsilon)\backslash S. \tag{30}$$

Proof. Set $\kappa := 1$ in the previous corollary and combine it with the corollary 2.3. □

4 Error Bounds for Subdifferential Stationary Sets

Let $f : X \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function that is bounded from below. We consider the system

$$U := \{x \in X : 0 \in \partial f(x)\} \tag{31}$$

where ∂f is the abstract subdifferential of f as defined in the second section. We use $(\partial f)^{-1}(v^*)$ to represent the inverse mapping of the subdifferential of f at $v^* \in X^*$: $(\partial f)^{-1}(v^*) := \{x \in X : v^* \in \partial f(x)\}$. The set that we are particularly interested in is the set of all subdifferential stationary points $(\partial f)^{-1}(0)$, which is equal to the set U defined in (31). Before stating our main results, we pause to make some preliminary definitions.

Let F be a set-valued map from a Banach space Y to a Banach space X. We use the following conventions: the domain of F is $\mathrm{dom}F := \{v \in Y : F(v) \neq \emptyset\}$; the graph of F is $\mathrm{gph}F := \{(v,x) \in X \times Y : x \in F(v), v \in \mathrm{dom}F\}$; the closed ball at $x \in X$ with radius ε is $\mathbb{B}(x,\varepsilon)$ and the unit ball at 0 is simply $\mathbb{B}$.

Given a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\phi(0) = 0$. We introduce the following ϕ-calmness definition for a set-valued map.

Definition 4.1. *Let $\bar{x} \in F(\bar{v})$ with $\bar{v} \in \mathrm{dom}F$. We say F is ϕ-calm at $(\bar{v},\bar{x})$ if there exist $\varepsilon > 0$ and $\delta > 0$ such that*

$$\phi(\mathrm{d}(x,F(\bar{v}))) \leq \|v-\bar{v}\|_* \quad \forall v \in \mathbb{B}(\bar{v},\delta) \cap \mathrm{dom}F,\, \forall\, x \in F(v) \cap \mathbb{B}(\bar{x},\varepsilon). \tag{32}$$

Choose $\phi(t) = t^{\kappa}$, we introduce the following calmness condition for set-valued maps of order κ.

Definition 4.2. *Let $\bar{x} \in F(\bar{v})$ with $\bar{v} \in \operatorname{dom} F$ and $\kappa > 0$. We say F is calm of order κ at $(\bar{v}, \bar{x})$ if there exist $\varepsilon > 0$, $\delta > 0$ and $\tau > 0$ such that*

$$[\,\mathrm{d}(x, F(\bar{v}))]^{\kappa} \leq \tau \|v - \bar{v}\|_* \quad \forall\, v \in \mathbb{B}(\bar{v}, \delta) \cap \operatorname{dom} F \,\forall\, x \in F(v) \cap \mathbb{B}(\bar{x}, \varepsilon), \tag{33}$$

or equivalently

$$F(v) \cap \mathbb{B}(\bar{x}, \varepsilon) \subset F(\bar{v}) + \tau \|v - \bar{v}\|_*^{1/\kappa} \mathbb{B} \quad \forall v \in \mathbb{B}(\bar{v}, \delta) \cap \operatorname{dom} F. \tag{34}$$

When $\kappa = 1$ and for a finite dimensional space, the above definition coincide with the calmness for set-valued maps given in R.T. Rockafellar and J.B. Wets [20]. We will restate the definition here for the convenience of discussions.

Definition 4.3. *Let $\bar{x} \in F(\bar{v})$ with $\bar{v} \in \operatorname{dom} F$. We say F is calm at $(\bar{v}, \bar{x})$ if there exist $\varepsilon > 0$, $\delta > 0$ and $\tau > 0$ such that*

$$\mathrm{d}(x, F(\bar{v})) \leq \tau \|v - \bar{v}\|_* \quad \forall\, v \in \mathbb{B}(\bar{v}, \delta) \cap \operatorname{dom} F \quad \forall\, x \in F(v) \cap \mathbb{B}(\bar{x}, \varepsilon), \tag{35}$$

or equivalently

$$F(v) \cap \mathbb{B}(\bar{x}, \varepsilon) \subset F(\bar{v}) + \tau \|v - \bar{v}\|_* \mathbb{B} \quad \forall v \in \mathbb{B}(\bar{v}, \delta) \cap \operatorname{dom} F. \tag{36}$$

The property that F is calm at $\bar{v}$ to $\bar{x}$ is also termed as pseudo-upper Lipschitz continuity of F at $(\bar{v}, \bar{x})$ by J.J. Ye [25] and locally upper Lipschitz continuity of F at $(\bar{v}, \bar{x})$ by D. Klatte [12].

The following theorem gives a sufficient condition for the existence of a ψ-error bound for the system (31).

Theorem 4.4. *Suppose $U = (\partial f)^{-1}(0) \neq \emptyset$, $\bar{x} \in U$ and ϕ is nondecreasing. If the inverse mapping $(\partial f)^{-1}$ is ϕ-calm at $(0, \bar{x})$, namely there exist $\varepsilon > 0$ and $\delta > 0$ such that*

$$\phi(\,\mathrm{d}(x, U)) \leq \|v\|_* \tag{37}$$

for all v and x satisfying

$$\|v\|_* \leq \delta,\ \|x - \bar{x}\| \leq 2\varepsilon,\ v \in \partial f(x) \neq \emptyset.$$

Then the system (31) has a ψ-error bound property

$$\psi(\,\mathrm{d}(x, U)) \leq f(x) - \inf f \quad \forall\, x \in \mathbb{B}(\bar{x}, \varepsilon) \tag{38}$$

where ψ is a function satisfying $\psi(0) = 0$ and for each $t \in (0, \varepsilon]$ there exists $\lambda \leq t$ such that $\psi(t) \leq \lambda \phi(t - \lambda)$.

Proof. Set $C := (\partial f)^{-1}(0)$ in Theorem 2.1 □

Corollary 4.5. *Suppose $U = (\partial f)^{-1}(0) \neq \emptyset$, $\bar{x} \in U$ and $\kappa > 0$. If the inverse mapping $(\partial f)^{-1}$ is calm of order κ at $(0, \bar{x})$, namely there exist $\varepsilon > 0$, $\delta > 0$ and $\tau > 0$ such that*

$$[\,\mathrm{d}(x, U)]^{\kappa} \leq \tau \|v\|_* \tag{39}$$

for all v and x satisfying

$$\|v\|_* \leq \delta,\ \|x - \bar{x}\| \leq 2\varepsilon,\ v \in \partial f(x) \neq \emptyset.$$

Then the system (31) has an error bound of order $\kappa + 1$

$$[\,\mathrm{d}(x, U)]^{\kappa+1} \leq \sigma[f(x) - \inf f] \quad \forall\, x \in \mathbb{B}(\bar{x}, \varepsilon) \tag{40}$$

where $\sigma = \tau(\kappa + 1)^{\kappa+1}/\kappa^{\kappa}$.

Proof. In Theorem 4.4, choose $\phi(t) := t^\kappa/\tau$, $\lambda := t/(\kappa+1)$ and

$$\psi(t) := \lambda\phi(t-\lambda) = \kappa^\kappa t^{\kappa+1}/(\tau(\kappa+1)^{\kappa+1}) = t^{\kappa+1}/\sigma.$$

We conclude the proof. □

Corollary 4.6. *Suppose $U = (\partial f)^{-1}(0) \neq \emptyset$ and $\bar{x} \in U$. If the inverse mapping $(\partial f)^{-1}$ is calm at $(0, \bar{x})$, namely there exist $\varepsilon > 0$, $\delta > 0$ and $\tau > 0$ such that*

$$\mathrm{d}(x, U) \leq \tau \|v\|_* \tag{41}$$

for all v and x satisfying

$$\|v\|_* \leq \delta,\ \|x - \bar{x}\| \leq 2\varepsilon,\ v \in \partial f(x) \neq \emptyset.$$

Then the system (31) has a quadratic error bound property

$$[\,\mathrm{d}(x, U)]^2 \leq 4\tau[f(x) - \inf f] \quad \forall\, x \in \mathbb{B}(\bar{x}, \varepsilon). \tag{42}$$

Proof. In Corollary 4.5, set $\kappa = 1$, we have $\sigma = 4\tau$. □

In the following corollary, we will show that if the set of subdifferential $\partial f(x)$ near U is bounded away from 0 by a constant, a linear error bound exists for the system (31).

Corollary 4.7. *Suppose $U = (\partial f)^{-1}(0) \neq \emptyset$, $\bar{x} \in U$ and there exist $\mu > 0$, $\varepsilon > 0$ and $\delta > 0$ such that*

$$\|v\|_* \geq \mu \tag{43}$$

for all v and x satisfying

$$\|v\|_* \leq \delta,\ \|x - \bar{x}\| \leq 2\varepsilon,\ v \in \partial f(x) \neq \emptyset,\ x \notin U.$$

Then the system (31) has a linear error bound

$$\mathrm{d}(x, U) \leq \mu^{-1}[f(x) - \inf f] \quad \forall\, x \in \mathbb{B}(\bar{x}, \varepsilon). \tag{44}$$

Proof. In Theorem 4.4, set $\phi(t) \equiv \mu$ for $t > 0$ and $\phi(0) = 0$ and choose $\psi(t) := \mu t$. □

5 Nonlinear Weak Sharp Minima

Consider a lower semicontinuous function f that is bounded from below. Let E be the set of all minimizers of f: $E := \{x \in X : f(x) \leq f(x')\ \forall\, x' \in X\}$. We assume $E \neq \emptyset$.

Given a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\psi(0) = 0$.

Definition 5.1. *Let $\bar{x} \in E$. The function f is said to have ψ-weak sharp minima around $\bar{x}$ if there exists $\varepsilon > 0$ such that*

$$f(x) - f(\bar{x}) \geq \psi(\,\mathrm{d}(x, E)) \quad \forall\, x \in \mathbb{B}(\bar{x}, \varepsilon).$$

We say E is the set of ψ-weak sharp minima of f if the above is satisfied with $\mathbb{B}(\bar{x}, \varepsilon)$ replaced by X.

For $\psi(t) = at^\kappa$ for some constants $a > 0$ and $\kappa > 0$, we have the following definitions of weak sharp minima of order κ.

Definition 5.2. *Let $\bar{x} \in E$. For a given constant $\kappa > 0$. The function f is said to have weak sharp minima of order κ around $\bar{x}$ if there exist $\varepsilon > 0$ and $a > 0$ such that*

$$f(x) - f(\bar{x}) \geq a[\,\mathrm{d}(x, E)]^{\kappa} \quad \forall\, x \in \mathbb{B}(\bar{x}, \varepsilon). \tag{45}$$

We say E is the set of weak sharp minima of f of order κ if the above is satisfied with $\mathbb{B}(\bar{x}, \varepsilon)$ replaced by X. The weak sharp minima of order one corresponds to the usual notion of weak sharp minima.

Notice that in the definitions above, κ needs not to be an integer.

Ever since the notions of weak sharp minima was first extensively explored by M.C. Ferris [11] and J. Burke and M.C. Ferris [5], there has been an explosive amount of interests among researchers on the subject. The property of weak sharp minima plays important roles in nonsmooth mathematical programming, the finite termination of the proximal point algorithms is just one of them. Nonlinear order weak sharp minima have been studied by J.-F. Bonnans and A.-D. Ioffe [3], M. Studniarski and Ward [22] and J. Burke and S. Deng [6].

Now we provide a sufficient condition for the existence of ψ-weak sharp minima for a lower semicontinuous function f defined on the Banach space X.

Theorem 5.3. *Let $\bar{x} \in E$. Suppose there exists $\varepsilon > 0$ and a function $\phi : \mathbb{R} \to \mathbb{R}$ nondecreasing on $[0, 2\varepsilon]$ such that*

$$\phi(\,\mathrm{d}(x, E)) \leq \|v\|_{*} \quad \forall\, v \in \partial f(x), \forall\, x \in \mathbb{B}(\bar{x}, 2\varepsilon). \tag{46}$$

Then for any function $\psi : \mathbb{R}_{+} \to \mathbb{R}_{+}$ satisfying $\psi(0) = 0$ and for each $t \in (0, \varepsilon]$ there exists $\lambda \leq t$ such that

$$\psi(t) \leq \lambda\phi(t - \lambda), \tag{47}$$

f has ψ-weak sharp minima around $\bar{x}$:

$$f(x) - f(\bar{x}) \geq \psi(\,\mathrm{d}(x, E)) \quad \forall\, x \in \mathbb{B}(\bar{x}, \varepsilon). \tag{48}$$

Proof. Set $C := E$ in Theorem 2.1. □

Corollary 5.4. *Let $\bar{x} \in E$. Suppose there exist $\varepsilon > 0$, $\mu > 0$ and $\kappa > 0$ such that*

$$\mu[\,\mathrm{d}(x, E)]^{\kappa} \leq \|v\|_{*} \quad \forall\, v \in \partial f(x), \forall\, x \in \mathbb{B}(\bar{x}, 2\varepsilon).$$

Then f has weak sharp minima of order $\kappa + 1$:

$$f(x) - f(\bar{x}) \geq \sigma[\,\mathrm{d}(x, E)]^{\kappa+1} \quad \forall\, x \in \mathbb{B}(\bar{x}, \varepsilon). \tag{49}$$

where $\sigma = \mu\kappa^{\kappa}/(\kappa + 1)^{\kappa+1}$.

Proof. In Theorem 5.3, select $\phi(t) := \mu t^{\kappa}$, $\lambda := t/(\kappa + 1)$ and

$$\psi(t) := \mu\kappa^{\kappa}/(\kappa + 1)^{\kappa+1} t^{\kappa+1}.$$

The conclusion is apparent. □

6 Conclusion

Our main focus is to study the existence of nonlinear error bounds for systems involving a lower semicontinuous function. We first established several theorems dealing with the error bounds for an arbitrary subset $C \subset X$ based on the properties of a function defined on X. Our results are mainly in terms of any subdifferential or directional derivative satisfying a set of basic assumptions. By applying these theorems to a functional system $g(x) \leq 0$ and the subdifferential stationary set $\{x \in X : 0 \in \partial f(x)\}$, we were able to derive various sufficient conditions for error bounds of these systems. We have also presented several necessary conditions for error bounds of a convex functional inequality system. We discovered that a number of conditions on error bounds existed in the literature are special cases of our results. Applying our main theorems to a set of all minimizers of a lower semicontinuous function we obtained some sufficient conditions for the existence of a general type of nonlinear weak sharp minima.

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